محاضرات مادة التبولوجي / المرحلة الرابعة

الكورس الاول والثاني

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Contents

References

[1] Colin Adams, Robert Franzosa," *Introduction to Topology: Pure and Applied* " ,1st Edition, Paperback – January 1, 2011.

[2] Seymour Lipschutz, "*Theory and Problems of General Topology* (*Schaum's Outline Series*)", Paperback – Import, 16 Jan 1965.

[3] William J. Pervin , Ralph P. Boas, " *Foundations of General Topology*", Paperback – January 1, 1964.

[4] أ.د أحمد القادر رمضان، د. طه مرسي العدوي ،" *التبولوجي العام* " ، سنة النشر 2015 :م / ١٤٣٦هـ .

[5] ا.م.د. سمير بشير حديد ، " *مقدمة في التبولوجيا العامة* " ، سنة النشر :١٩٨٨ م / ١٤٠٨هـ .

Chapter One

Topological Spaces

1.1 Topological space

1.1.1 Definition:-

Let X be a non empty set .A class τ of subsets of X is a ***topology*** on X iff τ satisfies the following axioms

1) X and $\emptyset$ are members of τ.
2) The intersection of any finite number of members of τ is a member of τ.
3) The union of any family of members of τ is again in τ.

The pair (X, τ) is called a ***topological space*** and the members of τ are called ***τ- open sets*** or simply **open sets**.

1.1.2 Example:-

If X is any set, then the collection $\{X, \emptyset\}$ of subsets of X also forms a topology on X. This topology is called the ***trivial*** (***indiscrete***) topology on X.

1.1.3 Example:-

If X is any set, then the family of all subsets of X forms a topology on X. This topology is called the ***discrete topology*** on X.

Notice that the discrete topology contains the maximum possible number of open sets since, relative to the discrete topology, every subset of X is open.

1.1.4 Example:-

Let τ be a class of all open sets of a metric space (X, *d*) then τ is a topology on X ,called the ***usual topology*** on X.

1.1.5 Example:-

Let τ be a class of all subsets of X whose complements are finite together with the empty set $\emptyset$. This class τ is a topology on X which is called the co-finite topology.

1.1.6 Example:-

Consider the following classes of subsets of $X = \{a,b,c\}$

$\tau_1 = \{X,\emptyset,\{a\},\{b\},\{a,b\}\}$

$\tau_2 = \{X,\emptyset,\{a\},\{b\}\}$

$\tau_3 = \{X,\emptyset,\{a,c\},\{b,c\}\}$

Observe that τ_1 is a topology on X since it satisfies the necessary three axioms. But τ_2 is not a topology on X since the unions $\{a\} \cup \{b\} = \{a,b\}$ of two members of τ_2 does not belongs to τ_2 ,i.e does not satisfy the axiom 3. Also τ_3 is not a topology on X since the intersection $\{a,c\} \cap \{b,c\} = \{c\}$ of two sets in τ_3 does not belongs to τ_3,i.e τ_3 does not satisfy the axiom 2.

1.1.7 Example:-

Let τ be a class of all subsets of N consisting of ∅ ,X and all subsets of ℕ of the form E_n ={1,2,…,n}with $n \in \mathbb{N}$ then the class τ is a topology on X.

1.1.8 Theorem:-

Let $\{\tau_i : i \in I\}$ be a collection of topologies on a set X. Then the intersection $\bigcap_i \tau_i$ is also a topology on X.

Note that the union of two topologies for *X* need not be a topology on X, for example $\tau_1 =$ {X,∅,{a}} , $\tau_2 = \{X,\emptyset,$ {b}} is two topologies on $X =$ {a,b,c} but the union $\tau_1 \cup \tau_2$is not a topology on X.

1.1.9 Definition:-

Let X be a non-empty set and $\tau_1$and τ_2 be two topologies on X. If $\tau_1 \subset \tau_2$ then τ_2 is said to be ***finer*** than τ_1,and $\tau_1$is said to be the ***courser*** than τ_2.

1.1.10 Example:-

Let X be a non-empty set then the discrete topology is finer of all topologies on X and the indiscrete topology is courser of all topologies on X.

Notice that the class $\{T_i\}$ of all topologies on X i partially ordered by class inclusion :

$$\tau_1 \lesssim \tau_2 \quad for \quad \tau_1 \subseteq \tau_2.$$

And we say that two topologies on X are not comparable if neither is coarser than the other.

Exercises:-

1. Let τ be a topology on a set X consisting of four sets ,i.e. $\tau =$ {A,∅,B,C}, where A and B are non-empty disjoint proper subsets of X .What conditions must A and B satisfy?
2. Determine all of the possible topologies on $X =$ {a,b,c} .
3. List all topologies on $X = \{a,$b,c} which consist of exactly four members.
4. Show that the class τ of all subsets of X whose complements are finite together with the empty set ∅ is a topology on X.
5. Let X be a set and assume $p \epsilon X$.Show that the collection τ consisting of $\emptyset, X$, and all subsets of X containing p, is a topology on X. This topology is called the ***particular point topology*** on X.
6. Let X be a set and assume $p \epsilon X$.Show that the collection τ consisting of ∅,X, and all subsets of X that exclude p, is a topology on X. This topology is called the ***excluded point topology*** on X.
7. Let τ consist of ∅,R, and all intervals (-∞,p) for $p \epsilon \mathbb{R}$ .Prove that τ is a topology on ℝ.
8. Let $f: X \to Y$ be a function fromm a non – empty set X into a topological space (Y, τ_Y) and let $\tau_X \tau$ be the class of intervals of open subsets of Y,i.e. $\tau_X = \{f^{-1}(G): G \epsilon \tau_Y\}$.Show that $\tau_X$is a topology on X.
9. Let τ be a class of all subsets of N consisting of ∅ and all subsets of ℕ of the form E_n ={n,n+1,n+2,…}with $n \in \mathbb{N}$.

   a) Show that τ is a topology on N. **b)** List the open sets containing the positive integer 6.

1.2 limit points

1.2.1 Definition:-

Let A be a subset of a topological space (X,τ).A point $p\epsilon X$ is ***an accumulation point*** or ***a limit point*** of A if every open set G containing p contains a point of A different from p, i.e.

$$G\ open\ \ ,p\epsilon G\ \ \rightarrow A\cap(G/\{p\})\neq\emptyset.$$

The set of accumulation points of A, denoted by d(A) (or A`).

Notice that a limit point p of a set A may or may n ot lie in the set A. Notice also that in every topology, the point p is not a limit point of the set {x}.

1.2.2 Example:-

Consider $A\subset\mathbb{R}$ with the usual topology on $\mathbb{R}$ then :

a) $d(A=\{\frac{1}{n}\epsilon\mathbb{R}: n\epsilon\mathbb{Z}^{+}\})=\{0\}$.

b) $d([a,b])=d\big((a,b]\big)=d([a,b))=d\big((a,b)\big)=[a,b]$.

c) $d(\mathbb{Q})=\mathbb{R}$.

d) $d(\mathbb{Z})=\emptyset$.

1.2.3 Example:-

Let $X=\{a,b,c,d,e\}$ and $\tau=\{\emptyset,\{a\},\{b,d\},\{a,b,d\},\{b,c,d,e\},X\}$ then

$$d(\{a,b,c\})=\{c,d,e\},\ d(\{b,c,d\})=\{b,c,d,e\}$$

1.2.4 Theorem:-

If A,B and E are subsets of the topological space (X,τ), then the derived set has the following properties:

a) $d(\emptyset)=\emptyset$.

b) If $A\subseteq B$ then $d(A)\subseteq d(B)$.

c) If $x\in d(E)$, then $x\in d(E\backslash\{x\})$.

d) $d(A\cup B)=d(A)\cup d(B)$.

Note that $d(A\cap B)\neq d(A)\cap d(B)$, for example let $X=\{a,b,c\}$ and let $A=\{a,c\}$,B={b,c} ,define the topology τ on X by $\tau=\{X,\emptyset,\{b\},\{a,b\}$ then $d(A\cap B)=d(\{c\})=\emptyset\neq d(A)\cap d(B)=\{c\}\cap\{a,c\}=\{c\}$.

Exercises: -

1. Let A be a subset of a topological space (X,τ) .When will a point $p\in X$ not be a limit point of A?

2. Let A be any subset of a discrete topological space X. Show that $d(A)=\emptyset$.

3. Consider the topological space $(\mathbb{R},\tau)$, where τ consists of of $\emptyset$,$\mathbb{R}$, and all open intervals $E_p=(a,\infty)$,$a\in\mathbb{R}$. Find the derived set of

a) The interval (4,10]; **b)** $\mathbb{Z}$ the set of integers.

4. Determine the set of limit points of [0,1] in the complement topology on $\mathbb{R}$.

5. Let τbe the topology on $\mathbb{N}$ which consists of $\emptyset$ and all subsets of $\mathbb{N}$ of the form E_n ={n,n+1,n+2,…}were $n\in\mathbb{N}$.

a) Find the limit points of the set $A = \{4,13,28,37\}$.

b) Determine those subsets E of ℕ for which $d(E) = \mathbb{N}$.

6. Let τ_1 and τ_2 be topologies on X such that $\tau_1 \subset \tau_2$ and let A be any subset of X. Show that every τ_2- limit point of A is also a τ_1- limit point of A.

1.3 Closed Sets

1.3.1 Definition:-

Let (X, τ) be a topological space. A subset A of X is ***closed set*** if it contains all its limit points, i.e. $d(A) \subseteq A$.

1.3.2 Example:-

Let X={a,b,c,d} and $\tau = \{\emptyset,\{a\},\{b,c\},\{a,b,c\},X\}$ then $A = \{a,d\}$ is a closed set since $d(A) = \{d\} \subseteq A = \{a,d\}$.

1.3.3 Theorem:-

If $x \notin A$, where A is a closed subset of a topological space (X, τ) then there exists an open set G such that $x \in G \subseteq A^c$.

1.3.4 Corollary:-

Let (X, τ) be a topological space. A subset A of X is closed set iff its complement A^c is open.

1.3.5 Example:-

Let $X = \{a,b,c,d,e\}$ and $\tau = \{\emptyset,\{a\},\{b,c\},\{a,b,c\},\{b,c,d,e\},X\}$ then

1) ∅,{a},{b,c},{a,b,c},{b,c,d,e},X are open sets.

2) X,{b,c,d,e},{a,d,e},{d,e},{a},∅ are closed sets.

3) ∅,X,{a},{b,c,d,e} are both open and closed sets.

4) {b,c},{a,b,c} are open not closed sets.

5) {d,e},{a,d,e} are closed not open sets.

6) {e},{c},{d},{c,d} are not open and closed sets.

1.3.6 Example:-

In a discrete topology all subsets are both open and closed.

1.3.7 Corollary:-

Let $\mathcal{F}$ be a family of closed subsets in a topological space (X, τ) then it has the following property:

a) *The intersection of any number of members of $\mathcal{F}$ is a member of $\mathcal{F}$ ($X \in \mathcal{F}$).*

b) *The union of any finite number of members of $\mathcal{F}$ is a member of $\mathcal{F}$ ($\emptyset \in \mathcal{F}$).*

Note that if A is a closed set then $d(A)$ is also a closed set (since A is closed then $d(A) \subseteq A$, i.e. $d(d(A)) \subseteq d(A)$,so $d(A)$ is a closed set) but the converse is not true for example in the usual topology (ℝ,u) the set (a,b) is an open set but d(a,b)=[a,b] is a closed set.

1.4 The Closure of Sets

1.4.1 Definition:-

Let A be a subset of a topological space (X,τ) the ***closure*** of A ,denote by $\bar{A}$ is the intersection of all closed subsets of X containing A , i.e.

$$\bar{A} = \cap_i F_i \ , \ A \subseteq F_i, F_i \text{ is closed set.}$$

Notice that $\bar{A}$ is closed set since its equals to intersection of closed sets (corollary 1.3.7 part a) . Also $\bar{A}$ is the smallest closed set containing A, i.e. if F is any closed set contain A then $\subseteq \bar{A} \subseteq F$.

1.4.2 Example:-

From example 1.3.5 we have $\overline{\{b,c\}}=\{b,c,d,e\} \cap X=\{b,c,d,e\}$, $\overline{\{d,e\}} = \{d,e\}\cap\{a,d,e\}\cap X=\{d,e\}$ and $\overline{\{a,b\}} = X$.

1.4.3 Exmaple:-

Let A be a subset of the cofinite topological space (X,τ) then

$$\bar{A} = \begin{cases} A & if\ A\ is\ finite \\ X & if\ A\ is\ infinite \end{cases}$$

Notice that the following theorem define the closure sets in terms of its limit points

1.4.4 Theorem:-

Let A be a subset of a topological space (X,τ) the closure of A is the union of A and its set of limit points, i.e.

$$\bar{A} = AUd(A).$$

1.4.5 Example:-

Let $(\mathbb{R},\tau)$ be the usual topology then $\overline{(a,b)}=\overline{[a,b)}=\overline{(a,b]}=\overline{[a,b]}=[a,b]$.

1.4.6 Example:-

Let $(\mathbb{R},\tau)$ be the usual topology then

a) If $A = \{1,\frac{1}{2},\frac{1}{3},\dots\} \subset \mathbb{R}$ then

$\bar{A} = A \cup d(A) = \{1,\frac{1}{2},\frac{1}{3},\dots\} \cup \{0\} = \{1,\frac{1}{2},\frac{1}{3},\dots,0\}$.

b) If $\mathbb{Q} \subset \mathbb{R}$ the set of rational numbers then

$$\overline{\mathbb{Q}} = \mathbb{Q} \cup \boldsymbol{d}(\mathbb{Q}) = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}.$$

1.4.7 Theorem (Closure Axioms):-

If A and B are subsets of a topological space (X,τ) then

a) $\bar{\emptyset} = \emptyset$, $\bar{X} = X$.

b) $A \subseteq \bar{A}$.

c) $A = \bar{A}$ *iff A is closed.*

d) $\bar{\bar{A}} = \bar{A}$.

e) $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$.

Notice that $\overline{(A \cap B)} \neq \bar{A} \cap \bar{B}$ as the following example:

1.4.8 Example:-

Let $X = \{a,b,c,d,e\}$,$\tau = \{\emptyset,X,\{a\},\{a,b\}\}$. If $A = \{a,c\}$, $B = \{b,c\}$ then $A \cap B = \{c\}$, ,$\overline{A}=X$,$\overline{B}=B$,$\overline{A\cap B}=\{c\}$,So $\overline{A \cap B} = \{c\} \neq \bar{A} \cap \bar{B} = X \cap B = B = \{b,c\}$

1.4.9 Example:-

If E is a subset of a topological space (X,τ),and if $d(F) \subseteq E \subseteq F$ for some subset $F \subseteq X$,show that E is a closed set.

1.4.10 Definition:-

A subset A of a topological space (X,τ) is called ***dense*** in X if $\bar{A} = X$.

1.4.11 Example:-

Let (X,τ) be the indiscrete topology. If $\emptyset \neq A \subseteq X$ then A is dense in X , i.e. $\bar{A} = X$ (since X the only closed set contain A).

1.4.12 Example:-

In discrete topology (X,τ) every proper subset of X is not dense in X ,i.e. $\forall A \subset X, \bar{A} = A$.

1.4.13 Example:-

In topological space $(\mathbb{R},\tau)$ where $\tau = \{\mathbb{R},\emptyset,E_a=(a,\infty): a \in \mathbb{R}\}$ the sets $A = \{2,4,6,...\}$, $B=\{1,3,5,...\}$ are dense in $\mathbb{R}$ while the set $C = \{-2,-4,-6,...\}$ is not dense in $\mathbb{R}$.

1.4.14 Example:-

The set of rational numbers $\mathbb{Q} \subset \mathbb{R}$ in the usual topology $(\mathbb{R},\tau)$ is dense in $\mathbb{R}$.

Exercises: -

1. Consider the following topology on $X = \{a,b,c,d,e\}$,$\tau = \{X,\emptyset,\{a\},\{a,b\},\{a,c,d\},\{a,b,c,d\},\{a,b,e\}\}$

- **a)** List the closed subsets of X.
- **b)** Determine the closure of the sets $\{a\}$,$\{b\}$ and $\{c\}$.
- **c)** Which sets in b) are dense in X.

2. Let τ be the topology on $\mathbb{N}$ which consists of $\emptyset$ and all subsets of $\mathbb{N}$ of the form $E_n = \{n,n+1,n+2,\dots\}$were $n \in \mathbb{N}$.

- **a)** Determine the closed subsets of $(\mathbb{N},\tau)$.
- **b)** Determine the closure of the sets $\{7,24,47,85\}$ and $\{3,6,9,12,\dots\}$.
- **c)** Determine those subsets of $\mathbb{N}$ which are dense in $\mathbb{N}$.

3. Let τ be the topological $\mathbb{R}$ consists of of $\emptyset$,$\mathbb{R}$, and all open infinite intervals $E_p = (a,\infty), a \in \mathbb{R}$.

- **a)** Determine the closed subsets of $(\mathbb{R},\tau)$.
- **b)** Determine the closure of the sets $[3,7)$,$\{7,24,47,85\}$,$\{3,6,9,12,\dots\}$.

4. Prove: If F is a closed contain any set A, then $\bar{A} \subset F$.

5. If $A \cap B \neq \emptyset$ prove that $\bar{A} \cap \bar{B} = \overline{A \cap B}$.

6. If F is a closed set ,prove that $\forall A \subseteq X; \overline{F \cap A} \subseteq F \cap \bar{A}$.

7. If U is an open set, prove that $\forall A \subseteq X; U \cap \bar{A} \subseteq \overline{U \cap A}$.
8. If U is an open set and A is dense in X ,prove that $U \subseteq \overline{U \cap A}$.
9. Prove that, A is dense in X iff $A^c \cap (A')^c = \emptyset$.
10. Show that every non-finite subset of an infinite cofinite spae X is dense in X.

1.5 The Interior,Exterior and Boundary points of a Set

1.5.1 Definition:-

Let A be a subset of a topological space (X, τ) the ***interior*** of A ,denote by A° is the union of all open subsets of X contained in A , i.e.

$$A^\circ = \bigcup_i G_i \ , \ G_i \subseteq A \ ,G_i \text{ is an open set.}$$

1.5.2 Example:-

Let $X = \{a,b,c,d,e\}$ and $\tau = \{\emptyset,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\},X\}$ then $\{a,b,e\}^\circ = \emptyset \cup \{a\} = \{a\}$ and$\{a,c,d\}^\circ = \emptyset \cup \{a\} \cup \{c,d\} \cup \{a,c,d\}=\{a,c,d\}$.

1.5.3 Theorem:-

Let A be a subset of a topological space (X, τ) then $A^\circ = \overline{A^c}^c$.

1.5.4 Theorem (Interior Axioms):-

If A and B are subsets of a topological space(X, τ) then

a) $X^\circ = X$.
b) *$A^\circ$the largest open set contained in A.*
c) *$A^\circ$is open iff $A^\circ = A$.*
d) $A^\circ \subseteq A$
e) $A^{\circ\circ} = A^\circ$.
f) $(A \cap B)^\circ = A^\circ \cap B^\circ$

Notice that$(A \cup B)^\circ \neq A^\circ \cup B^\circ$ as the following example:

1.5.5 Example:-

In example 1.5.2 $A \cup B = \{a,b,e\} \cup \{a,c,d\}=\{a,b,c,d,e\}$ then $A^\circ \cup B^\circ = \{a\} \cup \{a,c,d\}=\{a,c,d\}$ and $(A \cup B)^\circ = \{a,b,c,d,e\}$,i.e. $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

1.5.6 Definition:-

Let A be a subset of a topological space (X, τ) the ***exterior*** of A ,denote by A^e is the set of all points interior to the complement, i.e. $A^e = A^{c^\circ}$.

1.5.7 Theorem (Exterior Axioms):-

If A and B are subsets of a topological space(X, τ) then

a) $X^e = \emptyset \, , \emptyset^e = X$.
b) $A^e \subseteq A^c$

c) $A^e = A^{e^{c^e}}$.

d) $(A \cup B)^e = A^e \cap B^e$

1.5.8 Definition:-

Let A be a subset of a topological space (X, τ) the ***boundary*** of A ,denote by $b(A)$ is the set of all points interior to neither A nor A^c, i.e. $b(A) = \left(A^{\circ} \cup A^{c^{\circ}}\right)^c$.

1.5.9 Example:-

Let $X = \{a,b,c,d,e\}$,$\tau = \{\emptyset, X, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ and let $A = \{b,c,d\}$ then $A^{\circ} = \{c,d\}$, $A^e = \{a\}$, $b(A) = \{b,e\}$.

1.5.10 Example:-

Let A be a non-empty proper subset of an indiscrete space X. Then $A^{\circ} = \emptyset$, $A^e = \emptyset$, $b(A) = X$.

1.5.11 Example:-

Let A be a non-empty proper subset of discrete space X. Then $A^{\circ} = A$, $A^e = A^c$, $b(A) = \emptyset$.

