Harmonic Oscillator

What is a Harmonic Oscillator?

When a body oscillates about its location along a linear straight line under the influence of a force that is pointed towards the mean location, and is proportional to the displacement at any moment from this location, the motion of the body is considered to be simple harmonic, and the swinging body is known as a linear harmonic oscillator or simple harmonic oscillator. This form of oscillation is the best example of periodic motion.

A harmonic oscillator in classical physics is a body that is being exerted by a restoring force proportional to its displacement from its equilibrium location.

In the case of motion in one dimension,

$F = -kx$

Hooke's law is generally applied to real springs for small displacements; the restoring force is usually proportional to the displacement (compression or stretching) from the equilibrium position.

Simple Harmonic Oscillator

A simple harmonic oscillator is a type of oscillator that is either damped or driven. It generally consists of a mass' m', where a lone force 'F' pulls the mass in the trajectory of the point x = 0, and relies only on the position 'x' of the body and a constant k. The Balance of forces is,

$$F = ma$$

$$= m\frac{d^2x}{dt^2}$$

$$= m\ddot{x}$$

$$= -kx$$

$$\omega = \sqrt{\frac{k}{m}}$$

Solving the differential equation, the function that describes the motion is,

$(x(t)=A\cos(\omega t+\varphi))$

The motion is periodic, recurring itself in a sinusoidal fashion with fixed amplitude ‘A’. The movement of simple harmonic oscillators is characterised by its period

$T = 2\pi/\omega$, the time for one oscillation or its frequency

$f = 1/T$

, cycles per unit time. The location at a given time ‘t’ also relies on the phase φ, which decides the beginning point of the sine wave. The frequency and period are determined by the dimensions of the mass ‘m’ and the constant ‘k’. The phase and amplitude are determined by the velocity and starting position. The acceleration and velocity of a simple harmonic oscillator periodically change with the identical frequency as the position, with shifted phases. Interestingly, the velocity is highest for zero displacements, and the acceleration is in the opposite trajectory to the displacement. The potential energy in a simple harmonic oscillator at location ‘x’ is given by, $U = \frac{1}{2}kx^2$

Driven Oscillator Examples

If a damped oscillator is driven by an external force, the solution to the motion equation has two parts, a transient and a steady-state part, which must be used together to fit the physical boundary conditions of the problem.

Constant applied force

If a constant force is applied to a damped oscillator, it will stretch out to a final position determined by its spring constant. However, depending upon the initial conditions, it may oscillate about that final position and then settle down to the final position. This is the practical implication of the transient solution to the motion equations.

Resonant excitation

If a sinusoidal driving force is applied at the resonant frequency of the oscillator, then its motion will build up in amplitude to the point where it is limited by the damping forces on the system. If the damping forces are small, a resonant system can build up to amplitudes large enough to be destructive to the system. Such was the famous case of the Tacoma Narrows Bridge, which was blown down by the wind when it responded to a component in the wind force which excited one of its resonant frequencies.

Changing initial conditions

The effect of the transient solution on the behavior of a damped oscillator is strongly dependent upon the initial conditions. Although a driven system will ultimately settle down to a behavior determined by the driving force (steady-state solution), the early part of the motion can show a lot of variety.

Complex exciting forces

It is the nature of a resonant system to respond strongly to influences which have frequencies close to its resonant frequency. If a complex exciting force is applied, i.e., one which will have many frequency components, then the system will tend to pick out the components which are close to its resonant frequency. The example given is that of a square-wave driving force

Damped oscillator

All real oscillators undergo frictional forces. Frictional forces dissipate energy, transforming work into heat that is removed out of the system. As a consequence, the motion is damped, except if some external force supports it. If the damping is greater than a critical value, the system does not oscillate, but returns to the equilibrium position. The return velocity depends on the damping and we can find two different cases: over damping and critical damping. When the damping is lower than the critical value, the system realizes under damped motion, similar to the simple harmonic motion, but with an amplitude that decreases exponentially with time.

Motion Equation Notation

Motion equations for constant mass systems are based on Newton's 2nd Law, which can be expressed in terms of derivatives:

$$F_{net\ external} = ma \quad \Rightarrow \quad m\frac{d^2x}{dt^2} - F = 0$$

In many advanced mechanics texts, derivatives with respect to time are represented by a dot over the position variable which is being differentiated.

