Linear algebra2

Chapter -1-

Real Vector Space

(1.1) Vectors in the plane

We draw a pair of perpendicular lines intersecting at a point **O,** called the **origin.** One of the lines, the **x-axis,** is usually taken in a horizontal position.

The other line, the **y-axis,** is then taken in a vertical position. The x- and y-axes together are called **coordinate axes,** and they form a **rectangular coordinate system** or a **Cartesian coordinate system.**

We now choose a point on the x-axis to the right of O and a point on the y-axis above O to fix the units of length and **positive direction** on the x- and y- axes. Frequently, but not always these point are chosen so that they are both equidistant from O-that is ,so that the same unite of length is used for both axes.

With each point **p** in the plane we associate an order pair (x,y) of real numbers, its **coordinate .**Conversely ,we can associate a point in the plane with each ordered pair of real numbers .Point **p** with coordinate (x,y) is denoted by **p(x,y)** or simply **(x,y).**

The set of all points in the plane is denoted byR^2 ; it is called **2-space.**

Remark: Consider the 2×1 matrix X=$\begin{bmatrix} x \\ y \end{bmatrix}$

Where x,y are real numbers . with x we associate the directed line segment with the **initial point the origin O** and **terminal point p(x,y) .**

The direct line segment from O to P is denoted $\overrightarrow{OP}$

O is called its **tail** and P its **head** .we distinguishes tail and head by placing an arrow at the head. A directed line segment has a **direction,** indicated by the arrow at its head

The **magnitude** of a directed line segment is its length. Thus a directed line segment can be used to describe force, velocity or acceleration. Conversely, with the direct line segment $\overrightarrow{OP}$ with tail O(0,0) and head P(x,y) we can associate the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$

<u>Def.</u> A vector in the plane is a 2×1 matrix X= $\begin{bmatrix} x \\ y \end{bmatrix}$

Where x and y are real numbers, called the **components (or entries)** of X .we refer to a vector in the plane merely as a **vector** or as a **2-vector.**

<u>Remark</u> Since a vector is a matrix, the vectors $u = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $v = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

Are said to be **equal** if $x_1 = x_2$ and $y_1 = y_2$ .That is, two vectors are equal if their respective components are equal.

<u>Ex.</u> Find a,b where the vectors $\begin{bmatrix} a+b \\ 2 \end{bmatrix}$ **,** $\begin{bmatrix} 3 \\ a-b \end{bmatrix}$ **are equal**

<u>Solution:</u> $\begin{bmatrix} a+b \\ 2 \end{bmatrix}$ = $\begin{bmatrix} 3 \\ a-b \end{bmatrix}$

Then a+b=3

a-b=2

by solve the linear system obtain $a = \frac{5}{2}$ and $b = \frac{1}{2}$

<u>Def.</u> Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Be two vectors in the plane .The **sum** of the vectors **u and v** is the vector

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$

<u>Remark</u> observes that vector addition is a special case of matrix addition.

<u>Ex.</u> Find u+v where $\boldsymbol{u} = \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \end{bmatrix}$ **,v=** $\begin{bmatrix} \mathbf{3} \\ \mathbf{-4} \end{bmatrix}$

Solution: $u + v = \begin{bmatrix} 2+3 \\ 3+(-4) \end{bmatrix}$ = $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$

Def. If $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \end{bmatrix}$ is a vector and **c** is a scalar (a real number) ,then the **scalar multiplication cu** of **u** by **c** is the vector $\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}$ .Thus the scalar **cu** is obtained by multiplying each component of **u** by **c** .If c>0 then **cu** is in the same direction as **u**,whereas if d<0 then **du** is in the opposite direction.

Ex. Find cu,du if c=2,d=-3 and $\boldsymbol{u} = \begin{bmatrix} \mathbf{2} \\ \mathbf{-3} \end{bmatrix}$

Solution $cu = 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$

$du = -3\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$

Remark

(1) The vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is called the **zero vector** and is denoted by **0.**if **u** is any vector then **u+0=u**

(2) **(-1)u=-u** it is called the **negative of u** and **u+(-1)u=u-u=0**

(3) If u and v are any vectors then **u+(-1)v=u-v** it is called **the difference between u and v**

Vectors in Space

We first fix a **coordinate system** by choosing a point called **the origin** and three lines called **the coordinate axes** each passing through the origin so that each line is perpendicular to other two. These lines are individually called the x,y and z-axes.

With each point P in space we associate an order triple(x,y,z) of real numbers its coordinates .conversely, we can associate a point in space with each ordered triple of real numbers.

The point P with coordinates x,y and z is denoted by P(x,y,z) or (x,y,z)

The set of all points in space is called **3-space** and is denoted by R^3

A **vector in space, or 3-vector, or** simple a **vector** is a **3×1 matrix** $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ \boldsymbol{z} \end{bmatrix}$

Where x,y,z are real numbers called **the components** of vector **X .**

Two vectors in space are said to be **equal** if their **respective components are equal .**

With the vector $\mathrm{X} = \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \\ \mathrm{z} \end{bmatrix}$ we associate the directed line segment $\overrightarrow{OP}$, whose tail O(0,0,0) and whose head is P(x,y,z);conversely ,with each directed line segment we associate the vector X.

Remark as in the plane, in physical application we often deal with a directed line segment $\overrightarrow{PQ}$ from point P(x,y,z) (not the origin) to the point Q(x', y', z')

The components of such a vector are $(x' - x, y' - y, z' - z)$

Remark

(1) if $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $and\ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are vectors in R^3 then the **sum u+v** is define

$$u + v = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}$$

(2) if $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is vector in R^3 then the **scalar multiple cu** is define $cu = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$

(3) The **zero** vector in R^3 is denoted by **0** where $\mathbf{0} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$

If **u** is any vector in R^3 then **u+0=u**

(4) The **negative** of the vector $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $is\ the\ vector - u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and

$$\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$$

<u>Remark</u> a vector in plane as an ordered pair of real numbers or as 2×1 matrix.

A vector in space is an ordered triple of real numbers or 3×1 matrix.

<u>Ex.</u> Let $\boldsymbol{u} = \begin{bmatrix} \mathbf{2} \\ \mathbf{3} \\ \mathbf{-1} \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} \mathbf{3} \\ \mathbf{-4} \\ \mathbf{2} \end{bmatrix}$ compute: (a) u+v; (b)-2u; (c) 3u-2v

<u>Solution</u>

(a) $\mathrm{u} + \mathrm{v} = \begin{bmatrix} 2+3 \\ 3+(-4) \\ -1+2 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 1 \end{bmatrix}$

(b) $-2\mathrm{u} = \begin{bmatrix} -2(2) \\ -2(3) \\ -2(-1) \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \\ 2 \end{bmatrix}$

(c) $3u - 2v = \begin{bmatrix} 3(2) \\ 3(3) \\ 3(-1) \end{bmatrix}$ - $\begin{bmatrix} 2(3) \\ 2(-4) \\ 2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 17 \\ -7 \end{bmatrix}$

<u>Theorem 1.1</u>

If u,v and w are vectors in R^2or $R^3$and c and d are real scalars then the following properties are valid:

(a) u+v=v+u

(b) u+(v+w)=(u+v)+w

(c) u+0=0+u=u

(d) u+(-u)=0

(e) c(u+v)=cu+cv

(f) (c+d)u=cu+du

(g) c(du)=(cd)u

(h) 1u=u

Exercises

(1) Sketch line segment in R^2,representing each of the following vectors:

(a) $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ $\quad$ $(\boldsymbol{b})\ \boldsymbol{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$

(2) For what values of a,b are vectors $\begin{bmatrix} a+b \\ 2 \end{bmatrix}$ **and** $\begin{bmatrix} 6 \\ a-b \end{bmatrix}$ **equal?**

(3) For what values of a,b,c are vectors $\begin{bmatrix} 2a-b \\ a-2b \\ 6 \end{bmatrix}$ **and** $\begin{bmatrix} -2 \\ 2 \\ a+b-2c \end{bmatrix}$ **equal?**

(4) Determine the components of each vector $\overrightarrow{\boldsymbol{PQ}}$

(a) P(2,3),Q(4,5)