1.5.12 Example:-

Let $(\mathbb{R}, \tau)$ be the usual topology then

1) $[a,b]^{\circ} = [a,b)^{\circ} = (a,b]^{\circ} = (a,b)^{\circ} = (a,b)$, $\mathbb{Q}^{\circ} = \emptyset$.

2) $[a,b]^e = [a,b)^e = (a,b]^e = (a,b)^e = (-\infty,a) \cup (b,\infty)$, $\mathbb{Q}^e = \emptyset$.

3) $b([a,b]) = b\big([a,b)\big) = b\big((a,b]\big) = b\big((a,b)\big) = \{a,b\}$, $b(\mathbb{Q}) = \mathbb{R}$.

1.5.13 Example:-

The function f which assigns to each set its interior ,i.e. $f(A) = A^{\circ}$,does not commute with the function g which assigns to each set to its closure ,i.e. $g(A) = \bar{A}$,since if we take $\mathbb{Q}$ the set of rational numbers as a subset of $\mathbb{R}$ with the usual topology. Then

$$(g \circ f)(\mathbb{Q}) = g\big(f(\mathbb{Q})\big) = g(\mathbb{Q}^{\circ}) = g(\emptyset) = \overline{\emptyset} = \emptyset.$$

$$(f \circ g)(\mathbb{Q}) = f\big(g(\mathbb{Q})\big) = f(\overline{\mathbb{Q}}) = f(\mathbb{R}) = \mathbb{R}^{\circ} = \mathbb{R}.$$

1.5.14 Example:-

Let $(\mathbb{N}, \tau)$ be a topological space, $\tau = \{\emptyset, N, A_n = \{1,2,\ldots,n\}$, $\mathbb{N}$ the set of natural numbers then

1) $\{1,2,4,6\}^{\circ} = \{1,2\}$, $\{1,2,4,6\}^e = \emptyset$, $b(\{1,2,4,6\} = \{3,4,5,\ldots\}$.

2) $\{5,7,9,20\}^{\circ} = \emptyset$, $\{5,7,9,20\}^e = \{1,2,3,4\}$, $b(5,7,9,20\}) = \{5,6,7,\ldots\}$.

1.5.15 Example:-

Let A be a subset of a co-finite topological space (X, τ)then

a) If A is finite then $A^{\circ} = \emptyset$, $A^e = A^c$, $b(A) = A$.

b) If A is infinite then

either $A^c$is finite, i.e. A is open set then $A^{\circ} = A$, $A^e = \emptyset$, $b(A) = A^c$.

nor A is infinite then $A^{\circ} = \emptyset$, $A^e = \emptyset$, $b(A) = X$.

1.5.16 Example:-

Consider the topological space $(\mathbb{R}, \tau)$, where τ consists of $\emptyset$, $\mathbb{R}$, and all open intervals $E_a = (a,\infty)$, $a \in \mathbb{R}$ then $[7,\infty)^{\circ} = (7,\infty)$, $[7,\infty)^e = \emptyset$, $b([7,\infty) = (-\infty,7]$.

Exercises: -

1. Let A be a subset of a topological space (X, τ) then prove that:

a) $b(A) = \bar{A} \cap \overline{A^c}$.

b) $b(A)$ is a closed set.

c) $b(A) = b(A^c)$.

d) $b(A) = \bar{A} - A^{\circ}$.

e) $\bar{A} = b(A) \cup A^{\circ}$.

f) $b(A) \cap A^{\circ} = \emptyset$.

g) $b(A) \cap A^e = \emptyset$.

h) $A^{\circ} \cap A^e = \emptyset$.

i) $A^{\circ} \cup A^e \cup b(A) = X$.

2. Let A be a subset of a topological space (X, τ),show that $\bar{A} = A^{\circ} \cup b(A)$.

3. Prove that A is closed and open iff $b(A) = \emptyset$.

4. Prove that in any topological space A subset A is closed iff $b(A) \subseteq A$ and A subset A is open iff $b(A) \subseteq X - A$.

5. Give an example to show that $b(A \cup B) \neq b(A) \cup b(B)$ for any A and B subsets of a topological space (X, τ).

6. Let τ_1 and $\tau_2$be topologies on X with $\tau_1$coarser than τ_2 ,i.e. $\tau_1 \subset \tau_2$ and let $A \subset X$.Then

a) The τ_1 $-$interior of A is subset of the τ_2- interior of A.

b) The τ_2 $-$boundary of A is subset of the τ_1- boundary of A.

1.6 Bases and subbases

1.6.1 Definition:-

Let (X,τ) be a topological space. A class $\mathcal{B}$ of open subsets of X, i.e. $\mathcal{B} \subset \tau$,is ***a base for the topology*** τ iff every open set $G \epsilon \tau$ is the union of members of $\mathcal{B}$, (equivalently for any point p belonging to an open set G there exists $B \in \mathcal{B}$ with $p \in B \subset G$.

1.6.2 Example:-

The class of open intervals $\mathcal{B} = \{(a,b): a,b \in \mathbb{R}\}$ is a base for the usual topology $(\mathbb{R}, \tau)$.Similarly, the class of open discs form a base for the usual topology $(\mathbb{R}^2, \tau)$.

1.6.3 Example:-

The class $\mathcal{B} = \{\{a\}: a \in X\}$ of all singleton subsets of X is a base for the discrete topology τ on X.

1.6.4 Example:-

Let (X,τ) be a topological space where X={a,b,c,d} ,τ={X,∅,{a,b}, {c,d}} then $\mathcal{B}_1 =$ {{a,b},[c,d}}, $\mathcal{B}_2 = \{X$,{a,b},{c,d}} are bases for the topology τ while $\mathcal{B}_3 =$ {X,{a,b}} is not a base for the topology τ ,since {c,d}is an open set but it is not a union of members of $\mathcal{B}_3$.

Note that it is not necessary to include the empty set in a base for a topology, since $\emptyset = \bigcup\{B_\lambda : \lambda \in \emptyset\}$,also it is not every family of subsets of a set ***X*** is a base for a topology for X for example let X={a,b,c} then the class $\mathcal{B}$={{a,b},{b,c}} is not a base for any topology on X ,since {a,b},{b,c} are open sets and their intersection {a,b}∩{b,c}={b} is also an open set but {b}is not a union of members of $\mathcal{B}$.

The following theorem gives the necessary and sufficient conditions for a family of subsets to be a base for a topology.

1.6.5 Theorem:-

Let $\mathcal{B}$ be a class of subsets of a non- empty set X. Then $\mathcal{B}$ is a base for some topology on X iff it possesses the following two properties :

1) $X = \cup \{B : B \in \mathcal{B}\}$**.**

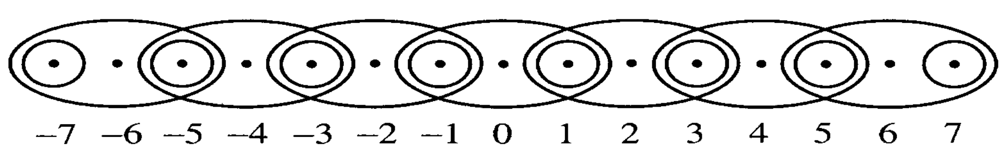
2) For any B_1,$B_2 \in \mathcal{B}$,$B_1 \cap B_2$ is a union of members of $\mathcal{B}$ or equivalently , if $p \in B_1 \cap B_2$ then $\exists B_p \in \mathcal{B}$ such that $p \in B_p \subset B_1 \cap B_2$.

1.6.6 Example:-

Let $\mathcal{B}$ be a class of open –closed intervals in the real line ℝ, i.e. $\mathcal{B}$={(a,b]:a,b∈ℝ ,a<b} then $\mathcal{B}$ is a base for a topology τ on ℝ .This topology τ is called the upper limit topology on ℝ (this topology is not equals to the usual topology). Similarly, the class of closed – open intervals , $\mathcal{B}^*$={[a,b):a,b∈ℝ ,a<b} is a base for a topology τ^* on ℝ called lower limit topology on ℝ.

1.6.7 Example:-

For each $n \in \mathbb{Z}$,define $B(n) = \begin{cases} \{n\} & if\ n\ is\ odd \\ \{n-1, n, n+1\} & if\ n\ is\ even \end{cases}$ .The collection



The collection $\mathcal{B} = \{B(n) : n \in \mathbb{Z}\}$ is a basis for a topology on ℤ ,this topology is called the digital line topology ,also ℤ with this topology is the digital line.

1.6.8 Definition:-

Let (X,τ) be a topological space,A class Ψ of open subsets of X, i.e. $\Psi \subset \tau$ is ***a subbase*** for the topology τ on X iff finite intersection of members of Ψ form a base for τ.

1.6.9 Example:-

Let $X =$ {a,b,c,d} ,$\tau =$ {∅,X,{a},{a,c},{a,d},{a,c,d}} and let $S = \{\{a$,c},{a,d}} so finite intersection of members of S is $\mathscr{B} =$ {{a},{a,c},{a,d},X} which is a base for τ therefore, S is a subbase for τ .

1.6.10 Example:-

Every open interval (a,b) in the real line ℝ is the intersection of two infinite open intervals (a,∞) and $(-\infty$,b) ,i.e. (a,b)=(a,∞)∩(-∞,b). But the open intervals form a base for the usual topology on ℝ , hence the class of all infinite open intervals ($S = \{(a,\infty)$, $(-\infty$,b):a,bϵℝ}) is a subbase for ℝ.

1.6.11 Example:-

Let (X,τ) be the discrete topology then the family $S = \{\{a,b\}\}: a,b\epsilon X\}$ is a subbase for the discrete topology.

1.6.12 Example:-

The family S of all infinite open strips is a subbase for $\mathbb{R}^2$.

1.6.13 Remark:-

Let S be any family of subsets of a non-empty set X. S may not be a base for a topology on X. However S is always generates a topology on X in the following sense:

1.6.14 Theorem:-

Any family S of subsets of a non-empty set X is the subbase for a unique topology τ on X. That is, finite intersection of members of S form a base for topology τ on X.

1.6.15 Example:-

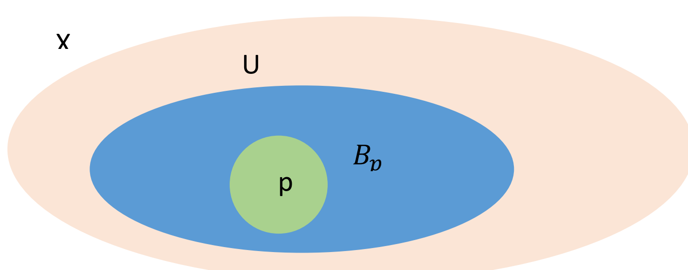
Let $X =$ {a,b,c,d} then the family S={{a,b},{b,c},{d}} is a subbase for a topology on X.

1.6.16 Theorem:-

Let S be a class of subsets of a non – empty set X. Then the topology τ on X generated by S is the intersection of all topologies on X which contain S.

1.6.17 Definition:-

Let p be any arbitrary point in a topological space (X,τ). A class $\mathcal{B}_p$ of open sets containing p is called ***a local base at p*** iff for each open set U contained p , $\exists B_p \in \mathcal{B}_p$ with the property $p \in B_p \subset U$.



1.6.18 Example:-

Let $X = \{a,b,c,d\}$ and $T = \{X,\emptyset,\{a\},\{a,b\},\{a,b,c\}\}$ then

$\mathcal{B}_a = \{\{a\}\}$ (or $\mathcal{B}_a = \{\{a\},\{a,b\},\{a,b,c\},X\}$) ,

$\mathcal{B}_b = \{\{a,b\}\}$ (or $\mathcal{B}_b = \{\{a,b\},\{a,b,c\},X\}$) ,

$\mathcal{B}_c = \{\{a,b,c\}\}$ (or $\mathcal{B}_c = \{\{a,b,c\},X\}$) ,

$\mathcal{B}_d = \{X\}$.

1.6.19 Example:-

Consider the topological space $(\mathbb{R},\tau)$,where τ is the usual topology of open intervals on $\mathbb{R}$. Consider the point $0 \in \mathbb{R}$. The local base of 0 is the $\mathcal{B}_0 = \{(a,b):a,\ b\in\mathbb{R}, a<0<b\}$. Now if we take any $x \in \mathbb{R}$ then the local base of x is $\mathcal{B}_x = \{(a,b):a,\ b\in\mathbb{R}, a<x<b\}$.

1.6.20 Example:-

Consider the topological space $(\mathbb{R}^2,\tau)$ where τ is the usual topology on $\mathbb{R}^2$. Consider the point $p \in \mathbb{R}^2$. Then the class $\mathcal{B}_p$ of all open discs centered at p is a local base at *p*.

1.6.21 Theorem:-

Let $\mathcal{B}$ be a base for a topology τ on X and let $p \in X$,. Then the members of the base $\mathcal{B}$ which contain *p* from a local base at the point *p*.

1.6.22 Theorem:-

A point p in a topological space X is a limit point of $A \subset X$ iff each members of some local base $\mathcal{B}_p$ at *p* contains a point of *A* different from *p*.

1.6.23 Example:-

Consider the lower limit topology τ on the real line $\mathbb{R}$ which has as a base the class of closed-open intervals $[a,b)$, and let $A = (0,1)$. Note that $G = \{1,2)$ is a τ-open set containing $1 \in \mathbb{R}$ for which $G \cap A = \emptyset$ hence 1 is not a limit point of A. On the other hand , $0 \in \mathbb{R}$ is a limit point of A since any open base set $[a,b)$ containing 0 ,i.e. for which $a \le 0 < b$ contains points of A other than 0.

1.6.24 Example:-

Every point p in a discrete topology has a finite local base.

Exercises: -

1. Let $\mathcal{B} = \{(a,b):a,b \in \mathbb{Q}\}$ be the class of open intervals in $\mathbb{R}$ with rational endpoints . Show that

(1) $\mathcal{B}$ is a basis for some topology on $\mathbb{R}$.

(2) The topology generated by $\mathcal{B}$ is the usual Euclidean topology on $\mathbb{R}$.

2. Let $\mathcal{B} = \{[a,b]: a,b \in \mathbb{R}\}$ be the class of all closed intervals in $\mathbb{R}$. Can $\mathcal{B}$ be a basis of some (not necessarily standard) topology on $\mathbb{R}$? Why or why not?
3. Show that the class of closed intervals [a,b], where a and b are rational and $a<b$ is not a base for a topology on the real line $\mathbb{R}$.
4. Show that the class of closed intervals [a,b],where a is rational and b is irrational and $a<b$ is a base for a topology on the real line $\mathbb{R}$.
5. Let $\mathcal{B},\mathcal{B}'$ be two bases for X, satisfy the following conditions:
   (1) For every $B \subset \mathcal{B}$ and every $x \in B$,there exists a $B' \in \mathcal{B}'$ s.t. $x \in B' \subset B$.
   (2) For every $B' \subset \mathcal{B}'$ and every $x \in B'$,there exists a $B \subset \mathcal{B}$ s.t. $x \in B \subset B'$.
   Show that $\mathcal{B}$ and $\mathcal{B}'$ generate the same topology on X.
6. Let $\mathcal{B}$ and $\mathcal{B}^*$ be bases , respectively ,for topologies τ and τ^* on a set X. Suppose that $B \in \mathcal{B}$ is the union of members of $\mathcal{B}^*$ .Show that τ is coarser than τ^*, i.e. $\tau \subset \tau^*$.
7. Show that the usual topology τ on the real line $\mathbb{R}$ is coarser than the upper limit topology τ^* on $\mathbb{R}$ which has as a base the class of open – closed intervals $(a,b]$.
8. Determine which of the following collection of subsets of $\mathbb{R}$ are bases:

   (1) $\tau_1 = \{(n,n+2) \subset \mathbb{R}: n \in \mathbb{Z}\}$.

   (2) $\tau_2 = \{[a,b) \subset \mathbb{R}: a \leq b\}$.

   (3) $\tau_3 = \{(-x,x) \subset \mathbb{R}: x \in \mathbb{R}\}$.

   (4) $\tau_4 = \{(a,b) \cup \{b+1\} \subset \mathbb{R}: a < b\}$.

Chapter Two

Creating New Topological Spaces

2.1 The Subspace Topology

Let (X,τ) be a topological space, A be a proper subset of X. Let $\tau^* = \{G^* = G \cap A : G \in \tau\}$,i.e. $G^* \in \tau^* \Leftrightarrow \exists G \in \tau , G^* = G \cap A$. The following theorem shows that τ^* is a topology on A called the ***Relative Topology*** **(or** ***Induced Topology*****)** and (A,τ^*) is called the ***Subspace Topology*** of topological space (X,τ).

2.1.1 Theorem:

Let (X,τ) be a topological space A be a proper subset of X. Then $\tau^* = \{G^* = G \cap A : G \in \tau\}$ is a topology on A.

Proof:

1) $\emptyset = \emptyset \cap A \Rightarrow \emptyset \in \tau^*$

$A = X \cap A \Rightarrow A \in \tau^*$.

2) Let $G_1^*, G_2^* \in \tau^*$ then $\exists G_1, G_2 \in \tau$ s. t. $G_1^* = G_1 \cap A, G_2^* = G_2 \cap A$ then

$G_1^* \cap G_2^* = (G_1 \cap A) \cap (G_2 \cap A) = (G_1 \cap G_2) \cap A \in \tau^*$ since $(G_1 \cap G_2) \in \tau$.

3) Let $\{G_i^* : i \in I\} \subseteq \tau^*$ then $\exists G_i \in \tau$ s. t. $G_i^* = G_i \cap A, \forall i \in I$. So

$\bigcup_i G_i^* = \bigcup_i (G_i \cap A) = \bigcup_i G_i \cap A \in \tau^*$, since $\bigcup_i G_i \in \tau$.

So τ^* is a topology on A.□

2.1.2 Example:

Let (X,τ) be a topological space where $X = \{a,b,c,d,e\}$, $\tau = \{X, \emptyset, \{a\}, \{c,d\}, \{a,c,d\}, \{b,c,d,e\}\}$ .Find τ_A, τ_B, τ_C, $A = \{a,d\}$, $B = \{a,b,c\}$, $C = \{a\}$.

Solution:

$X \cap A = A$, $\emptyset \cap A = \emptyset$, $\{a\} \cap A = \{a\}$, $\{c,d\} \cap A = \{d\}$, $\{a,c,d\} \cap A = A$, $\{b,c,d,e\} \cap A = \{d\}$

So $\tau_A = \{A, \emptyset, \{a\}, \{d\}\}$. Similar $\tau_B = \{B, \emptyset, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$, $\tau_C = \{C, \emptyset\}$.

2.1.3 Remark:

In example 2.1.2 , τ_A is the discrete topology on A , τ_C is the indiscrete topology on C but τ is not discrete or indiscrete topology on X. Also we can find {d}∈τ_A but {d}$\notin\tau$.

2.1.4 Example:

The subspace of discrete topology (indiscrete topology) is also a discrete topology (indiscrete topology).

2.1.5 Example:

Let (X,τ) be a co-finite topology and let $A \neq \emptyset$ be a subset of X the τ_A is the discrete topology.

Solution:

Let p be any point in A then the set $X\backslash\{A\backslash\{p\}\}$ is open in X and their intersect with A is $\{p\}$ i.e. $A \cap (X\backslash\{A\backslash\{p\}\}) = \{p\}$ is open in A .Since p be any point in A then the subspace topology on A is the discrete topology.

2.1.6 Example:

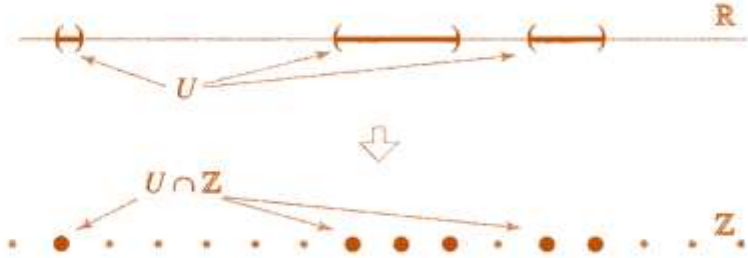
Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$ then the subspace topology ($\mathbb{N}$,$D_{\mathbb{N}}$) is the discrete topology.

Solution:

Let $n \in \mathbb{N}$ then $(n-\frac{1}{2},n+\frac{1}{2})$ is an open interval contain n and $\mathbb{N} \cap (n-\frac{1}{2},n+\frac{1}{2})$ ={n}.So every {n} contain a natural number in the subspace ($\mathbb{N}$,$D_{\mathbb{N}}$) ,so every subset of $\mathbb{N}$ is an open set i.e. $D_{\mathbb{N}}$ is the discrete topology.

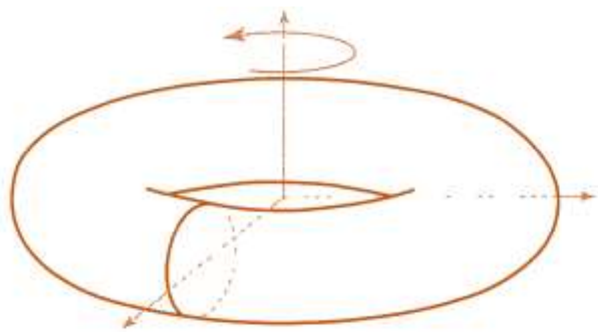
2.1.7 Example:

Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$ then the subspace topology ($\mathbb{Z}$,$D_{\mathbb{Z}}$) is the discrete topology.



