Linear algebra

Chapter -1-

System of linear equations and matrices

(1.1) System of linear equations

A **linear equation** in n unknowns is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots \dots + a_nx_n = b$$

Where $a_1, a_2, \dots \dots, a_n \; and \; b$ are real numbers and $x_1, x_2, \dots \dots, x_n$

Are variables.

Remark: linear equations have no products or roots of variables and no variables involved in trigonometric , exponential or logarithmic functions.

Variables appear only to the first power.

Ex.(1) The following equations are linear

(a) $6x_1 - 3x_2 + 4x_3 = -13$

(b) $-3x_1 + 5x_2 + x_3 = 6$

(c) 8x-7y-6z=0

(2) The following equations are non linear

(a) $x_1x_2 + sinx_3 = 0$

(b) $e^x - y = 3$

(c) ${x_1}^2 - lnx_2 = 5$

Remark: a **solution** of a linear equation in n variables

$a_1x_1 + a_2x_2 + \cdots \dots + a_nx_n = b$ is a sequences of n numbers

which has the property that satisfies equation when $x_1 = s_1, s_2, \dots \dots, s_n$
$s_1, x_2 = s_2, \dots \dots, x_n = s_n$ are substituted in the equation.

System of linear equations

A linear system of m equations in n unknowns is then a system of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n = b_2$$

$$. \qquad (1)$$

$$.$$

$$.$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots + a_{mn}x_n = b_m$$

Where a_{ij} and b_i are all real numbers

If $b_1 = b_2 = \cdots \ldots . = b_m = 0$

Then the system (1) is **homogeneous** system

We will refer to system of the form (1) as m×n linear system.

Ex.

(1) The system

$$x_1 + 2x_2 = 5$$

$$2x_1 + 3x_2 = 8$$

Is a 2x2 system

(2) The system

$$x_1 - x_2 + x_3 = 2$$

$$2x_1 + x_2 - x_3 = 4$$

Is a 2x3 system

(3) The system

$$x_1 + x_2 = 2$$

$$x_1 - x_2 = 1$$

$$x_1 \qquad = 4$$

Is a 3x2 system

Remark: a solution to an mxn system we mean an order n-tuple of numbers $(x_1, x_2 \ldots \ldots \ldots x_n)$that satisfies all the equations of the system.

It is possible for a system of linear equation to have exactly **one solution, an infinite number of solutions or no solution**.

Ex.

(1) prove that the order pair (1,2) is a solution to system

$$\boldsymbol{x_1 + 2x_2 = 5}$$

$$\boldsymbol{2x_1 + 3x_2 = 8}$$

Solution: 1+2.(2)=5

2.(1)+3.(2)=8

(2) Prove that the order triple (2,0,0) is a solution to system

$$\boldsymbol{x_1 - x_2 + x_3 = 2}$$

$$\boldsymbol{2x_1 + x_2 - x_3 = 4}$$

Solution: 2-1.(0)+1.(0)=2

2+1.(0)-1.(0)=4

(3) is the system

$$\boldsymbol{x_1 + x_2 = 2}$$

$$\boldsymbol{x_1 - x_2 = 1}$$

$$\boldsymbol{x_1 \qquad = 4}$$

Has a solution?

Solution: the system (3) has no solution.

(4) Is (1,1,1) the solution for the system

$$\boldsymbol{x_1 + x_2 - x_3 = 1}$$

$$\boldsymbol{2x_1 + x_2 = 2}$$

$$\boldsymbol{x_1 - x_2 + 2x_3 = 4}$$

Solution: (1,1,1) is not the solution for this system.

Remark:

1- The set of all solution to a linear system is called the **solution set** of the system

2- A system is **inconsistent** if its solution set is **empty**.

3- A system is **consistent** if its solution set is **nonempty** ,to solve a consistent system we must find its solution set.

4- $x_1 = x_2 \ldots \ldots \ldots = x_n = 0$ is always a solution to **homo. system** ; it is called the **trivial solution** .

5- A solution to a homo. system in which not all of $x_1, x_2, \ldots \ldots, x_n$ are **zero** is called a non **trivial solution**.

Def. we say that the two linear system equations are **equivalent** if the both have exactly **the same solution set** .

Ex. The linear system

$$2x_1 + 3x_2 - 4x_3 = 12$$

$$x_1 + 2x_3 = -1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$$

$$-3x_1 + 5x_2 + x_3 = 6$$

Has only solution (1,2,-1)

$x_1 = 1, x_2 = 2,\ x_3 = -1$

The linear system

$$8x_1 - 7x_2 - 6x_3 = 0$$

$$x_1 + 2x_3 = -1$$

..................................(3)$x_1 + 5x_2 + 9x_3 = 2$

$$3x_2 - 8x_3 = 14$$

Also has only solution $x_1 = 1, x_2 = 2,\ x_3 = -1$

Thus (2) and (3) are **equivalent**.

2x2 linear system

The general form of 2x2 linear system

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

Each equation can be represented graphically as a line in the plane.

The order pair (x_1, x_2) will be solution to the system iff it lies on both lines.

Ex.

(1)

$$x_1 + x_2 = 2$$

$$x_1 - x_2 = 2$$

The two lines in the system **intersect** at the point (2,0).

Thus (2,0) is the solution set of this system.

(2)

$$x_1 + x_2 = 2$$

$$x_1 + x_2 = 1$$

The two lines in the system **parallel** .Therefore this system inconsistent and hence its solution set is empty.

(3)

$$x_1 + x_2 = 2$$

$$-x_1 - x_2 = 2$$

The two equations in the system both represent the **same line**.

any point on this line will be a solution to the system.

Solution for system of linear equations

Def. a system is said to b in **strict triangular form** if in the kth equation the coefficients of the first k-1 variables are all zero and the coefficient of x_k is nonzero (k=1,…..,n)

Ex. Solve the following systems

(1)

$$\mathbf{3x_1 + 2x_2 + x_3 = 1}$$

$$\mathbf{x_2 - x_3 = 2}$$

$$\mathbf{2x_3 = 4}$$

Solution: using back substitution, we obtain:

$\therefore x_3 = 2$

$\therefore x_2 - 2 = 2 \rightarrow x_2 = 4$

Then

$3x_1 + 2.4 + 2 = 1 \rightarrow x_1 = -3$

The solution of the given system is (-3, 4, 2)

(2)

$$\mathbf{2x_1 - x_2 + 3x_3 - 2x_4 = 1}$$

$$\mathbf{x_2 - 2x_3 + 3x_4 = 2}$$

$$\mathbf{4x_3 + 3x_4 = 3}$$

$$\mathbf{4x_4 = 4}$$

Solution: using back substitution, we obtain:

$4x_4 = 4 \rightarrow x_4 = 1$

$4x_3 + 3.1 = 3 \rightarrow x_3 = 0$

$x_2 - 2.0 + 3.1 = 2 \rightarrow x_2 = -1$

$2x_1 - 1 + 3.0 - 2.1 = 1 \rightarrow x_1 = 1$

The solution is (1,0,-1,1)

(3)

$$\boldsymbol{x_1 + 2x_2 + x_3 = 3}$$

$$\boldsymbol{3x_1 - x_2 - 3x_3 = -1}$$

$$\boldsymbol{2x_1 + 3x_2 + x_3 = 4}$$

Solution:

$-3x_1 - 6x_2 - 3x_3 = -9$

$\underline{3x_1 - x_2 - 3x_3 = -1}$

$-7x_2 - 6x_3 = -10$

and

$-2x_1 - 4x_2 - 2x_3 = -6$

$\underline{2x_1 + 3x_2 + x_3 = 4}$

$-x_2 - x_3 = -2$

We obtain the equivalent system

$x_1 + 2x_2 + x_3 = 3$

$-7x_2 - 6x_3 = -10$

$-x_2 - x_3 = -2$

$-7x_2 - 6x_3 = -10$

$\underline{7x_2 + 7x_3 = 14}$

$x_3 = 4$

We obtain the equivalent system

$x_1 + 2x_2 + x_3 = 3$

$-7x_2 - 6x_3 = -10$

$\boldsymbol{x_3 = 4}$

Using back substitution, we get

$-7x_2 - 6(4) = -10 \rightarrow \boldsymbol{x_2 = -2}$

$x_1 + 2(-2) + 4 = 3 \rightarrow \boldsymbol{x_1 = 3}$

The solution is (3,-2,4)

(4)

$$2\boldsymbol{x_1} + \boldsymbol{x_2} = \boldsymbol{4}$$

$$\boldsymbol{4x_1 + 2x_2 = 6}$$

<u>Solution:</u>

$-4x_1 - 2x_2 = -8$

$\underline{4x_1 + 2x_2 = 6\quad}$

0=-2

This system has no solution

(5)

$$\boldsymbol{x_1 + 2x_2 - 3x_3 = -4}$$

$$\boldsymbol{2x_1 + x_2 - 3x_3 = 4}$$

<u>Solution:</u>

$-2x_1 - 4x_2 + 6x_3 = 8$

$\underline{2x_1 + x_2 - 3x_3 = 4\quad}$

$-3x_2 + 3x_3 = 12$

$\therefore x_2 - x_3 = -4$

We obtain ;

$x_1 = x_3 + 4$

$x_2 = x_3 - 4$

$x_3 = \text{any real number}$

This linear system has infinitely many solutions.

Exercises:

(1)Solve the following system of linear equations:

(1)

$$\mathbf{x_1 + 2x_2 = 8}$$

$$\mathbf{3x_1 - 4x_2 = 4}$$

(2)

$$\mathbf{x_1 + 4x_2 - x_3 = 12}$$

$$\mathbf{3x_1 + 8x_2 - 2x_3 = 4}$$

(3)

$$\mathbf{3x_1 + 2x_2 + x_3 = 2}$$

$$\mathbf{4x_1 + 2x_2 + 2x_3 = 8}$$

$$\mathbf{x_1 - x_2 + x_3 = 4}$$

(4)

$$\mathbf{x - 5y = 6}$$

$$\mathbf{3x + 2y = 1}$$

$$\mathbf{5x + 2y = 1}$$

(5)

$$\mathbf{2x + 3y - z = 6}$$

$$\mathbf{2x - y + 2z = -8}$$

$$\mathbf{3x - y + z = -7}$$

(6)

$$2x + y - 2z = -5$$

$$3y + z = 7$$

$$z = 4$$

(7)

$$4x = 8$$

$$-2x + 3y = -1$$

$$3x + 5y - 2z = 11$$

2) Given the system

$$2x + 3y - z = 0$$

$$x - 4y + 5z = 0$$

(a) Prove that $x_1 = 1, y_1 = -1, z_1 = -1$ *is a solution*
(b) Prove that $x_2 = -2, y_2 = 2, z_2 = 2$ *is a solution*

(c) *is* $x = x_1 + x_2 = -1, y = y_1 + y_2 = 1,$

$z = z_1 + z_2 = 1$ *a solution to the linear system*?,

(1.2) Matrices

Def. an m×n **matrix** A is a rectangular array of mn real or complex numbers arranged in m horizontal rows and n vertical columns.

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} \cdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} (1)$$

The **ith row** of A is $[\ a_{i1} \quad a_{i1} \ldots\ldots\ldots\ldots \quad a_{in}]$ $\quad (1 \le i \le m)$

The **jth column** of A is $\begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{mj} \end{bmatrix} \quad (1 \le j \le n)$

We shall say that A is **m by n** (written as mxn)

Remark

(1) If m=n we say that A is **square matrix of order n** and that numbers $a_{11}, a_{22}, \ldots \ldots a_{nn}$ form the **main diagonal** of A.

(2) We refer to the number a_{ij} which is in the i th row and j th column of A

(3) The i,j element of A or the (i,j) entry of A and we often write (1) as A=$[a_{ij}]$

(4) If all the numbers in matrix A are zero then it is called the zero matrix.

Ex. Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1+i & 4i \\ 2-3i & -3 \end{bmatrix}, C = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, D = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & -1 & 2 \end{bmatrix},$$

$$E = [3]\ , F = [-1 \quad 0 \quad 2]\ , G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$u = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} 4 - vectoer \ \ , v = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} 3 - vector$$

Def. Two m×n matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if $a_{ij} = b_{ij}$ for

$$i = 1,2,\ldots\ldots,m \text{ and } j = 1,2,\ldots\ldots,n$$

Ex. The matrices $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 4 \\ 0 & -4 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & w \\ 2 & x & 4 \\ y & -4 & z \end{bmatrix}$ are equal

if $w = -1, x = -3, y = 0, z = 5$

Matrices operations

1- Matrices addition

Def. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then the sum

$A + B$ is an $m \times n$ matrix $C = [c_{ij}]$ define by $c_{ij} = a_{ij} + b_{ij}$,

$$i = 1,2\ldots\ldots,m, j = 1,2,\ldots\ldots,n.$$

Thus to obtain the sum of A , and B ,we merely add corresponding entries.