(b) P(-2,2,3),Q(-3,5,2)

(5) Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$

C=-2 and d=3.compute each the following:

(a) v+u

(b) cu+dw

(c) u-v+w

(d) cu+dv+w

(6) Let $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

compute each the following:

(a) u+v

(b) u-v

(c) 2u

(d) 2u-3v

(7) Let $x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, y = \begin{bmatrix} -3 \\ 4 \end{bmatrix}, z = \begin{bmatrix} r \\ 4 \end{bmatrix}, u = \begin{bmatrix} -2 \\ s \end{bmatrix}$

Find r,s where

(a) z=2x

(b) z+u=x

(8) If possible, find scalars r,s where $r\begin{bmatrix} 1 \\ -2 \end{bmatrix} + s\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \end{bmatrix}$

(9) If possible, find scalars x,y,z ,not all zero ,so that

$$x\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + z\begin{bmatrix} 3 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(1.2) vector spaces

Def. A real vector space is a set V of elements on which we have two operation (+) and(·)define with the following properties:

(a) If **u and v** are any elements in V, then **u+v in V** (we say that Vis **closed** under the operation(+))

(1) **u+v=v+u** for all u,v in V

(2) **u+(v+w)=(u+v)+w** for all u,v and w in V

(3) There exists an element **0** in V such that **u+0=0+u =u** for any u in V

(4) For each u in V there exists an element **–u** in V such that **u+(-u)=-u+u=0**

(b) If **u** is any element in V and **c** is any real number then c.u in V (i.e Vis **closed** under the operation (.))

(5) **c.(u+v)=c.u+c.v** for any u,v in V and any real number c

(6) **(c+d).u=c.u+d.u** for any u,v in V and any real numbers c,d

(7) **c.(d.u)=(cd).u** for any u in V and any real numbers c,d

(8) **1.u=u** for any u in V

Remark

(1) The elements of V are called **vectors**

(2) The elements of the set of real number R are called **scalars**

(3) The operation **(+)** is called **vector addition**

(4) The operation **(.)** is called **scalar multiplication**

(5) The vector **0** is called **zero vector**

(6) The vector **–u** is called **a negative of u**

(7) The vector **0** and **–u** are **unique**

Remark In order to specify **a vector space**, we must be given a set V and two operation (+) and (.) satisfying all the properties of the definition we shall often refer to real vector space merely as **a vector space.**

Ex.1 Consider R^{n}, the set of all matrices $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ with real entries.

Let the operation (+) be matrix addition and let the operation (.) by multiplication of matrix by a real number (scalar multiplication) thenR^{n}, is a vector space.

Thus the matrix $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ as an element of $\mathrm{R}^{\mathrm{n}}$is called n-vector or vector.

Ex.2 The set of all m×n matrices with matrix addition as (+) and multiplication of a matrix by a real number as (.) is a vector space .We denoted this vector space by M_{mn}

Ex.3 The set of all real numbers with (+) as the usual addition of real numbers and (.) the usual multiplication of real numbers is a vector space.

Ex.4 Let $\mathrm{R_n}$ be the set of all $1 \times \mathrm{n}$ matrices $[\mathrm{a_1} \quad \mathrm{a_2} \quad \cdots \quad \mathrm{a_n}]$

where we define $(+)$

by $[\mathrm{a_1} \quad \mathrm{a_2} \quad \cdots \quad \mathrm{a_n}](+)[\mathrm{b_1} \quad \mathrm{b_2} \quad \dots \quad \mathrm{b_n}]$
$$= [\mathrm{a_1} + \mathrm{b_1} \quad \mathrm{a_2} + \mathrm{b_2} \quad \dots \quad \mathrm{a_n} + \mathrm{b_n}]$$

and define $(.)$by $\mathrm{c}(.)[\mathrm{a_1} \quad \mathrm{a_2} \quad \cdots \quad \mathrm{a_n}] = [\mathrm{ca_1} \quad \mathrm{ca_2} \quad \cdots \quad \mathrm{ca_n}]$

then $\mathrm{R_n}$ is a vector space.

Ex.5 Let V be the set of all 2×2 matrices with trace equal to zero ;that is A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in V provided $\mathrm{Tr(A)} = \mathrm{a} + \mathrm{d} = 0$

The operation (+) is standard matrix addition and the operation (.) is standard multiplication of matrices then V is a vector space.

Ex.6 The set P_n of all polynomials of degree ≤ n is a vector space

A polynomial in t is a function that is expressible as $P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots \dots + a_1 t + a_o$

Where $a_n, a_{n-1}, \dots \dots, a_1, a_o$ are real numbers and n is a nonnegative integer.

If $a_n \neq 0$ then P(t) is said to have degree n. thus the degree of a polynomial is the highest power of a term having a nonzero coefficient

P(t)=2t+1 has degree 1

P(t)=3 has degree 0

P(t)=0 has no degree denoted by 0

Ex.7 Let V be the set of all real-valued continuous functions define in R.

If f,g are in V and c is a scalar, we define

(f (+) g)(t)=f(t) (+) g(t)

(c (.) f)(t)=cf(t)

The vector space which is denoted by $C(-\infty, \infty)$

Ex.8 Let V be the set of all real numbers with the operation

u (+) v=u-v and c (.) u=cu

V is not vector space because property (6) does not hold since

(c+d)(.)u=(c+d)u=cu+du

Whereas c (.) u (+) d (.) u=cu(+)du=cu-du

are not equal in general

Ex.9 Let V be the set of all order triples of real numbers (x,y,z) with the operation $(x, y, z)\ (+)\ (x', y', z') = (x', y + y', z + z')$

$$and\ c\ (.)(x, y, z) = (cx, cy, cz)$$

V is not vector space because property 1,3,4,6 fails to hold.

Ex.10 Let V be the set of all integer with the operation (+)as ordinary addition and(.) as ordinary multiplication.

V is not vector space because if u is any nonzero vector in V and $c = \sqrt{3}$

Then c (.)u is not in V

Theorem 1.2 If V is vector space, then

(a) 0(.)u=0 for any vector u in V

(b) c(.)0=0 for any scalar c

(c) If c(.)u=0 ,then either c=0 or u=0

(d) (-1)(.)u=-u for any vector u in V

Remark The following notation and the descriptions of the set:

$R^n the\ set\ of\ n \times 1\ matroces$

$R_n\ the\ set\ of\ 1 \times n\ matrices$

$M_{mn}\ the\ set\ of\ m \times n\ matrices$

P the set of polynomials

P_n the set of all polynomials of degree n

or less together with the zero polynomial

$\mathrm{C}(-\infty, \infty)$the set of all

real − valued continuous functions with domain all real numbers

Exercise

(1) Let V be the set of all polynomials of degree 2 with the def. of addition and scalar multiplication as in Ex.6

(a) Show that V is not closed under addition

(b) Is V closed under scalar multiplication?

(2) Let V be the set of all 2×2 matrices $A = \begin{bmatrix} a & b \\ 3b & d \end{bmatrix}$ **let the operation (+) be stander addition of matrices and the operation (.) be stander multiplication of matrices**

(a) Is V closed under addition?

(b) Is V closed under scalar multiplication?

(c) What is the zero vector in the set V?

(d) Does every matrix A in V have a negative that is in V?

(e) Is V a vector space?

(3) The set of all order triples of real numbers with the operations

$$(x, y, z)(+)(x', y', z') = (x + x', y + y', z + z')$$

And r(.)(x,y,z)=(x,2,z).is the set a vector space?

(4) The set of all 2×1 matrices $\begin{bmatrix} x \\ y \end{bmatrix}$ **where x≤0, with the usual operations in** R^2

is the set a vector space?

(5) The of all order pairs of real numbers with the operations

$$(x, y)(+)(x', y') = (x + x', y + y')\ and\ r(.)(x, y) = (rx, y)$$

Is the set a vector space?

1.3 Sup Spaces

Def. Let V be a vector space and W a nonempty sub set of V.If W is a vector space w.r.t the operations in V,then W is called a **sup space** of V.