2.1.8 Example:

In $\mathbb{R}^3$, let C be the circle of radius 1 in the xy-plane with center at the point (2,0,0).Consider the subspace of $\mathbb{R}^3$swept out as C is rotated about the z-axis the resulting space is called the torus and denoted by T which is a subspace of $\mathbb{R}^3$.



2.1.9 Theorem:

Let (A,τ_A) be a subspace of (X,τ) then the subset E of A is closed in (A,τ_A) iff there exist a closed set F in (X,τ) such that $E = F \cap A$.

Proof:

$\Rightarrow$

Let E be a closed in (A,τ_A) the E^c is an open set in (A,τ_A) .By definition of subspace $\exists G \in \tau$ s.t. $E^c = A \cap G = A\backslash E$. So

$$E = A\backslash E^c = A\backslash(A \cap G) = A \cap (A \cap G)^c = A \cap G^c.$$

Put $E^c = F$ which is the closed set we want to find.

$\Leftarrow$

Assume there exist a closed set F in (X,τ) such that $E = F \cap A$ we want to prove that E is closed in (A,τ_A) i.e. E^c is an open set in (A,τ_A)

$$E^c = A\backslash E = A\backslash(A \cap F) = A \cap (A \cap F)^c = A \cap (A^c \cup F^c) = (A \cap A^c) \cup (A \cap F^c) = A \cap F^c.$$

So E^c is an open set in (A,τ_A).□

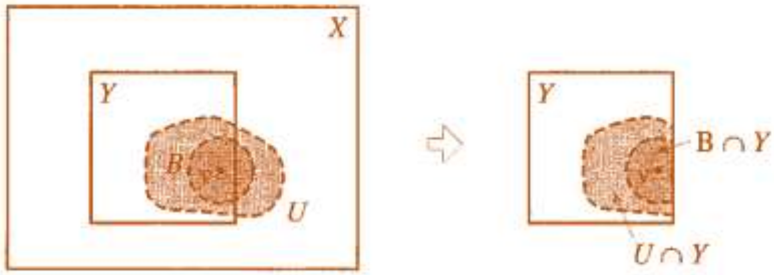
2.1.10 Corollary:

If A is a non-empty open (closed) subset of (X,τ) then the subset B of A is open (closed) in (A,τ_A) iff B an open set F in (X,τ).

2.1.11 Theorem:

Let (Y,τ_Y) be a subspace of (X,τ). If $\mathcal{B}=\{B_i\}_{i\in I}$ is a base for (X,τ) then $\mathcal{B}^*=\{B_i\cap Y\}_{i\in I}$ is a base for (Y,τ_Y).

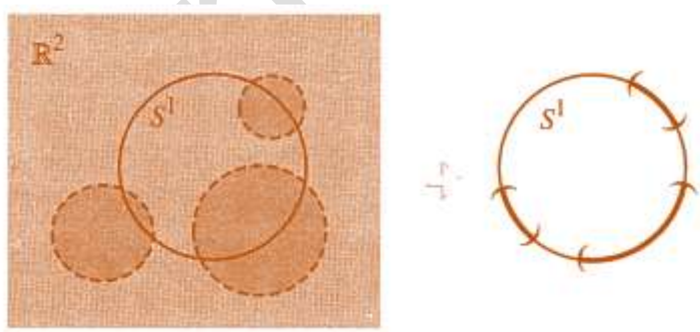
Proof:



Assume $\mathcal{B}=\{B_i\}_{i\in I}$ is a base for (X,τ) then $\forall U\in\tau$, $y\in U\Rightarrow\exists B\in\mathcal{B}, y\in B\subseteq U$. From definition of subspace the family $\{B_i\cap Y\}_{i\in I}$ is open in (Y,τ_Y). If $y\in Y$ then $y\in B\cap Y\subseteq U\cap Y$ where $U\cap Y\in\tau_Y$ then $\{B_i\cap Y\}_{i\in I}$ is a base for (Y,τ_Y).□

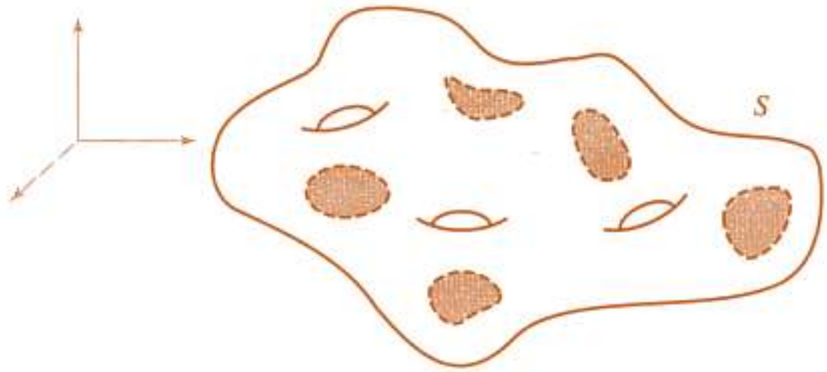
2.1.12 Example:

Let the circle $S^1\subseteq\mathbb{R}^2$ with the usual topology. Since the class of open balls form a basis for the usual topology on $\mathbb{R}^2$ then their intersection with S^1 are class of open intervals in the circle consisting of all points between two angles in the circle .This class form a base for the usual topology on S^1.



2.1.13 Example:

If S is a surface in $\mathbb{R}^3$ then the collection of open patches in S obtained by intersecting open balls in $\mathbb{R}^3$ with S is a basis for the standard topology on S.



2.1.14 Remark:

The following theorem gives the relation between the limit and interior points and the closure of sets in subspaces and spaces .we denote $d(A_Y)$,A_Y°,$\overline{A_Y}$ for limit ,interior ,closure for a set A in subspace.

2.1.15 Theorem:

Let (Y,τ_Y) be a subspace of (X,τ). If $A \subseteq Y$ then :

1) $d(A_Y) = d(A) \cap Y$.
2) $A^\circ = A_Y^\circ \cap Y^\circ$, $A^\circ \cap Y = A_Y^\circ$.
3) $\overline{A_Y} = \bar{A} \cap Y$.

Proof:

1) Assume $x \in d(A_Y)$ then $\forall U \in \tau_Y, x \in U$,U∩A≠∅ then $\exists W \in \tau, x \in U$,U=W∩Y. So for any $W \in \tau$ s.t. $x \in W$ we find W∩Y≠∅ therefore we get (W∩Y)∩A=W∩A≠∅ i.e. $x \in d(A)$,SO

$$d(A_Y) \subseteq d(A) \qquad \ldots\ldots. \qquad (1)$$

Let $x \in d(A)$ then $\forall U \in \tau, x \in U$,U∩A≠∅.Its clear that W=U∩Y $\in \tau_Y$ is an open set in (Y,τ_Y),SO $W \cap A = (U \cap Y) \cap A = Y \cap (U \cap A \neq \emptyset \; i.e. \; x \in d(A_Y)$

$$d(A) \subseteq d(A_Y) \qquad \ldots\ldots.. \qquad (2)$$

From (1) and (2) we get $d(A_Y) = d(A) \cap Y$.

2) Let $p \in A^\circ$them $\exists H \in \tau$ s.t. p∈H⊆A⊆Y ,so $p \in Y \cap A \subseteq A$, p∈$Y^\circ \Rightarrow p \in A_Y^\circ$, $p \in Y^\circ \Rightarrow p \in Y^\circ \cap A_Y^\circ$,so

$$A^\circ \subseteq A_Y^\circ \cap Y^\circ \qquad \ldots\ldots\ldots \qquad (1)$$

Let $x \in Y^{\circ} \cap A_Y^{\circ} \Rightarrow \exists H_1, H_2 \in \tau$ s.t. $x \in H_2 \subseteq Y, x \in Y \cap H_1 \subseteq A$, SO

$x \in H_1 \cap H_2 \subseteq A \Rightarrow x \in A^{\circ}$,so

$$A_Y^{\circ} \cap Y^{\circ} \subseteq A^{\circ} \quad \ldots\ldots\ldots \quad (2)$$

From (1) and (2) we get $A^{\circ} = A_Y^{\circ} \cap Y^{\circ}$.

3) $\overline{(A_Y)} = d(A_Y) \cup A = (d(A) \cap Y) \cup A$, $A \subseteq Y$

$$= (d(A) \cup Y) \cap (A \cup Y) = (d(A) \cup A) \cap Y = \bar{A} \cap Y. \square$$

2.1.16 Example:

Show that if $d(A) = \emptyset$ in a topological space (X, τ) then τ_A is the discrete topology.

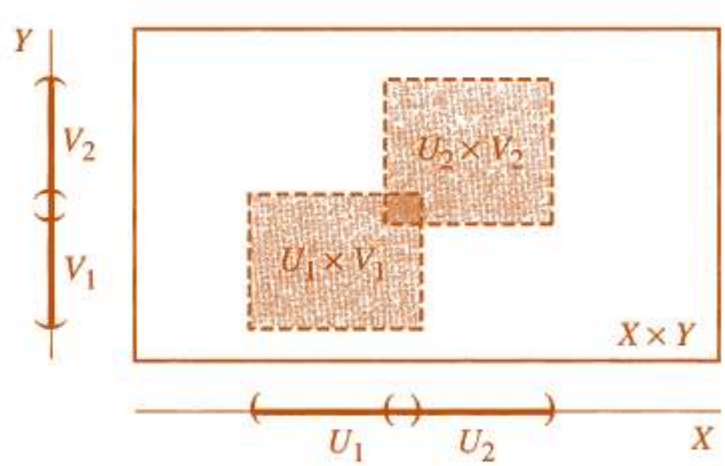
Solution:

In order to prove that τ_A is the discrete topology we shall show that every subset of A is closed.

If $B \subseteq A$ then $d(B) \subseteq d(A)$,so $d(B) \subseteq \emptyset$ (since $d(A) = \emptyset$),so B is a closed set in X and then B is closed in A (since $B = B \cap A$).

2.2 The Product Topology

Given two topological spaces X and Y ,we would like to generate a natural topology on the product $X \times Y$. Our first inclination might be to take as the topology on $X \times Y$ the collection $\mathcal{C}$ of sets of the form $U \times V$ where U is open in X and V is open in Y. But $\mathcal{C}$ is not a topology since the union of two sets $U_1 \times V_1$ and $U_2 \times V_2$ need not be in the form $U \times V$ for some $U \subset X$ and $V \subset Y$. However, if we use $\mathcal{C}$ as a basis, rather than as the whole topology, we can proceed.



2.2.1 Definition:

Let (X,τ_X) and (Y,τ_Y) be topological spaces and $X \times Y$ be their product. The ***product topology*** on $X \times Y$ is the topology generated by the basis

$$\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}.$$

2.2.2 Remark:

We shall verify that $\mathcal{B}$ actually is a basis for a topology on the product, $X \times Y$.

2.2.3 Theorem:

The collection $\mathcal{B} = \{U \times V : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ is a basis for a topology on $X \times Y$.

Proof:

1- Every point (x, y) is in $X \times Y$, and $X \times Y \in \mathcal{B}$. Therefore, the first condition for a basis is satisfied.

2- Assume that (x, y) is in the intersection of two elements of $\mathcal{B}$. That is, $(x, y) \in (U_1 \times V_1) \cap (U_2 \times V_2)$ where U_1 and U_2 are open sets in X, and V_1 and

V_2 are open sets in Y. Let $U_3 = U_1 \cap U_2$ and $V_3 = V_1 \cap V_2$. Then U_3 is open in X, and V_3 is open in Y, and therefore $U_3 \times V_3 \in \mathcal{B}$. Also,

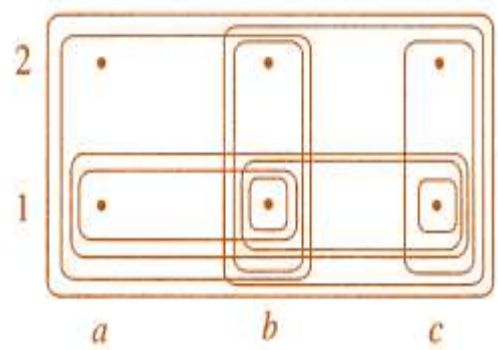
$$U_3 \times V_3 = (U_1 \cap U_2) \times (V_1 \cap V_2) = (U_1 \times V_1) \cap (U_2 \times V_2)$$

and thus $(x, y) \in U_3 \times V_3 \subset (U_1 \times V_1) \cap (U_2 \times V_2)$. It follows that the second condition for a basis is satisfied.

Therefore $\mathcal{B}$ is a basis for a topology on $X \times Y$.□

2.2.4 Example:

Let $X = \{a, b, c\}$ and $Y = \{1, 2\}$ with topologies {ø,{b},{c},{a,b},{b,c}, X} and {ø, {1}, Y},respectively. A basis for the product topology on $X \times Y$. Each nonempty open set in the product topology on $X \times Y$ is a union of the basis elements.



2.2.5 Remark:

As with open sets, products of closed sets are closed sets in the product topology. But here too, this does not account for all of the closed sets because there are closed sets in the product topology that cannot be expressed as a product of closed sets. For instance, the set $\{(a, 2), (c, 1), (c, 2)\}$ is a closed set in the product topology in Example 2.2.4, but it is not a product of closed sets.

2.2.6 Remark:

In Definition 2.2.1, the basis B that we use to define the product topology is relatively large since we obtain it by pairing up every open set U in X with every open set V in Y. Fortunately, as the next theorem indicates, we can find a smaller basis for the product topology by using bases for the topologies on X and Y, rather than using the whole topologies themselves.

2.2.7 Theorem:

If $\mathcal{B}_X$ is a basis for X and $\mathcal{B}_Y$ is a basis for Y, then

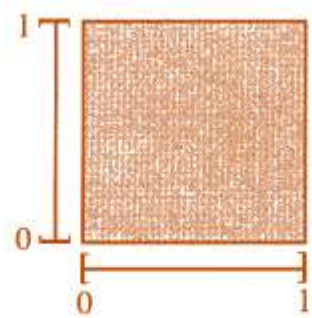
$$\mathcal{B} = \{C \times D : C \in \mathcal{B}_X \text{ and } D \in \mathcal{B}_Y\}$$

is a basis that generates the product topology on $X \times Y$.

Proof:

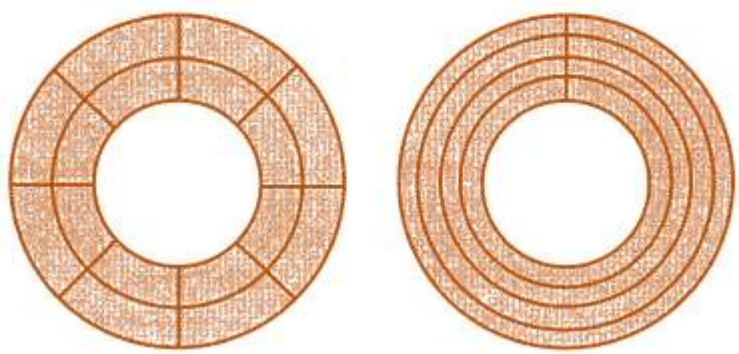
Each set $C \times D \in \mathcal{B}$ is an open set in the product topology; therefore, by definition 1.6.1, it suffices to show that for every open set W in $X \times Y$ and every point $(x, y) \in W$, there is a set $C \times D \in \mathcal{B}$ such that $(x, y) \in C \times D \subset W$. But since W is open in X, we know that there are open sets U in X and V in Y such that $(x, y) \in U \times V \subset W$. So $x \in U$ and $y \in V$. Since U is open in X, there is a basis element $C \in \mathcal{B}_X$ such that $x \in C \subset U$. Similarly, since V is open in Y, there is a basis element $D \in \mathcal{B}_Y$ such that $y \in D \subset V$. Thus $(x, y) \in C \times D \subset W$. Hence, by definition 1.6.1, it follows that $\mathcal{B} = \{C \times D : C \in \mathcal{B}_X \text{ and } D \in \mathcal{B}_Y\}$ is a basis for the product topology on $X \times Y$.□

2.2.8 Example:



Let I = [0, 1] have the slandered topology as a subspace of $\mathbb{R}$. The product space $I \times I$ is called the unit square. The product topology on $I \times I$ is the same as the standard topology on $I \times I$ as a subspace of $\mathbb{R}^2$.

2.2.9 Example:

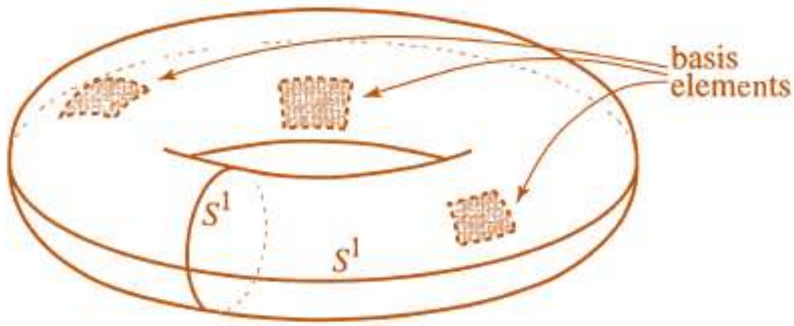


Let S^1 be the circle, and let I = [0, 1] have the standard topology.Then $S^1 \times I$ can think of it as a circle with intervals perpendicular at each point of the circle. Seen this way, it is a circle's worth of intervals. Or it can be thought of as an interval with perpendicular circles at each point. Thus it is an interval's worth of circles. The resulting topological space is called the ***annulus***.

The product space $S^1 \times (0, 1)$ is the annulus with the inner most and outermost circles removed. We refer to it as the ***open annulus***.

2.2.10 Example:

Consider the product space $S^1 \times S^1$, where S^1 is the circle. For each point in the first S^1. there is a circle corresponding to the second S^1.Since each S^1 has a topology generated by open intervals in the circle, it follows by Theorem 2.2.7 that $S^1 \times S^1$ has a basis consisting of rectangular open patches. The resulting space resembles the torus introduced in Example 2.1.8; in fact, they are topologically equivalent.



2.2.11 Example:

Let D be the disk as a subspace of the plane. The product space $S^1 \times D$ is called the ***solid torus***. If we think of the torus as the surface of a doughnut, then the solid torus is the whole doughnut itself.

2.2.12 Remark:

Let A and B be subsets of topological spaces X and Y, respectively. We now have two natural ways to put a topology on $A \times B$. On the one hand, we can view $A \times B$ as a subspace of the product $X \times Y$. On the other hand, we can view $A \times$

B as the product of subspaces, $A \subset X$ and $B \subset Y$. The next theorem indicates that both approaches result in the same topology.

2.2.13 Theorem:

Let (X,τ_X) and (Y,τ_Y) be topological spaces, and assume that $A \subset X$ and $B \subset Y$. Then the topology on $A \times B$ as a subspace of the product $X \times Y$ is the same as the product topology on $A \times B$, where A has the subspace topology inherited from X, and B has the subspace topology inherited from Y.

Proof: Left as exercise.

2.2.14 Remark:

The approach used to define a product of two spaces extends to a product $X_1 \times \cdots \times X_n$ of n topological spaces. It is straightforward to see that the collection $\mathcal{B} = \{U_1 \times \cdots \times U_n : U_i \text{ open in } X_i \text{ for each } i\}$ is a basis for a topology on $X_1 \times \cdots \times X_n$.The resulting topology is called the ***product topology*** on $X_1 \times \cdots \times X_n$. We have an analog to Theorem 2.2.7 for this case. Specifically, if $\mathcal{B}_i$ is a basis for X_i for each $i = 1, \cdots, n$, then the collection

$$\mathcal{B}' = \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i \text{ for } i = 1, \cdots, n\}$$

is a basis for $X_1 \times \cdots \times X_n$.

2.2.15 Remark:

We note that the standard topology on $\mathbb{R}^n$ is the topology generated by the basis of open balls defined by the Euclidean distance formula on We also pointed that the same topology results from taking a basis made up of products of open intervals in $\mathbb{R}$ It follows that the standard topology on $\mathbb{R}^n$ is the same as the product topology that results from taking the product of n copies of $\mathbb{R}$ with the standard topology.

2.2.16 Example:

The ***n-torus***, T^n is the topological space obtained by taking the product of n copies of the circle, S^1.

2.2.17 Remark:

The next theorem indicates that the interior of a product is the product of the interiors.