Ex. Find A+B where

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{2} & \mathbf{-1} & \mathbf{4} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{-4} \end{bmatrix}$$

Solution:

$$A + B = \begin{bmatrix} 1+0 & -2+2 & 3+1 \\ 2+1 & -1+3 & 4-4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 3 & 2 & 0 \end{bmatrix}$$

Remark: we now make the convention that when A+B is written both A and B are of the same size.

Theorem 1.1 properties of matrix addition

Let A,B and C be m×n matrices

(a) A+B=B+A (commutative property)

(b) A+(B+C)=(A+B)+C (associative property)

(c) There is a unique m×n matrix O such that A+O=O+A=A

For any m×n matrix A.The matrix O is called the m×n **zero matrix** .

(d) For each m×n matrix A,there is a unique m×n matrix D such that

A+D =D+A=O we shall write D as (-A) so can be written as A+(-A)=O

The matrix **–A is called the negative of A**.we also note that –A is (-1)A

Ex. (1) If $A = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Then $A + O = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix}$

(2) If $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & 4 & 3 \end{bmatrix}, then - A = \begin{bmatrix} -1 & -3 & 2 \\ 2 & -4 & -3 \end{bmatrix}$

$$and\ A + (-A) = -A + A = O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2)Scalar multiplication

Def. If $\mathrm{A} = [\mathrm{a_{ij}}]$ is an $\mathrm{m} \times \mathrm{n}$ matrix and r is a real number , then

the **scalar multiple** of A by r , rA , is the $\mathrm{m} \times \mathrm{n}$ matrix $\mathrm{C} = [\mathrm{c_{ij}}]$

where $\mathrm{c_{ij}} = \mathrm{ra_{ij}}$, $\mathrm{i} = 1,2, \dots, \mathrm{m}$ and $\mathrm{j} == 1,2, \dots, \mathrm{n}$;

that is the matrix C is obtained by multipling each entry of A by r.

Ex. Find $-2\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix}$

Solution: $-2\begin{bmatrix} 4 & -2 & -3 \\ 7 & -3 & 2 \end{bmatrix} = \begin{bmatrix} (-2)(4) & (-2)(-2) & (-2)(-3) \\ (-2)(7) & (-2)(-3) & (-2)(2) \end{bmatrix} =$
$\begin{bmatrix} -8 & -4 & 6 \\ -14 & 6 & -4 \end{bmatrix}$

Remark: If A and B are $\mathrm{m} \times \mathrm{n}$ matrices, we write A+(1-)B as A-B and call this the **difference between A and B**.

Ex. Find A-B where $\begin{bmatrix} 2 & 3 & -5 \\ 4 & 2 & 1 \end{bmatrix}$ **and** $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 5 & -2 \end{bmatrix}$

Solution:

$$A - B = \begin{bmatrix} 2-2 & 3-(-1) & -5-3 \\ 4-3 & 2-5 & 1-(-2) \end{bmatrix} = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -3 & 3 \end{bmatrix}$$

Remark:

1- We shall sometimes use the **summation notation,** and we now review this useful and compact notation by $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots .. + a_n$

The letter i is called the **index of summation**; it is a dummy variable that can be replaced by another letter. Hence we can write $\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$

Ex. $\sum_{i=1}^{4} a_i = a_1 + a_2 + a_3 + a_4$

The summation notation satisfies the following properties:

$$1- \sum_{i=1}^{n} (r_i + s_i) a_i = \sum_{i=1}^{n} r_i a_i + \sum_{i=1}^{n} s_i a_i$$

$$2- \sum_{i=1}^{n} c(r_i a_i) = c \sum_{i=1}^{n} r_i a_i$$

$$3- \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} \right) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} \right)$$

2- If $A_1, A_2, \dots \dots, A_k$ $are\ m \times n\ matrices$

$and\ c_1, c_2, \dots .., c_k\ are\ real\ numbers, then\ an\ expression\ of\ the\ form$

$$c_1 A_1 + c_2 A_2 + \cdots .. + c_k A_k \qquad (*)$$

Is called a **linear combination** of $A_1, A_2, \dots \dots, A_k$ $and c_1, c_2, \dots .., c_k$

Are called **coefficients**.

The linear combination in (*) can also expressed

$$\sum_{i=1}^{k} c_i A_i = c_1 A_1 + c_2 A_2 + \cdots .. + c_k A_k$$

<u>Ex.</u> The following are linear combination of matrices

(1) $3\begin{bmatrix}0 & -3 & 5\\ 2 & 3 & 4\\ 1 & -2 & -3\end{bmatrix} - \frac{1}{2}\begin{bmatrix}5 & 2 & 3\\ 6 & 2 & 3\\ -1 & -2 & 3\end{bmatrix} = \begin{bmatrix}\frac{-5}{2} & -10 & \frac{27}{2}\\ 3 & 8 & \frac{21}{2}\\ \frac{7}{2} & -5 & \frac{-21}{2}\end{bmatrix}$

(2) $2[3 \quad -2] - 3[5 \quad 0] + 4[-2 \quad 5] = [-17 \quad 16]$

(3) $-0.5\begin{bmatrix}1\\ -4\\ -6\end{bmatrix} + 0.4\begin{bmatrix}0.1\\ -4\\ 0.2\end{bmatrix} = \begin{bmatrix}-0.46\\ 0.4\\ 3.08\end{bmatrix}$

<u>Def.</u> If A=$[a_{ij}]$ is an $m \times n$ matrix , then the **transpose** of A,

$A^T = [a_{ij}{}^T]$ *is the* $n \times m$ *matrix define by* $a_{ij}{}^T = a_{ji}$.

thus the transpose of A is obtain from A by interchanging the rows and columns

Of A.

<u>Remark:</u> $(A^T)^T = A$

<u>Ex.</u> Find A^T, B^T, C^T, D^T, E^T for the following matrices

$$A = \begin{bmatrix}4 & -2 & 3\\ 0 & 5 & -2\end{bmatrix}, B = \begin{bmatrix}6 & 2 & -4\\ 3 & -1 & 2\\ 0 & 4 & 3\end{bmatrix}, C = \begin{bmatrix}5 & 4\\ -3 & 2\\ 2 & -3\end{bmatrix},$$

$$D = [3 \quad -5 \quad 1], E = \begin{bmatrix}2\\ -1\\ 3\end{bmatrix}$$

<u>Solution:</u>

$$A^T = \begin{bmatrix}4 & 0\\ -2 & 5\\ 3 & -2\end{bmatrix}, B^T = \begin{bmatrix}6 & 3 & 0\\ 2 & -1 & 4\\ -4 & 2 & 3\end{bmatrix}, C^T = \begin{bmatrix}5 & -3 & 2\\ 4 & 2 & -3\end{bmatrix},$$

$$D^T = \begin{bmatrix}3\\ -5\\ 1\end{bmatrix}, E^T = [2 \quad -1 \quad 3]$$

Exercises:

(1) Let $A = \begin{bmatrix} 3 & -2 & 4 \\ 8 & 5 & 7 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 2 \\ -3 \end{bmatrix}, C = \begin{bmatrix} 4 & 2 & 1 \\ 3 & -2 & 5 \\ 2 & 6 & -6 \end{bmatrix}$

(a) What is a_{21}, a_{13}, a_{23}?

(b) What is b_{11}, b_{31}?

(a) What is c_{31}, c_{23}, c_{12}?

(2) If $\begin{bmatrix} a+2b & 2a-b \\ 2c+d & c-2d \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 4 & -3 \end{bmatrix}$ $then\ find\ a, b, c, d$

(3) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{bmatrix}, D = \begin{bmatrix} 3 & -2 \\ 2 & 4 \end{bmatrix},$

$$E = \begin{bmatrix} 2 & -4 & 5 \\ 0 & 1 & 4 \\ 3 & 2 & 1 \end{bmatrix}, F = \begin{bmatrix} -4 & 5 \\ 2 & 3 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then compute

(1) C+E ,E+C

(2) 3A+2A,5A

(3) 2(D+F),2D+2F,2B+F

(4) -3C+5O

(5) $(A+B)^T, (A-B)^T$

(6) $A^T, -A^T, -(A^T), (2A)^T$

(7) $(2D+F)^T, D-D^T$

(4) Is the matrix

$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ **a linear combination of the matrices** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ **and** $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$?

(5) Is the matrix $\begin{bmatrix} -1 & -3 & 2 \\ 0 & -3 & 2 \\ 2 & 0 & -3 \end{bmatrix}$ **a linear combination of**

the matrices $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ **and** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$?

(3) Matrix Multiplication

Matrix multiplication has some properties that distinguish it form multiplication of real numbers.

Def. The **dot product; or inner product** .of the n-vectors in R^n.

$$\boldsymbol{a} = \begin{bmatrix} \boldsymbol{a_1} \\ \vdots \\ \boldsymbol{a_n} \end{bmatrix} \ \ \boldsymbol{and\ b} = \begin{bmatrix} \boldsymbol{b_1} \\ \vdots \\ \boldsymbol{b_n} \end{bmatrix}$$

Is defined as $a.b = a_1b_1 + a_2b_2 + \cdots \ldots . + a_nb_n = \sum_{i=1}^{n} a_nb_n$

Ex.

(1) Find u.v where u $= \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ **and v** $= \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

Solution: u.v=(1)(2)+(-2)(3)+(3)(-2)=-10

(2) Let a $= \begin{bmatrix} x \\ 2 \\ 3 \end{bmatrix}$ **and b** $= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ **. if** $\mathbf{a.b = -4}$ **find x**.

Solution: **a.b**=4x+2+6=-4

4x+8=-4

x=-3

Def. if $A = [a_{ij}]$ $is\ an\ \ m \times p\ matrix\ and\ B = [b_{ij}] is = \ a\ \ p \times n\ matrix\ ,$

$$then\ the\ product\ of\ A\ and\ B\ , denoted\ AB\ is\ the\ matrices\ m \times n$$

$$C = [c_{ij}], define\ by$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots \ldots + a_{ip}b_{pj} \quad = \sum_{k=1}^{p} a_{ik}b_{kj} \quad (1 \le i \le m , 1 \le j \le n)$$

<u>Ex.</u> Find the following

(1) $\mathbf{AB}\ \boldsymbol{where\ A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{1} & \mathbf{4} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{-2} & \mathbf{5} \\ \mathbf{4} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$

<u>Solution:</u> AB $= \begin{bmatrix} 4 & -2 \\ 6 & 16 \end{bmatrix}$

(2) $\boldsymbol{c_{32}\ if\ AB = C\ where\ A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} & \mathbf{3} \\ \mathbf{4} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-2} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{4} \\ \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{2} \end{bmatrix}$

<u>Solution:</u> $\boldsymbol{c_{32}} = (\mathbf{0})(\mathbf{4}) + (\mathbf{1})(-\mathbf{1}) + (-\mathbf{2})(\mathbf{2}) = -\mathbf{5}$

(3) x and y if AB=$\begin{bmatrix} \mathbf{12} \\ \mathbf{6} \end{bmatrix}$ $\boldsymbol{where\ A} = \begin{bmatrix} \mathbf{1} & \boldsymbol{x} & \mathbf{3} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{2} \\ \mathbf{4} \\ \boldsymbol{y} \end{bmatrix}$

Then 2+4x+3y=12

4-4+y=6

So x=-2 and y=6

(4) second column of AB where $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \\ \mathbf{-1} & \mathbf{5} \end{bmatrix}$ $\boldsymbol{and\ B} = \begin{bmatrix} \mathbf{-2} & \mathbf{3} & \mathbf{4} \\ \mathbf{3} & \mathbf{2} & \mathbf{1} \end{bmatrix}$

<u>Solution:</u> The second column of AB $= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \\ 7 \end{bmatrix}$

<u>Remark:</u>

(1) BA may not be defined;will take place if $n \neq m$

(2) If BA is defined which means that m=n,then BA is p × p while AB is

$\mathrm{m \times m}$; thus if $\mathrm{m \neq p}$, AB and BA are of different sizes

(3) If AB and BA are both of the same size ,they may be equal.

(4) If AB and BA are both of the same size ,they may be unequal.

(5) If u and v are n-vectors (n×1 matrices) then $u.v = u^T v$

Ex.

(1) If Ais a 2×3 matrix and B is a 3×4 matrix then AB is a 2×4 matrix while BA is undefined

(2) Let A be 2×3 and let B be 3×2 then AB is 2×2 while BA is 3×3.