Theorem 1.3 Let V be a vector space with operations (+) and (.) and let W be a nonempty sub set of V.Then W is a sub space of V iff the following conditions hold:

(a) If **u and v** are any vectors in W,then **u(+)v** is in W

(b) If **c** is any real number and **u** is any vector in W then **c(.)u** is in W

Ex.1 Every vector space has at least two sub space itself and the sup space {0} (Recall **0(+)0=0** and **c(.)0=0** is any vector space)

Thus **{0}** is closed for both operations and hence **sup space of V**

The sup space **{0}** is called the **zero sup space** of V

Ex.2 Let P_2 be the set consisting of all polynomials of degree ≤2 and the zero polynomial; $P_2$is a sub set of P, the vector space of all polynomials.

Is a sup space of P P_2

In general the set P_n consisting of all polynomials of degree ≤ n and the zero polynomial is a sub space of P.Also P_n is a sub spacse of P_{n+1}

Ex.3 Let V be the set of all polynomials of degree 2;V is a sub set of P,the vector space of all polynomials ;but V is not a sub space of P because the sum of the polynomials $2t^2 + 3t + 1 \; and - 2t^2 + t + 2$ is not in V,since it is a polynomial of degree 1.

Ex.4 Let W be the set of all vectors in $R^3 \; of \; the \; form \begin{bmatrix} a \\ b \\ a+b \end{bmatrix}$

Where a and b are any real numbers.

We let $u = \begin{bmatrix} a_1 \\ b_1 \\ a_1 + b_1 \end{bmatrix}$ $and\ v = \begin{bmatrix} a_2 \\ b_2 \\ a_2 + b_2 \end{bmatrix}$

$$u(+)v = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ (a_1 + b_1) + (a_2 + b_2) \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ (a_1 + a_2) + (b_1 + b_2) \end{bmatrix}$$

And c(.) $u = \begin{bmatrix} ca_1 \\ cb_1 \\ c(a_1 + b_1) \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ ca_1 + cb_1 \end{bmatrix}$

<u>Remark</u> we shall denoted **u(+)v** and **c(.)u** in a vector space V as **u+v** and **cu** ,respectively.

<u>Def.</u> Let $v_1, v_2, \dots \dots, v_k$ be vectors in a vector space V.A vector **v** in V is called a **linear combination** of $v_1, v_2, \dots \dots, v_k$ if

$$v = a_1 v_1 + a_2 v_2 + \cdots \dots + a_k v_k = \sum_{j=1}^{k} a_j v_j$$

For some real numbers $a_1, a_2, \dots \dots, a_k$

<u>Remark</u> The previous def. was stated for a finite set of vectors but it also applies to an infinite set s of vectors in a vector space using corresponding notation for infinite sums.

<u>Ex.1</u> Let W be the set of all vectors in R^3 $of\ the\ form$ $\begin{bmatrix} a \\ b \\ a + b \end{bmatrix}$

Where a, b are any real numbers, is a sub space of R^3

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $and\ v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ then every vector in W is a linear combination of $v_1$and v_2 since $av_1 + bv_2 = \begin{bmatrix} a \\ b \\ a + b \end{bmatrix}$

<u>Ex.2</u> Let P_2 be the set consisting of all polynomials of degree ≤2 and the zero polynomial; every vector in P_2has the form $at^2 + bt + c$,so each vector in $P_2$is a linear combination of $t^2, t and\ 1$.

Ex.3 In R^3 $let\ v_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\0\\2\end{bmatrix}\ and\ v_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$

$$the\ vector\ v = \begin{bmatrix}2\\1\\5\end{bmatrix}\ is\ a\ linear\ combination\ of\ v_1, v_2\ and\ v_3$$

$if\ we\ can\ find\ real\ numbers\ a_1, a_2\ and\ a_3$

$so\ that\ \ a_1v_1 + a_2v_2 + a_3v_3 = v$

Substituting for $v_1, v_2\ and\ v_3 we\ have\ a_1\begin{bmatrix}1\\2\\1\end{bmatrix} + a_2\begin{bmatrix}1\\0\\2\end{bmatrix} +\ a_3\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}2\\1\\5\end{bmatrix}$

Leads to the linear system

$$a_1 + a_2 +\ a_3 = 2$$

$$2a_1 +\ a_3 = 1$$

$$a_1 + 2a_2 = 5$$

Solving this system obtain $a_1 = 1, a_2 = 2\ and\ a_3 = -1$

Then $v = v_1 + 2v_2 -\ v_3$

Exercises

(1) The set W consisting of all points in R^2 of the form (x,x) is a straight line Is W is a subspace of $\boldsymbol{R^2}$?

(2) Let W be the set of all points in $\boldsymbol{R^3}$ that lie in xy-plane. Is W a subspace of $\boldsymbol{R^3}$?

(3) Is the set of all vectors of the following form a subspace of $\boldsymbol{R^3}$?

$$(\boldsymbol{a})\ \begin{bmatrix}\boldsymbol{a}\\\boldsymbol{b}\\\boldsymbol{2}\end{bmatrix}\quad (\boldsymbol{b})\ \begin{bmatrix}\boldsymbol{a}\\\boldsymbol{b}\\\boldsymbol{a+3b}\end{bmatrix}\quad (\boldsymbol{c})\begin{bmatrix}\boldsymbol{a}\\\boldsymbol{0}\\\boldsymbol{0}\end{bmatrix}\quad (\boldsymbol{d})\begin{bmatrix}\boldsymbol{a}\\\boldsymbol{b}\\\boldsymbol{c}\end{bmatrix}\boldsymbol{where\ a - 2b + c = 0}$$

(4) Is the set of all vectors of the following form a subspace of$\boldsymbol{R_4}$?

$(\boldsymbol{a})\ [\boldsymbol{a}\quad \boldsymbol{b}\quad \boldsymbol{c}\quad \boldsymbol{d}]\ \boldsymbol{where\ a + b = 3}\quad (\boldsymbol{b})[\boldsymbol{a}\quad \boldsymbol{b}\quad \boldsymbol{c}\quad \boldsymbol{d}]\boldsymbol{where\ a = 0\ , b = 2d}$

(5) Let W be the set of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ s.t a+b+c+d=0 .Is W a subspace of M_{22}?

(6) Is the set of all 2×3 matrices $\begin{bmatrix} a & b & c \\ d & 0 & 0 \end{bmatrix}$ where c<0 subspace of M_{23}?

1.4 Span

Def. If S={ $v_1, v_2, \dots\dots, v_k$} Is a set of vectors in a vector space V then the set of all vectors in V that are linear combination of the vectors in S is denoted by span S or span { $v_1, v_2, \dots\dots, v_k$}

Remark the definition is stated for a finite set of vectors but it also applies to an infinite set S of vectors in a vector space

Ex.1 Consider the set S of all 2×3 matrices given by

$$S = \{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\}$$

Then the span S is the set in M_{23}consisting of all vectors of the form

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$$

Where a,b,c,d are real number

That is span S is the sub set of M_{23} consisting of all matrices of the form $\begin{bmatrix} a & b & 0 \\ 0 & c & d \end{bmatrix}$

Where a,b,c,d are real numbers

Ex.2 Let $S = \{t^2, t, 1\}$ be a sub set of p_2 we have span S =P_2

$$P_2(t) = at^2 + bt + c \text{ where } a, b, c \text{ are real numbers}$$

Ex.2 Let $S = \left\{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\}$

be a sub set of R^3.span S is the set of all vectors in R^3 of the form

$$a\begin{bmatrix}2\\0\\0\end{bmatrix}+b\begin{bmatrix}0\\-1\\0\end{bmatrix}+c\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}2a\\-b\\0\end{bmatrix}$$

Where a,b,c are real numbers

Theorem 1.4 Let $S=\{v_1,v_2,\ldots\ldots,v_k\}$ be a set of vectors in a vector space V then span S is a sub space of V.