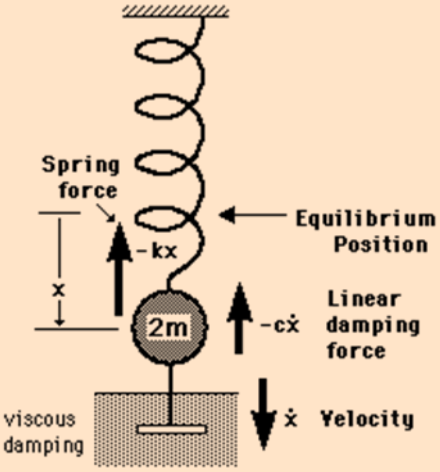
Acceleration $a = \frac{d^2x}{dt^2} = \ddot{x}$ Velocity $v = \frac{dx}{dt} = \dot{x}$

This makes it simpler to write equations where the forces are position or velocity dependent. For example, the damped oscillator has forces:

$F_{spring} = -kx$ $F_{damping} = -c\dot{x}$

and the motion equation can be written

$$m\ddot{x} + c\dot{x} + kx = 0$$



The motion equation of the form

$$m\ddot{x} + c\dot{x} + kx = 0$$

may be solved in the form $x = e^{qt}$

$$mq^2 + cq + k = 0$$

The roots of the auxiliary equation are

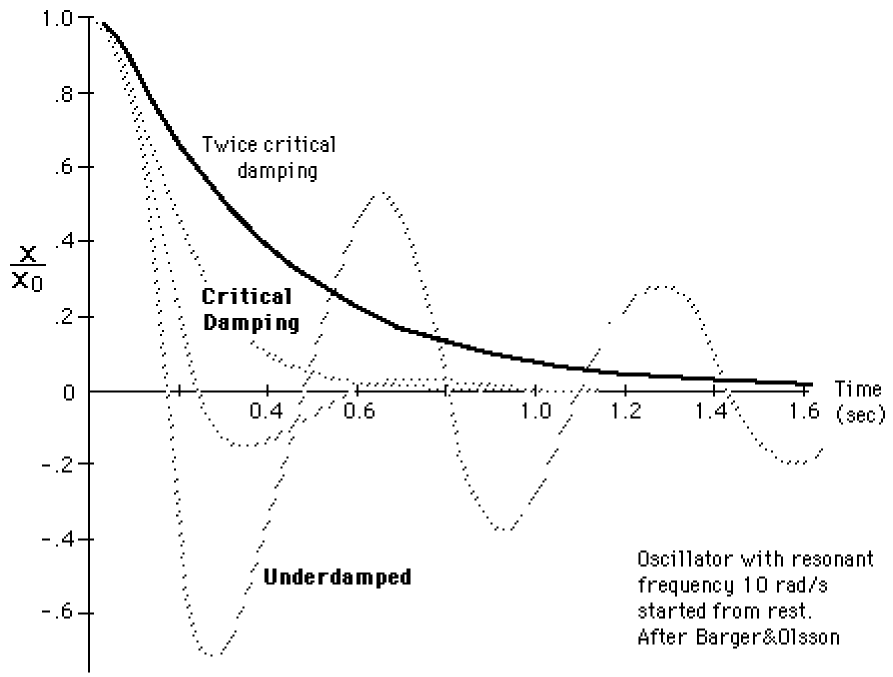
$$q = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

which give the three cases:

$c^2 - 4mk = 0$ **Critical damping**

$c^2 - 4mk < 0$ **Underdamped**

$c^2 - 4mk > 0$ **Overdamped**



Three- Dim Isotropic Harmonic oscillator

$m\ddot{x} = -kx$, $m\ddot{y} = -ky$, $m\ddot{z} = -kz$

$x = A_1 \cos\omega t + B_1 \sin\omega t$

$y = A_2 \cos\omega t + B_2 \sin\omega t$

$z = A_3 \cos\omega t + B_3 \sin\omega t$

The six constants of Integration are determined From the Intial position and velocity

$\therefore r = A\cos\omega t + B\sin\omega t$

So In the motion of Isotropic oscillation. the restoring force is Independent of direction of the displacement.

NoN Isotropic oscillator

If the magnitudes of components of the restoring force depend on the direction and displacement It's the case of non Isotropic case.