(3) Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ $then\ AB = \begin{bmatrix} 2 & 3 \\ -2 & 2 \end{bmatrix} and\ BA = \begin{bmatrix} 1 & 7 \\ -1 & 3 \end{bmatrix}$

$$thus\ AB \neq BA$$

Remark: The product an m×n matrix A=$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} \dots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ and an n×1 matrix c= $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ (the matrix –vector product)can be written a linear combination of the column of A where the coefficients are the entries in the matrix c

Ac=$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} \dots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$ $=c_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots \dots + c_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$

Ex. Find Ac where $\boldsymbol{A = \begin{bmatrix} 2 & -1 & -3 \\ 4 & 2 & -2 \end{bmatrix}}$ $\boldsymbol{and\ c = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}}$

Solution: Ac=$2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} -3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ -6 \end{bmatrix}$

Theorem (1.2)properties of matrix multiplication

If A,B ,C are matrices of the appropriate sizes, then

(a) A(BC)=(AB)C

(b) (A+B)C=AC+BC

(c) C(A+B)=CA+CB

Ex. Prove that (A+B)C=AC+BC where $\boldsymbol{A = \begin{bmatrix} 2 & 2 & 3 \\ 3 & -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 3 & -1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 2 & 2 \\ 3 & -1 \end{bmatrix}}$

Solution: (A+B)C=AC+BC=$\begin{bmatrix} 18 & 0 \\ 12 & 3 \end{bmatrix}$

Theorem (1.3)properties of scalar multiplication

If r ,s are real numbers and A,B are matrices of the appropriate sizes ,then

(a) r(sA)=(rs)A

(b) (r+s)A=rA+sA

(c) r(A+B)=rA+rB

(d) A(rB)=r(AB)=(rA)B

Ex. Find 2(3A) and A(2B) where $\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{-3} & \mathbf{4} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{-2} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} \end{bmatrix}$

Solution:

$$2(3A) = 2\begin{bmatrix} 12 & 6 & 9 \\ 6 & -9 & 12 \end{bmatrix} = \begin{bmatrix} 24 & 12 & 18 \\ 12 & -18 & 24 \end{bmatrix} = 6A$$

$$A(2B) = \begin{bmatrix} 4 & 2 & 3 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 6 & -4 & 2 \\ 4 & 0 & -2 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 32 & -10 & 16 \\ 0 & 0 & 26 \end{bmatrix} = 2(AB)$$

Theorem (1.4)properties of Transpose

If r is a scalar and A ,B are matrices of the appropriate sizes, then

(a) $(A^T)^T = A$

(b) (A+B)$^T = A^T + B^T$

(c) $(AB)^T = B^T A^T$

(d) $(rA)^T = rA^T$

Ex. (1) Prove property (b) where $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{3} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$

Solution:

$$A^T = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}, B^T = \begin{bmatrix} 3 & 3 \\ -1 & 2 \\ 2 & -1 \end{bmatrix}, A + B = \begin{bmatrix} 4 & 1 & 5 \\ 1 & 2 & 0 \end{bmatrix}, (A+B)^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 4 & 1 \\ 1 & 2 \\ 5 & 0 \end{bmatrix} = (A+B)^T$$

(2) Prove property (c) where $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} \\ \mathbf{3} & \mathbf{-1} \end{bmatrix}$

Solution:

$AB = \begin{bmatrix} 12 & 5 \\ 7 & -3 \end{bmatrix}, (AB)^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 2 & -1 \end{bmatrix},$

$B^T A^T = \begin{bmatrix} 12 & 7 \\ 5 & -3 \end{bmatrix} = (AB)^T$

Remark: If A,B,C are matrices of the appropriate sizes ;then

(a) AB need not equal BA

(b) AB may be the zero matrix with A≠0 and B≠0

(c) AB may equal AC with B≠C

Ex.

(1) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}, A \neq 0, B \neq 0 \ but \ AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(2)If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} -2 & 7 \\ 5 & -1 \end{bmatrix} \ then \ AB = AC = \begin{bmatrix} 8 & 5 \\ 16 & 10 \end{bmatrix}$

Exercises

(1) Find a.b for the following:

(a) $\boldsymbol{a} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \boldsymbol{b} = \begin{bmatrix}1\\0\\1\end{bmatrix}$

(b) $\boldsymbol{a} = \begin{bmatrix}2\\-3\\1\end{bmatrix}, \boldsymbol{b} = \begin{bmatrix}3\\2\\4\end{bmatrix}$

(2) Find x if v.u =22 where $\boldsymbol{v} = \boldsymbol{u} = \begin{bmatrix}x\\-2\\3\end{bmatrix}$

(3) Determent values of x ,y if v.w=0 and v.u=0 where

$\boldsymbol{v} = \begin{bmatrix}x\\1\\y\end{bmatrix}, \boldsymbol{w} = \begin{bmatrix}2\\-2\\1\end{bmatrix}, \boldsymbol{u} = \begin{bmatrix}1\\8\\2\end{bmatrix}$

(4) Find x,y where $A = \begin{bmatrix}2 & 3 & x\\-1 & 2 & 3\end{bmatrix}, B = \begin{bmatrix}y\\x\\2\end{bmatrix}$ $and\ AB = \begin{bmatrix}6\\2\end{bmatrix}$

(5) Consider the following matrices

$$A = \begin{bmatrix}2 & 1 & -2\\3 & 4 & 5\end{bmatrix}, B = \begin{bmatrix}3 & -1\\-2 & 3\\4 & 5\end{bmatrix}, C = \begin{bmatrix}3 & -1 & 3\\4 & 1 & 5\\2 & 1 & 3\end{bmatrix}, D = \begin{bmatrix}-2 & 1\\3 & 4\end{bmatrix}$$

If possible, compute the following:

(a) AB (b) BA (c) AB+D (d) B^TC $(e) 2(AB)^T$ $(f)\ B^TA^T$ $(g)\ AB + D^2$

(i) find the second column and second row in AC and find the (2,2) ,(2,3) entries in AC

(6) Find a value of r if $AB^T = 0$ **where** $A = [3 \quad 2 \quad r]$ **and** $B = [-2 \quad 5 \quad 2]$

(7) Find $A^2 where\ A = \begin{bmatrix}cos\theta & sin\theta\\-sin\theta & cos\theta\end{bmatrix}$

(8) Find r where Ax=rx where A=$\begin{bmatrix}1 & 2 & -1\\1 & 0 & 1\\4 & -4 & 5\end{bmatrix}, x = \begin{bmatrix}\frac{-1}{2}\\\frac{1}{4}\\1\end{bmatrix}$

(1.3)Special types of Matrices and partitioned matrices

(1) Zero matrix

$O = [a_{ij}]$ if $a_{ij} = 0$ for all i, j

Ex. $O = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(2) square matrix

An n×n matrix $A = [a_{ij}]$

Ex. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & -4 & 0 \\ -7 & 1 & 3 \\ 5 & 0 & -2 \end{bmatrix}$

(3) Diagonal matrix

An n×n matrix $A = [a_{ij}]$ if $a_{ij} = 0$ for $i \neq j$ (the terms off the main diagonal are all zero)

Ex. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(4) Scalar matrix

Is a diagonal matrix whose diagonal elements are equal.

Ex. $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(5) Identity matrix

The n×n scalar matrix (diagonal matrix) $I_n = [d_{ij}] where\ d_{ij} = 1\ and$

$d_{ij} = 0\ for\ i \neq j$

Ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Remark:

(1) If A is any m×n matrix then $AI_n = I_m A = A$

(2) If A is a scalar matrix then A=r I_n for some scalar r

(3) If A is a square matrix then $A^p = A.A \dots \dots A$ *where p isa positive integer*

(4) If A is n×n then $A^0 = I_n$

(5) If A is a square matrix and p,q are non negative integer then $A^p A^q = A^{p+q}$ *and* $(A^p)^q = A^{pq}$

(6) $(AB)^p = A^p A^p$ *doesnot hold for square matrix unless* $AB = BA$

(6) <u>upper triangular matrix</u>

An n×n matrix $\mathrm{A} = [\mathrm{a_{ij}}]$ if $\mathrm{a_{ij}} = 0$ for $\mathrm{i} > j$

<u>Ex.</u> $D = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 5 \\ 0 & 0 & 2 \end{bmatrix}$

(7) <u>Lower triangular matrix</u>

An n×n matrix $\mathrm{A} = [\mathrm{a_{ij}}]$ if $\mathrm{a_{ij}} = 0$ for $\mathrm{i} < j$

<u>Ex.</u> $C = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 5 & 2 \end{bmatrix}$

(8) <u>Symmetric matrix</u>

A matrix is symmetric(square matrix) if $A^T = A$

The entries of A are symmetric w.r.t the main diagonal of A.

A is symmetric iff $a_{ij} = a_{ji}$

<u>Ex.</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

(9) Skew symmetric matrix

A matrix is skew symmetric (square matrix) if $A^T = -A$

The entries on the main diagonal of A are all zero

A is skew symmetric iff $a_{ij} = -a_{ji}$

Ex. $A = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & -4 \\ -3 & 4 & 0 \end{bmatrix}$

Remark: If A is an n×n matrix then we can show that A=S+K where S is symmetric and K is skew symmetric. Moreover this decomposition is unique.

(10) Nonsingular matrix

an n×n matrix A (square matrix) is called nonsingular or invertible if there exists an n×n matrix B s.t $AB = BA = I_n$

B is called an inverse of A.

A is called singular or non invertible.

Remark: If $AB = I_n$ then $BA = I_n thus\ B\ is\ an\ inverse\ of\ A$.

Ex. Let $A = \begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix} and\ B = \begin{bmatrix} -1 & \frac{3}{2} \\ 1 & -1 \end{bmatrix}\ then\ B\ is\ an\ inverse\ of\ amatrix$

$$scince\ AB = BA = I_2$$

Theorem (1.5) The inverse of a matrix if it is exists ,is unique.

Proof: Let B and C be inverses of A, then

AB=BA= $I_n\ and\ AC = CA = I_n$

We have B= $BI_n = B(AC) = (BA)C = I_nC = C$

The inverse of a matrix if it is exists ,is unique∴

Remark: We write the inverse of a nonsingular n×n matrix A as A^{-1}.

$$thus\ AA^{-1} = A^{-1}A = I_n$$

<u>Ex.</u>(1) Find $\boldsymbol{A^{-1}}$ *if there exists where* $\boldsymbol{A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$

<u>Solution:</u> Suppose $\boldsymbol{A^{-1}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $AA^{-1} =$
$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So that $\begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

a+c=1 and b+d=0

a+2c=0 b+2d=1

by solution the linear system obtain a=2,c=-1,b=-1,d=1

$\therefore A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$\therefore A^{-1}A = AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(2) Find $\boldsymbol{A^{-1}}$ *if there exists where* $\boldsymbol{A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}$

<u>Solution:</u> Suppose $\boldsymbol{A^{-1}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then $AA^{-1} =$
$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So that $\begin{bmatrix} a+2c & b+2d \\ 2a+4c & 2b+4d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

a+2c=1 and b+2d=0

2a+4c=0 2b+4d=1

These linear systems have no solution and there is no exist A^{-1}

$\therefore$ *the matrix A is singular.*

properties of inverses of matrices

Theorem (1.6) If A and B are both nonsingular n×n matrices, then AB is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$

Corollary (1.1) If
$A_1, A_2, \ldots \ldots, A_r$ are nonsingular matrices , then $A_1, A_2, \ldots \ldots A_r$ is nonsingular

And $(A_1 A_2 \ldots \ldots A_r)^{-1} = {A_1}^{-1}{A_2}^{-1} \ldots \ldots . {A_r}^{-1}$

Theorem (1.7) If A is a nonsingular matrix then A^{-1} is non singular and

$(A^{-1})^{-1} = A$

Theorem (1.8) If A is a nonsingular matrix then A^T is non singular and

$(A^{-1})^T = (A^T)^{-1}$

Ex. If $\mathbf{A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$ **and** $\mathbf{A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}}$ **then prove theorem (1.8)**

Solution: $(A^{-1})^T = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{bmatrix}, A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, (A^T)^{-1} = \begin{bmatrix} -2 & \frac{3}{2} \\ 1 & \frac{-1}{2} \end{bmatrix}$

Remark:

(1) Suppose that A is nonsingular .then AB=AC implies that B=C ,and AB=0 implies B=0.