Proof $u=\sum_{j=1}^{k}a_jv_j \quad and\ w=\sum_{j=1}^{k}b_jv_j$

For some real numbers $a_1,a_2,\ldots\ldots,a_k\ and\ \ b_1,b_2,\ldots\ldots,b_k$

$u+w=\sum_{j=1}^{k}a_jv_j\ +\sum_{j=1}^{k}b_jv_j=\sum_{j=1}^{k}(a_j\ +b_j)v_j$

for any real number c$cu=c\left(\sum_{j=1}^{k}a_jv_j\right)=\sum_{j=1}^{k}(ca_j)v_j$

u+w and cu are linear combination of the vectors in S.

Then span S is a sub space of V.

Ex.1 Let S=$\{t^2,t\}$ be a number of the vector space P_2 then span S is the sub space of all polynomials of the form a t^2+bt where a,b are real numbers.

Ex.2 Let $S=\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix},\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

Be subset of the vector space M_{22} then span S is the subspace of all 2×2 diagonal matrices.

Def. Let S be a set of vectors in a vector space V. if every vector in V is a linear combination of the vectors in S then the set S is said to **span V** or V is spanned by the set S that is span S=V.

Remark If span S =V ,S is called a **spanning set** V .A vector space can have many spanning sets.

Ex.1 Let P be the vector space of all polynomials. Let S=$\{1,t,t^2,\ldots\ldots\}$ that is the set of all (nonnegative integer)powers of t.then span S=P .

every spanning set for P will have infinitely many vectors.

Ex.2 In $R^3, let\ v_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\ and\ v_2 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$

Determine whether the vector $v = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}\ belong\ to\ span\{v_1, v_2\}$

Solution If we can find scalars a, b s.t $av_1 + bv_2 = v$

$$a\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \end{bmatrix}$$

We obtain the linear system

2a+b=1

a-b=5

a+3b=-7

Solve this linear system obtain a=2,b=3

is belong to span $\{v_1, v_2\} \therefore v$

Ex.3 In P_2 let $v_1 = 2t^2 + t + 2, v_2 = t^2 - 2t, v_3 = 5t^2 - 5t + 2,$

$$v_4 = -t^2 - 3t - 2$$

$determine\ whether\ the\ vector$

$v = t^2 + t + 2\ belongs\ to\ span\{v_1, v_2, v_3, v_4\}$

Solution If we can find scalars $a_1, a_2, a_3\ and\ a_4\ so\ that$

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = v$$

$$a_1(2t^2 + t + 2) + a_2(t^2 - 2t) + a_3(5t^2 - 5t + 2) + a_4(-t^2 - 3t - 2) = t^2 + t + 2$$

$$(2a_1 + a_2 + 5a_3 - a_4)t^2 + (a_1 - 2a_2 - 5a_3 - 3a_4)t + (2a_1 + 2a_3 - a_4) = t^2 + t + 2$$

Thus we get the linear system

$$2a_1 + a_2 + 5a_3 - a_4 = 1$$

$$a_1 - 2a_2 - 5a_3 - 3a_4 = 1$$

$$2a_1 + 2a_3 - 2a_4 = 2$$

Thus linear system has no solution hence v does not belong to

Span $v_1, v_2, v_3, v_4\}$

Ex.4 Let V be the vector space P_2

let $v_1 = t^2 + 2t + 1, v_2 = t^2 + 2. Does \{v_1, v_2\}$ $span$ V?

Solution Let v= $at^2 + bt + c$

Where a,b,c are real numbers, then

$$a_1 v_1 + a_2 v_2 = v$$

$$a_1(t^2 + 2t + 1) + a_2(t^2 + 2) = at^2 + bt + c$$

$$(a_1 + a_2)t^2 + (2a_1)t + (a_1 + 2a_2) = at^2 + bt + c$$

Thus we get the linear system

$$a_1 + a_2 = \text{a}$$

$$2a_1 = \text{b}$$

$$a_1 + 2a_2 = \text{c}$$

Thus linear system has no solution hence$v_1, v_2\}$ v does not Span V

Exercises

(1) Explain the set S is not a spanning set for the vector space V

(a) $\boldsymbol{S} = \left\{\begin{bmatrix} \mathbf{1} \\ \mathbf{2} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}\right\}, \boldsymbol{V} = \boldsymbol{R}^2$

(b) $\boldsymbol{S} = \left\{\begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{0} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{0} \end{bmatrix}\right\}, \boldsymbol{V} = \boldsymbol{M}_{22}$

(2) Determine whether the given vector p(t) in
$\boldsymbol{p_2\ belong\ to\ span\{p_1(t), p_2(t), p_3(t)\}\ where}$

$$p_1(t) = t^2 + 2t + 1, p_2(t) = t^2 + 3, p_3(t) = t - 1$$

(a) $p(t) = t^2 + t + 2$ (b) $p(t) = -t^2 + t - 4$

(3) Determine whether the given vector A in

M_{22} *belong to span*$\{A_1, A_2, A_3\}$ *where* $A = \begin{bmatrix} 5 & 1 \\ -1 & 9 \end{bmatrix}$

(4) Is the following set of vectors spanR^4?

$$\left\{\begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}\right\}$$

(5) Is the following set of vectors spanR_4?

$$\{[1 \quad -2 \quad 3 \quad 0], [1 \quad 2 \quad -1 \quad 0], [0 \quad 0 \quad 0 \quad 3]\}$$

Linear Independence

Def. the vectors $v_1, v_2, \dots \dots, v_k$ in a vector space V are said to be **linearly dependent** if there exist constants $a_1, a_2, \dots \dots, a_k$ not all zero s.t

$$a_1v_1 + a_2v_2 + \cdots \dots + a_kv_k = 0$$

Other wise $v_1, v_2, \dots \dots, v_k$ are called **linearly independent** that is $v_1, v_2, \dots \dots, v_k$ are linearly independent if whether $a_1v_1 + a_2v_2, \dots + a_kv_k = 0$

$= 0 a_1 = a_2 = \cdots \dots = a_k$

If S= $\{v_1, v_2, \dots \dots, v_k\}$ then we also say that the set S is **linearly dependent** or **linearly independent** if the vectors have the corresponding property.

Ex.1 Determine whether the vectors $v_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ **are linearly independent.**

Solution $a_1v_1 + a_2v_2 + a_3v_3 = 0$

$$a_1\begin{bmatrix}3\\2\\1\end{bmatrix}+a_2\begin{bmatrix}1\\2\\0\end{bmatrix}+a_3\begin{bmatrix}-1\\2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We obtain the homo. linear system

$3a_1+a_2-a_3$=0

=0$2a_1+2a_2+2a_3$

=0a_1-a_3

Solve this system obtain $\begin{bmatrix}k\\-2k\\k\end{bmatrix}, k\neq 0$

The vectors are linearly dependent

<u>**Ex.2**</u> **Are the vectors $v_1=[1 \quad 0 \quad 1 \quad 2], v_2=[0 \quad 1 \quad 1 \quad 2]$,**

in R_4 linearly dependent or linearly $v_3=[1 \quad 1 \quad 1 \quad 3]$ **independent?**

<u>**Solution**</u> $a_1v_1+a_2v_2+a_3v_3$=0

We obtain the homo. linear system

=0a_1+a_3

=0$a_1+a_2+a_3$

=0$2a_1+2a_2+3a_3$

Solve this system obtain the only solution is the trivial solution $a_1=a_2=a_3$=0

So the vectors are linearly independent.

<u>**Ex.3**</u> **Are the vectors $v_1=\begin{bmatrix}2&1\\0&1\end{bmatrix}, v_2=\begin{bmatrix}1&2\\1&0\end{bmatrix}, v_3=\begin{bmatrix}0&-3\\-2&1\end{bmatrix}$ in M_{22} linearly independent?**

<u>**Solution**</u> $a_1v_1+a_2v_2+a_3v_3$=0

=0$a_1\begin{bmatrix}2&1\\0&1\end{bmatrix}+a_2\begin{bmatrix}1&2\\1&0\end{bmatrix}+a_3\begin{bmatrix}0&-3\\-2&1\end{bmatrix}$

$$\begin{bmatrix} 2a_1 + a_2 & a_1 + 2a_2 - 3a_3 \\ a_2 - 2a_3 & a_1 + a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We have the linear system

$$2a_1 + a_2 = 0$$

$$a_1 + 2a_2 - 3a_3 = 0$$

$$a_2 - 2a_3 = 0$$

$$a_1 + a_3 = 0$$

Solve this linear system obtain nontrivial solution $\begin{bmatrix} -k \\ 2k \\ k \end{bmatrix}, k \neq 0$

So the vectors are linearly dependent.