2.2.13 Theorem:

Let A and B be subsets of topological spaces X and Y, respectively. Then

$$(A \times B)^{\circ} = A^{\circ} \times B^{\circ}.$$

Proof: $\Rightarrow$

Since A° is an open set contained in A, and B° is an open set contained in B, it follows that $A^{\circ} \times B^{\circ}$ is an open set in the product topology and is contained in $A \times B$. Thus $A^{\circ} \times B^{\circ} \subset (A \times B)^{\circ}$

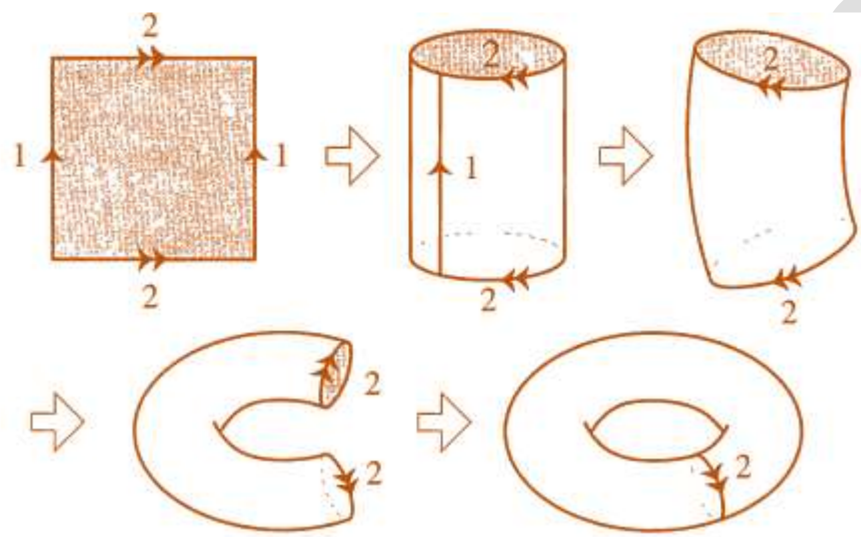
$\Leftarrow$

Now suppose $(x, y) \in (A \times B)^{\circ}$. We will prove that $(x, y) \in A^{\circ} \times B^{\circ}$. Since $(x, y) \in (A \times B)^{\circ}$, it follows that (x, y) is contained in an open set contained in $A \times B$ and therefore is also contained in a basis element contained in $A \times B$. So there exists a U and V open in X and Y, respectively, such that $(x, y) \in U \times V \subset A \times B$. Thus, x is in an open set U contained in A, and y is in an open set V contained in B, implying that $x \in A^{\circ}$ and $y \in B^{\circ}$. Therefore $(x, y) \in A^{\circ} \times B^{\circ}$. It follows that $(A \times B)^{\circ} \subset A^{\circ} \times B^{\circ}$.

Since we have both $A^{\circ} \times B^{\circ} \subset (A \times B)^{\circ}$ and $(A \times B)^{\circ} \subset A^{\circ} \times B^{\circ}$ then $(A \times B)^{\circ} = A^{\circ} \times B^{\circ}$.□

<u>2.3 The Quotient Topology</u>

The concept of a quotient topology allows us to construct a variety of additional topological spaces from the ones that we have already introduced. Put simply, we create a topological model that mimics the process of gluing together or collapsing parts of one or more objects. One of the most well-known examples is the torus, as obtained from a square sheet by gluing together the opposite edges.



2.3.1 Definition:

Let X be a topological space and A be a set (that is not necessarily a subset of X). Let $p: X \to A$ be a surjective map. Define a subset U of A to be open in A if and only if $p^{-1}(U)$ is open in X. The resultant collection of open sets in A is called the ***quotient topology induced by p***, and the function p is called a ***quotient map***. The topological space A is called a ***quotient space***.

2.3.2 Theorem:

Let $p: X \to A$ be a quotient map. The quotient topology on A induced by p is a topology.

Proof:

We verify each of the three conditions for a topology.

1- The set $p^{-1}(\emptyset) = \emptyset$, which is open in X. The set $p^{-1}(A) = X$, which is open in

X. So $\emptyset$ and A are open in the quotient topology.

2- Suppose each of the sets U_i , $i = 1, \cdots, n$, is open in the quotient topology on A. Then $p^{-1}(\bigcap_{i=1}^{n} U_i) = \bigcap_{i=1}^{n} p^{-1}(U_i)$, which is a finite intersection of open sets in X, and therefore is open in X. Hence, $\bigcap_{i=1}^{n} U_i$ is open in the quotient topology, and it follows that the finite intersection of open sets in the quotient topology is an open set in the quotient topology.

3- Suppose each of the sets in the collection $\{U_i\}_{i\in I}$ is open in the quotient topology on A. Then $p^{-1}(\bigcup_i U_i) = \bigcup_i p^{-1}(U_i)$, which is a union of open sets in X, and therefore is open in X. Thus, $\bigcup_i U_i$ is open in the quotient topology, implying that the arbitrary union of open sets in the quotient topology is an open set in the quotient topology.

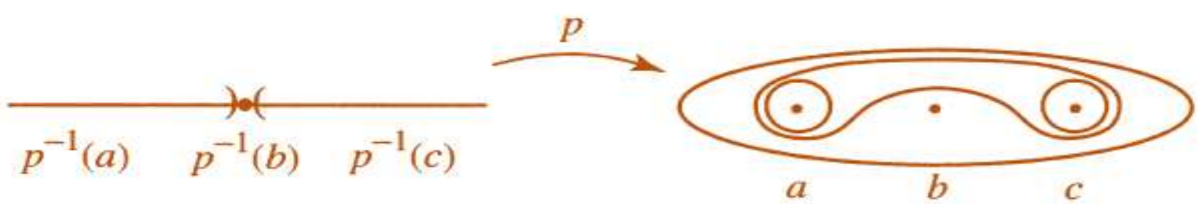
Hence, the quotient topology is a topology on A.

<u>2.3.3 Example:</u>

Give $\mathbb{R}$ the standard topology, and define $p: \mathbb{R} \to \{a,b,c\}$ by

$$p(\mathrm{x}) = \begin{cases} a & if\ x < 0 \\ b & if\ x = 0 \\ c & if\ x > 0 \end{cases}$$

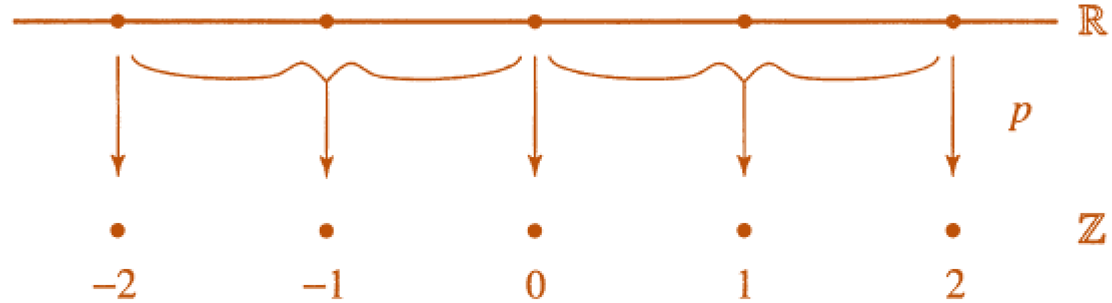
The resulting quotient topology on $\{a,b,c\}$ is $\{\{a\},\{c\},\{a,c\},\{a,b,c\}\}$. The subsets $\{a\}$, $\{c\}$, and $\{a,c\}$ are all open since their preimages are open in $\mathbb{R}$. But $\{b\}$ is not open since its preimage is $\{0\}$, which is not open in $\mathbb{R}$.



<u>2.3.4 Example:</u>

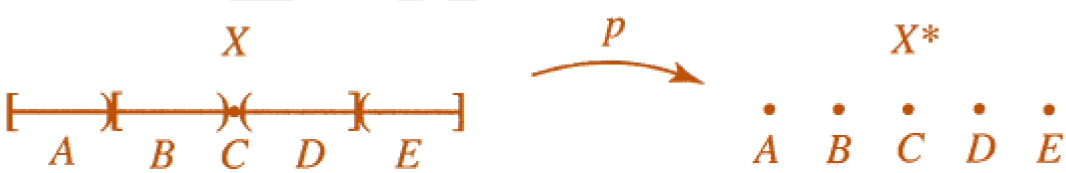
Let $\mathbb{R}$ have the standard topology, and define $p: \mathbb{R} \to \mathbb{Z}$ by $p(x) = x$ if x is an integer, and $p(x) = n$ if $x \in (n-1, n+1)$ and n is an odd integer. So p is the identity on the integers, and p maps non integer values to the nearest odd integer. In the resulting quotient topology on $\mathbb{Z}$, if n is an odd integer, then $\{n\}$ is an open set since $p^{-1}(\{n\}) = (n-1, n+1)$, an open set in $\mathbb{R}$. If n is an even integer, then $\{n\}$ is not an open set since $p^{-1}(\{n\})$ is not open in $\mathbb{R}$. In the quotient topology, the smallest open set containing an even integer n is the set

$\{n - 1, n, n + 1\}$. It follows that the quotient topology induced by p on Z is the digital line topology.



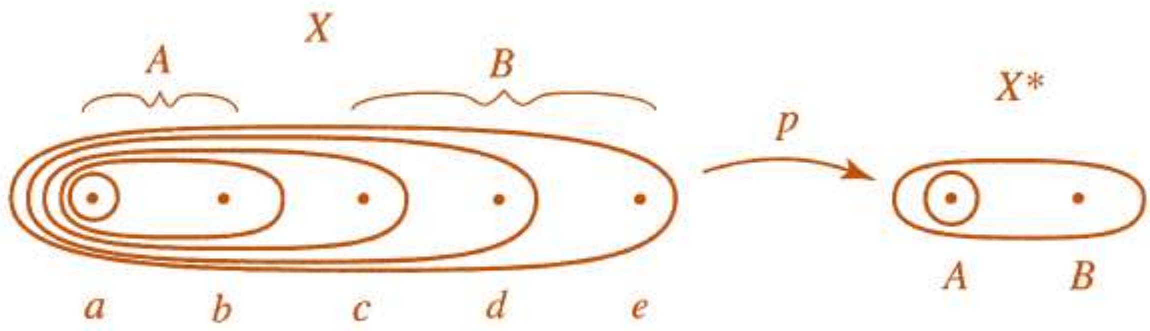
2.3.5 Remark:

Let (X,τ) be a topological space. We are particularly interested in quotient spaces defined on partitions of X. Specifically, let X^* be a collection of mutually disjoint subsets of X whose union is X, and let $p: X \to X^*$ be the surjective map that takes each point in X to the corresponding element of X^* that contains it. Then p induces a quotient topology on X^*. We think of the process of going from the topology on X to the quotient topology on as taking each subset S in the partition and identifying all of the points in S with one another, thereby collapsing S to a single point in the quotient space. A set U of points in is open in the quotient topology on exactly when the union of the subsets of X, corresponding to the points in U, is an open subset in X.



2.3.6 Example:

Let $X = \{a, b, c, d, e)$ with topology $\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}, X\}$. With $A = \{a , b\}$ and $B = \{c , d , e\}$, let X^* be the partition of X given by $X^* = \{A, B\}$. Note that X^* is a two-point set. Since $\{a, b\}$ is open in X and $\{c, d, e\}$ is not, the only open sets in the quotient topology on are $\emptyset$, $\{A\}$, and X^* itself.

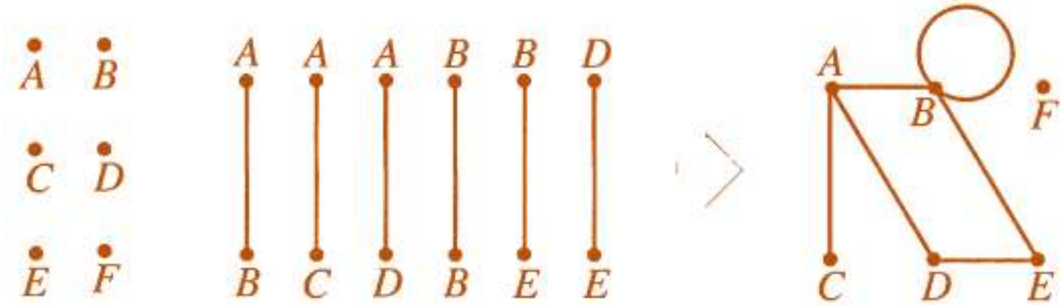


2.3.7 Example:

Let $X = [0, 1]$, and consider the partition X^* that is made up of the single-point sets $\{x\}$, for $0 < x < 1$, and the double-point set $D = \{0, 1\}$. Then, in the quotient topology on we think of D as a single point, as if we had glued the two endpoints of [0, 1] together. A subset of X^* that does not contain D is a collection of single-point subsets, and it is open in X^* exactly when the union of those single-point sets is an open subset of (0, 1). A subset of X^* that contains D is open in X^* when the union of all the sets making up the subset is an open subset of [0, 1]. Such an open subset must contain 0 and 1, and therefore must contain intervals [0, a) and (b, 1], which are open in the subspace topology on [0, 1]. The resulting space is topologically equivalent to the circle, S^1.

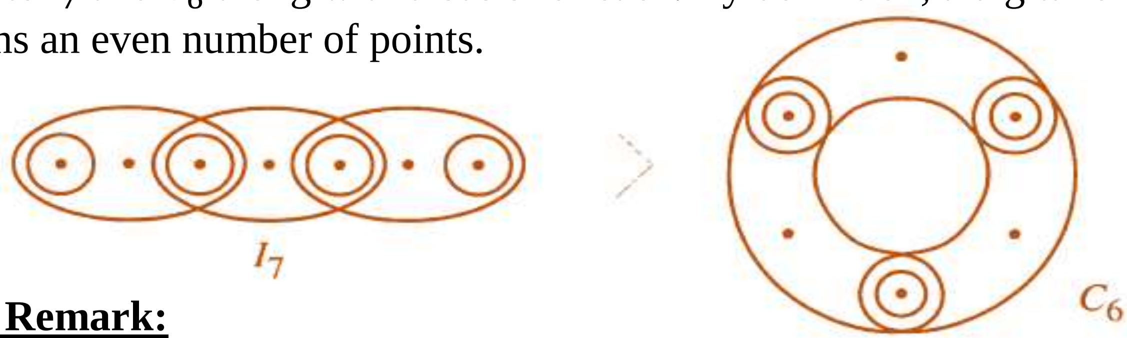
2.3.8 Example:

In the previous example 2.3.7, we glued the endpoints of an interval together to obtain a single point. That is an example of a more general construction that results in a space known as a topological graph. Specifically, a ***topological graph*** G is a quotient space constructed by taking a finite set of points, called the **vertices** of G, along with a finite set of mutually disjoint closed bounded intervals in $\mathbb{R}$. and gluing the endpoints of the intervals to the vertices in some fashion. The glued intervals are called the ***edges*** of G.



2.3.9 Example:

In Example 2.3.7 we obtained a circle by identifying endpoints of an interval in the real line. We describe a similar process here, using the digital line, that yields spaces we call digital circles. Specifically, a ***digital interval*** is a subset $\{m, m+1, \cdots, n\}$ of $\mathbb{Z}$ with the subspace topology inherited from the digital line topology. Let I_n be the digital interval in the form $\{1, 2, \cdots. n - 1, n\}$. If $n \geq 5$ is an odd integer, then the topological space C_{n-1} resulting from identifying the endpoints 1 and n in I_n is called a ***digital circle***. The digital circle C_{n-1} is a quotient space of the digital interval I_n.The following Figure we illustrate I_7 and C_6 along with a basis for each. By definition, a digital circle contains an even number of points.



2.3.10 Remark:

The following examples 2.3.11 and 2.3.12 gives two different quotient spaces defined on $I \times I$.

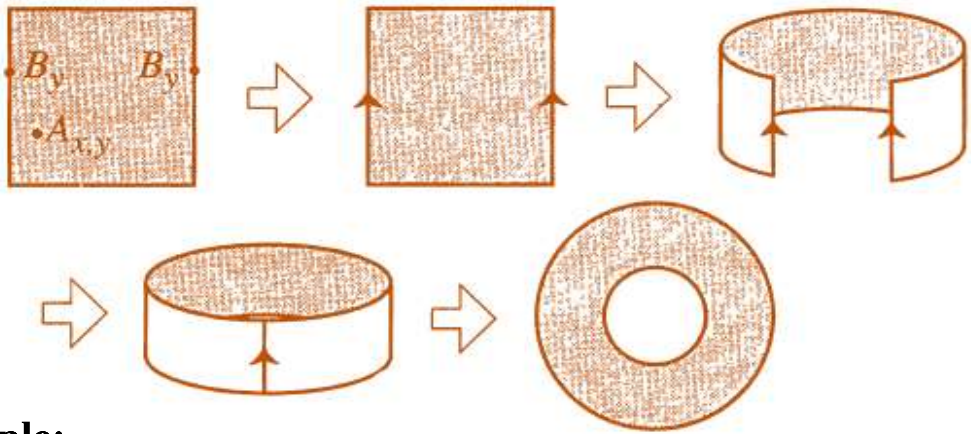
2.3.11 Example:

Define a partition on $I \times I$ by taking subsets of the following form:

i) $A_{x,y} = \{(x,y)\}$ for every x and y such that $0 < x < 1$ and $0 \leq y \leq 1$.

ii) $B_y = \{(0,y),(1,y)\}$ for every y such that $0 < y < 1$.

In the quotient topology, the subsets B_y cause the left and right edges of the square to be glued. The result is a space that is topologically equivalent to the ***annulus***.



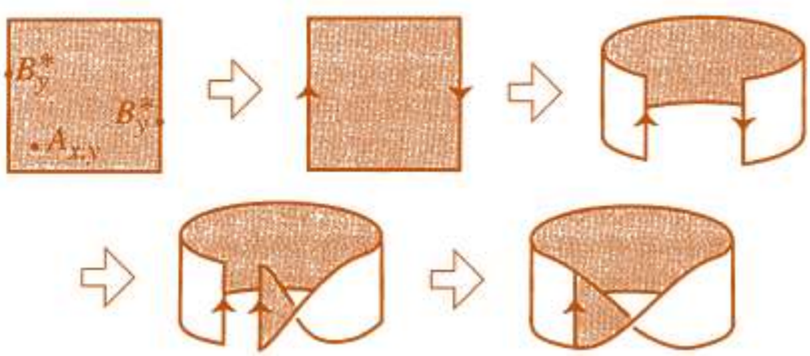
2.3.12 Example:

Define a partition on $I \times I$ by taking subsets of the following form:

i) $A_{x,y} = \{(x,y)\}$ for every x and y such that $0 < x < 1$ and $0 \leq y \leq 1$.

ii) $B_y^* = \{(0,y),(1,1\text{-}y)\}$ for every y such that $0 < y < 1$.

Here the subsets B_y^* also cause the left and right edges of the square to be glued. But in order to accomplish the gluing, we need to perform a half twist so that the identified points on the edges can be properly brought together. The result is the well-known ***Möbius band***.



2.3.13 Example:

Define a partition of $I \times I$ by taking subsets of the following form:

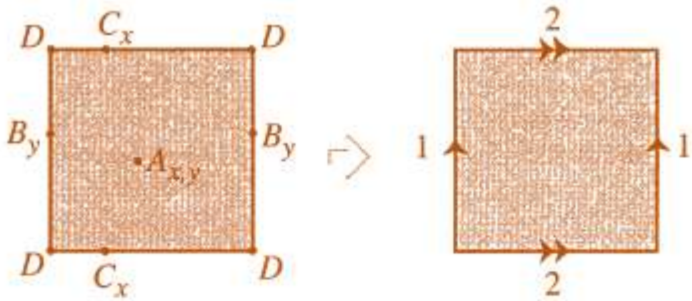
i) $A_{x,y} = \{(x,y)\}$for every x and y such that $0 < x < 1$ and $0 < y < 1$.

ii) $B_y = \{(0,y),(1,y)\}$ for every y such that $0 < y < 1$.

iii) $C_y = \{(x,0),(x,1)\}$ for every x such that $0 < x < 1$.

iv) $D = \{(0,0),(0,1),(1,0),(1,1)\}$.

In the quotient topology, the two-point subsets in (ii) cause the gluing of the left edge of the square to the right edge, and the two-point subsets in (iii) cause the gluing of the top edge of the square to the bottom edge. Furthermore, the four-point subset causes the gluing of the four corners of the square to a single point. The topological space we obtain is therefore the result of taking a square and gluing together its opposite edges. Such a construction results in a ***torus***.



Chapter Three

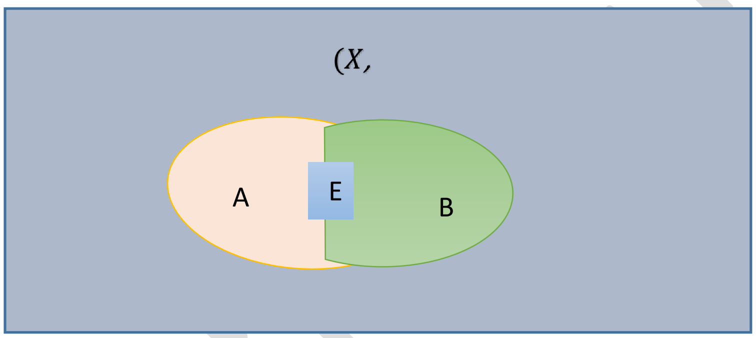
Connected and Compact Spaces

3.1 Connected Sets

3.1.1 Definition:

Two subsets A and B form ***a separation*** or ***partition*** of a set E in a topological space (X,τ) denote by $E = A|B$ iff they satisfy the followings:

1) $A \neq \emptyset$, $B \neq \emptyset$.
2) $E = A \cup B$.
3) $A \cap B = \emptyset$.
4) $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$.