(2) If follows from Theorem (1.8) that if A is symmetric nonsingular matrix then A^{-1} is symmetric

Exercises

(1) Show that the matrix A=$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$ **is singular**

(2) If D=$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ **then find** $\mathbf{D^{-1}}$

(3) If A is a nonsingular matrix whose inverse is $\begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$ **findA**

(4) Find the inverse of each the following matrices :

(a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{-1} & \mathbf{4} \end{bmatrix}$ $\quad\quad (\boldsymbol{b})\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$

(5) If $\mathbf{A^{-1}} = \begin{bmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$ **and** $\mathbf{B^{-1}} = \begin{bmatrix} \mathbf{2} & \mathbf{5} \\ \mathbf{3} & \mathbf{-2} \end{bmatrix}$ **then find** $(\mathbf{AB})^{-1}$

<u>**Linear system**</u>

Consider the linear system of m equations and n unknown,

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots \ldots + a_{1n}x_n = b_1$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (*) \; a_{21}x_1 + a_{22}x_2 + \cdots \ldots \ldots + a_{2n}x_n = b_2$$

.

.

.

$$a_{m1}x_1 + a_{m2}x_2 + \cdots \ldots \ldots + a_{mn}x_n = b_m$$

Now define the following matrices:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} , \quad x = \begin{bmatrix} x_1 \\ : \\ x_n \end{bmatrix} , \quad b = \begin{bmatrix} b_1 \\ : \\ b_n \end{bmatrix}$$

The linear system (*) can be written in matrix form as **Ax=b**

The matrix A is called the **coefficient matrix** of the linear system (*).

Then Ax= $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ : \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ : \\ b_n \end{bmatrix} = b$

$$= \begin{bmatrix} a_{11}x_1 + & \cdots & +a_{1n}x_n \\ \vdots & \ddots & \vdots \\ a_{m1}x_1 + & \cdots & +a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ : \\ b_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 \\ : \\ a_{m1}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ : \\ a_{m2}x_2 \end{bmatrix} + \cdots \ldots \begin{bmatrix} a_{1n}x_n \\ : \\ a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ : \\ b_2 \end{bmatrix}$$

$$=x_1\begin{bmatrix}a_{11}\\ \vdots\\ a_{m1}\end{bmatrix}+x_2\begin{bmatrix}a_{12}\\ \vdots\\ a_{m2}\end{bmatrix}+\cdots\ldots+x_n\begin{bmatrix}a_{1n}\\ \vdots\\ a_{mn}\end{bmatrix}=\begin{bmatrix}b_1\\ \vdots\\ b_n\end{bmatrix}$$

The matrix $\begin{bmatrix}a_{11} & \cdots & a_{1n}:b_1\\ \vdots & \ddots & \vdots\\ a_{m1} & \cdots & a_{mn}:b_n\end{bmatrix}$

Obtained by adjoining column **b** to A is called the augmented matrix of the linear system (*) and written as [A: b].

Conversely any matrix with more than one column can be thought of as the augmented matrix of a linear system.

The coefficient and augmented matrices play key roles in our method for solving linear systems.

Remark:

(1)If $b_1 = b_2 = \cdots \ldots . = b_n$ in the linear system $(*)$,

then the linear system is called a **homogeneous system**

(2) a homogeneous system can be written as $\mathbf{Ax} = \mathbf{0}$

(3) **Ax=b** is consistent iff b can be expressed as a linear combination of the columns of the matrix A.

Ex. (1) Consider the linear system

-2x+z=5

2x+3y-4z=7

3x+2y+2z=3

$$A=\begin{bmatrix}-2 & 0 & 1\\ 2 & 3 & -4\\ 3 & 2 & 2\end{bmatrix},x=\begin{bmatrix}x\\ y\\ z\end{bmatrix},b=\begin{bmatrix}5\\ 7\\ 3\end{bmatrix}$$

A is the coefficients matrix

Writing Ax=b as a linear combination of the columns of A:

$$x\begin{bmatrix}-2\\2\\3\end{bmatrix}+y\begin{bmatrix}0\\3\\2\end{bmatrix}+z\begin{bmatrix}1\\-4\\2\end{bmatrix}=\begin{bmatrix}5\\7\\3\end{bmatrix}$$

The augmented matrix is:

$$\begin{bmatrix}\mathbf{-2} & \mathbf{0} & \mathbf{1:5}\\ \mathbf{2} & \mathbf{3} & \mathbf{-4:7}\\ \mathbf{3} & \mathbf{2} & \mathbf{2:3}\end{bmatrix}$$

(2) The augmented matrix $\begin{bmatrix}2 & -1 & 3:4\\3 & 0 & 2:5\end{bmatrix}$

Of the linear system

2x-y+3z=4

3x+2z=5

<u>Exercises</u>

(1) Consider the following linear system

$$\mathbf{2x_1+3x_2-3x_3+x_4+x_5=7}$$

$$\mathbf{3x_1+2x_3+3x_5=-2}$$

$$\mathbf{2x_1+3x_2-4x_4=3}$$

$$\mathbf{x_3+x_4+x_5=5}$$

(a) Find the coefficient matrix

(b) Write the linear system in matrix form

(c) Find the augmented matrix

(2) Write the linear system whose augmented matrix is

$$\begin{bmatrix}\mathbf{3} & \mathbf{2} & \mathbf{3:0}\\ \mathbf{-2} & \mathbf{-1} & \mathbf{2:3}\end{bmatrix}$$

(3) Consider the linear system:

4x+2y=0

-3y+5z=0

2x-y=0

(a) Write the linear system in matrix form.

(b) Write the linear system as a linear combination of the columns of the coefficient matrix.

(4) Write each of the following linear combinations columns as a linear system

(a) $x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \end{bmatrix}$

(b) $x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

(5) Construct a coefficient matrix A so that $x = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ *is a solution to the system* $Ax = b$ *where* $b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(6) Write the linear system as a linear combination of the columns of the coefficient matrix

2x-y+3z=-5

-3x+2y-z=8

<u>**Linear system and inverses**</u>

If A is an n×n matrix then the linear system Ax=b is a system of n equation in n unknown .suppose that A is nonsingular. Then A^{-1} exists and we can multiply Ax=b byA^{-1} on the left on both sides, yielding

$A^{-1}(Ax) = A^{-1}b$

$(A^{-1}A)x = A^{-1}b$

$I_n x = A^{-1}b$

x=$A^{-1}b$

is a solution of the given linear system

if A is nonsingular ,we have a **unique solution**.

Remark: If A is an n×n matrix .then the linear system Ax=b has the unique solution x=$A^{-1}b$

If b=0 then the unique solution to the homogeneous system Ax=0 is x=0.

Ex. Find the solution of the system Ax=b where $\boldsymbol{A^{-1}} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$

Solution

$$x = A^{-1}b = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} -10 \\ 9 \end{bmatrix}$$

$If\ b = \begin{bmatrix} 10 \\ 20 \end{bmatrix}\ \ then\ x = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$

Exercise

Consider the linear system Ax=b where A=$\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$**, then find a solution if**

(a) $\boldsymbol{b} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$ (**b**) $\boldsymbol{b} = \begin{bmatrix} -5 \\ 11 \end{bmatrix}$

Chapter -2-

Solving linear systems

(2.1) Echelon form of a matrix

Def. An m×n matrix A is said to be in **reduced row echelon form** if it is satisfies the following properties:

(a) All zero rows, if there are any; appear at the bottom of the matrix

(b) The first nonzero entry from the left of a nonzero row is a 1.This entry is called a **leading one** of its row**.**

(c) For each nonzero row, the leading one appears to the right and below any leading ones in preceding rows.

(d) If a column contains a leading one, then all other entries in that column are zero.

Remark:

(1) A matrix in reduced row echelon form appears as a staircase (echelon) pattern of leading ones descending from the upper left corner of the matrix.

(2) An m×n matrix satisfying properties (a),(b) and (c) is said to be in row echelon form.

(3) In def. there may be no zero rows

(4) A similar def. can be formulated in the obvious manner for reduced column echelon form and column form

Ex. (1) The following are matrices in reduced row echelon form:

$$, B = \begin{bmatrix} 1 & 0 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & 1 & 7 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} c = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad , \ I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) The matrices that follow are not in reduced row echelon form:

$$D = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \ , \ E = \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \ , G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Def. An elementary row (column) operation on a matrix A is any one of the following operations:

(a) Type I: Interchange any two rows (columns)

$$r_i \leftrightarrow r_j \ \ (c_i \leftrightarrow c_j)$$

(b) Type II: Multiply a row (column) by a nonzero number

$$Kr_i \rightarrow r_i \ \ \ (Kc_i \rightarrow c_i)$$

(c) Type III: Add a multiple of one row (column)to another

$)Kr_i + r_j \rightarrow r_j \quad (Kc_i \rightarrow c_j$

Ex.

(1) $\boldsymbol{A = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 3 & 3 & 6 & -9 \end{bmatrix}}$

Solution:

$$\mathrm{B} = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 3 & 3 & 6 & -9 \\ 2 & 3 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\mathrm{C} = \mathrm{A}_{\frac{1}{3}\mathrm{r}_3 \rightarrow \mathrm{r}_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & 1 & 2 & -3 \end{bmatrix}$$

$$\mathrm{D} = \mathrm{A}_{-2\mathrm{r}_2 + \mathrm{r}_3 \rightarrow \mathrm{r}_3} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 2 & 3 & 0 & -2 \\ 1 & -3 & 6 & -5 \end{bmatrix}$$

(2) $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{1} & \mathbf{-2} & \mathbf{2} & \mathbf{3} \end{bmatrix}$

Solution:

$$B = A_{2r_2+r_3 \to r_3} = \begin{bmatrix} 1 & 2 & 4 & 3 \\ 2 & 1 & 3 & 2 \\ 1 & -2 & 2 & 3 \end{bmatrix}$$

$$C = B_{r_1 \to r_3} = \begin{bmatrix} 5 & 0 & 8 & 7 \\ 2 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \end{bmatrix}$$

$$D = C_{2r_3 \to r_3} = \begin{bmatrix} 5 & 0 & 8 & 7 \\ 2 & 1 & 3 & 2 \\ 2 & 4 & 8 & 6 \end{bmatrix}$$

Def. An m×n matrix B is said to be **row (column) equivalent** to an m×n matrix A if B can be applying a finite sequence of elementary row (column).

Theorem (2.1)

Every nonzero m×n matrix A is row (column) equivalent to a matrix in row (column) equivalent to a matrix in row (column) echelon form.

Ex. $A = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{3} & \mathbf{-4} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{2} & \mathbf{-5} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{0} & \mathbf{-6} & \mathbf{9} & \mathbf{7} \end{bmatrix}$

Solution:

$$B = A_{r_1 \leftrightarrow r_3} = \begin{bmatrix} 2 & 2 & -5 & 2 & 4 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 2 & 0 & -6 & 9 & 7 \end{bmatrix}$$

$$C = B_{\frac{1}{2}r_1 \to r_1} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \frac{\mathbf{-5}}{\mathbf{2}} & \mathbf{1} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} & \mathbf{-4} & \mathbf{1} \\ \mathbf{2} & \mathbf{0} & \mathbf{-6} & \mathbf{9} & \mathbf{7} \end{bmatrix}$$

$$D = C_{-2r_1+r_4\to r_4} = \begin{bmatrix} 1 & 1 & \frac{-5}{2} & 1 & 2 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 2 & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

Identify A_1as the sub matrix of D obtained by deleting the first row of D: do not erase the first row of D.repeat the preceding steps withA_1 instead of A.

$$A_1 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{\frac{-5}{2}} & 1 & \mathbf{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & \mathbf{2} & 3 & -4 & 1 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{\frac{-5}{2}} & 1 & \mathbf{2} \\ 0 & 2 & 3 & -4 & 1 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{\frac{-5}{2}} & 1 & \mathbf{2} \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & -2 & -1 & 7 & 3 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{\frac{-5}{2}} & 1 & \mathbf{2} \\ 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

Deleting the first row of D_1 *yilds the matrix* A_2 .

We repeat the procedure with A_2 *instead of* A.

Now rows of A_2 *have to be interchanged*.

$$A_2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{\frac{-5}{2}} & 1 & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{\frac{3}{2}} & \mathbf{-2} & \mathbf{\frac{1}{2}} \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \frac{-5}{2} & 1 & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \frac{3}{2} & -\mathbf{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 2 & 3 & 4 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \frac{-5}{2} & 1 & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \frac{3}{2} & -\mathbf{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $H = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \frac{-5}{2} & 1 & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \frac{3}{2} & -\mathbf{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Is in row echelon form and equivalent to A.

<u>Theorem(2.2)</u>

Every nonzero m×n matrix A=$[a_{ij}]$ is row (column) equivalent to a unique matrix in reduced row(column) echelon form.

<u>Remark:</u> it should be noted that a row echelon form of a matrix is not unique.