<u>Ex.4</u> Are the vectors $v_1 = t^2 + t + 2$, $v_2 = 2t^2 + t$ and $v_3 = 3t^2 + 2t + 2$ in P_2 linearly dependent or linearly independent?

<u>Solution</u> we have

$$a_1 + 2a_2 + 3a_3 = 0$$

$$a_1 + a_2 + 2a_3 = 0$$

$$2a_1 + 2a_3 = 0$$

Which has infinitely many solutions .A particular $a_1 = 1, a_2 = 1, a_3 = -1$

So $v_1 + v_2 - v_3 = 0$

Hence the given vectors are linearly dependent.

Theorem 1.5 Let $S = \{v_1, v_2, \dots.., v_n\}$ be a set of n vectors in $R^n(R_n)$ .Let A be the matrix whose columns(rows) are elements of S.Then S is linearly independent iff det(A)≠0

Ex. Is $\boldsymbol{S} = \{[\mathbf{1} \quad \mathbf{2} \quad \mathbf{3}], [\mathbf{0} \quad \mathbf{1} \quad \mathbf{2}], [\mathbf{3} \quad \mathbf{0} \quad -\mathbf{1}]\}$

$\textbf{a linearly independent set of vectors in} \boldsymbol{R}^3?$

Solution we form the matrix A whose rows are the vectors in S

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix} \text{ since } \det(\mathrm{A}) = 2 \text{ then S is linearly independent}$$

Theorem 1.6 Let $S_1$and S_2be finite subsets of a vector space and let S_1be a subset of S_2 then the following statements are true:

(a) If S_1 is linearly dependent so is S_2

(b) If S_2 is linearly independent so is S_1

Remark

(1) The set **S={0}** is linearly dependent. If S is any set of vectors that contains **0** then S must be linearly dependent.

(2)) A set of vectors consisting of a single nonzero vector is linearly

(3) I f $v_1, v_2, \dots.., v_k$ are vectors in a vector space V and any two of them are equal then$v_1, v_2, \dots.., v_k$ are linearly dependent

Theorem1.7 The nonzero vectors $v_1, v_2, \dots.., v_n$ in a vector space V are linearly dependent iff if one of the vectors $v_j (j \geq 2)$ is a linear combination of the preceding vectors $v_1, v_2, \dots.., v_{j-1}$

Ex. Let $V = R_3 \; and \; also \; v_1 = [1 \quad 2 \quad -1], v_2 = [1 \quad -2 \quad 1]$

$$, v_3 = [-3 \quad 2 \quad 1], v_4 = [2 \quad 0 \quad 0] we \; find \; that$$

$$v_1 + v_2 + 0v_3 - v_4 = 0$$

$$so \; v_1, v_2, v_3, v_4 \; are \; linearly \; dependent \; we \; then \; have \; v_4 = v_1 + v_2 + 0v_3$$

Remark

(1) Does not say that every vector v is a linear combination of the preceding vectors.

(2) We can prove that if S=$\{v_1, v_2, \dots\dots, v_k\}$ is a set of vectors in a vector space V,then S is linearly dependent iff one of the vectors in S is a linear combination of all other vectors in S

(3) Observe that if $v_1, v_2, \dots\dots, v_k$ are linearly independent vectors in a vector space, then they must be distinct and nonzero.

Exercises

(1) Determinate whether $\left\{\begin{bmatrix}1\\2\\1\\-1\end{bmatrix}, \begin{bmatrix}4\\3\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\1\\3\end{bmatrix}\right\}$ **is a linearly independent set in** $\boldsymbol{R^4}$

(2) Determinate whether $\{[3 \quad 1 \quad 2], [3 \quad 8 \quad -5], [-3 \quad 6 \quad -9]\}$ **is a linearly independent set in** $\boldsymbol{R_3}$

(3) Which of the given vectors in $\boldsymbol{R_3}$ **are linearly dependent? For those which are express one vector as a linear combination of the rest**

(a) $[2 \quad -1 \quad 0], [0 \quad 3 \quad 2], [2 \quad 4 \quad 3], [3 \quad 6 \quad 6]$

(b) $[1 \quad 1 \quad 0], [3 \quad 4 \quad 2]$

1.6 Basis and Dimension

Def. The vectors $v_1, v_2, \dots\dots, v_k$ in a vector space V are said to form a **basis** for V if

(a) $v_1, v_2, \dots\dots, v_k$ span V

(b) $v_1, v_2, \dots\dots, v_k$ are linearly independent

Remark

(1) If $v_1, v_2, \dots\dots, v_k$ form a basis for a vector space V, then they must be distinct and non zero

(2) in definition a finite set of vectors but it also applies to an infinite set S of vectors in a vector space

Ex.1 Let V=R^3 the vectors $\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$ form a basis for R^3,called **the natural basis** or **standard basis** for$\boldsymbol{R^3}$

Similarly the vectors $[1 \quad 0 \quad 0],[0 \quad 1 \quad 0],[0 \quad 0 \quad 1]$ is **the natural basis** for $\boldsymbol{R_3}$

Remark

(1) The natural basis for $\boldsymbol{R^n}$ is denoted by $\{e_1, e_2, \ldots\ldots, e_n\}$,where

$$e_i = \begin{bmatrix}0\\ \vdots\\ 0\\ 1\\ 0\\ \vdots\\ 0\end{bmatrix} \leftarrow i\ th\ row$$

That is e_i is an n×1 matrix with a (1) in the i th row and zeros elsewhere.

(2) The natural basis for $R^3$is also often denoted by

$$i = \begin{bmatrix}1\\0\\0\end{bmatrix}, j = \begin{bmatrix}0\\1\\0\end{bmatrix}, k = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Thus any vector $v = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ in R^3can be written as $v = a_1 i + a_2 j + a_3 k$

Ex.2 Show that $S = \{t^2 + 1, t - 1, 2t + 2\}$ is a basis for the vector space P_2

Solution To do this we must show that S spans V and is linearly independent.

To show that it spans V we take any vector in V that is a polynomial

$$at^2 + bt + c \ where\ a, b, c\ are\ real\ numbers$$

And find a_1, a_2, a_3 s.t $at^2 + bt + c = a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2)$

$= a_1 t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3)$

We get the linear system

$a_1 = a$

$a_2 + 2a_3 = b$

$a_1 - a_2 + 2a_3 = c$

Solving, we have $a_1 = a$, $a_2 = \frac{a+b-c}{2}$, $a_3 = \frac{c+b-a}{4}$

$\therefore S \; span \; V$

For example suppose that we are given the vector $2t^2 + 6t + 13$

Substituting, we find that $a_1 = 2$, $a_2 = \frac{-5}{2}$, $a_3 = \frac{17}{4}$

$\therefore \; 2t^2 + 6t + 13 = 2(t^2 + 1) + \frac{-5}{2}(t - 1) + \frac{17}{4}(2t + 2)$

To show that S is linearly independent, we form

$a_1(t^2 + 1) + a_2(t - 1) + a_3(2t + 2) = 0$

$a_1 t^2 + (a_2 + 2a_3)t + (a_1 - a_2 + 2a_3) = 0$

We get the linear system $a_1 = 0$

$$a_2 + 2a_3 = 0$$

$$a_1 - a_2 + 2a_3 = 0$$

The only solution to this homo. system is

$a_1 = 0, a_2 = 0, a_3 = 0$

$\therefore S \; is \; linearly \; independent$

Thus S is a basis for P_2

Remark The set of vectors $\{t^n, t^{n-1}, \dots \dots, t, 1\}$ form a basis for the vector space P_n called the natural or stander basis for P_n

Ex.3 Show that the set $S = \{v_1, v_2, v_3, v_4\}$

Where $v_1 = [1 \quad 0 \quad 1 \quad 0], v_2 = [0 \quad 1 \quad -1 \quad 2], v_3 = [0 \quad 2 \quad 2 \quad 1],$
$v_4 = [1 \quad 0 \quad 0 \quad 1]$

Solution To show that S is linearly independent

We form the equation $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

We get the linear system $a_1 + a_4 = 0$

$$a_2 + 2a_3 = 0$$

$$a_1 - a_2 + 2a_3 = 0$$

$$2a_2 + a_3 + a_4 = 0$$

The only solution to this homo. system is

$a_1 = 0, a_2 = 0, a_3 = 0, a_4 = 0$

To show that S spans R_4 we let $v = [a \quad b \quad c \quad d]$be any vector in R_4

Then $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = v$

Substituting$v_1, v_2, v_3, v_4 and\ v$ for we find a solution for a_1, a_2, a_3, a_4 to the resulting linear system

$\therefore S\ spans\ R_4$ and is a basis for R_4

Remark A vector space V is called **finite-dimensional** if there is a finite subset of V that is a basis for V .If there is no such finite subset of V,then V is called **infinite-dimensional** .