$m\ddot{x} = -k_1 x$, $m\ddot{y} = -k_2 y$,

$m\ddot{z} = -k_3 z$

So we have three different frequency

$w_1 = \sqrt{\frac{k_1}{m}}$, $w_2 = \sqrt{\frac{k_2}{m}}$ $w_3 = \sqrt{\frac{k_3}{m}}$

The equation of motion

$$x = A\cos(\omega_1 t + \alpha)$$
$$y = B\cos(\omega_2 t + \beta)$$
$$z = C\cos(\omega_3 t + \gamma)$$

{ The energy consideration }

$$V(x,y,z) = \frac{1}{2}K_1 x^2 + \frac{1}{2}K_2 y^2 + \frac{1}{2}K_3 z^2$$

$$F_x = -\frac{\partial u}{\partial x} = -K_1 x \qquad F_y = -\frac{\partial u}{\partial y} = -K_2 y ,$$

$$F_z = -\frac{\partial u}{\partial z} = -K_3 z$$

The General potential energy

$$V(x,y,z) = \frac{1}{2}K(x^2+y^2+z^2) = \frac{1}{2}Kr^2$$

Total energy $E = \frac{1}{2}mv^2 + \frac{1}{2}Kr^2$

Ex The particle of mass (m) moves in two dimension under the following potential energy function

$$V(r) = \frac{1}{2}K(x^2+4y^2)$$

Find the resulting motion. given the Intial condition at $t=0$: $x=a$, $y=0$, $\dot{x}=0$, $\dot{y}=V_0$.

Sol: This oscillator is Isotropic

The force $F = -\nabla u = -iKx - j4Ky$

$F = ma$, $F = m\ddot{r}$ $\quad\rightarrow\quad \frac{1}{2}(Kx^2+Ky^2)$

$\therefore m\ddot{r} = -Kx - j4Ky$

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The component of Differential equation of motion

$m\ddot{x} = -Kx \Rightarrow m\ddot{x} + Kx = 0$

$m\ddot{y} = -4Ky \Rightarrow m\ddot{y} + 4Ky = 0$

x-motion has angular frequency $\omega_x = \left(\frac{K}{m}\right)^{1/2}$

while the y motion has angular frequency

$$W_y = \left(\frac{4K}{m}\right)^{1/2} = 2\left(\frac{K}{m}\right)^{1/2} = 2W_x .$$

$W_y = 2W$

The equation of motion

$x = A_1 \cos\omega t + B_1 \sin\omega t$

$y = A_2 \cos 2\omega t + B_2 \sin 2\omega t$

$\dot{x} = -A_1 w \sin wt + B_1 w \cos wt$

$\dot{y} = -2A_2 w \sin 2wt + w 2B_2 \cos 2wt$

at $t=0$ $x=a$ $\therefore A_1 = a$, $A_2 = 0$

$B_1 w = 0$ $B_1 = 0$ $V_0 = 2B_2 W \Rightarrow B_2 = \frac{V_0}{2W}$

The final equation

$x = a \cos\omega t$

$y = \frac{V_0}{2W} \sin 2\omega t$

Find the force for each of the following potential energy functions. (4)

(a) $V = cxyz + C$

$F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$

$F_x = -cyz + 0$, $F_y = -cxz$, $F_z = -cxy$

$F = -cyz\,i - cxz\,j - cxy\,K$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

$f_x = -\frac{\partial V}{\partial x} = -2x\alpha + 0 + 0 + 0$

$f_y = -\frac{\partial V}{\partial y} = -0 - 2\beta y + 0 + 0$

$f_z = -\frac{\partial V}{\partial z} = -2\gamma z + 0$

$\therefore F = -2\alpha x i - 2\beta y j - 2\gamma z K$

(c) $V = c e^{-(\alpha x - \beta y - \gamma z)}$ H.W

Ex

By finding the curl. determine which of the following forces are conservative.

(a) $f = ix + jy + Kz$

$$\nabla \times f = \begin{vmatrix} i & j & K \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)i + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)j + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

(b) $F = iy + jx + kx^2$

(c) $F = iy + jx + kz^3$

Ex Find the value of constant (c) such that the force is conservative

$$F = ixy + jcx^2 + kz^3$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & cx^2 & z^3 \end{vmatrix} =$$

$$= \left(\frac{\partial z^3}{\partial y} - \frac{\partial cx^2}{\partial z}\right)i + \left(\frac{\partial xy}{\partial z} - \frac{\partial z^3}{\partial x}\right)j +$$

$$\left(\frac{\partial cx^2}{\partial x} - \frac{\partial xy}{\partial y}\right)k$$

$$= (0-0)i + (0-0)j + (2cx - x)k$$

$$0 = 2(cx - x)k$$

$$2cx = x \Rightarrow \boxed{c = \frac{1}{2}}$$