<u>Ex.</u> Find a matrix in reduced row echelon form that is row equivalent to the matrix A of previous example.

$$j_1 = H_{\frac{-3}{2}r_3+r_2\to r_2} = \begin{bmatrix} 1 & 1 & \frac{-5}{2} & 1 & 2 \\ 0 & 1 & 0 & \frac{-17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$j_2 = (j_1)_{\frac{5}{2}r_3+r_1\to r_1} = \begin{bmatrix} 1 & 1 & 0 & \frac{19}{4} & 7 \\ 0 & 1 & 0 & \frac{-17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K=(j_2)_{-r_2+r_1\to r_1} = \begin{bmatrix} 1 & 0 & 0 & 9 & \frac{19}{2} \\ 0 & 1 & 0 & \frac{-17}{4} & \frac{-5}{2} \\ 0 & 0 & 1 & \frac{3}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is reduced row echelon form and is row equivalent to A.

Exercises

(1) Find a row echelon of each the given matrices:

(a) $A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} 3 & 4 & -1 \\ 6 & 7 & -4 \\ 1 & 1 & -1 \\ 5 & 6 & -3 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -1 & 1 & 0 & -3 \\ -1 & 2 & 3 & 1 & 4 \\ 3 & 4 & 1 & 1 & 10 \end{bmatrix}$

(2) Find the reduced row echelon form of each of the given matrices:

(a) $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ -2 & 9 & 4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -2 \\ -5 & 4 & -7 \\ -2 & 6 & -5 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 3 & 5 & -2 \\ 1 & 4 & 6 & -2 \\ -1 & -1 & -3 & 2 \end{bmatrix}$

(3) Let $A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ -1 & 0 & 3 & 4 \\ 0 & 1 & 2 & -1 \\ 2 & 3 & 0 & -3 \end{bmatrix}$

(a) Find a matrix in column echelon form that is column equivalent to A.

(b) Find a matrix in reduced column echelon form that is column equivalent to A.

(4) Determinate the reduced row echelon form of $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

(2.2) Solving linear systems

Theorem (2.3)

Let **Ax=b** and **Cx=d** be two linear systems, each of m equations in n unknowns .if the augmented matrices **[A:b]** and **[C:d]** are row equivalent ,then the linear systems are equivalent ;that is ,they have exactly the same solutions.

Corollary (2.1)

If A and C are row equivalent m×n matrices, then the homogenous system Ax=0 and Cx=0 are equivalent.

Remark

1-The set of solutions to this system gives precisely the set of solutions to Ax=b; that is, the linear systems Ax=b and Cx=d are equivalent.

2-The method where [C:d] is in row echelon form is called **Gaussian elimination.**

3-The method where [C:d] is in reduced row echelon form is called **Gauss-Jordan reduction.**

Gaussian elimination consists of two steps:

Step 1. The transformation of the augmented matrix [A:b] to the matrix [C:d] in row echelon form using elementary row operations.

Step 2. Solution of the linear system corresponding to the augmented matrix [C:d] using **back substitution**.

4-For the case in which A is in n×n, and the linear system **Ax=b** has a unique solution, the matrix **[C:d]** has the following form:

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} & \vdots\, d_1 \\ 0 & 1 & c_{23} & \cdots & c_{2n} & \vdots\, d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots \\ 0 & 0 & 0 & 1 & c_{n-1\,n} & \vdots\, d_{n-1} \\ 0 & 0 & 0 & 0 & 1 & \vdots\, d_n \end{bmatrix}$$

This augmented matrix represents the linear system:

$$x_1 + c_{12}x_2 + c_{13}x_3 + \cdots + c_{1n}x_n = d_1$$

$$x_2 + c_{23}x_3 + \cdots + c_{2n}x_n = d_2$$

.

.

$$x_{n-1} + c_{n-1\,n}x_n = d_{n-1}$$

$$x_n = d_n$$

Back substitution proceeds from the nth equation upward, solving for one variable from each equation:

$$x_n = d_n$$

$$x_{n-1} = d_{n-1} = c_{n-1\,n}x_n$$

.

.

$$x_2 = d_2 - c_{23}x_3 - \cdots - c_{2n}x_n$$

$$x_1 = d_1 - c_{12}x_2 - c_{13}x_3 - \cdots - c_{1n}x_n$$

5- The general case in which A is m×n is handled in a similar fashion, but we need to elaborate upon several situations that can occur. We thus consider Cx=d,where C is m×n ,and [C:d] is in row echelon form.

For example [C:d] might be of the form:

$$\begin{bmatrix} 1 & c_{12} & c_{13} & \cdots & c_{1n} & \vdots\, d_1 \\ 0 & 0 & 1 & \cdots & c_{2n} & \vdots\, d_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots \\ 0 & 0 & 0 & 1 & c_{k-1\,n} & \vdots\, d_{k-1} \\ 0 & \cdots & \vdots & 0 & 1 & :d_k \\ 0 & 0 & \cdots & \vdots & 0 & \vdots\ :d_{k+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & \cdots & :d_m \end{bmatrix}$$

This augmented matrix represented the linear system

$$
\begin{array}{cccccc}
x_1 & c_{12}x_2 & c_{13}x_3 & \cdots & c_{1n}x_n & = d_1 \\
0 & 0 & x_3 & \cdots & c_{2n}x_n & = d_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & x_{n-1} & c_{k-1\,n}x_n & = d_{k-1} \\
0 & \cdots & \vdots & 0 & x_n & = d_k \\
0 & 0 & \cdots & \vdots & 0 & = d_{k+1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & = d_m
\end{array}
$$

If $d_{k+1} = 1$ then

$\mathrm{Cx} = \mathrm{d}$ has no solution , since at least one equation is not satisfied

If $d_{k+1} = 0$ *which implies that* $d_{k+2} = \cdots \ldots = d_m = 0$

Since [C:d] was assumed to be in row echelon form, we then obtain

$$x_n = d_k, x_{n-1} = d_{k-1} - c_{k-1\,n}x_n = d_{k-1} - c_{k-1\,n}d_k$$

And continue using back substitution to find the remaining unknowns corresponding to the leading entry in each row.

In the solution some of the unknowns may be expressed in terms of others that can take on any values whatsoever.

This merely indicates that Cx=d has infinitely many solution.

Every unknown may have a determined value, indicating that the solution is unique.

<u>Examples</u>

1-Solve the following linear system

X+2y+3z=9

2x-y+z=8

3x-z=3

<u>Solution</u>

The linear system has the augmented matrix

$$[A:b] = \begin{bmatrix} 1 & 2 & 3 & :9 \\ 2 & -1 & 1 & :8 \\ 3 & 0 & -1 & :3 \end{bmatrix}$$

Transforming the matrix to row echelon form, we obtain

$$[C:d] = \begin{bmatrix} 1 & 2 & 3 & :9 \\ 2 & 1 & 1 & :2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

Using back substitution, we now have

z=3

y=2-z=2-3=-1

x=9-2y-3z=9+2-9=2

Thus the solution is x=2, y=-1, z=3 which is unique.

2- Let [C:d] = $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} & \mathbf{5} & :\mathbf{6} \\ \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{-1} & :\mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & \mathbf{3} & :\mathbf{7} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{2} & :\mathbf{9} \end{bmatrix}$

then $\mathrm{x}_4 = 9 - 2x_5$

$x_3 = 7 - 2x_4 - 3x_5 = -11 + x_5$

$x_2 = 7 - 2x_3 - 3x_4 + x_5 = 2 + 5x_5$

$x_1 = 6 - 2x_2 - 3x_3 - 4x_4 - 5x_5 = -1 - \ 10x_5$

$x_5 = any\ real\ number$

The system is consistent and all solutions are of the form

$x_1 = -1 - \ 10r$

$x_2 = 2 + 5r$

$x_3 = -11 + r$

$\mathrm{x}_4 = 9 - 2r$

$x_5 = r\ any\ real\ number$

The given linear system has infinitely many solutions

3- If [C:d] = $\begin{bmatrix} 1 & 2 & 3 & 4 & :5 \\ 0 & 1 & 2 & 3 & :6 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Cx=d has no solution since the last equation is

$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$

4- If [C:d] = $\begin{bmatrix} 1 & 0 & 0 & 0 & :5 \\ 0 & 1 & 0 & 0 & :6 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}$

Then the unique is

$x_1 = 5$

$x_2 = 6$

$x_3 = 7$

$x_4 = 8$

5- If [C:d] = $\begin{bmatrix} 1 & 1 & 2 & 0 & \frac{-5}{2} & :\frac{2}{3} \\ 0 & 0 & 0 & 1 & \frac{1}{2} & :\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then $x_4 = \frac{1}{2} - \frac{1}{2} x_5$

$x_1 = \frac{2}{3} - x_2 - 2x_3 + \frac{5}{2} x_5$

Where x_2, x_3, x_5 *can take on any real numbers* .

The system has infinitely many solutions of the form

$x_1 = \frac{2}{3} - r - 2s + \frac{5}{2} t$

$x_2 = r$

$x_3 = s$

$x_4 = \frac{1}{2} - \frac{1}{2} t$

$x_5 = t$

6- Consider the linear system

x+2y+3z=6

2x-3y+2z=14

3x+y-z=-2

Then solve this system

<u>Solution</u>

To solve the system by Gaussian elimination by the following:

We form the augmented matrix

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 2 & -3 & 2 & :14 \\ 3 & 1 & -1 & :-2 \end{bmatrix}$$

Add (-2) times the first row to the second row:

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 3 & 1 & -1 & :-2 \end{bmatrix}$$

Add (-3) times the first row to the third row:

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & -7 & -4 & :2 \\ 0 & -5 & -10 & :-20 \end{bmatrix}$$

Multiply the third row by$\left(-\frac{1}{5}\right)$ and interchange the second row and third rows

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & 1 & 2 & :4 \\ 0 & -7 & -4 & :2 \end{bmatrix}$$

Add 7 times the second row to the third row:

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & 1 & 2 & :4 \\ 0 & 0 & 10 & :30 \end{bmatrix}$$

Multiply the third row by$\left(\frac{1}{10}\right)$

$$[A:b]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & 1 & 2 & :4 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

This matrix is in row echelon from.

Z=3

Y=4-2z=4-2(3)=-2

X+2y+3z=6

X=6-2(-2)-3(3)=1

The solution is x=1,y=-2,z=3

To solve the given linear system by Gauss-jordan reduction, we transform the last matrix to [C:d],which is in reduced row echelon form by the following steps:

Add (-2) times the third row to the second row:

$$[C:d]=\begin{bmatrix} 1 & 2 & 3 & :6 \\ 0 & 1 & 0 & :-2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

Add (-3) times the third row to the first row:

$$[C:d]=\begin{bmatrix} 1 & 2 & 0 & :-3 \\ 0 & 1 & 0 & :-2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

Add (-2) times the second row to the first row:

$$[C:d]=\begin{bmatrix} 1 & 0 & 0 & :1 \\ 0 & 1 & 0 & :-2 \\ 0 & 0 & 1 & :3 \end{bmatrix}$$

The solution is x=1,y=-2,z=3,as before.

Remarks

1- As we perform elementary row operations, we may encounter a row of the augmented matrix being transformed to reduce row echelon form whose first n entries are zero and whose n+1 entry is not zero, in this case, we can stop our computations and conclude that the given linear system is inconsistent.

2- In both Gaussian elimination and Gauss-Jordan reduction, we can use only row operation .do not try to use any column operations.

Homogeneous systems

A homo. system Ax=0 of m linear equations in n unknowns.

Ex. Consider the homo. System whose augmented matrix is

$$\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{2} & :\mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{3} & :\mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{4} & :\mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & :\mathbf{0} \end{bmatrix}$$

Solve this system.

Solution

Since the augmented matrix is in reduced row echelon form, the solution is seen to be

$x_1 = -2r$

$x_2 = s$

$x_3 = -3r$

$\mathrm{x}_4 = -4r$

$x_5 = r$

Where r and s are any real numbers.

Theorem 2.4

A homo. system of m linear equations in n unknowns always has a nontrivial solution if $m < n$,that is if the number of unknowns exceeds the number of equations.

Ex. Consider the homo. system

X+y+z+w=0

X+w=0

X+2y+z=0

Solve this system.

Solution

The augmented matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & :0 \\ 1 & 0 & 0 & 1 & :0 \\ 1 & 2 & 1 & 0 & :0 \end{bmatrix}$

Is row equivalent to $\begin{bmatrix} 1 & 0 & 0 & 1 & :0 \\ 0 & 1 & 0 & -1 & :0 \\ 0 & 0 & 1 & 1 & :0 \end{bmatrix}$

Hence the solution is

x=-r,y=r,z=-r,w=r any real number.

Remark

If $x_p$is a particular solution to the linear system Ax=b, b≠0 and x_h is a solution to the associated Ax=0 then $x_p + x_h$ is a solution to Ax=b.

Moreover every solution x to the Ax=b can be written as $x_p + x_h$ where x_p is a particular solution to Ax=b, b≠0 and is x_h a solution to the associated Ax=0.