We now establish some results about finite-dimensional vector space

(1) If $\{v_1, v_2, \dots, v_k\}$ is basis for a vector space V then $\{cv_1, v_2, \dots, v_k\}$is also a basis when c≠0

(2) A basis for a nonzero vector space is never unique.

Theorem 1.8 If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V then every vector in V can be written in one and only one way as a linear combination of the vectors in S .

Theorem 1.9 Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of nonzero vectors in a vector space V and let W=span S.Then some subset of S is a basis is a basis for W.

Theorem 1.10 If $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V and T=$\{w_1, w_2, \ldots, w_r\}$ is a linear independent set of vectors in V ,then r≤n.

corollary 1.1 If $S = \{v_1, v_2, \ldots, v_n\}$ and T=$\{w_1, w_2, \ldots, w_m\}$ are bases for a vector space V,then n=m.

Proof Since S is a basis and T is linearly independent ,from theorem 1.10 that m≤n .Similarly, we obtain n≤m because T is basis and S is linearly independent

Hence n=m.

Def. The **dimension** of a nonzero vector space V is the number of vector in a basis for V.We often write **dim** V for the dimension of V .we also define the dimension of the trivial vector space **{0}** to be zero.

Ex1. The set $\{t^2, t, 1\}$ is a basis for P_2 so dimp_2=3

Ex2. Let V be the subspace of R_3 spanned S=$\{v_1, v_2, v_3\}$ where

$$v_1 = [0 \quad 1 \quad 1], v_2 = [1 \quad 0 \quad 1]$$

$$v_3 = [1 \quad 1 \quad 2] thus\ every\ vector\ in\ V\ is\ of\ the\ form\ a_1v_1 + a_2v_2 + a_3v_3$$

Where a_1, a_2, a_3 are arbitrary real numbers.

We find that S is linearly dependent and $v_3 = v_1 + v_2$ thus $S_1 = \{v_1, v_2\}$ also spans V.since $S_1\ is$ linearly independent. we conclude that is a basis for V.

Hence dimV=2.

Def. Let S be a set of vectors in a vector space V .A subset T of S is called a **maximal independent subset** of S if T is a linearly independent set of vectors that is not properly contained in any other linearly independent subset of S.

Ex. Let V be R^3 and consider the set S=$\{v_1, v_2, v_3, v_4\}$ where

Maximal independent subset of $v_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, v_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}, v_4 = \begin{bmatrix}1\\1\\1\end{bmatrix}$

S are

$\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}$

$\{v_2, v_3, v_4\}$

Corollary1.2 If the vector space V has dimension n,then a maximal independent subset of vectors in V contains n vectors.

Corollary1.3 If a vector space V has dimension n,then a maximal spanning set for V contains n vectors.

Corollary1.4 If a vector space V has dimension n,then any subset of m$>n$ vectors must be linearly dependent.

Corollary1.5 If a vector space V has dimension n,then any subset of m$<n$ vectors cannot span V.

Theorem 1.11 If S is a linearly independent set of vectors in a finite-dimensional vector space V.Then there is a basis T for V that contains S.

Theorem 1.12 Let V be an n-dimensional vector space

(a) If S=$\{v_1, v_2, v_3, \dots \dots, v_n\}$ *is a linearly independent set of vectors in* V,

Then S is a basis for V

(a) If S=$\{v_1, v_2, v_3, \dots \dots, v_n\}$ *spans* V,

Then S is a basis for V

Theorem 1.13 Let S be a finite subset of the vector space V that spans V .A maximal independent subset T of S is a basis for V.

Exercises

(1) The set W of all 2×2 matrices with trace equal to zero is a subspace of M_{22} show that the set S={v_1, v_2, v_3} where

is basis for W.$v_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, v_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(2) Find a basis for the subspace V of P_2 consisting of all vectors of the form $at^2 + bt + c$ $where$ c=a-b

(3) Which of the following sets of vectors are bases for R^2?

$$\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right\}$$

(4) Which of the following sets of vectors are bases for R^3?

$$\left\{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

(5) Which of the following sets of vectors are bases for R_4?

$$\{[3 \quad -2 \quad 0 \quad 3], [5 \quad -1 \quad 3 \quad 1], [1 \quad 0 \quad 0 \quad 1]\}$$

(6) Which of the following sets of vectors are bases for P_2?

$$\{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}$$

(7) Show that the set of matrices from a basis for the vector space M_{22}

$$\left\{\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right\}$$

(8) Find a basis for the subspace W of R^3 spanned by

$$\left\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -7 \\ 4 \end{bmatrix}\right\}$$

what is the dimension of W?

Chapter-2-

Inner product spaces

2.1 Length and direction in R^2 and R^3

Length

The length or magnitude of the vector denoted by $\|v\|$ is:

(1) The length of the vector $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in R^2 ,is by the Pythagorean theorem $\|v\| = \sqrt{v^2{}_1 + v^2{}_2}$

(2) Let $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $be\ a\ vector\ in\ R^3$

Using the Pythagorean theorem the length of v is $\|v\| = \sqrt{v^2{}_1 + v^2{}_2 + v^2{}_3}$

Ex. Find the length of v where

(1) $v = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$

Solution $\|v\| = \sqrt{(2)^2 + (-5)^2} = \sqrt{4 + 25} = \sqrt{29}$

(1) $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution $\|v\| = \sqrt{(1)^2 + (2)^2 + (3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$

Remark

(1) If the points $P_1 = (u_1, u_2), P_2 = (v_1, v_2)\ in\ R^2$

The distance from P_1 to P_2 the length of the line from P_1 to P_2 is given by $\sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2}$

If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ $are\ vectors\ in\ R^2$

Define the distance between the vectors u and v as the distance between the points P_1 and P_2.

The distance between **u and v** is given by $\|\boldsymbol{v}-\boldsymbol{u}\| = \sqrt{(\boldsymbol{v_1}-\boldsymbol{u_1})^2+(\boldsymbol{v_2}-\boldsymbol{u_2})^2}$

(2) If the points $P_1 = (u_1, u_2, u_3), P_2 = (v_1, v_2, v_3)\ in\ R^3$

The distance between P_1 and P_2 is given by $\sqrt{(v_1-u_1)^2+(v_2-u_2)^2+(v_3-u_3)^2}$

If $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ $are\ vectors\ in\ R^3$

The distance between **u and v** is given by $\|\boldsymbol{v}-\boldsymbol{u}\| = \sqrt{(\boldsymbol{v_1}-\boldsymbol{u_1})^2+(\boldsymbol{v_1}-\boldsymbol{u_1})^2+(\boldsymbol{v_3}-\boldsymbol{u_3})^2}$

(3) The zero vector has length zero. the zero vector is the only vector whose length is zero.