3.1.2 Remark:

We can replace condition 4) by $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$.

3.1.3 Example:

Let (X,τ) be a topological space where $X = \{a,b,c,d,e\}$, $\tau=\{X,\emptyset,\{c\},\{a,b,c\},\{c,d,e\}\}$, $E = \{a,d,e\}$, $F=\{b,c,e\}$, $A=\{a\}$, $B=\{d,e\}$, $C=\{b\}$ and $D = \{c,e\}$. Show that $E = A|B$ and $F = C \nmid D$.

Solution:

1. $A \neq \emptyset$, $B \neq \emptyset$, 2. $E = A \cup B$, 3. $A \cap B = \emptyset$, 4. $\bar{A} \cap B = \{a,b\} \cap \{d,e\} = \emptyset$, $A \cap \bar{B} = \{a\} \cap \{d,e\} = \emptyset$, so $E = A|B$ but $C \cap \bar{D} = \{b\} \cap X = \{b\} \neq \emptyset$ i.e. $F = C \nmid D$.

3.1.4 Example:

Let $(\mathbb{R}, D)$ be the usual topology on $\mathbb{R}$. If $A = (1,2)$, $B=(2,3)$ & $C=[3,4)$ then the sets A,B are separation since $\bar{A}=[1,2]$, $\bar{B}=[2,3]$ then $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$ but

C,B are not separation since $3 \in C$ and 3 is a limit point of B i.e. $\bar{B} \cap C = [2,3] \cap [3,4) = \{3\} \neq \emptyset$.

3.1.5 Definition:

Let E be a subset of topological (X, τ) is ***connected*** set if there does not exist a separation for E and E is ***disconnected*** set if there exist a separation for E.

3.1.6 Example:

Consider the two topologies $\tau_1 = \{\{b\},\{a,b\},\{b,c\},X,\emptyset\}$, $\tau_2 = \{\{b\},\{c\},\{a,b\},\{b,c\},X,\emptyset\}$ On the set $X = \{a,b,c\}$ then X is connected in τ_1 and X is disconnected in τ_2 since there is $U = \{a,b\}$, $V=\{c\}$ s.t. $X = U|V$.

3.1.7 Example:

If a set X consists of more than one point and it has a discrete topology, then it is disconnected.

Solution:

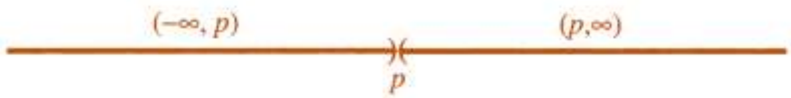
If A is any nonempty proper subset of X then the pair of sets A and X/A is a separation of X.

3.1.8 Example:

If $p \in \mathbb{R}$ then $\mathbb{R}/\{p\}$ is a disconnected topological space.

Solution:

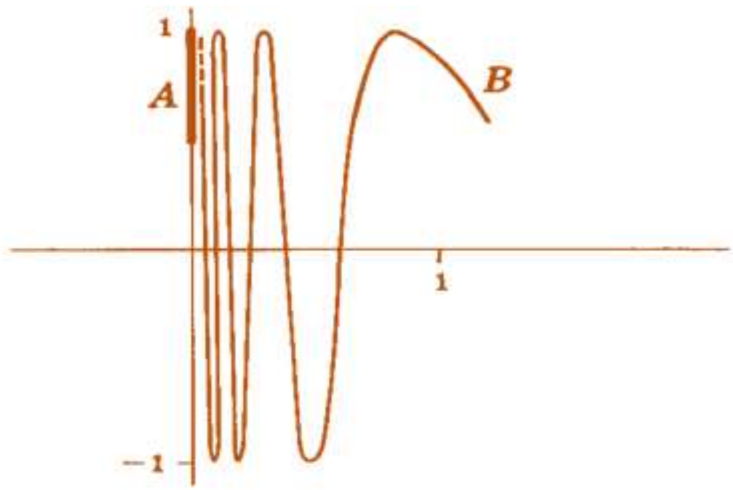
The pair $U = (-\infty, p)$ and $V = (p, \infty)$ is a separation of $\mathbb{R}/\{p\}$.



3.1.9 Example:

Consider the following subsets of the plane $\mathbb{R}^2$ is connected

$$A = \{(0,y): \tfrac{1}{2} \leq y \leq 1\},\ B = \{(x,y): y=\sin\left(\tfrac{1}{x}\right), 0 < x \leq 1\}$$



Solution:

Each point in A is a limit point of B then A and B are not separation i.e. they are connected.

3.1.10 Example:

Assume $X = (\text{-}1{,}0) \cup (0{,}1)$ is disconnected then there exists $\mathbb{R}$ is disconnected since the pair of sets (-1,0) and (0,1) is a separation of X.

3.1.11 Theorem:

If E is a subset of a subspace (Y,τ_Y) of a topological space (X,τ) then E is $\tau_Y - \text{connected}$ iff it is $\tau - \text{connected}$.

Proof:

In order to have a separation of ***E*** with respect to either topology, we must be able to write ***E*** as the union of two nonempty, disjoint sets. If ***A*** and ***B*** are two nonempty, disjoint sets whose union is ***E*** then $A{,}B \subseteq E \subseteq Y \subseteq X$.

$$(A \cap \bar{B}) \cup (\bar{A} \cap B) = \big((A \cap Y) \cap \bar{B}\big) \cup \big(\bar{A} \cap (Y \cap B)\big) = (A \cap \overline{B_Y}) \cup (\overline{A_Y} \cap B)$$

Thus if the condition is satisfied with respect to one topology, it is satisfied with respect to the other. □

3.1.12 Theorem:

Let (X,τ) be a topological space . X is disconnected iff there exists a non-empty proper subset of X which is both open and closed.

Proof:

$\Rightarrow$

Suppose $X = G \cup H$ where G and H are non-empty and open then G is a non-empty proper subset of X and since $G = H^c$, G is both open and closed.

$\Leftarrow$

Suppose A is a non-empty proper subset of X which is both open and closed. Then A^c is also non-empty and open and $X = A \cup A^c$. Accordingly, X is disconnected.□

3.1.13 Example:

The indiscrete topology (X, τ) is connected topology since X and $\emptyset$ are only subsets of X which are both open and closed.

2.1.14 Example:

Let (X, τ) be a co-finite topology where X is infinite is connected space.

Solution:

Assume X is disconnected then there exists A,B are nonempty open subset of X and $A \cap B = \emptyset$ separation for X then A^c,B^c are finite sets and $A^c \cup B^c = X$ this implies that X is finite and this is contradiction since X is infinite ,so X is connected.

2.1.15 Exercise:

Let (X, τ) be a co-finite topology where X is finite is disconnected space.

2.1.16 Example:

In $\mathbb{R}$ with the lower limit topology then $\mathbb{R}$ is disconnected since every intervals [a,b) are open and closed sets.

3.1.17 Theorem:

If C is a connected subset of a topological space (X, τ) which has a separation $X = A|B$ then either $C \subseteq A$ or $C \subseteq B$.

Proof:

Suppose that $X = A|B$ then

$$C = C \cap X = C \cap (A \cup B) = (C \cap A) \cup (C \cap B)$$

$$(C \cap A) \cap (C \cap B) = C \cap (A \cap B) = C \cap \emptyset = \emptyset$$

$$\left((C \cap A) \cap \overline{(C \cap B)}\right) \cup \left(\overline{(C \cap A)} \cap (C \cap B)\right) \subseteq (A \cap \bar{B}) \cup (\bar{A} \cap B) = \emptyset$$

Thus we see that if we assume that both $C \cap A = \emptyset$ and $C \cap B = \emptyset$ we have a separation for $C = (C \cap A)|(C \cap B)$.Hence, either $C \cap A$ is empty so that $C \subseteq B$ or or $C \cap B$ is empty so that $C \subseteq A$.□

3.1.18 Corollary(1):

If C is a connected set in a topological space (X, τ) and $C \subseteq E \subseteq \bar{C}$ then E is a connected set.

Proof:

If *E* is not a connected set, it must have a separation $E = A|B$ .By theorem 3.1.17 must be contained in *A* or contained in *B*. Assume $C \subseteq A$ it follows that $\bar{C} \subseteq \bar{A}$ and hence $\bar{C} \cap B \subseteq \bar{A} \cap B = \emptyset$. On the other hand,$B \subseteq E \subseteq \bar{C}$ and so $B \cap \bar{C} = B$,so that we must have $B = \emptyset$,which contradicts our hypothesis that $E = A|B$.□

3.1.19 Corollary(2):

If every two points of a set E are contained in some connected subset of E, then E is a connected set.

Proof:

If *E* is not connected, it must have a separation $E = A|B$.Since *A* and *B* must be nonempty, let us choose points $a \in A$ and $b \in B$.From the hypothesis we know that *a* and *b* must be contained in some connected subset *C* contained in ***E*.** By theorem 3.1.17 requires that *C* be either a subset of *A* or a subset of ***B*.** Since *A* and *B* are disjoint, this is a contradiction then *E* is connected. □

3.1.20 Corollary (3):

The union E of any family $\{C_\lambda\}$ of connected sets having a nonempty intersection ($\bigcap_\lambda C_\lambda \neq \emptyset$) is a connected set.

Proof:

If *E* is not connected, it must have a separation $E = A|B$.By hypothesis, we may choose a point $x \in \bigcap_\lambda C_\lambda$. The point x must belong to either *A* or ***B*.** Let us suppose $x \in A$.Since x belongs to C_λ for every λ, $C_\lambda \cap A \neq \emptyset$ for every λ.By theorem 3.1.17, however, each C_λ must be either a subset of *A* or a subset of ***B*.** Since *A* and *B* are disjoint sets we must have $C_\lambda \subseteq A$ for all λ , and so$E \subseteq A$**.** From this we obtain the contradiction that $B = \emptyset$.□

3.1.21 Remark:

1. The structure of the connected subsets of the real line is deceptively simple. For example, if the removal of a single point x from a connected set *C* leaves a disconnected set, then $C/\{x\}$ is the union of two disjoint connected sets.
2. Another geometrically reasonable property of connected sets is given in the following theorem:

3.1.22 Theorem:

If a connected set C has a nonempty intersection with both a set E and the complement of E in a topological space (X, τ),then C has a nonempty intersection with the boundary of E (i.e. $C \cap b(E) \neq \emptyset$).

Proof:

We will show that if we assume that *C* is disjoint from $b(E)$ we obtain the contradiction that $C = (C \cap E)|(C \cap E^c)$.

From the equation $C = C \cap X = C \cap (E \cup E^c) = (C \cap E) \cup (C \cap E^c)$ we see that *C* is the union of the two sets. These two sets are nonempty by hypothesis. If we calculate

$$(C \cap E) \cap \overline{(C \cap E^c)} \subseteq (C \cap E) \cap \overline{E^c} = C \cap (E \cap \overline{E^c}) = C \cap b(E),$$

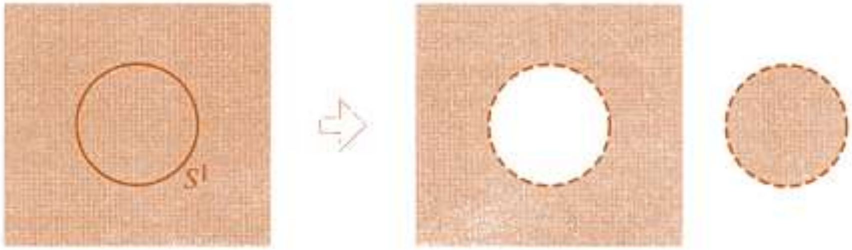
we see that the assumption that $C \cap b(E) = \emptyset$ leads to the conclusion that $(C \cap E) \cap \overline{(C \cap E^c)} = \emptyset$ **.** In the same way we may show that $\overline{(C \cap E)} \cap (C \cap E^c) = \emptyset$ **,** and we have a separation of ***C***.□

3.1.23 Definition:

Let (X, τ) be a connected topological space . A ***cutset*** of X is a subset of X such that X/S is disconnected . A ***cutpoint*** of X is a point $p \in X$ such that $\{p\}$ is a cutset of X. A cutset or cutpoint of X is said to ***separate*** X.

3.1.24 Example:

The plane $\mathbb{R}^2$ is connected . If we remove the circle S^1, we are left with two disjoint nonempty open sets.

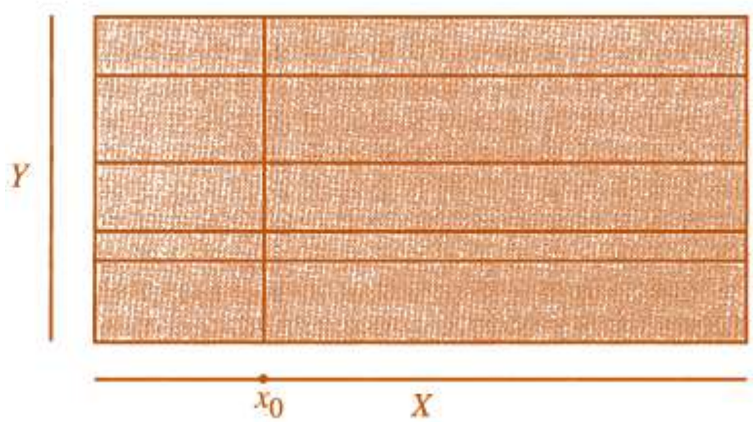


3.1.25 Theorem:

Let $X_1, \cdots, X_n$ be connected spaces. Then the product space $X_1 \times \cdots \times X_n$ is connected.

Proof:

We shall prove the product of two spaces. The general result can then be shown by induction. Assume that X and Y are connected topological spaces. For every $x \in X$, the subspace $\{x\} \times Y$ of $X \times Y$ is homeomorphic to Y and is therefore connected. Similarly, for every $y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is connected. Thus, by Corollary 3.1.20, for every $x \in X$ and $y \in Y$ the set $(\{x\} \times Y)\, U\, (X \times \{y\})$ is connected in $X \times Y$.



Now fix $x_0 \in X$ and let y vary. Each set $(\{x_0\} \times Y)\ U\ (X \times \{y\})$ Contains the set $\{x_0\} \times Y$. It then follows by Corollary 3.1.20 that $\bigcup_{y \in Y}((\{x_0\} \times Y) \cup (X \times \{y\}))$ is connected in $X \times Y$. Furthermore, $\bigcup_{y \in Y}((\{x_0\} \times Y) \cup (X \times \{y\})) = X \times Y$, implying that $X \times Y$ is connected.□

3.2 Components

3.2.1 Definition:

A ***component*** E of a topological space (X,τ) is a maximal connected subset of X i.e. E is connected and E is not A proper subset of any connected subset of X.

3.2.2 Example:

If X is connected then X has one component X itself . Also $(\mathbb{R},\tau)$ the usual topology has one component $\mathbb{R}$ itself.

3.2.3 Example:

Consider the following topology on $X=\{a,b,c,d,e\}$, $\tau=\{X,\emptyset,\{a\},\{c,d\},\{a,c,d\},\{b,c,d,e\}\}$ then the components of X are {a} and {b,c,d,e}.Any other connected subset of X such that $\{b,d,e\}$ is a subset of one of the components.

3.2.4 Theorem:

The components of a topological space(X,τ) are closed subsets of X.

Proof:

If *C* is a component of *X***,** choose a point $x\in C$ and suppose that $y\in\bar{C}$**.** Since $\bar{C}$ is a connected set by Corollary 1,y is in a connected subset of *X* which contains x**.** Hence$\bar{C}\subseteq C$, and so *C* must be closed.□

3.2.5 Theorem:

Every connected subset of a topological space (X,τ) is contained in a connected component.

Proof:

Assume A is a connected subset of a topological space (X,τ) . If $\{A_i : i\in I\}$ is a family of connected contained A i.e. $A_i\subseteq A$; $\forall i\in\mathbb{N}$ then $A\neq\emptyset$,so $\bigcap_i A_i\neq\emptyset$ by Corollary (3) we get $C=\bigcup_i A_i$ is a connected contain A .If E is connected contain C then E also contain A , so E=C then C is a component contain A.

3.2.6 Corollary:

Every point in a topological space (X,τ) is contained in a connected component.

Proof:

Since for every $p \in X$ the set $\{p\}$ is connected then by theorem 3.2.5 Every point in a topological space (X, τ) is contained in a connected component.□

3.2.7 Theorem:

The component of a topological space (X, τ) forms a partition of X.

Proof:

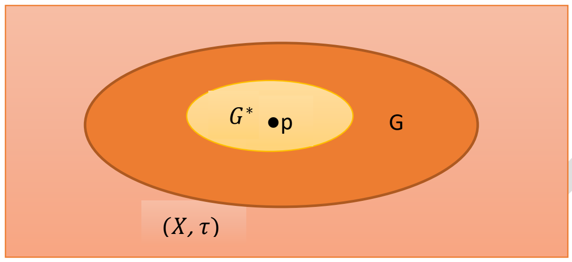
Let $\{C_i\}_{i \in \mathbb{N}}$ be a family of connected component in a topological space (X, τ) then

1. $C_i \cap C_j = \emptyset, \forall i \neq j$ since if $C_i \cap C_j \neq \emptyset$ then by corollary (3) we get $C_i \cup C_j$ is connected contain the sets C_i,C_j and since C_i,C_j are connected component then $C_i = C_i \cap C_j = C_j$ and this is contradiction.
2. It's clear that $X = \bigcup_{i \in \mathbb{N}} C_i$.□

3.3 Locally Connected Spaces

3.3.1 Definition:

A topological space (X, τ) is ***locally connected*** at $p \in X$ iff every open set G containing p, there exists a connected open set G^* containing p and contained in G. Thus a space is ***locally connecte***d iff the family of all open connected sets is a base for the topology for the space.



3.3.2 Remark:

A locally connected set need not be connected. For example, a set consisting of two disjoint open intervals is locally connected but not connected. The connected subsets of the real numbers are locally connected, but this implication need not hold in general i.e. in topological spaces The connected subsets need not be a locally connected set.

locally connected set	$\nRightarrow$	connected sets	in topological spaces
connected sets	$\nRightarrow$	locally connected set	in topological spaces
connected sets	$\Rightarrow$	locally connected set	in real numbers

3.3.3 Example

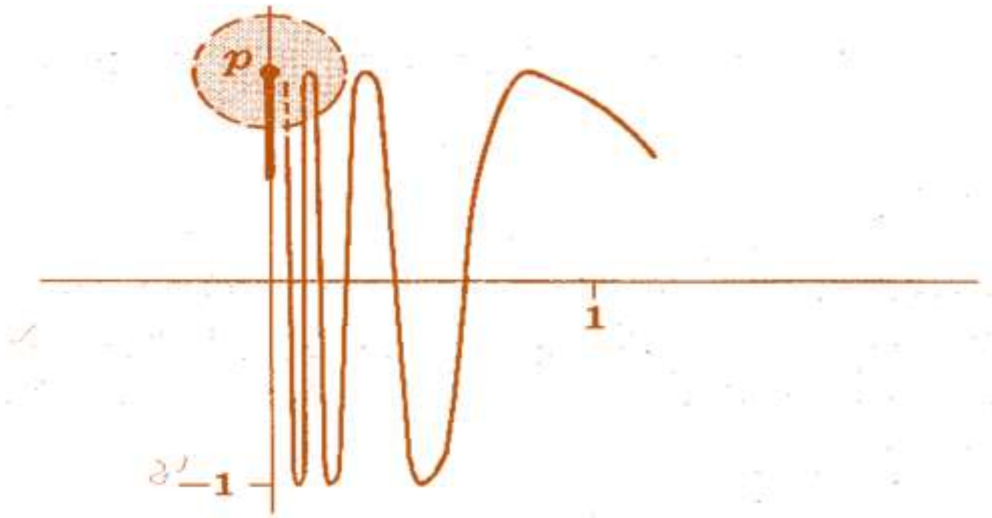
Every discrete topological space (X, τ) is locally connected.

Solution:

If $p \in X$ then $\{p\}$ is an open connected set containing p which is contained in every open set containing p (Note that X is not connected if X contains more than one point).

3.3.4 Example:

Let A and B be subsets of the plane $\mathbb{R}^2$ of example 3.1.9 , $A \cup B$ is a connected set but $A \cup B$ is not locally connected at $p = (0,1)$. For example the open disc with center p and radius $\frac{1}{4}$ does not contain any connected open set contain p.



3.3.5 Theorem:

Let E be a component in locally connected space (X, τ) *then E is open.*

Proof:

Let $p \in E$. Since X is locally connected space then p belongs to at least one connected set G_p but E is the component of p hence $p \in G_p \subset E$ and so $E = \cup \{G_p : p \in E\}$. Therefore, E is open since it is the union of open sets. □

3.3.6 Theorem:

Let (X, τ) be a locally connected space and let Y be an open subset of X then the subspace (Y, τ_Y) is locally connected.