Exercises

1- Each of the given linear system is in row echelon form. solve the system:

(a) x+3y+z=6
y+z=5
z=4

(b) x-4y+5z+w=0
z-2w=4
w=1

(c) x+2y-z+3w=4
w=5

(d) x+2y=2
z+w=-3

2- (a) find all solutions, if any exist ,by using the Gaussian elimination method

(b) find all solutions, if any exist ,by using the Gauss-Jordan reduction method

For the following linear system:

(a) x+2y+3z=-1
x-2y-z=-1
3x+y+z=3

(b) x+y+3z+2w=13
x-2y+2z+w=8
3y+z-2w=1

(c) x+y-2z=1
x-2y-3z=4
2x+y+z=3

(3) Solve the linear system with the given augmented matrix, if it is consistent:

(a) $\begin{bmatrix} 1 & 1 & 1 & :0 \\ 1 & 1 & 0 & :3 \\ 0 & 1 & 1 & :1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 3 & :0 \\ 1 & 2 & 1 & :0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 3 & -3 & :0 \\ 0 & 2 & 1 & -3 & :3 \\ 1 & 0 & 2 & -1 & :-1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 & :0 \\ 2 & 3 & 0 & :0 \\ 0 & 1 & 2 & :0 \\ 2 & 1 & 4 & :0 \end{bmatrix}$

(4) in the following linear systems ,determine all values of a for which the resulting linear system has :

(a) no solution (b) a unique solution (c) infinitely solution

(a) x+y+z=3

X+2y+z=3

x+y+(a^2-8)z=a

(b) x+y+z=2

2x+3y+2z=5

2x+3y+(a^2--1)z=a+1

(c) x+y=4

x+(a^2-15)y=a

(2.3) Elementary matrix; finding A^{-1}

Def. An n×n elementary matrix of type I, type II or type III is a matrix obtained from the identity I_n matrix by performing a single elementary row or elementary column operation of type I ,type II or type III ,respectively.

Ex. The following are elementary matrices:

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_4 = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Theorem2.5 Let A be any m×n matrix and let an elementary row(column) operation of type I ,type II or type III be performed on A to yield matrix B .Let E be the elementary matrix obtained from $I_m(I_n)$ by performing the same elementary row(column) operation as was performed on A. Then B=EA (B=AE).

Theorem2.6 If A and B are m×n matrices, then A is row (column) equivalent to B iff there exist elementary matrices $E_1, E_2 \ldots\ldots E_k$ s.t B= $E_1E_2 \ldots\ldots E_kA$

$$(B = AE_1E_2 \ldots\ldots E_k)$$

Theorem2.7 A n elementary matrix E is nonsingular, and its inverse is an elementary matrix of the same type.

Lemma 2.1 Let A be an n×n matrix and let and let the homo. system Ax=0 have only the trivial solution x=0 .then A is row equivalent to I_n .(that is the reduced row echelon form of A isI_n)

Theorem2.8 A is nonsingular iff A is a product of elementary matrices.

Corollary2.2 A is nonsingular iff A is row equivalent to I_n .(that is the reduced row echelon form of A is I_n.)

Theorem2.9 The homo. system of n linear equations in n unknowns Ax=0 has a nontrivial solution iff A is singular .(that is the reduced row echelon form of

A $\neq I_n$.)

Ex. Let $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$

Consider the homo. system Ax=0;that is:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The reduced row echelon form of the augmented matrix is:

$$\begin{bmatrix} 1 & 2 & :0 \\ 0 & 0 & :0 \end{bmatrix}$$

So a solution is x=-2r

Y=r is any real number

Thus the homo.system has a nontrivial solution and A is singular.

Remark at this point we have shown that the following statements are equivalent for an n×n matrix A:

1- A is nonsingular

2- Ax=0 has only the trivial solution

3- A is row (column) equivalent to I_n (the reduced row echelon form of A is I_n)

4- The linear system Ax=b has a unique solution for every n×1 matrix b.

5- A is a product of elementary matrices.

Finding A^{-1}

For A nonsingular matrix n×n, we transform the partition matrix [A: I_n] to reduce row echelon form, obtaining [I_n: A^{-1}].

Ex. Let $A = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{5} & \mathbf{5} & \mathbf{1} \end{bmatrix}$ **Then find $\mathbf{A}^{-1}$**

Solution

$$[A: I_3] = \begin{bmatrix} 1 & 1 & 1 & :1 & 0 & 0 \\ 0 & 2 & 3 & :0 & 1 & 0 \\ 5 & 5 & 1 & :0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} 1 & 1 & 1 & :1 & 0 & 0 \\ 0 & 2 & 3 & :0 & 1 & 0 \\ 5 & 5 & 1 & :0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 & :1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & :0 & \frac{1}{2} & 0 \\ 0 & 0 & -4 & :-5 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 1 & 1 & :1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & :0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & :\frac{5}{4} & 0 & \frac{-1}{4} \end{matrix}$$

$$\begin{matrix} 1 & 1 & 0 & :\frac{-1}{4} & 0 & \frac{1}{4} \\ 0 & 1 & 0 & :\frac{-15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & :\frac{5}{4} & 0 & \frac{-1}{4} \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 & :\frac{13}{8} & \frac{-1}{2} & \frac{-1}{8} \\ 0 & 1 & 0 & :\frac{-15}{8} & \frac{1}{2} & \frac{3}{8} \\ 0 & 0 & 1 & :\frac{5}{4} & 0 & \frac{-1}{4} \end{matrix}$$

Hence $A^{-1} = \begin{bmatrix} \frac{13}{8} & \frac{-1}{2} & \frac{-1}{8} \\ \frac{-15}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{5}{4} & 0 & \frac{-1}{4} \end{bmatrix}$

A$A^{-1} = A^{-1}A = I_3$

<u>Remark:</u> If A is 2×2 matrix, where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $then\ A\ is\ invertible\ iff$

$$ad - bc \neq 0, then\ the\ inverse\ of\ A\ is A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Ex.</u> If possible,find the inverse of each matrix:

(1) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{2} \end{bmatrix}$ $\quad$ $(\mathbf{2})\mathbf{B} = \begin{bmatrix} \mathbf{4} & \mathbf{-1} \\ \mathbf{-8} & \mathbf{2} \end{bmatrix}$

Solution

(1) ad-bc=6-2=4 then A is invertible

$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

(2) ad-bc=0 then B is noninvertible

Theorem2.10 An n×n matrix A is singular iff A is row equivalent to a matrix B that has a row of zeros (the reduced row echelon form of A has a row of zeros)

Remark to find A^{-1} ,we do not have to determine, in adance,whether it exists.we merely start to calculate A^{-1} ;if at any point in the computation we find a matrix B that is row equivalent to A and has a row of zeros,then A^{-1}does not exists.that is,we transform the partitioned matrix [A: I_n] to reduced row echelon form,obtaining [C:D].if C= I_n,then D=A^{-1}.if C≠I_n,then C has a row of zeros and we conclude that A is singular.

Ex. Find A^{-1} **where A=** $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} \\ \mathbf{5} & \mathbf{-2} & \mathbf{-3} \end{bmatrix}$

Solution

$$\begin{matrix} 1 & 2 & -3 & :1 & 0 & 0 \\ 1 & -2 & 1 & :0 & 1 & 0 \\ 5 & -2 & -3 & :0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 2 & -3 & :1 & 0 & 0 \\ 0 & -4 & 4 & :-1 & 1 & 0 \\ 5 & -2 & -3 & :0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 2 & -3 & :1 & 0 & 0 \\ 0 & -4 & 4 & :-1 & 1 & 0 \\ 0 & -12 & 12 & :-5 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 2 & -3 & :1 & 0 & 0 \\ 0 & -4 & 4 & :-1 & 1 & 0 \\ 0 & 0 & 0 & :-2 & -3 & 1 \end{matrix}$$

At the point is row equivalent to **A=** $\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{0} & \mathbf{-4} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

The last matrix under A. since B has a row of zeros; we stop and conclude that A is a singular matrix.

Theorem 2.11 if A and B are n×n matrices s.t. AB=I_n then BA=I_n .thus B= A^{-1}

Remark theorem 2.11 implies that if we want to check whether a given matrix B is A^{-1} we need merely check whether AB=$I_n$or BA =I_n .thus is we do not have to check both equalities.

Def. If A and B are two m×n matrices then **A is equivalent to** B if we obtain B from A by a finite sequences of elementary row or elementary column operations.

Solving a system of linear equations by using an inverse matrix

Consider the linear system $a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_1$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n = b_2$$

.

.

.

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots + a_{nn}x_n = b_n$$

Then the coefficient matrix

n×n A is A=$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} \ldots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and b=$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$, x=$\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}$

If A is invertible then we can solve the given system of linear equations by using an inverse matrix and the solution is $\mathbf{x} = \mathbf{A^{-1}b}$

<u>Ex.</u> Use an inverse matrix to solve the following system:

(1) 2x+3y+z=-1

3x+3y+z=1

2x+4y+z=-2

<u>Solution:</u>

The coefficient matrix is A= $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

and $b = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. find A^{-1} (H.W)

= $\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} \mathrm{A}^{-1}$

Then the solution of given system is $\mathrm{x} = \mathrm{A}^{-1}\mathrm{b} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

x=2 ,y=-1,z=-2

(2) 2x+3y+z=0

3x+3y+z=0

2x+4y+z=0

<u>Solution:</u>

The coefficient matrix is A= $\begin{bmatrix} 2 & 3 & 1 \\ 3 & 3 & 1 \\ 2 & 4 & 1 \end{bmatrix}$

And $b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. find A^{-1} (H.W)

$$= \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 6 & -2 & -3 \end{bmatrix} A^{-1}$$

Then the solution of given system is $x = A^{-1}b = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

The solution is trivial x=0 ,y=0,z=0

Exercises

1- Let A= $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{-2} & \mathbf{0} & \mathbf{1} \end{bmatrix}$

a- Find a matrix C in reduced row echelon form that is row equivalent to A.recored the row operations used.

b- Apply the same operations to I_3that were used to obtain C.denoted the resulting matrix by B.

c- How are A and B related?(find AB and BA)

2- Find the inverse for the following matrices

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$

3- Which of the given matrices are singular? for the non singular ones ,find the inverse

$$\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

4- Find the inverse, if it exists, of each of the following:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

5- Prove that the following matrices are nonsingular

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \mathrm{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix}$$

6- If $A^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}, find\ A$

7- Which of the following homo. system have a non trivial solution?

2x+z=0	3x+y+2z=0
2x+2y+2z=0	-x+y+z=0
4x+2y+3z=0	X+2y+z=0

8- Find all values of a for which the inverse of $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{bmatrix}$ $is\ exists. what\ is\ A^{-1}?$

9- Use an inverse matrix to solve the following systems:

(a) 2x-y=-3

2x+y=7

(b) x+2y+z=1

X+2y-z=3

x-2y+z=-3

Chapter (3)

Determinants

Def. **the trace** of a square (n×n) matrix A= $[a_{ij}]$ is $\mathbf{Tr(A)} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{a_{ii}}$

Ex. **Find Tr(A) where** $(\boldsymbol{a}) A = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{3} & \mathbf{4} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} \end{bmatrix}$ $(\boldsymbol{b})\ A = \begin{bmatrix} \mathbf{5} & \mathbf{1} \\ \mathbf{-2} & \mathbf{3} \end{bmatrix}$

Solution (a) Tr(A)=3 (b) Tr(A)=8

Def.(3.1) Let S={1,2,3,…..,n} be the set of integers from1 to n,arranged in ascending oder.A rearranged $j_1, j_2, \dots \dots, j_n$of the elements of S is called a **permutation** of S.We can consider a permutation of S to be a one-to-one mapping of S onto itself.

Remark A permutation $j_1, j_2, \dots \dots, j_n$ is said to have an **inversion** if a large integer j_r ,precedes a smaller one j_s.A permutation is called **even** if the total number of inversions in it even ,or **odd** if the total number of inversions in it odd .if $n \geq 2$,there are $\frac{n!}{2}$ even and $\frac{n!}{2}$ odd permutation in S_n .

Examples

(1) Let S={1,2,3} the set S_3 of all permutations of S consists of the 3!=6 permutations 123 ,231,312 which are even and 132,213,321 which are odd

(2) S_1 has only 1!=1 permutation:1, which is even because there are no inversions

(2) S_2 has only 2!=2 permutations:21 , which is even (no inversions) and 12 which is odd (one inversion)

Def. Let $A = [a_{ij}]$ be n×n matrix.the **determinant** function,denoted by **det** is

$$\det(A) = \sum(\pm) a_{1j_1} a_{2j_2} \dots \dots \dots a_{nj_n}$$

Where the summation is over all permutation $j_1, j_2, \dots \dots, j_n$ of the set

S ={1,2,3,….,n} the sign is taken as + or – according to whether the permutation $j_1, j_2, \dots \dots, j_n$ even or odd.