<u>Ex.</u> Compute the distance between the vectors

$$(\mathbf{1})\ \boldsymbol{u} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{5} \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} \mathbf{3} \\ \mathbf{2} \end{bmatrix}$$

<u>Solution</u> $\|v-u\| = \sqrt{(3+1)^2+(2-5)^2} = \sqrt{16+9} = \sqrt{25} = 5$

$$(\mathbf{2})\ \boldsymbol{u} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} \mathbf{-4} \\ \mathbf{3} \\ \mathbf{5} \end{bmatrix}$$

<u>Solution</u> $\|v-u\| = \sqrt{(-4-1)^2+(3-2)^2+(5-3)^2} = \sqrt{30}$

<u>Direction</u>

(1) The direction of a vector in R^2 is given by specifying its angle of inclination or slope.

(2) The direction of a vector v in R^3 is given by specifying by giving the cosine of the angles that the vector v makes with the positive x,y and z-axes these are called **direction cosines.**

(3) The zero vector on R^2 or R^3 has no specific direction

Remark

(1) If $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ are nonzero vectors in R^2 and θ is the angle between u and v,then:

$$cos\theta = \frac{u_1v_1 + u_2v_2}{\|u\|\|v\|} \quad ,0 \leq \theta \leq \pi$$

(2) If $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ are nonzero vectors in R^3 and θ is the angle between u and v,then:

$$cos\theta = \frac{u_1v_1 + u_2v_2 + u_3v_3}{\|u\|\|v\|} \quad ,0 \leq \theta \leq \pi$$

Ex. Find the angle between the vectors $\boldsymbol{u} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \\ \mathbf{0} \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

Solution

$$cos\theta = \frac{(1)(0) + (1)(1) + (0)(1)}{\sqrt{1^2 + 1^2 + 0^2}\sqrt{0^2 + 1^2 + 1^2}} = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

Def. The stander inner product or dot product

On R^2 or R^3 is the function that assigns to each ordered pair of vectors

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ in R^2 or $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ in R^3

The number **u.v**

u.v $=u_1v_1+u_2v_2$ in R^2

u.v$=u_1v_1+u_2v_2+u_3v_3$ in R^3

$\therefore \|v\|=\sqrt{v.v}$ $\; v$ *is a vector* in R^2 *or* R^3

$\therefore cos\theta=\frac{u.v}{\|u\|\|v\|}$,$0\le\theta\le\pi$,*u and v are nonzero vectors* in R^2 *and* R^3

<u>Remark</u> The two vectors **u** and **v** *in* R^2 *or* R^3 are **orthogonal or perpendicular** iff **u.v=0**

<u>Ex.</u>are the two vectors $u=\begin{bmatrix}2\\-4\end{bmatrix}$,$v=\begin{bmatrix}4\\2\end{bmatrix}$ **orthogonal?**

<u>Solution</u> u.v=(2)(4)+(-4)(2)=0

The two vectors orthogonal

<u>Theorem 2.1</u>

Let u ,v and w be vectors *in* R^2 *or* R^3 and let c be scalar .the stander inner product on R^2 *or* R^3has the following properties:

(a) u.u≥0;u.u=0 iff u=0

(b) u.v=v.u

(c) (u+v).w=u.w+v.w

(d) cu.v=c(u.v) for any real scalar c

<u>Unit vectors</u>

A unit vector in R^2 *or* R^3 is a vector whose length is **1**.

If x is any nonzero vector, then the vector $u=\frac{1}{\|x\|}x$ *is a unit vector in the direction of x.*

<u>Ex.</u> Find a unit vector from the vector $\boldsymbol{x}=\begin{bmatrix}\mathbf{-3}\\\mathbf{4}\end{bmatrix}$

Solution $\|x\| = \sqrt{(-3)^2+(4)^2} = \sqrt{25} = 5$

The unit vector is $u = \frac{1}{5}\begin{bmatrix}-3\\4\end{bmatrix} = \begin{bmatrix}\frac{-3}{5}\\ \frac{4}{5}\end{bmatrix}$

$$\|u\| = \sqrt{\left(\frac{-3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9+16}{5}}$$

$$= 1\,, u\ points\ in\ the\ direction\ of x.$$

Remark

(1) There are two vectors in R^2 that are of special important.

These are $\boldsymbol{i} = \begin{bmatrix}\mathbf{1}\\\mathbf{0}\end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix}$ the unit vectors along the positive x and y-axes respectively.

i and j are orthogonal ,since i and j form the natural basis for R^2,every vector in R^2 can be written uniquely as a linear combination of the orthogonal vectors **i and j**.

If $u = \begin{bmatrix}u_1\\u_2\end{bmatrix}$ is a vector in R^2 then $\boldsymbol{u} = \boldsymbol{u_1}\begin{bmatrix}\mathbf{1}\\\mathbf{0}\end{bmatrix} + \boldsymbol{u_2}\begin{bmatrix}\mathbf{0}\\\mathbf{1}\end{bmatrix} = \boldsymbol{u_1 i} + \boldsymbol{u_2 j}$

i.i=j.j=1 ; i.j=0

(2) Similarly,the vector in the natural basis for R^3

$$\boldsymbol{i} = \begin{bmatrix}\mathbf{1}\\\mathbf{0}\\\mathbf{0}\end{bmatrix}, \boldsymbol{j} = \begin{bmatrix}\mathbf{0}\\\mathbf{1}\\\mathbf{0}\end{bmatrix}\ \boldsymbol{and}\ \boldsymbol{k} = \begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{1}\end{bmatrix}$$

Are unit vectors that are mutually orthogonal.

If $u = \begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}$ is a vector in R^3 then

$$u = u_1\begin{bmatrix}1\\0\\0\end{bmatrix} + u_2\begin{bmatrix}0\\1\\0\end{bmatrix} + u_3\begin{bmatrix}0\\0\\1\end{bmatrix} = u_1 i + u_2 j + u_3 k$$

i.i=j.j=k.k=1 ; i.j=i.k=j.k=0

Exercises

(1) Find the length of each vector

(a) $\begin{bmatrix}-1\\-3\\-4\end{bmatrix}$ (b) $\begin{bmatrix}4\\-2\\-1\end{bmatrix}$ $(c)\begin{bmatrix}1\\-1\end{bmatrix}$ $(d)\begin{bmatrix}3\\2\end{bmatrix}$

(2) Compute $\|u-v\|$

(a) u= $\begin{bmatrix}-1\\0\\-4\end{bmatrix}$ $,v=\begin{bmatrix}-4\\-5\\-6\end{bmatrix}$ $(b)\ u=\begin{bmatrix}1\\0\end{bmatrix}$ $,v=\begin{bmatrix}2\\1\end{bmatrix}$

(3) Find distance between u and v and find the cosine of the angle between u and v

(a) u= $\begin{bmatrix}0\\1\\-1\end{bmatrix}$ $,v=\begin{bmatrix}1\\2\\0\end{bmatrix}$ $(b)\ u=\begin{bmatrix}1\\2\end{bmatrix}$ $,v=\begin{bmatrix}4\\-5\end{bmatrix}$

(3) Find all values of c where $\|u\|=3\ for\ \mathbf{u}=\begin{bmatrix}2\\c\\1\end{bmatrix}$

(4) Which of the following vectors are orthogonal?

(a) u= $\begin{bmatrix}1\\-1\\-2\end{bmatrix}$ $,v=\begin{bmatrix}3\\-1\\2\end{bmatrix},w=\begin{bmatrix}2\\4\\-1\end{bmatrix}$

(5) Find c so that the vector $v=\begin{bmatrix}1\\c\end{bmatrix}$ is orthogonal to $w=\begin{bmatrix}2\\-1\end{bmatrix}$

(6) Let P(3,-1,2) , Q(4,2,-3) are points in R^3.Find length the segment PQ.

2.2 Cross product in R^3

Let $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$ are $vectors$ in R^3 $, then$

The cross product of u and v is denoted by u×v .