Proof:

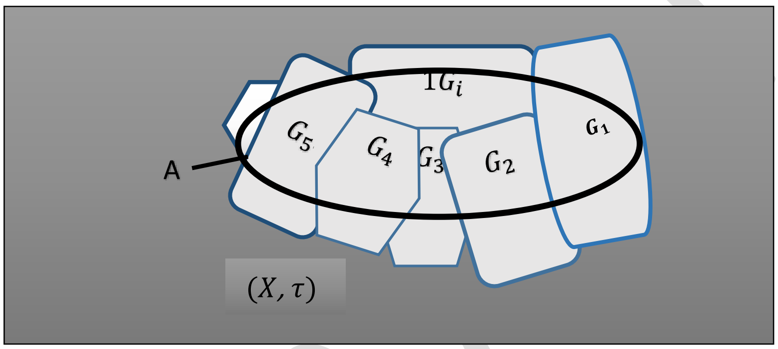
Assume $p \in Y$, N *is an open* set in (Y, τ_Y) contain p so there exist an open set U in X such that $Y \cap U = N$ but Y is an open set in X ,so N is an open set in X contain p and X is locally connected then there exists a connected set W in X such that $p \in W \subseteq U$. Now we have $V = W \cap Y \subseteq Y \cap U = N$ where V is a connected set in Y contain p so (Y, τ_Y) is locally connected.□

3.4 Compact Spaces

3.4.1 Definition:

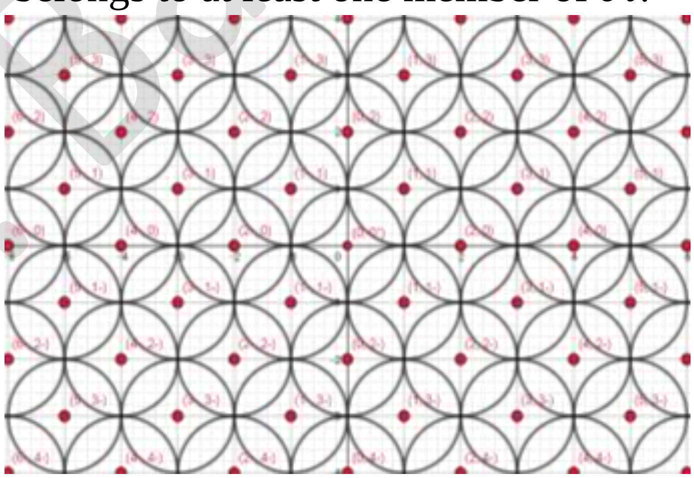
Let A be a subset of a topological space (X, τ) and let $\mathcal{A} = \{G_i\}_i$ be a collection of subsets of X then:

1. The collection $\mathcal{A}$ is said to ***cover*** A or to be a ***cover*** of A is contained in the union of sets in $\mathcal{A}$,(i.e. $A \subseteq \bigcup_i G_i$).
2. If $\mathcal{A}$ covers and each set in $\mathcal{A}$ is open then we call $\mathcal{A}$ an ***open cover*** of A.
3. If $\mathcal{A}$ covers A ,and $\mathcal{A}'$ is a subcollection of $\mathcal{A}$ that also covers A, then $\mathcal{A}'$ is called a ***subcover*** of $\mathcal{A}$.



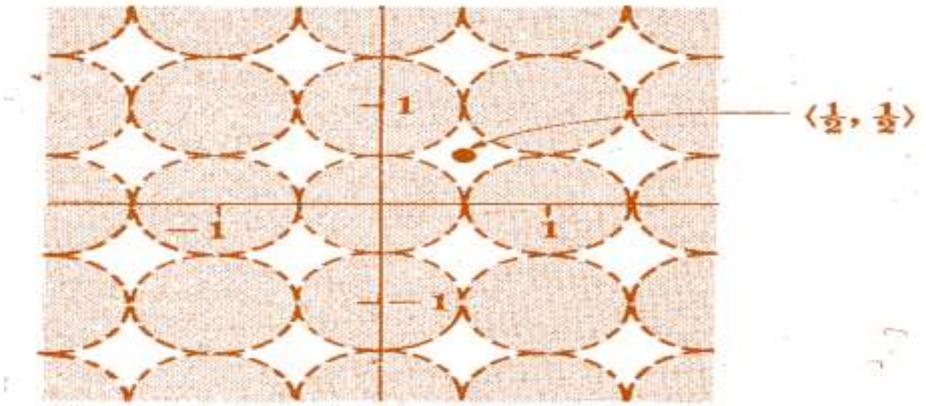
3.4.2 Example:

Consider the class $\mathcal{A} = \{D_p : p \in \mathbb{Z} \times \mathbb{Z}\}$, where D_p is the open disc in the plane $\mathbb{R}^2$with radius 1 and center $p = (m,n)$,m and n integers.Then $\mathcal{A}$ is a cover of $\mathbb{R}^2$, i.e. every point in $\mathbb{R}^2$ belongs to at least one member of $\mathcal{A}$.



3.4.3 Remark:

In example 3.4.2 if we take the collection of open discs $\mathcal{B} = \{D_p^* : p \in \mathbb{Z} \times \mathbb{Z}\}$, where D_p^* has center p and radius $\frac{1}{2}$, is not a cover of $\mathbb{R}^2$.For example the point $(\frac{1}{2},\frac{1}{2}) \in \mathbb{R}^2$ does not belong to any member of $\mathcal{B}$.



3.4.4 Definition:

A topological space (X, τ) is ***compact*** iff every open cover of X has finite subcover,(i.e. if $\mathcal{A} = \{G_i\}_i$ is an open cover for X ($X \subseteq \bigcup_i G_i$) then there exists $\{G_1, G_2, \dots, G_n\}$ finite subcover s.t. $X \subseteq \bigcup_{i=1}^{n} G_i$.

3.4.5 Example:

Let A be any finite subset of a topological space (X, τ) then A is compact.

Solution:

Let $A = \{a_1, a_2, \dots, a_n\}$ be a finite subset of a topological space (X, τ) and let $\mathcal{A} = \{G_i\}_i$ be an open cover for A, i.e. $A \subseteq \bigcup_i G_i$ then

$$\because a_1 \in A \longrightarrow \exists\, G_1 \in \mathcal{A}, \text{ s.t. } a_1 \in G_1$$
$$\because a_2 \in A \longrightarrow \exists\, G_2 \in \mathcal{A}, \text{ s.t. } a_2 \in G_2$$
$$\vdots$$
$$\because a_n \in A \longrightarrow \exists\, G_n \in \mathcal{A}, \text{ s.t. } a_n \in G_n$$

Then $A = \{a_1, a_2, \dots, a_n\} \subseteq \{G_1, G_2, \dots, G_n\} = \bigcup_{i=1}^{n} G_i$,A is compact.

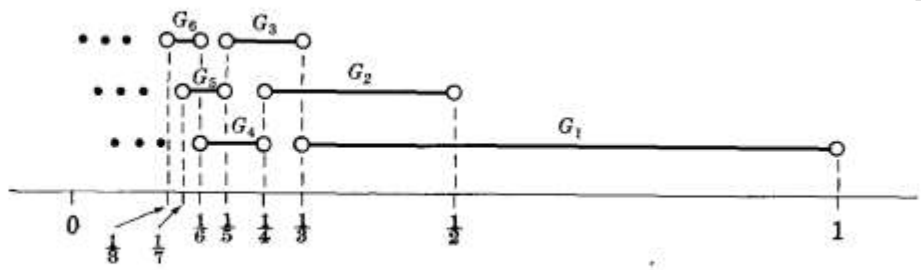
3.4.6 Example:

The open interval $A = (0,1)$ on the real line $\mathbb{R}$ with the usual topology is not compact.

Solution:

Assume A is compact and let $\mathcal{A} = \{G_n = (\frac{1}{n+2},\frac{1}{n}): n \in \mathbb{N}\} = \{(\frac{1}{3},1),(\frac{1}{4},\frac{1}{2}),(\frac{1}{5},\frac{1}{3}),..\}$ be an open cover for A such that $A \subseteq \bigcup_{n=1}^{\infty} G_n$ then $\mathcal{A}$ has finite subcover $\mathcal{A}' = \{(a_1,b_1),(a_2,b_2),...,(a_n,b_n)\}$ for A .

Let $\in= \min\{a_1,a_2,..., a_n\}$ then $\in> 0$ and $(a_1,b_1)\cup(a_2,b_2)\cup\ldots\cup(a_n,b_n) \subseteq (\in ,1)$.But $(0, \in]$ and $(\in ,1)$ are disjoint hence $\mathcal{A}'$is not a cover of A and A is not compact



3.4.7 Example:

The subset $A = \{0\} \cup \{\frac{1}{n}: n \in \mathbb{N}\}$ is compact in $\mathbb{R}$ with the usual topology.

Solution:

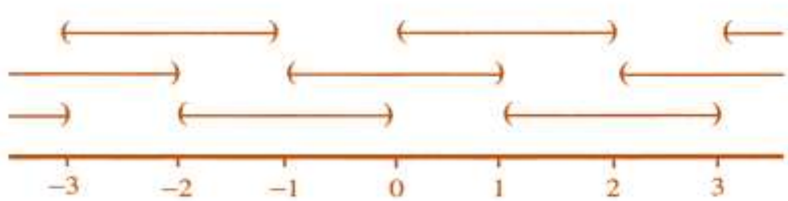
Let $\mathcal{A}$ be an open cover for A. Since $0 \in A$ then there exists at least one open set $U_0 \in \mathcal{A}$, $0 \in U_0$.Let $\varepsilon > 0$, s.t. $0\in(-\varepsilon,\varepsilon)\subseteq U_0$. By Archimedes theorem $\exists k \in \mathbb{N}$, s.t. $\frac{1}{k} < \mathcal{E} \longrightarrow \frac{1}{n}\in(-\varepsilon,\varepsilon)\subseteq U_0$,n>k .Now since $\frac{1}{n} \in A$,1≤n≤k⟶$\exists U_n \in \mathcal{A}$, s.t. $\frac{1}{n} < U_n$,1≤n≤k , so $\{U_0,U_1,U_2,\ldots,U_k\}$ is a finite subcover of $\mathcal{A}$ for A .Then A is compact.

3.4.8 Example:

Consider (0,1] as a subspace of $\mathbb{R}$ then (0,1] is not compact , since $\mathcal{A} = \{(\frac{1}{n},2): n \in \mathbb{Z}^+\}$ is an open cover for (0,1] has no finite subcover of $\mathcal{A}$ that cover (0,1].

3.4.9 Example:

The real line $\mathbb{R}$ with the usual topology is not compact since $\mathcal{A}$={…,(-1,1),(0,2),(1,3),…} is an open cover has no finite subcover for.



3.4.10 Example:

Let (X,τ) be the co-finite topology then X is compact.

Solution:

Let $\mathcal{A}=\{G_i\}$ be an open cover of X .Choose $G_0\in\mathcal{A}$. Since τ is the co-finite topology, G_0^c is a finite set , i.e. $G_0^c=\{a_1,a_2,\dots,a_m\}$.Since $\mathcal{A}$be an open cover of X, for each $a_k\in G_0^c\ \exists G_{i_k}\in\mathcal{A}$ such that $a_k\in G_{i_k}$.Hence $G_0^c\subseteq G_{i_1}\cup G_{i_2}\cup\dots\cup G_{i_m}$ and $X=G_0\cup G_0^c=G_0\cup G_{i_1}\cup G_{i_2}\cup\dots\cup G_{i_m}$.Thus X is compact.

3.4.11 Example:

Every infinite subset A of a discrete topological space (X,τ) is not compact.

Solution:

Let $\mathcal{A}=\{\{a\}:a\in A\}$ be a collection of singleton subsets of A,i.e. $A=\cup\{\{a\}:a\in A\}$ then $\mathcal{A}$ is an open cover of A since every subsets of a discrete topology are open. $\mathcal{A}$ is infinite since A is infinite ,so $\mathcal{A}$ has no finite subcover for A.

3.4.12 Remark:

From examples 3.4.5 and 3.4.11we get a subset of a discrete topology is compact iff it is finite.

3.4.13 Example:

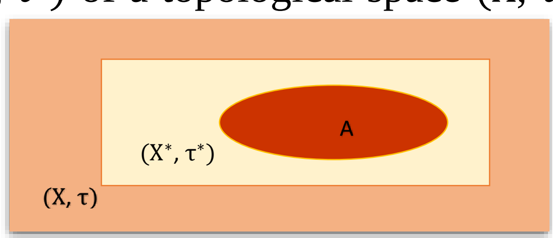
The indiscrete topology (X, τ) is compact.

Solution:

Since $\tau=\{\emptyset,X\}$ then any open cover for X must be of the form $\mathcal{A}=\{X\}$ which is finite cover since it contain X only , X is compact.

3.4.14 Theorem:

If A is a subset of a subspace (X^*,τ^*) of a topological space (X,τ) then A is τ^*-compact iff it is τ -compact.



Proof:

$\Longrightarrow$

Suppose A is τ^*-compact and $\{G_i\}$ is some τ -open covering of A. The family of sets $\{X^*\cap G_i\}$ clearly forms a τ^*-open covering for A since $A=X^*\cap A\subseteq X^*\cap(\cup_i G_i)=\cup_i(X^*\cap G_i)$. Since A is τ^*-compact, there is a finite subcovering $A\subseteq\cup_{i=1}^n(X^*\cap G_i)\subseteq\cup_{i=1}^n G_i$ of A which yields a finite subcovering of A from $\{G_i\}$.

$\Longleftarrow$

Now suppose that A is τ -compact and $\{{G_i}^*\}$ is some τ^*-open covering of A. From the definition of the induced topology, each ${G_i}^* = X^* \cap G_i$ for some τ - open set G_i. The family $\{G_i\}$ is clearly a τ -open covering of A and so there must be some finite subcovering $A \subseteq \bigcup_{i=1}^{n} G_i$. But then we have $A = X^* \cap A \subseteq X^* \cap (\bigcup_{i=1}^{n} G_i) = \bigcup_{i=1}^{n}(X^* \cap G_i) = \bigcup_{i=1}^{n} {G_i}^*$.and so a finite subcovering of A from $\{{G_i}^*\}$.□

3.5 Finite Intersection Property

3.5.1 Definition:

A family $\{A_i\}$ of sets will be said to have the ***Finite Intersection Property*** (denote by F.I.P.) iff every finite subfamily $\{A_i\}_{i=1}^{n}$ of the family has a nonempty intersection $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

3.5.2 Example:

$The\ family\ \mathcal{A} = \{(0,\frac{1}{n}) : n \in \mathbb{N}\} = \{(0,1),(0,\frac{1}{2}),(0,\frac{1}{3}).(0,\frac{1}{4}),\ldots\}\ has\ F.I.P.$

Solution:

Let $\{(0,a_1), (0,a_2), (0,a_3), \ldots, (0,a_n)\}$ be a finite subfamily of $\mathcal{A}$ and let $b = \min\{a_1,a_2,a_3,\ldots,a_n\} > 0$ then $(0,a_1)\cap(0,a_2)\cap(0,a_3)\cap\ldots\cap(0,a_n) = (0,b) \neq \emptyset$,so $\mathcal{A}$ has F.I.P.

3.5.3 Remark:

In example 3.5.2 we have $\bigcap_{n\in\mathbb{N}}(0,\frac{1}{n}) = \emptyset$.

3.5.4 Example:

The family $\mathfrak{B} = \{(-\infty,n] : n \in \mathbb{Z}\} = \{\ldots,(-\infty,-2],(-\infty,-1],(-\infty,0],(-\infty,1],(-\infty,2],\ldots\}$ has F.I.P.

Solution:

Let $\{(-\infty,a_1], (-\infty,a_2], (-\infty,a_3], \ldots, (-\infty,a_n]\}$ be a finite subfamily of $\mathfrak{B}$ and let $b = \min\{a_1,a_2,a_3,\ldots,a_n\} > 0$ then $(-\infty,a_1] \cap(-\infty,a_2]\cap (-\infty,a_3] \cap \ldots\cap(-\infty,a_n] = (-\infty,b] \neq \emptyset$,so $\mathfrak{B}$ has F.I.P. Note that $\bigcap_{n\in\mathbb{N}}(-\infty,n] = \emptyset$.

3.5.5 Theorem:

A topological space (X, τ) is compact iff any family of closed sets having the finite intersection property has a nonempty intersection.

Proof:

$\Longrightarrow$

Let us suppose that (X,τ) is compact and $\{F_i\}$ is a family of closed sets whose intersection is empty. Since $\bigcap_i F_i = \emptyset$, we may take the complement of each side of the equation and, using DeMorgan's Law, obtain $X = \emptyset^c = (\bigcap_i F_i)^c = \bigcup_i {F_i}^c$. Thus the family $\{{F_i}^c\}$ is an open covering of the compact space X, and so there must exist some finite subcovering. But if $X = \bigcup_{i=1}^{n} {F_i}^c$ then $\emptyset = X^c =$

$(\bigcup_{i=1}^{n} {F_i}^c)^c = \bigcap_{i=1}^{n} F_i$ so that the family $\{F_i\}$ cannot have the finite intersection property.

$\Leftarrow$

Now suppose (X,τ) is not compact. From the definition this means that there must be some open covering $\{G_i\}$ of X which has no finite subcovering. To say that there is no finite subcovering means that the complement of the union of any finite number of members of the cover is nonempty. By DeMorgan's Law, the family $\{{G_i}^c\}$ is then a family of closed sets with the finite intersection property. Since $\{G_i\}$ is a covering of X, however, $\bigcap_i {G_i}^c = \emptyset$ since $\emptyset = X^c = (\bigcup_i G_i)^c = \bigcap_i {G_i}^c$. Thus this family of closed sets with the finite intersection property has an empty intersection. □

3.5.6 Theorem:

Every closed subset of a compact space is compact.

Proof:

Let $\mathcal{A} = \{G_i\}$ be an open cover of F the closed subset of a compact space (X,τ) , i.e. $F = \bigcup_i G_i$.Then $X = F \cup F^c = (\bigcup_i G_i) \cup F^c$, i.e. $\mathcal{A}^* = \{G_i\} \cup \{F^c\}$ is a cover of X . But F^c is open since F is closed , so $\mathcal{A}^*$is an open cover of X. By hypotheses, X is compact ; hence $\mathcal{A}^*$has a finite subcover of X i.e.

$$X = G_1 \cup G_2 \cup \dots \cup G_n \cup F^c,\ G_i \in \mathcal{A}, \text{ i=1,2,...,}n$$

But F andF^c are disjoint ; hence

$$F \subseteq G_1 \cup G_2 \cup \dots \cup G_n\ , G_i \in \mathcal{A}, \text{ i=1,2,…,}n.$$

WE have shown that any open cover $\mathcal{A} = \{G_i\}$ of F contains a finite subcover, i.e. F is compact.□

3.6 Sequentially compact sets

3.6.1 Definition:

A subset A of a topological space (X,τ) is ***sequentially compact*** iff every sequence in A contains a subsequence which converges to a point in A.

3.6.2 Example:

Let A be a finite subset of a topological space (X,τ) then A is sequentially compact.

Solution:

Let $\langle a_1,a_2,a_3,\ldots\rangle$ be a sequence in A then at least one of the elements in A say a_0 must appears an infinite number of times in the sequence ,hence $\langle a_0,a_0,a_0,\ldots\rangle$ is a subsequence of $\langle a_n\rangle$ it converges to $a_0 \in A$.

3.6.3 Example:

The open interval $A=(0,1)$ in $\mathbb{R}$ with the usual topology is not sequentially compact.

Solution:

Consider the sequence $\langle a_n\rangle = \langle \frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\rangle$ in A which converge to 0 then every subsequence is also converge to 0. But $0 \notin A$, i.e. the sequence $\langle a_n\rangle$ does not contain a subsequence converge to a point in A. So A is not sequentially compact.

3.6.4 Remark:

In general, there exists compact sets which are not sequentially compact and vise versa although in metric spaces they are equivalent.

3.6.5 Example:

Let $\tau = \{\emptyset, U \subseteq X : U^c \text{is countable}\}$ be a topology on a non-empty set X then every infinite subset of X is not sequentially compact.

Solution:

The sequence $\langle a_n\rangle = \langle a_1,a_2,a_3,\ldots\rangle$ in X converge to $b \in X$ iff THE sequence of the form $\langle a_1,a_2,a_3,\ldots,a_n,\text{b},\text{b},\ldots\rangle$, i.e.the set A consisting of the terms of $\langle a_n\rangle$

different from b is finite. Now A is countable and so A^c is an open set containing b. Hence if $a_n \longrightarrow b$ then A^c contain all except a finite number of the terms of the sequence and so A is finite .Hence if A is an infinite subset of X , there exists a sequence $\langle b_n \rangle$ in A with distinct terms. Thus $\langle b_n \rangle$ does not contain any convergent subsequence and A is not sequentially compact.

3.6.6 Theorem:

Let A be a sequentially compact subset of a topological space (X,τ) then every countable open cover of A has a finite subcover.