We can find the determinant for a square (n×n) matrix A (det(A)=$|A|$) by the following:

1- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

Then det(A)=$|A| = a_{11}a_{22} - a_{12}a_{21}$

<u>**Ex.**</u> **Find** $|A|$ $\boldsymbol{where}$ $A = \begin{bmatrix} 2 & -3 \\ 4 & 5 \end{bmatrix}$

<u>**Solution**</u> $|A| = (2)(5) - (-3)(4) = 22$

2- If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then det(A)=$|A|$= $\begin{matrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{matrix} =$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

<u>**Ex.**</u> **Find** $|A|$ $\boldsymbol{where}$ $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

<u>**Solution**</u> $|A| = \begin{matrix} 1 & 2 & 3 & 1 & 2 \\ 2 & 1 & 3 & 2 & 1 \\ 3 & 1 & 2 & 3 & 1 \end{matrix}$

=(1)(1)(2)+(2)(3)(3)+(3)(2)(1)-(1)(3)(1)-(2)(2)(2)-(3)(1)(3)=6

<u>**Exercises**</u>

(1) Determine whether each of the following permutation of S={1,2,3,4} is even or odd:

(a) 3214 (b) 3124 (c) 2143

(2) Evaluate:

(a) $\begin{vmatrix} 3 & 2 \\ -1 & 4 \end{vmatrix}$

(b) $\begin{vmatrix} 5 & 1 \\ -2 & 3 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 1 & 3 \\ -3 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix}$

(d) det$\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\right)$

(e) det$\left(\begin{bmatrix} 2 & 1 & 3 \\ 3 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}\right)$

(f) $\begin{vmatrix} 3 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 2 \end{vmatrix}$

(3) If det=0 then find the value of t

(a) det$\left(\begin{bmatrix} t & -3 \\ 2 & t-1 \end{bmatrix}\right)$

(b) $\begin{vmatrix} t & 4 \\ 5 & t-8 \end{vmatrix}$

Properties of Determinants

Theorem 3.1 If A is a matrix then det(A)=det(A^T)

Ex. **Consider the matrix** $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$

Then prove that det(A)=det(A^T)

Solution $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 3 & 2 \end{bmatrix}$

det(A)=det(A^T)=6

Theorem 3.2 If matrix B results from A by interchange two different rows(columns) of A .then det(B)=-det(A)

Ex. Let A= $\begin{bmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} \end{bmatrix}$

Then $|A| = \begin{vmatrix} \mathbf{2} & \mathbf{-1} \\ \mathbf{3} & \mathbf{2} \end{vmatrix} = -\begin{vmatrix} \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{-1} \end{vmatrix} = \mathbf{7}$

Theorem 3.3 If two rows (columns) of A are equal then det(A)=0

Ex. Find $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{-1} & \mathbf{0} & \mathbf{7} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{vmatrix}$

Solution $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 7 \\ 1 & 2 & 3 \end{vmatrix} = 0$

Theorem 3.4 If a row (column) of A consists entirely of zeros then det(A)=0

Ex. Find $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \end{vmatrix}$

Solution $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 0$

Theorem 3.5 If B is obtained from A by multiplying a row (column) of A by a real number k then det(B)=kdet(A)

Ex. Let $A = \begin{bmatrix} \mathbf{2} & \mathbf{6} \\ \mathbf{1} & \mathbf{12} \end{bmatrix}$ **.Find** $|A|$

Solution $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 18$

Or $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 2(12-3) = 18$

Or $\begin{vmatrix} 2 & 6 \\ 1 & 12 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = 6(4-1) = 18$

Ex. Let $A = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{1} & \mathbf{5} & \mathbf{3} \\ \mathbf{2} & \mathbf{8} & \mathbf{6} \end{bmatrix}.\boldsymbol{Find}\ |A|$

Solution $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 5 & 3 \\ 2 & 8 & 6 \end{vmatrix} = 0$

Or$\begin{vmatrix}1 & 2 & 3\\1 & 5 & 3\\2 & 8 & 6\end{vmatrix} = 2\begin{vmatrix}1 & 2 & 3\\1 & 5 & 3\\1 & 4 & 3\end{vmatrix} = 0$

Or$\begin{vmatrix}1 & 2 & 3\\1 & 5 & 3\\2 & 8 & 6\end{vmatrix} = 2\begin{vmatrix}1 & 2 & 3\\1 & 5 & 3\\1 & 4 & 3\end{vmatrix} = (2)(3)\begin{vmatrix}1 & 2 & 1\\1 & 5 & 1\\1 & 4 & 1\end{vmatrix} = 0$

<u>Theorem 3.6</u> If B=$[b_{ij}]$ is obtained from A=$[a_{ij}]$ by adding to each element of the rth row (column) of A, k times the corresponding element sth row (column),s≠r, of A then det(B)= det(A)

<u>Ex.</u> We have $\begin{vmatrix}1 & 2 & 3\\2 & -1 & 3\\1 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 0 & 9\\2 & -1 & 3\\1 & 0 & 1\end{vmatrix}$

Obtained by adding twice the second row to the first row.

Then $\begin{vmatrix}1 & 2 & 3\\2 & -1 & 3\\1 & 0 & 1\end{vmatrix} = \begin{vmatrix}5 & 0 & 9\\2 & -1 & 3\\1 & 0 & 1\end{vmatrix} = 4$

<u>Theorem 3.7</u> If a matrix A=$[a_{ij}]$ is upper (lower) triangular,then det(A)=$a_{11}a_{22}\dots\dots a_{nn}$that is,the determinant of a triangular matrix is the product of the elements on the main diagonal.

<u>Ex.</u> Compute $\begin{vmatrix}1 & 0 & 0\\2 & -1 & 0\\1 & 0 & 4\end{vmatrix}$

<u>Solution</u> $\begin{vmatrix}3 & 0 & 0\\2 & -1 & 0\\-1 & 0 & 4\end{vmatrix} = -12$

<u>Remark</u> we can find a determinant of a matrix A by transformation a matrix A to triangular matrix.

<u>Ex.</u> Compute det(A) where $\begin{bmatrix}\mathbf{4} & \mathbf{3} & \mathbf{2}\\\mathbf{3} & \mathbf{-2} & \mathbf{5}\\\mathbf{2} & \mathbf{4} & \mathbf{6}\end{bmatrix}$

<u>Solution</u> $\begin{bmatrix}4 & 3 & 2\\3 & -2 & 5\\2 & 4 & 6\end{bmatrix}$

$=2\begin{bmatrix} 4 & 3 & 2 \\ 3 & -2 & 5 \\ 1 & 2 & 3 \end{bmatrix}$

$=-2\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 5 \\ 4 & 3 & 2 \end{bmatrix}$

$=-2\begin{bmatrix} 1 & 2 & 3 \\ 0 & -8 & -4 \\ 0 & -5 & -10 \end{bmatrix}$

$=(-2)(-5)(-4)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$=-(-2)(-5)(-4)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

$=-(-2)(-5)(-4)\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -3 \end{bmatrix}$

=-(-2)(-5)(-4)(-3)=-120

Lemma 3.1 If E is an elementary matrix,then det(EA)=det(E)det(A),and det(AE)=det(A)det(E).

Theorem 3.8 If A is an n×n matrix,then A is nonsingular iff det(A)≠0 .

Corollary 3.1 If A is an n×n matrix,then Ax=0 has a nontrivial solution iff det(A)=0.

Ex. Let A be a 4×4 matrix with det(A)=-2

(a) Describe the set of all solutions to the homo. system Ax=0

(b) If A is transformed to reduced row echelon form B,what is B?

(c) Give an expression for a solution to the linear system Ax=b,where $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

(d) Can the linear system Ax=b have more than one solution?Explain

(e) Does A^{-1} $exist$?

Solution

(a) Since det(A)≠0,by Corollary 3.1 , the homo. system has only the trivial solution.

(b) Since det(A)≠0,A is a nonsingular matrix ,and B=I_n

(c) A solution to the given system is given by $x = A^{-1}b$

(d) No,the solution given in part (c) only one.

(e) Yes

Theorem 3.9 If A and B are n×n matrices, then det(AB)=det(A)det(B).

Ex. **Let** $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{2} & -\mathbf{1} \\ \mathbf{1} & \mathbf{2} \end{bmatrix}$ $\boldsymbol{then\ prove\ theorem\ 3.9}$

Solution AB=$\begin{bmatrix} \mathbf{4} & \mathbf{3} \\ \mathbf{10} & \mathbf{5} \end{bmatrix}$

det(AB)=-10 ,det(A)=-2 ,det(B)=5

det(AB)=det(A)det(B)=-10

Corollary 3.2 If A is nonsingular, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Corollary 3.2 If A and B are similar matrices, then det(A)=det(B).

Ex. $\boldsymbol{Consider A} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{4} \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & -\mathbf{2} & -\mathbf{4} \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} \mathbf{2} & \mathbf{2} & \mathbf{3} \\ \mathbf{0} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} \end{bmatrix}$

Solution $|A| = 8, |B| = -9\ and\ |C| = -1, so\ |C| = |A| + |B|$

Exercises

(1) Compute the following det.via reduction to triangular form

(a) $\begin{vmatrix} 3 & -1 \\ 2 & 2 \end{vmatrix}$ $(b)\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{vmatrix}$ $(c)\begin{vmatrix} 3 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 3 & 2 \end{vmatrix}$

(2) Verify that det(AB)=det(A)det(B) for the following:

$$A = \begin{vmatrix} 1 & 3 & -2 \\ -2 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix} \, , B = \begin{vmatrix} 1 & 0 & 2 \\ 3 & -2 & 5 \\ 2 & 1 & 3 \end{vmatrix}$$

(3) If det(A)=3,find det(A^2)

(4) Use Theorem 3.8 to determine which of the following matrices are nonsingular:

(a) $\begin{bmatrix} 2 & 3 & -1 \\ 3 & 6 & 1 \\ -1 & 0 & 3 \end{bmatrix}$ $(b)\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(5) Use corollary 3.1 to find out whether the following homo. system has a nontrivial solution(do not solve):

x-2y+z=0

2x+3y+z=0

3x+y+2z=0

Cofactor Expansion

Def.(3.3) Let A=$[a_{ij}]$be an n×n matrix.Let M_{ij} be the (n-1)×(n-1) sub matrix of A obtained by deleting the ith row and jth column of A .The determinant det(M_{ij}) is called the minor of a_{ij}.

Def.(3.4) Let A=$[a_{ij}]$be an n×n matrix. The **cofactor** A_{ij} of a_{ij} is define as A_{ij}=$(-1)^{i+j}$det(M_{ij}).

Ex. Let A= $\begin{bmatrix} \mathbf{3} & \mathbf{-1} & \mathbf{2} \\ \mathbf{4} & \mathbf{5} & \mathbf{6} \\ \mathbf{7} & \mathbf{1} & \mathbf{2} \end{bmatrix}$

$$\det(M_{12}) = \begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} = 8 - 42 = -34$$

$$\det(M_{23}) = \begin{vmatrix} 3 & -1 \\ 7 & 1 \end{vmatrix} = 3 + 7 = 10$$

$$\det(M_{31}) = \begin{vmatrix} -1 & 2 \\ 5 & 6 \end{vmatrix} = -6 - 10 = -16$$

$$A_{12} = (-1)^{1+2}\det(M_{12}) = (-1)(-34) = 34$$

$$A_{23} = (-1)^{2+3}\det(M_{23}) = (-1)(10) = -10$$

$$A_{31} = (-1)^{3+1}\det(M_{31}) = (1)(-16) = -16$$

Remark if we think of the sign $(-1)^{i+j}$ as being located in position (i,j) of an n×n matrix ,then the sign form a checkerboard pattern that has a + in the (1,1)position The patterns for n=3 and n=4 are as follows:

n=3 $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

n=4 $\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$

Theorem3.10 Let A=$[a_{ij}]$ be n×n matrix.Then

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots.. + a_{in}A_{in} (expansion\ of\ \det(A)\ along\ ith\ row)$$

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + .. + a_{nj}A_{nj} (expansion\ of\ \det(A)\ along\ jth\ column)$$

<u>Ex.</u> Evaluate $\begin{vmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} & \mathbf{4} \\ \mathbf{-4} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{-3} \\ \mathbf{2} & \mathbf{0} & \mathbf{-2} & \mathbf{3} \end{vmatrix}$