Let $\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$

the vector $u \times v$ is:

$$u \times v = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

u×v=$(u_2 v_3 - u_3 v_2)i + (u_3 v_1 - u_1 v_3)j + (u_1 v_2 - u_2 v_1)k$

$$= \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Ex. Find u×v where $\boldsymbol{u} = \boldsymbol{2i} + \boldsymbol{j} + \boldsymbol{2k}$ $\boldsymbol{and}$ $\boldsymbol{v} = \boldsymbol{3i} - \boldsymbol{j} - \boldsymbol{3k}$

Solution Let $\begin{bmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 2 \\ 3 & -1 & -3 \end{bmatrix}$

the vector $u \times v$ is:

$$u \times v = \begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} i - \begin{vmatrix} 2 & 2 \\ 3 & -3 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} k$$

u×v=(-3+2)i-(-6-6)j+(-2-3)k=-i+12j-5k

Remark

(1) (u×v).u=0 and (u×v).v=0 ((u×v) orthogonal to u and v)

(2) The cross product u×v is a vector while the dot product u.v is a number.

(3) The cross product is not define onR^n if n≠3.

(4) Let u,v and w be vectors in $R^3$and c a scalar ,then

(5) u×v=-(v×u)

(6) u×(v+w)= u×v + u×w

(7) (u+v)×w=u×w + v×w

(8) c(u×v)=(cu)×v=u×(cv)

(9) u×u=0

(10) 0×u=u×0=0

(11) u×(v×w)=(u.w)v-(u.v)w

(12) (u×v)×w=(w.u)-(w.v)u

(13) (u×v).w=u.(v×w)

(14) u and v are parallel iff u×v=0

(15) i×i=j×j=k×k=0 ; i×j=k , j×k=i , k×i=j ; j×i=-k , k×j=-i , i×k=-j

(16) $\|u \times v\| = \|u\|\|v\| sin\theta$,$0 \leq \theta \leq \pi$

$(sin\theta \ non\ negative\ since\ 0 \leq \theta \leq \pi)$

Ex. Let $\boldsymbol{u = 2i + j + 2k}$, $\boldsymbol{v = 3i - j - 3k}$ $\boldsymbol{and\ w = i + 2j + 3k}$ $\boldsymbol{then}$:

(1) Find u×v

(2) Show that (u×v).w=u.(v×w)

Solution u×v=-i+12j-5k , (u×v).w=8

v×w=3i-12j+7k ,u.(v×w)=8

Area of a Triangle

The area of the triangle is $\boldsymbol{A_T = \frac{1}{2}\|u\|\|v\| sin\theta = \frac{1}{2}\|u \times v\|}$

Ex. **Find the area of the triangle with vertices** $\boldsymbol{p_1(2,2,4), p_2(-1,0,5) and\ p_3(3,4,3)}$

Solution

$u = \overrightarrow{p_1p_2} = -3i - 2j + k$

$v = \overrightarrow{p_1p_3} = i + 2j - k$

Then the area of the triangle $\mathrm{A_T}$ is :

$\mathrm{A_T} = \frac{1}{2}\|(-3i - 2j + k) \times (i + 2j - k)\|$

$= \frac{1}{2}\|(-2j - 4k)\| = \|(-j - 2k)\| = \sqrt{\mathbf{5}}$

Area of a Parallelogram

The area $\mathrm{A_P}$ of the parallelogram with adjacent sides u and v is:

$$\mathbf{A_P} = \|\boldsymbol{u} \times \boldsymbol{v}\| = \mathbf{2\ A_{\mathit{T}}}$$

Ex. **Find the area of the Parallelogram with adjacent sides** $\overrightarrow{\boldsymbol{p_1p_2}}$ **and** $\overrightarrow{\boldsymbol{p_1p_3}}$ **where** $\boldsymbol{p_1(2,2,4), p_2(-1,0,5) and\ p_3(3,4,3)}$

Solution

$u = \overrightarrow{p_1p_2} = -3i - 2j + k$

$v = \overrightarrow{p_1p_3} = i + 2j - k$

Then the area of the triangle $\mathrm{A_T}$ is :

$\mathrm{A_T} = \frac{1}{2}\|(-3i - 2j + k) \times (i + 2j - k)\|$

$= \frac{1}{2}\|(-2j - 4k)\| = \|(-j - 2k)\| = \sqrt{\mathbf{5}}$

$\therefore A_P = 2\mathrm{A_T} = 2\sqrt{5}$

Exercises

(1) Compute u×v

(a) u=2i+3j+4k , v=-2i+j-3k

(b) u=j+k , v=2i+3j-k

(c) u=i-j+2k , v=3i+j+2k

(d) $\boldsymbol{u} = \begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \\ \mathbf{1} \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} \mathbf{-4} \\ \mathbf{2} \\ \mathbf{-1} \end{bmatrix}$

(2) Find the area of the triangle with vertices $\boldsymbol{p_1(1,-2,3), p_2(-3,1,4) and\ p_3(0,4,3)}$

(3) Find the area of the Parallelogram with adjacent sides u=i+3j-2k , v=3i-j-k

Inner product spaces

Def. Let V be a real vector space .An **inner product** on V is a function that assigns to each ordered pair of vectors **u,v** in V a real number (**u,v**) satisfying the following properties:

(a) (u,u)≥ 0 ;(u,u)=0 iff u=0_v

(b) (v,u)=(u,v) For any u,v in V

(c) (u+v,w)=(u,w)+(v,w) for any u,v,w in V

(d) (cu,v)=c(u,v) for u,v in V and c a real scalar

From these properties it follows that (u,cv)=c(u,v) because (u,cv)=(cv,u)=c(u,v)=c(v,u)

Also (u,v+w)=(u,v)+(u,w)

Ex.1 The standard inner product or dot product on R^n as the function that assigns to each ordered pair of vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\ in\ R^n$

The number, denoted by (u,v) ,given by

$$(u,v) = u_1v_1 + u_2v_2 + \cdots \dots \dots + u_nv_n$$

this function satisfies the properties in definition

<u>Ex.2</u> Let $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} \boldsymbol{v_1} \\ \boldsymbol{v_2} \end{bmatrix}$ **be vectors in** $\boldsymbol{R^2}$**.**

We define (u,v)= $\boldsymbol{u_1v_1 - u_2v_1 - u_1v_2 + 3\, u_2v_2}$

Show that that this gives an inner product on $\boldsymbol{R^2}$

<u>**Solution**</u> $(u,u) = {u_1}^2 - 2u_1u_2 + 3{u_2}^2 = {u_1}^2 - 2u_1u_2 + {u_2}^2 + 2{u_2}^2$

$$=(u_1 - u_2)^2 + 2{u_2}^2 \geq 0$$

If (u,u)=0 then $u_1 = u_2 \; and \; u_2 = 0 \; so \; u = 0$

Conversely if u=0 then (u,u)=0

The remaining three properties in definition are satisfying.

<u>**Ex.3**</u> Let V be vector space of all continuous real-valued functions on the interval [0,1]

$(f,g) = \int_0^1 f(t)g(t)\,dt \quad where\; f \; and \; g \; in \; V$

The properties of definition are satisfied

(a) $(f,f) = \int_0^1 (f(t))^2\,dt \geq 0$

If (f,f)=0 then f=0 conversely ,if f=0 then (f,f)=0

$$(b)\; (f,g) = \int_0^1 f(t)g(t)\,dt \; = \int_0^1 g(t)f(t)\,dt \; = (g,f)$$

$$(c)(f+g,h) = \int_0^1 \big(f(t) + g(t)\big)h(t)\,dt$$

$$= \int_0^1 f(t)h(t)\,dt + \int_0^1 g(t)h(t)\,dt = (f,h) + (g,h)$$

$$(d)(cf,g) = \int_0^1 (cf(t))g(t)\,dt \; = c\int_0^1 f(t)g(t)\,dt \; = c(f,g)$$

For example if f(t)=t+1 and g(t)=2t+3,then

$$(f,g) = \int_0^1 (t+1)(2t+3)\,dt \quad = \int_0^1 (2t^2+5t+3)\,dt \quad = \frac{37}{6}$$

<u>**Theorem**</u> Let $s = \{u_1, u_2, \dots., u_n\}$ be an ordered basis for a finite-dimensional vector space V,and assume that we are given an inner product on V.

Let $c_{ij} = (u_i, u_j)$and $C = [c_{ij}]$ .then

(a) C is a symmetric matrix

(b) C determines (v,w) for any v and w in V