Proof:

Assume A is infinite for otherwise the proof is trivial and assume there exists a countable open cover $\{G_i : i \epsilon \mathbb{N}\}$ with no finite subcover .Let n_1 be the smallest integer such that $A \cap G_{n_1} \neq \emptyset$. Choose

Let n_1 be the smallest integer s.t. $A \cap G_{n_1} \neq \emptyset$.Choose $a_1 \in A \cap G_{n_1}$

Let n_2 be the least positive integer larger than n_1 s.t. $A \cap G_{n_2} \neq \emptyset$. Choose $a_2 \in \left(A \cap G_{n_2}\right) \backslash (A \cap G_{n_1})$.

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We obtain the sequence $\langle a_1, a_2, a_3, \dots \rangle$ with the property that , for every $i \in \mathbb{N}$,

$$a_i \in A \cap G_{n_i} \text{ , } a_i \notin \bigcup_{j=1}^{n-1} \left(A \cap G_{n_j}\right) \text{ and } n_i > n_{i-1}$$

We claim that $\langle a_i \rangle$ has no convergent subsequence in A . Let $p \in A$ then

$$\exists G_{i_0} \in \{G_i\} \text{ s.t. } p \in G_{i_0}.$$

Now $A \cap G_{i_0} \neq \emptyset$ since $p \in A \cap G_{i_0}$, hence $\exists j_0 \in \mathbb{N}$ s.t. $G_{j_{n_0}} = G_{i_0}$.But by the choice of the sequence $\langle a_1, a_2, a_3, \dots \rangle, i > j_0 \Longrightarrow a_i \notin G_{i_0}$.Accordingly since G_{i_0} is an open set containing p, no subsequence of $\langle a_i \rangle$ converge to p. But p was arbitrary, so A is not sequentially compact and this is contradiction then every countable open cover of A has a finite subcover. □

3.7 Countable Compact Spaces

3.7.1 Definition:

A subset A of a topological space (X,τ) is ***countably compact*** iff every infinite subset B of A has at least one limit point in A.

3.7.2 Theorem (Bolzano-Weierstrass Theorem):

Every bounded infinite set of real numbers has a limit point.

3.7.3 Example:

Every bounded closed interval $A = [a,b]$ is countably compact.

Solution:

Assume B is an infinite subset of A .Since A is bounded and $B \subseteq A$ then by Bolzano-Weierstrass Theorem B has a limit point p .Since A is closed and $d(B) \subseteq d(A)$ then the limit point of B belongs to A, i.e. A is locally compact.

3.7.4 Example:

The open interval $A = (0,1)$ is not countably compact.

Solution:

Consider the infinite subset $B = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ of A .Observe that B has exactly one limit point which is 0 but $0 \notin A$,hence A is not countably compact.

3.7.5 Remark:

The general relationship between compact, sequentially compact and countably compact sets is given in the following diagram, theorems (3.7.6 , 3.7.7) and example 3.7.8.

Compact	$\Rightarrow$ $\not\Leftarrow$	countably compact	$\Leftarrow$ $\not\Rightarrow$	sequentially compact

3.7.6 Theorem:

A compact subset of a topological space is countably compact.

Proof:

Assume (X,τ) is a compact topological space and let A be infinite subset of ***X*** with $no\ limit\ points\ in\ X$, i.e. for each point $x \in X$ is not a limit point of A so there must $exist\ an\ open\ G_x\ containing\ x\ such\ that\ G_x \backslash \{x\} \cap A = \emptyset$. Clearly$G_x \cap A$ contains, at most, the one point x itself. Since the family $\{G_x\}_{x \in X}$ forms an open covering of the compact space ***X***, there must be some finite subcovering $X = \bigcup_{i=1}^{n} G_{x_i}$. From this it follows that $A = A \cap X = A \cap \left(\bigcup_{i=1}^{n} G_{x_i}\right) = \bigcup_{i=1}^{n}(A \cap G_{x_i})$ is a finite union of sets, each containing, at most, one element, and so A is finite and this is contradiction. Thus every infinite subset of ***X*** must have at least one limit point. □

3.7.7 Theorem:

A sequentially compact subset of a topological space is countably compact.

Proof:

Let A be any infinite subset of X. Then there exists a sequence $\langle a_1, a_2, a_3, \ldots \rangle$ in A with distinct terms. Since X is sequentially compact then the sequence $\langle a_n \rangle$ contains a subsequence $\langle a_{i_1}, a_{i_2}, a_{i_3}, \ldots \rangle$ (also with distinct terms) which converges to a point $p \in X$. Hence every open set G_p contain p contains an infinite number of points in A. Since $p \in X$ is a limit point of A, i.e. X is countably compact.□

3.7.8 Example:

Let τ be the topology on $\mathbb{N}$, the set of positive integers generated by sets {{1,2},{3,4},{5,6},...} .Let A be a non – empty infinite subset of $\mathbb{N}$, say $n_0 \in A$. If n_0 is odd then $n_0 + 1$ is a limit point of A , and if n_0 is even then $n_0 - 1$ is a limit point of A. In either case A has a limit point, so $(\mathbb{N},\tau)$ is countably compact.

On the other hand $(\mathbb{N},\tau)$ is not compact since $\mathcal{A} = \{\{1,2\},\{3,4\},\{5,6\},\ldots\}$ is an open cover of $\mathbb{N}$ with no finite subcove. Also $(\mathbb{N},\tau)$ is not sequentially compact since the sequence $\langle 1,2,3, \ldots \rangle$ contains no convergent subsequence.

3.7.9 Theorem:

A closed subset of countably compact is countably compact.

Proof:

Let F be a closed subset of countably compact space (X,τ) and let A be any infinite subset of F.

Since $A \subseteq F$ then $A \subseteq X$ but X is countably compact, so A has a limit point $p \in X$.Since $A \subseteq F$ and F is closed set then F is countably compact.□

3.8 Locally Compact Spaces

3.8.1 Definition:

A topological(X,τ) is ***locally compact*** iff each point of X is contained in a compact neighborhood.

3.8.2 Remark:

Since a compact space is a compact neighborhood of each of its points, it is clear that $every\ compact\ space\ is\ locally\ compact$, i.e. every compact space is locally compact but the converse is not true as the following example.

3.8.3 Example:

Let $(\mathbb{R},\tau)$ be the usual topology .For each point $p \in \mathbb{R}$ there exists a closed interval $[p-\mathcal{E},p+\mathcal{E}]$ contain p. Since every closed interval is closed and bounded then its compact by Heine-Borel Theorem (A subset of the real line is compact iff it is closed and bounded). Hence $\mathbb{R}$ $is\ a\ locally\ compact\ space$. On the other hand $\mathbb{R}$ is not compact since the class $\mathcal{A}=$ {..,(-3,-1),(-2,0),(-1,1),(0,2),(1,3),...} is an open cover of $\mathbb{R}$ but contains no finite subcover.

3.8.4 Example:

The discrete topology (X,τ) is locally compact since $\forall p \in X\ \exists\{p\}$ a compact neighborhood of p.

3.8.5 Example:

The indiscrete topology (X,τ) is locally compact since X is compact.

3.8.6 Theorem:

A closed subset of a locally compact space is locally compact space.

Proof:

Let A be a closed subset of locally compact space (X,τ) and let p ∈A then there exists a compact neighborhood H of p.Since A is closed then $F=A\cap H$ is compact (by let (X,τ) is a topological space and $F\subseteq X$ be a closed set. If A is compact then $A\cap F$ is compact) but $p\in H^{\circ}$then $p\in H^{\circ}\cap A\subseteq F$, where $H^{\circ}\cap A\in \tau_A$,so p has compact neighborhood $F=A\cap H$, i.e. A is locally compact. □

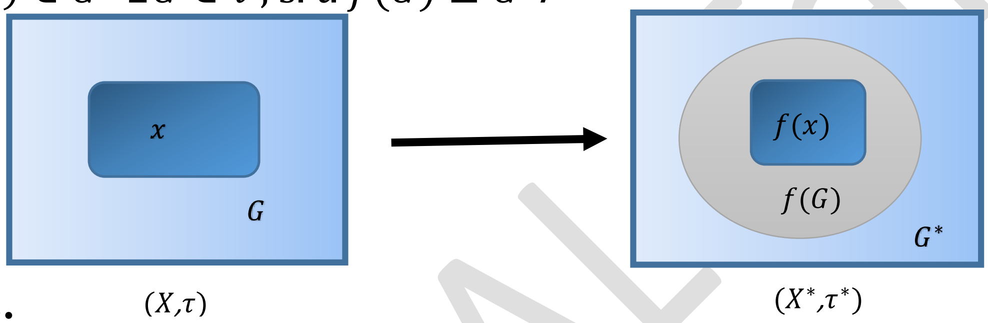
Chapter Four

Continuity and Topological Equivalence

4.1 Continuous Functions

4.1.1 Definition:

A function f mapping a topological space (X,τ) into a topological space(X^*,τ^*) will be said to be ***continuous at a point*** $\boldsymbol{x} \in \boldsymbol{X}$ iff for every open set G^* containing $f(x)$ there is an open set G containing x such that $f(G) \subseteq G^*$, i.e. $\forall G^* \in \tau^*, f(x) \in G^*\ \exists G \in \tau$, s. t. $f(G) \subseteq G^*$.

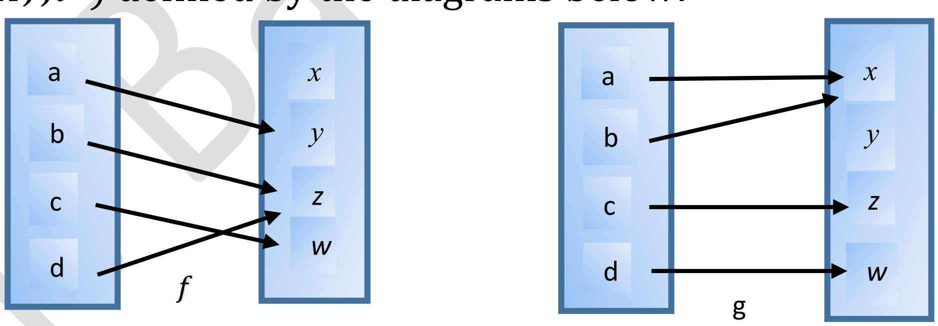


4.1.2 Remark:

We say that f is continuous on a set $E \subseteq X$ iff it is continuous at each point of E.

4.1.3 Example:

Let $X = \{a,b,c,d\}$ and $X^* = \{x,y,z,w\}$ have the topologies $\tau = \{X,\emptyset,\{a\},\{a,b\},\{a,b,c\}\}$,$\tau^* = \{X^*,\emptyset,\{x\},\{y\},\{x,y\},\{y,z,w\}\}$ respectively consider the functions f,g: $(X,\tau) \longrightarrow (X,,\tau^*)$ defined by the diagrams below:



The function f is continuous but the function g is not continuous on X.

Solution:

Take $a \in X$,$f(a)$=y the open sets in X^*contain y are X^*,$\{y\}$,$\{x,y\}$and$\{y,z,w\}$, so

$\exists$ X $\in \tau$, s. t. f(X) $\subseteq$ X*,
$\exists\{a\} \in \tau$, s. t. $f(\{a\}) \subseteq \{y\}$,
$\exists\{a\} \in \tau$, s. t. $f(\{a\}) \subseteq \{x , y\}$,
$\exists$ X $\in \tau$, s. t. f(X) $\subseteq \{y,z,w\}$

Thus the function f is continuous at a similar we can show that f is continuous at b , c and d ,so f is continuous on X but the function g is not continuous on X since it's not continuous on c, i.e. $g(c) = z, z \in \{y,z,w\} \in \tau^*, \nexists G \in \tau\ s.t.\ g(G) = \{y,z,w\}$.

4.1.4 Theorem:

If $f:(X,\tau) \longrightarrow (X^*, \tau^*)$ ***then the following conditions are each equivalent to the continuity of f on X:***

1) ***The inverse of every open set in*** X^* ***is an open set in X.***

2)***The inverse of every closed set in*** X^* ***is a closed set in X.***

3) $f(\bar{E}) \subseteq \overline{f(E)}$ ***for every*** $E \subseteq X$.

Proof:

Continuity ⟺ (1)

Suppose that f is continuous on X, and G^* is an open set in X^*.If x is any point of $f^{-1}(G^*)$ then f is continuous at x, and there must exist an open set G containing x such that $f(G) \subseteq G^*$. Thus G is contained in $f^{-1}(G^*)$, and hence $f^{-1}(G^*)$, is an open set in X. Conversely, if the inverses of open sets are open, we may choose the set $f^{-1}(G^*)$, let $x \in X$ and let G^*be an open set in X^* contain $f(x)$,i.e. $f(x) \in G^*$, so $x \in f^{-1}(G^*)$ which is an open set in X satisfy $ff^{-1}(G^*) \subseteq G^*$. Then $f$is continues at x and x is arbitrary so $f$is continues on X.

(1) ⟺ (2)

Suppose that the inverses of open sets are open and let F^*be a closed set in X^*,so $F^{*c}$is an open set in X^* then by (1) , $f^{-1}(F^{*c}) = (f^{-1}(F^*))^c$is open in X ,i.e. $f^{-1}(F^*)$ is closed set in X. Conversely, assume the inverses of closed sets are closed and let G^*be an open set in X^*,so $G^{*c}$is a closed set in X^* then by (2) , $f^{-1}(G^{*c}) = (f^{-1}(G^*))^c$ is closed in X ,i.e. $f^{-1}(G^*)$ is an open set in X.

(2) ⟺ (3)

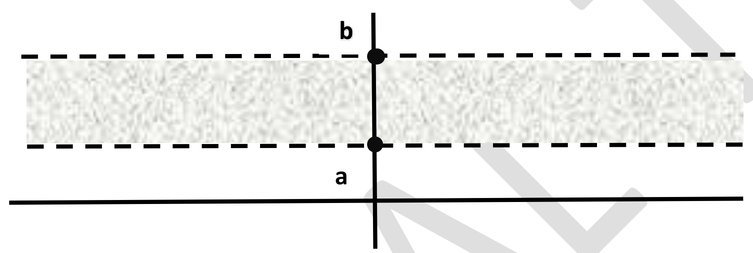
Suppose that the inverses of closed sets are closed, and $E \subseteq X$.Since $E \subseteq f^{-1}(f(E))$ for any function, $E \subseteq f^{-1}(\overline{f(E)})$.But $f^{-1}(\overline{f(E)})$ is the inverse under a continuous mapping of a closed set and hence is a closed set containing E. Therefore, $\bar{E} \subseteq f^{-1}(\overline{f(E)})$ and so $f(\bar{E}) \subseteq f(f^{-1}(\overline{f(E)})) \subseteq \overline{f(E)}$. Conversely, suppose the condition (3) holds for all subsets $E \subseteq X$, and F^*be a closed set in X^*, $f\left(\overline{f^{-1}(F^*)}\right) \subseteq \overline{ff^{-1}(F^*)} \subseteq \overline{F^*} = F^*$also $\overline{f^{-1}(F^*)} \subseteq f^{-1}(F^*)$, i.e. $f^{-1}(F^*) = \overline{f^{-1}(F^*)}$, so . $f^{-1}(F^*)$ is closed in X, i.e. the inverse of every closed set is a closed set. □

4.1.5 Example:

Consider (X,τ) any discrete topology and $(X^*, ,\tau^*)$ any topological space then every function $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ is continuous, since if H is any open subset of X^*its invers $f^{-1}(H)$ is open subset of X (every subset of a discrete topology is open).

4.1.6 Example:

The projection map $f:(\mathbb{R}^2,\tau) \longrightarrow (\mathbb{R}, ,\tau^*)$ defined by $f(x,y) = y$ is continuous relative to the relative topology. Since the inverse of any open interval (a,b) is an infinite open strip then by theorem 4.1.4 the inverse of every open subset of $\mathbb{R}$ is an open in $\mathbb{R}^2$,i.e. f is continuous.



4.1.7 Example:

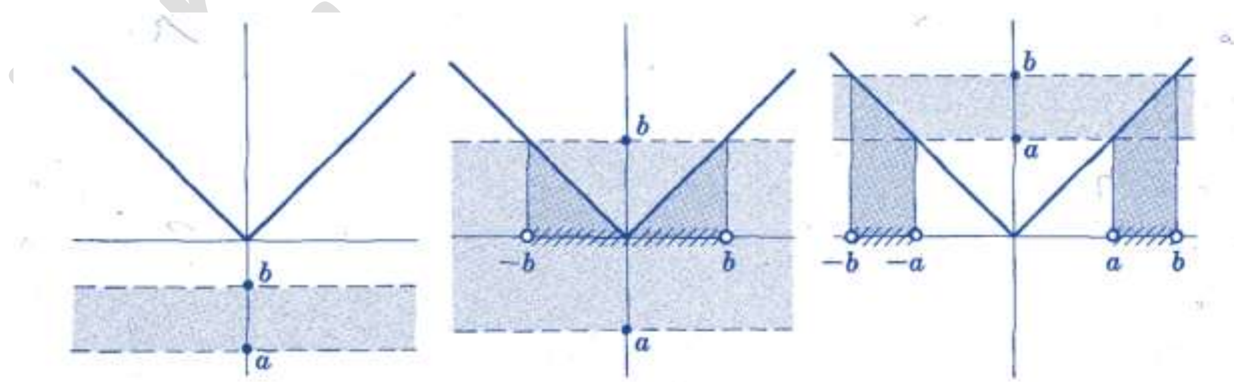
The absolute value function $f:(\mathbb{R},\tau) \rightarrow (\mathbb{R},\tau)$, *i.e.* $f(x) = |x|$ for every $x \in \mathbb{R}$ is continuous.

Solution:

Since if $G =$ (a,b) is an open interval in $\mathbb{R}$ then

$$f^{-1}(G) = \begin{cases} \emptyset & if\ a < b \leq 0 \\ (-b,b) & if\ a < o < b \\ (-b,-a)\cup(a,b) & if\ 0 \leq a < b \end{cases}$$

In each case $f^{-1}(G)$ is open, hence f is continuous.



$f^{-1}[A] = \emptyset$ $\quad$ $f^{-1}[A] = (-b, b)$ $\quad$ $f^{-1}[A] = (-b, -a) \cup (a, b)$

4.1.8 Example:

Let $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ be a constant function ,i.e. . $f(x) = c \in X^*$ for every $x \in X$. Then f is continuous relative to any topology τ on X and any topology τ^*on X^*.

Solution:

We need to show that the inverse image of any τ^* −open subset of Y is a τ −open subset of X. Let $G^* \in \tau^*$.Now $f(x) = c$ for every $x \in X$,so

$$f^{-1}(G^*) = \begin{cases} X & if\ c \in G^* \\ \emptyset & if\ c \notin G^* \end{cases}$$

In either case $f^{-1}(G^*)$ is an open subset of X since X and ∅ belong to every topology τ on X.

4.1.9 Example:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be any function. If (X^*,τ^*) is any indiscrete space then f is continuous for any τ.

Solution:

We want to show that the inverse image of every open subset of $X^*$is an open subset of X. Since $(X^*, ,\tau^*)$ is an indiscrete space, $X^*$and ∅ are the only open subset of X^*.But $f^{-1}(X^*) = X$, $f^{-1}(\emptyset) = \emptyset$ and $X, \emptyset \in \tau$ on X. Hence f is continuous for any τ.

4.1.10 Example:

Let $(\mathbb{R},\tau)$ be the real topology and let f,g,h: $(\mathbb{R},\tau) \longrightarrow (\mathbb{R}, ,\tau)$ be functions defined on ℝ as $f(x) = x + 2$, g(x)=2x and $h(x) = x^2$.Show that the all functions f,g and h are continuous.

Solution:

Since if $G =$ (a,b) is an open interval in ℝ then

$$f^{-1}((a,b)) = (a - 2, b - 2)$$

$$g^{-1}((a,b)) = (\tfrac{a}{2}, \tfrac{b}{2})$$

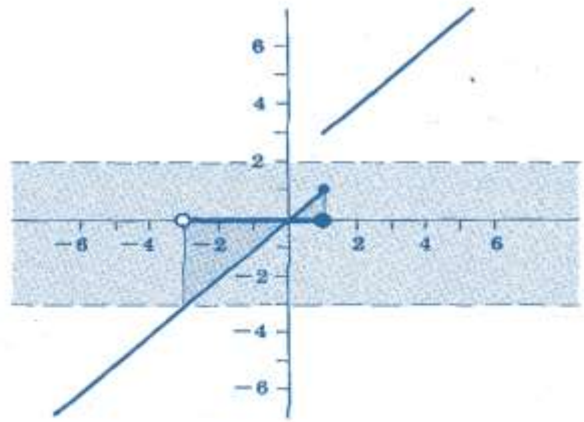
$$h^{-1}((a,b)) = \begin{cases} (-\sqrt{b}, -\sqrt{a}) \cup (\sqrt{a}, \sqrt{b}) & if\ a \geq 0, \\ (-\sqrt{b}, \sqrt{b}) & if\ a < 0\ and\ b > 0, \\ \emptyset & if\ b \leq 0. \end{cases}$$

In each case the preimage of an arbitrary G is an open set. Thus each function is continuous.