<u>Solution</u> expand along the third row, we have:

$$=3\begin{vmatrix} 2 & -3 & 4 \\ 2 & 1 & 3 \\ 0 & -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 1 & -3 & 4 \\ -4 & 1 & 3 \\ 2 & -2 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 & 4 \\ -4 & 2 & 3 \\ 2 & 0 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 2 & -3 & 4 \\ -4 & 2 & 1 & 3 \\ 3 & 0 & 0 & -3 \\ 2 & 0 & -2 & 3 \end{vmatrix}$$

$$(-3)\begin{vmatrix} 1 & 2 & -3 \\ -4 & 2 & 1 \\ 2 & 0 & -2 \end{vmatrix}$$

=3(20)+0+0-(-3)(-4)=48

<u>Ex.</u> Evaluate $\begin{vmatrix} \mathbf{0} & \mathbf{-2} & \mathbf{-3} \\ \mathbf{2} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{4} \end{vmatrix}$

<u>Solution</u> expand along the first column, we have:

$$\begin{vmatrix} 0 & -2 & -3 \\ 2 & 1 & -1 \\ 0 & -2 & 4 \end{vmatrix} = 0\begin{vmatrix} 1 & -1 \\ -2 & 4 \end{vmatrix} - 2\begin{vmatrix} -2 & -3 \\ -2 & 4 \end{vmatrix} + 0\begin{vmatrix} -2 & -3 \\ 1 & -1 \end{vmatrix}$$

=0-2(-8-6)+0=28

<u>Exercises</u>

(1) Let A=$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-3} \\ \mathbf{3} & \mathbf{1} & \mathbf{4} \\ \mathbf{4} & \mathbf{3} & \mathbf{-2} \end{bmatrix}$**.Find the following minors:**

(a) $\det(M_{23})$ $(\boldsymbol{b})\det(M_{13})$ $(\boldsymbol{c})\det(M_{31})$ $(\boldsymbol{d})\det(M_{32})$

(2) Let A=$\begin{bmatrix} \mathbf{-1} & \mathbf{2} & \mathbf{3} \\ \mathbf{-2} & \mathbf{5} & \mathbf{4} \\ \mathbf{0} & \mathbf{1} & \mathbf{-3} \end{bmatrix}$**.Find the following minors:**

(a) A_{13} $(\boldsymbol{b})A_{21}$ $(\boldsymbol{c})A_{32}$ $(\boldsymbol{d})A_{33}$

(3) Evaluate the given determinants by using theorem3.10

(a) $\begin{vmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{4} \end{vmatrix}$ $(\boldsymbol{b})$ $\begin{vmatrix} \mathbf{4} & \mathbf{2} & \mathbf{2} & \mathbf{0} \\ \mathbf{2} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \end{vmatrix}$ $(\boldsymbol{c})$ $\begin{vmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{2} & \mathbf{3} & \mathbf{0} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \end{vmatrix}$ $(\boldsymbol{d})$ $\begin{vmatrix} \mathbf{2} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{5} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{-1} \end{vmatrix}$

(4) Find all values of t for which:

(a) $\begin{vmatrix} \boldsymbol{t-2} & \mathbf{2} \\ \mathbf{4} & \boldsymbol{t-4} \end{vmatrix} = \mathbf{0}$ (b) $\begin{vmatrix} \boldsymbol{t+3} & \mathbf{-3} \\ \mathbf{2} & \boldsymbol{t-2} \end{vmatrix} = \mathbf{0}$

Theorem3.11 **If** A=[a_{ij}] is an n×n matrix, then

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0 \; for \; i \neq k$$

$$a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = 0 \; for \; j \neq k$$

Remark We may summarize our expansion result by writing

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \cdots + a_{in}A_{kn} = 0 \; if \; i \neq k$$

=det(A) if i=k

And $a_{1j}A_{1k} + a_{2j}A_{2k} + \cdots + a_{nj}A_{nk} = 0 \; if \; j \neq k$

=det(A) if j=k

Ex. Let A=$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{-2} & \mathbf{3} & \mathbf{1} \\ \mathbf{4} & \mathbf{5} & \mathbf{-2} \end{bmatrix}$

$\det(M_{21}) = \begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} = -4 - 15 = -19$

$\det(M_{22}) = \begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = -2 - 12 = -14$

$\det(M_{23}) = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 5 - 8 = -3$

$A_{21} = (-1)^{2+1}\det(M_{21}) = (-1)(-19) = 19$

$A_{22} = (-1)^{2+2}\det(M_{22}) = -14$

$A_{23} = (-1)^{2+3}\det(M_{23}) = (-1)(-3) = 3$

$a_{31}A_{21} + a_{32}A_{22} + a_{33}A_{23}$ =0

$a_{11}A_{21}+a_{12}A_{22}+a_{13}A_{23}$ =0

Def.(3.5) Let A=[a_{ij}]be an n×n matrix. The n×n matrix adj A, called the **adjoint** of A,is the matrix whose (i,j)th entry is the cofactor A_{ji} of a_{ji} thus

$$\mathbf{adjA}=\begin{bmatrix}\mathbf{A_{11}} & \mathbf{A_{21}} & \dots & \mathbf{A_{n1}}\\ \mathbf{A_{12}} & \mathbf{A_{22}} & \dots & \mathbf{A_{n2}}\\ \vdots & \vdots & \vdots & \vdots\\ \mathbf{A_{1n}} & \mathbf{A_{2n}} & \dots & \mathbf{A_{nn}}\end{bmatrix}$$

Ex. Let A=$\begin{bmatrix}\mathbf{3} & \mathbf{-2} & \mathbf{1}\\ \mathbf{5} & \mathbf{6} & \mathbf{2}\\ \mathbf{1} & \mathbf{0} & \mathbf{-3}\end{bmatrix}$ $\boldsymbol{.compute\ adj\ A}$

Solution

$A_{11}=(-1)^{1+1}\det(M_{11})=-18$

$A_{12}=(-1)^{1+2}\det(M_{12})=17$

$A_{13}=(-1)^{1+3}\det(M_{13})=-6$

$A_{21}=(-1)^{2+1}\det(M_{21})=-6$

$A_{22}=(-1)^{2+2}\det(M_{22})=-10$

$A_{23}=(-1)^{2+3}\det(M_{23})=-2$

$A_{31}=(-1)^{3+1}\det(M_{31})=-10$

$A_{32}=(-1)^{3+2}\det(M_{32})=-1$

$A_{33}=(-1)^{3+3}\det(M_{33})=28$

$$\therefore\ \text{adj A}=\begin{bmatrix}-18 & -6 & -10\\ 17 & -10 & -1\\ -6 & -2 & 28\end{bmatrix}$$

Theorem3.12 Let A=[a_{ij}]is an n×n matrix, then A(adj A)= (adjA)A=det(A) I_n

Ex. Let A=$\begin{bmatrix}\mathbf{3} & \mathbf{-2} & \mathbf{1}\\ \mathbf{5} & \mathbf{6} & \mathbf{2}\\ \mathbf{1} & \mathbf{0} & \mathbf{-3}\end{bmatrix}$

$$\text{adj A}=\begin{bmatrix}-18 & -6 & -10\\ 17 & -10 & -1\\ -6 & -2 & 28\end{bmatrix}$$

$$=\begin{bmatrix}-94 & 0 & 0\\ 0 & -94 & 0\\ 0 & 0 & -94\end{bmatrix}\begin{bmatrix}3 & -2 & 1\\ 5 & 6 & 2\\ 1 & 0 & -3\end{bmatrix}\begin{bmatrix}-18 & -6 & -10\\ 17 & -10 & -1\\ -6 & -2 & 28\end{bmatrix}$$

$$=-94\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=-94\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-18 & -6 & -10\\ 17 & -10 & -1\\ -6 & -2 & 28\end{bmatrix}\begin{bmatrix}3 & -2 & 1\\ 5 & 6 & 2\\ 1 & 0 & -3\end{bmatrix}$$

<u>**Corollary3.4**</u> If A is n×n matrix and det(A)≠0 then $\boldsymbol{A^{-1}} = \frac{\mathbf{1}}{\mathbf{det}(A)}(\boldsymbol{adjA})$

<u>**Ex.**</u> **Find A^{-1} by using adj(A) where A=**$\begin{bmatrix}\mathbf{1} & \mathbf{4} & \mathbf{-2}\\ \mathbf{0} & \mathbf{-3} & \mathbf{-1}\\ \mathbf{0} & \mathbf{0} & \mathbf{2}\end{bmatrix}$

<u>**Solution**</u> det(A)=1(-3)(2)=-6 (since the matrix A is upper triangular matrix)

adj(A)=$\begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix}$

$$A_{11} = (-1)^{1+1}\det(M_{11}) = \begin{vmatrix}-3 & -1\\ 0 & 2\end{vmatrix} = -6$$

$$A_{12} = (-1)^{1+2}\det(M_{12}) = \begin{vmatrix}0 & -1\\ 0 & 2\end{vmatrix} = 0$$

$$A_{13} = (-1)^{1+3}\det(M_{13}) = \begin{vmatrix}0 & -3\\ 0 & 0\end{vmatrix} = 0$$

$$A_{21} = (-1)^{2+1}\det(M_{21}) = \begin{vmatrix}4 & -2\\ 0 & 2\end{vmatrix} = -8$$

$$A_{22} = (-1)^{2+2}\det(M_{22}) = \begin{vmatrix}1 & -2\\ 0 & 2\end{vmatrix} = 2$$

$$A_{23} = (-1)^{2+3}\det(M_{23}) = \begin{vmatrix}1 & 4\\ 0 & 0\end{vmatrix} = 0$$

$$A_{31} = (-1)^{3+1}\det(M_{31}) = \begin{vmatrix}4 & -2\\ -3 & -1\end{vmatrix} = -4-6 = -10$$

$$A_{32} = (-1)^{3+2}\det(M_{32}) = \begin{vmatrix}1 & -2\\ 0 & -1\end{vmatrix} = -(-1) = 1$$

$$A_{33} = (-1)^{3+3}\det(M_{33}) = \begin{vmatrix}1 & 4\\ 0 & -3\end{vmatrix} = -3$$

adj(A)=$\begin{bmatrix} -6 & -8 & -10 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$∴

$$\therefore\ \boldsymbol{A^{-1}} = \frac{\mathbf{1}}{\mathbf{-6}}\begin{bmatrix} -6 & -8 & -10 \\ 0 & 2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \text{(upper triangular matrix)}$$

Exercises

(1) Let A= $\begin{bmatrix} 2 & 1 & 3 \\ -1 & 2 & 0 \\ 3 & -2 & 1 \end{bmatrix}$

(a) Find adjA

(b) Compute det(A)

(c) Find $\boldsymbol{A^{-1}}$

(d) Show that A(adjA)=(adjA)A=det(A)$\boldsymbol{I_3}$

(2) Find inverse if it exist, of by using corollary 3.4

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 2 \\ 3 & 2 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} -3 & 2 & 4 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{bmatrix}$

(d) $\begin{bmatrix} \mathbf{2} & \mathbf{3} \\ \mathbf{4} & \mathbf{-1} \end{bmatrix}$

Cramer's rule

Theorem 3.13 Let

$$a_{11}x_1 + a_{12}x_2 + \cdots \ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots \ldots + a_{2n}x_n = b_2$$

.

.

$$a_{n1}x_1 + a_{n2}x_2 + \cdots \ldots + a_{nn}x_n = b_n$$

Be a linear system of n equation in n unknowns, and let A=[a_{ij}] be the coefficients matrix so that we can write the given system as Ax=b,where $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$

if det(A)≠0, then the system has the unique solution

$$x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots \ldots \ldots \ldots, x_n = \frac{\det(A_n)}{\det(A)}$$

Where A_i is the matrix obtained from A by replacing the ith column of A by b.

Ex. Solve the linear system by using Cramer's rule

-2x+3y-z=1

X+2y-z=4

-2x-y+z=-3

Solution

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$$

$$x_1 = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{|A|} = \frac{-4}{-2} = 2$$

$$x_2 = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{|A|} = \frac{-6}{-2} = 3$$

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 2 \\ -2 & -1 & -3 \end{vmatrix}}{|A|} = \frac{-8}{-2} = 4$$

$\therefore$ the solution of given system is $x_1 = 2, x_2 = 3, x_3 = 4$

Remark:

We have shown that the following statements are equivalent for an n×n matrix A:

1- A is nonsingular

2- Ax=0 has only the trivial solution

3- A is row (column)equivalent to I_n

4- The linear system Ax=b has a unique solution for every n×1 matrix b

5- A is a product of elementary matrix

6- det(A)≠0

Exercises

Solve the following linear system by using Cramer's rule

(a) 3x+4y+6z=4

x-3y=4

x+3y-z=8

b) x-2y+3z=5

x-3y+2z =4

2y+z=3

(c) x+y-2z=1

x-2y-3z =4

2x+ y+z=3

(d) x+2y+4z=0

2x-4z=0

4y+4z=0