4.1.11 Example:

Let τ be the usual topology on $\mathbb{R}$ and let τ^* be the upper limit topology on $\mathbb{R}$ which generated by the open – closed intervals $(a,\text{b}]$.Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$f(x)=\begin{cases} x & if\ x\le 1\\ x+2 & if\ x>1\end{cases}$$



a) Show that $f:(\mathbb{R},\tau)\longrightarrow(\mathbb{R},,\tau)$ is not continuous.

b) Show that $f:(\mathbb{R},\tau^*)\longrightarrow(\mathbb{R},,\tau^*)$ is continuous.

Solution:

a) Let $A=(-3,2)\in\tau$ then $f^{-1}(\text{A})=(-3,1]\notin\tau$.

So f is not continuous.

b) Let $A=(a,\text{b}]\in\tau^*$then

$$f^{-1}(\text{A})=\begin{cases}(a,\text{b}] & if\ a<b\le 1\\ (a,1] & if\ a<1<b\le 3\\ (a,\text{b-2}] & if\ a<1<3<b\\ \emptyset & if\ 1\le a<b\le 3\\ (1,b-2] & if\ 1\le a<3<b\\ (a-2,\text{b-2}] & if\ 3\le a<b\end{cases}$$

In each case the$f^{-1}(\text{A})$ is a τ^*- open set. Hence f is τ^*continuous.

4.1.12 Example:

Let τ^* be the usual topology on $\mathbb{R}$ and let τ be the co-finite topology on $\mathbb{R}$. If $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by $f(x)=x$, $\forall x\in\mathbb{R}$ then f is not continuous.

Solution:

Since if $G=(\text{a,b})\in\tau$ then $f^{-1}((\text{a,b}))=(a,b)\in\tau^*$,since $(a,b)^c=(-\infty,\text{a}]\cup[\text{b},\infty)$ is finite, so f is not continuous.

4.1.13 Example:

Show that the identity function $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ is continuous iff τ is finer than τ^*, i.e. $\tau^*\subset\tau$.

Solution:

The identity function $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ is continuous iff $\forall G\in\tau^*\Longrightarrow f^{-1}(\text{G})\in\tau$. But $f^{-1}(\text{G})=\text{G}$,so f is continuous iff $\forall G\in\tau^*\Longrightarrow G\in\tau$, i.e. $\tau^*\subset\tau$.

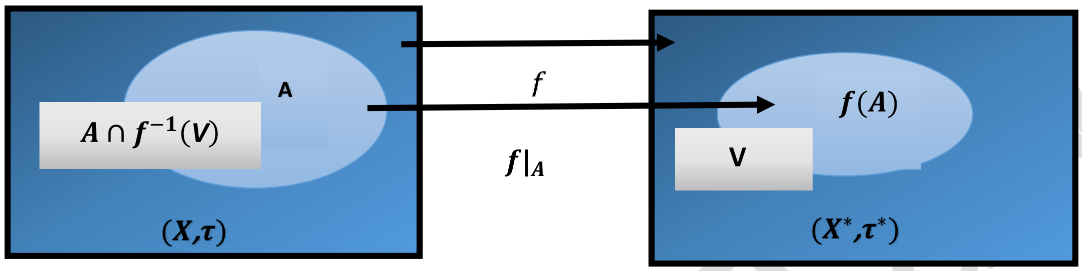
4.1.14 Example:

Let $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ be continuous then Prove that $f|_A:(X,\tau_A)\longrightarrow(X^*,{\tau^*}_A)$ is continuous, where $A\subset X$ and $f|_A$ is restriction of f to A.

Solution:

If $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is a function and $A \subset X$ then the restriction function $f|_A: (X,\tau_A) \longrightarrow (X^*,{\tau^*}_A)$ is defined as $f|_A(x) = f(x), \forall x \in A$.

Let $V \in \tau^*$, since f is continuous then $f^{-1}(\mathrm{V}) \in \tau$ then $A \cap f^{-1}(V) \in \tau_A$.Since ${f|_A}^{-1}(V) = A \cap f^{-1}(V)$ then $f|_A$ is continuous function.

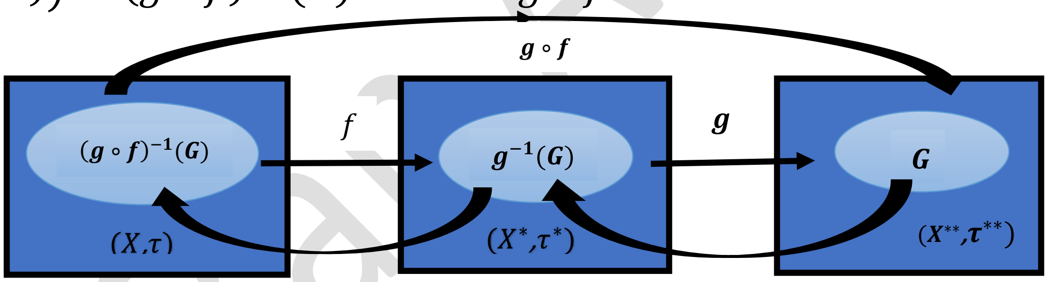


4.1.15 Corollary:

Let the functions $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ and $g: (X^*,\tau^*) \longrightarrow (X^{**},\tau^{**})$ be continuous then the composition $g \circ f: (X,\tau) \longrightarrow (X^{**},\tau^{**})$.

Proof:

Let $G \in \tau^{**}$ then $g^{-1}(G) \in \tau^*$since g is continuous .But f is also continuous, so $f^{-1}\big(g^{-1}(G)\big) = (g \circ f)^{-1}(G) \in \tau$ then $g \circ f$ is continuous.□



4.1.16 Theorem:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff the inverse of each member of a base $\mathcal{B}$ forX^* is an open subset of X.

Proof: $\Rightarrow$

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be a continuous function and let $\mathcal{B}$ be a base for the topology τ^*, i.e. $\mathcal{B} \subset \tau^*$. Now for every $B \in \mathcal{B}$ we have $f^{-1}(B) \in \tau$ so $f^{-1}(B)$ is an open subset of X function.

$\Leftarrow$

Let $G \in \tau$, since $\mathcal{B}$ is a base for τ^*then $G = \bigcup_i B_i, B_i \in \mathcal{B}$, so $f^{-1}(G) = f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i)$ and since $f^{-1}(B_i) \in \tau$ then $f^{-1}(G)$ is union of open sets and therefore its open ,so f is continuous.□

4.1.17 Theorem:

Let $\mathcal{S}$ be a subbase for a topological space (X^*,τ^*).Then a function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is continuous iff the inverse of each member of $\mathcal{S}$ is an open subset of X.

Proof: $\Rightarrow$

Suppose $f^{-1}(S) \in \tau$ for every $S \in \mathcal{S}$.We want to show that f is continuous, i.e. if $G \in \tau^*$ then $f^{-1}(G) \in \tau$.Let $G \in \tau^*$then by definition of subbase

$$G = \bigcup_i \left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right), \text{ where } S_{i_k} \in \mathcal{S}$$

Hence, $f^{-1}(G) = f^{-1}(\bigcup_i \left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right)) = \bigcup_i f^{-1}\left(S_{i_1} \cap S_{i_2} \cap \ldots \cap S_{i_{n_i}}\right)$

$$= \bigcup_i \left(f^{-1}(S_{i_1}) \cap f^{-1}(S_{i_2}) \cap \ldots \cap f^{-1}(S_{i_{n_i}})\right)$$

But $S_{i_k} \in \mathcal{S} \Rightarrow f^{-1}(S_{i_k}) \in \tau$.Hence $f^{-1}(G) \in \tau$ since it is the union of finite intersections of open sets.herefore f is continuous.

$\Leftarrow$

If f is contiuous then the inverse of all open sets, including the member of $\mathcal{S}$ are open. □

4.1.18 Example:

Let f be a function from a topological space (X,τ) into the unit interval [0,1]. Show that if $f^{-1}((a,1])$ and $f^{-1}([0,b))$ are open subsets of X for all $0 < a$,b<1, then f is continuous.

Solution:

Since the intervals $(a,1]$ and $[0,b)$ form a subbase for the unit interval [0,1] then by theorem 4.1.17 , f is continues.

4.1.19 Theorem:

Let $\{\tau_i\}$ be a collection of topologies on a set X. If a function $f:(X,\tau_i) \longrightarrow (X^*,\tau^*)$ is continuous with respect to each τ_i, then f is continuous with respect to the intersection topology $\tau = \bigcap_i \tau_i$.

Proof:

Let G be an open subset of X^*then by hypothesis $f^{-1}(G)$ belongs to each τ_i. Hence $f^{-1}(G)$ belongs to the intersection, i.e. $f^{-1}(G) \in \bigcap_i \tau_i = \tau$ and so f is continuous with respect to the intersection topology τ.□

4.1.20 Theorem:

A function $f:(X,\tau) \longrightarrow (X^*, ,\tau^*)$ be a continuous at a point $a_0 \in X$ if for every sequence $\langle a_n \rangle$ in X converges to a_0 the sequence $\langle f(a_n) \rangle$ in X^*converges to $f(a_0)$, i.e. $a_n \longrightarrow a_0 \Rightarrow f(a_n) \longrightarrow f(a_0)$.

4.1.21 Remark:

The following theorems show that some characteristics transfer by continuity.

4.1.22 Theorem:

If $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is a continuous function then f maps every connected subset of X onto a connected subset of X^*.

Proof:

Let E be a connected subset of X and suppose that $E^* = f(E)$ is not connected then there exists a separation $E^* = A^*|B^*$,where $A^*$and $B^*$are nonempty disjoint sets which are both and closed subsets of E^*.Let $A = f^{-1}(A^*) \cap E$ and $B = f^{-1}(B^*) \cap E$.

Since f is continuous function and A^*, $B^*$are both and closed subsets of E^*then by theorem 4.1.4 , A and B are nonempty disjoint sets which are both and closed subsets of E. Thus E has a separation $E = A|B$, i.e. E is not connected and this is contradiction , so $E^* = f(E)$ is connected. □

4.1.23 Theorem:

If $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is a continuous function then f maps every compact subset of X onto a compact subset of X^*.

Proof:

Let E be a compact subset of X and suppose that $\{G_i^*\}$ be an open cover of $f(E)$, i.e. $f(E) \subseteq \bigcup_i G_i^*$.Since $E \subseteq f^{-1}\big(f(E)\big) \subseteq f^{-1}(\bigcup_i G_i^*) = \bigcup_i f^{-1}(G_i^*)$. Since f is continuous function and by theorem 4.1.4 we get $\{f^{-1}(G_i^*)\}$ is an open covering of E. But E is compact then there exists a finite subcover $\{f^{-1}(G_i^*)\}_{i=1}^{n}$ of $\{f^{-1}(G_i^*)\}$ for E, i.e. $E \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i^*)$, so $f(E) \subseteq f(\bigcup_{i=1}^{n} f^{-1}(G_i^*)) \subseteq \bigcup_{i=1}^{n} f(f^{-1}(G_i^*)) \subseteq \bigcup_{i=1}^{n} G_i^*$.Then $f(E)$ is compact. □

4.1.24 Theorem:

If $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ is a continuous function then f maps every sequentially compact subset of X onto a sequentially compact subset of X^*.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*, \tau^*)$ be a continuous function and let E be a sequentially compact subset of X. We want to show that $f(E)$ is a sequentially compact subset of X^*.

Let $\langle b_1,b_2,... \rangle$ be a sequence in $f(E)$ then $\exists a_1,a_2,... \in E$ s.t. $f(a_n) = b_n, \forall n \in \mathbb{N}$. But E is a sequentially compact subset of X, so the sequence $\langle a_1,a_2,\ldots \rangle$ contains a subsequence $\langle a_{i_1},a_{i_2},\ldots \rangle$ which converges to a point $a_0 \in E$. Since f is continuous then $\langle f(a_{i_1}),f(a_{i_2}),\ldots \rangle = \langle b_{i_1},b_{i_2},... \rangle$ converges to $f(a_0) \in f(E)$. Thus $f(E)$ is sequentially compact. □

4.1.25 Example:

Show that :

a) A continuous image of a countably compact set need not be countably compact.

b) A continuous image of a locally compact set need not be locally compact.

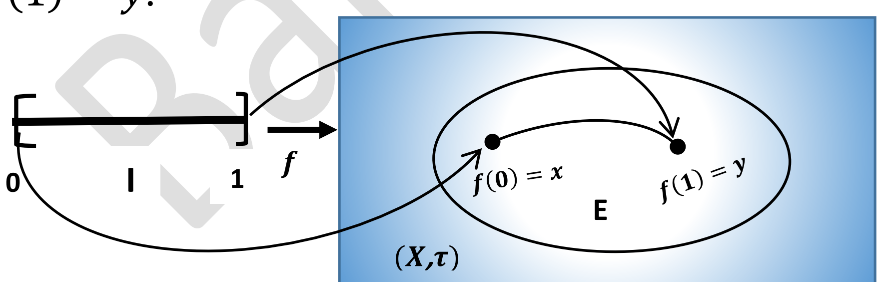
Solution:

a) Let τ be the topology on $\mathbb{N}$, the set of positive integers generated by sets $\{\{1,2\},\{3,4\},\{5,6\},\ldots\}$ by example 3.7.8, X is countably compact. Let $(\mathbb{N}, \tau^*)$ be the discrete topology on $\mathbb{N}$ which is not countably compact. The function $f: (\mathbb{N},\tau) \longrightarrow (\mathbb{N}, \tau^*)$ which maps $2n$ and $2n-1$ onto n for $n \in \mathbb{N}$ is continuous and maps the countably compact space $(\mathbb{N},\tau)$ onto the non – countably compact space $(\mathbb{N}, ,\tau^*)$.

b) Let $(\mathbb{Q},\tau)$ be the discrete topology which is locally compact and $(\mathbb{Q},\tau^*)$ be the usual topology which is not locally compact. Consider $f: (\mathbb{Q},\tau) \longrightarrow (\mathbb{Q}, \tau^*)$ to be the identity function which is continuous.

4.1.26 Definition:

If E is a subset of a topological space (X,τ) and we let $I = [0, 1]$, then a ***path*** in E joining two points x and y of E is a continuous function $f: I \longrightarrow E$ such that $f(0) = x$ and $f(1) = y$.



4.1.27 Definition:

A subset E of a topological space (X,τ) is said to be ***arcwise connected*** if for any two points $a,b \in E$ there is a path $f: I \longrightarrow E$ from a to b which is contained in E, i.e. $f(I) \subseteq E$.

4.1.28 Remark:

The relationship between connected and arcwise sets connected sets is given in the following diagram, theorem 4.1.29 and example 4.1.30.

arcwise connected $\Rightarrow$ $\nLeftarrow$ connected

4.1.29 Theorem:

A rcwise connected sets are connected.

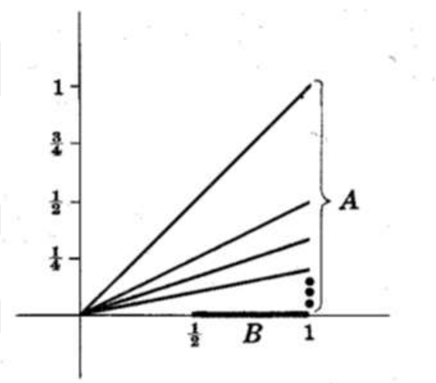
Proof:

Since I is connected, $f(I)$ is connected for any continuous function f. Thus any two points is an arcwise connected space belong to a connected subset $f(I)$ of the space, where f is a path joining the two points. By Corollary 3.1.19, any arcwise connected space must be connected. □

4.1.30 Example:

Consider the following subsets of the plane $\mathbb{R}^2$

$A=\{(x,y): 0\leq x\leq 1 , y=\frac{x}{n} , n\in\mathbb{N}\}$, $B=\{(x,0): \frac{1}{2}\leq x\leq 1\}$.



Here A consists of the points on the line segments joining the origin (0,0) to the points $\left(1,\frac{1}{n}\right)$, $n\in\mathbb{N}$ and B consists of points on the x - axis between $\frac{1}{2}$ and 1. Now A and B are both arcwise connected ,hence each also connected. Also A and B are not separated since each $p\in B$ is a limit point of A and so $A\cup B$ is connected.But $A\cup B$ is not arcewise connected since there is no path from any point in A to any point in B.

4.1.31 Theorem:

If $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ is a continuous function then f maps every arcwise connected subset of X onto an arcwise connected subset of X^*.

Proof:

Suppose E is an arcwise connected subset of X**,** and $x^*$and $y^*$are any two points of $f(E)$.There must exist points x and y in A such that f(x)= $x^*$and $f(y)=y^*$.Since E is arcwise connected, there exists a path g in E joining x and y , i.e. a continuous function g from I into E such that $g(0)=x$ and $g(1)=y$.By Corollary 4.1.15, we have $f\circ g$ is a continues function from I into $f(E)$ such that $(f\circ g)$(0)= $x^*$and $(f\circ g)$(1)= y^*.Thus $f\circ g$ is a path in $f(E)$ joining$x^*$and y^* and $f(E)$ **)** must be arcwise connected.□

4.1.32 Remark:

Although very few properties of sets are preserved by continuous transformations, many of the important properties are preserved if we put additional restrictions on the function. The following is an example of a property that is preserved if we merely add the restriction of one-to-oneness.

4.1.33 Definition:

A subset E of a topological space (X,τ) is ***dense-in-itself*** if every point of E is a limit point of E, i.e. $E \subseteq d(E)$.

4.1.34 Theorem:

If f is a one-to-one continuous function of (X,τ) into (X^*,τ^*) then f maps every dense-in-itself subset of X onto a dense-in-itself subset of X^*.

Proof:

Suppose E is a dense – in –itself subset of X. We want to show that $f(E)$ is dense – in –itself , i.e. $f(E) \subseteq d(f(E))$.

$Let\ x^* \in f(E)$, G^*open in X^*,s.t.$x^* \in G^*$ then $\exists x \in E$, s. t. $f(x) = x^*$.Now $x \in f^{-1}(\{x^*\}) \subseteq f^{-1}(G^*)$ and $f^{-1}(G^*)$ is an open set since f is continuous. But E is dense-in-itself, so $x \in E \subseteq d(E)$.Thus x is a limit point of the set E which is contained in the open set $f^{-1}(G^*)$, and so, by the definition of limit point,$E \cap f^{-1}(G^*)/\{x\} \neq \emptyset$**.** Since this set is nonempty, let us choose a point $z \in E \cap f^{-1}(G^*)/\{x\}$. Since z is in this intersection, it is in each part. Thus, $z \in E$, and so $f(z) \in f(E)$, while $z \in f^{-1}(G^*)$, and so $f(z) \in f(f^{-1}(G^*)) \subseteq G^*$**.** Finally, $z \neq x$, and so $f(z) \neq f(x) = x^*$ since f is one-to-one. This shows that $f(z) \in f(E) \cap G^*/\{x^*\}$, and so $f(E) \cap G^*/\{x^*\} \neq \emptyset$**,** as desired**.** □

Exercise:

Show that if D is a dense-in-itself set, $\overline{D}$ is dense-in -itself, and any set E such that $D \subseteq E \subseteq d(E)$ is also dense-in-itself. Furthermore, the union of any family of dense-in-itself sets is dense-in- itself.

4.1.35 Definition:

Let E be a subset of a topological space (X,τ), ***the nucleus*** of E is defined to be the union of all dense-in-itself subsets of E and is clearly the largest set contain in E and dense-in-itself.

4.1.36 Definition:

A subset E of a topological space (X,τ) is whose nucleus is empty is called ***scattered.***

4.1.37 Definition:

A subset E of a topological space (X,τ) is called ***perfect*** if it's both closed and dense-in-itself (i.e. $E = d(E)$).

4.1.38 Theorem:

If f is a one-to-one continuous function of (X,τ) into (X^*,τ^*) then f maps every scattered subset of X onto a scattered subset of X^*.

Proof:

Suppose E is a scattered subset of X. We want to show that $f(E)$ is scattered. Since E is scattered then their nucleus is empty set ,i.e, $\bigcup_i G_i = \emptyset$, where $\forall i, G_i \subseteq E$ is dense-in-itself .Since f is one-to-one and continuous then by theorem 4.1.34 we get $\forall i, f(G_i)$ is dense-in-itself . Since f is one-to-one and $\bigcup_i G_i = \emptyset$ then $\bigcup_i f(G_i) = \emptyset$, so the nucleus of $f(E)$ is empty set, i.e. $f(E)$ is scattered. □

4.2 Open and Closed Functions

4.2.1 Definition:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is called an ***open function*** if the image of every open set is open.

4.2.2 Definition:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is called a ***closed function*** if the image of every closed set is closed.

4.2.3 Remark:

In general, functions which are open(closed) need not be closed (open) even if they are continuous as the following example:

4.2.4 Example:

Let (X,τ) be any topological space and let (X^*,τ^*) be the space for which $X^*=\{a,b,c\}$ and $\tau^* = \{\emptyset,\{a\},\{a,c\},X^*\}$. The function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ defined by $f(x) = a$, $\forall x \in X$ is a continuous open map which is not closed. Since the image of every open set G in X is $\{a\}$open in X^*but the image of every closed set F in X is $\{a\}$which is not closed in X^*.

If $g: (X,\tau) \longrightarrow (X^*,\tau^*)$ defined by $g(x) = b, \forall x \in X$ is a continuous closed map which is not open. Since the image of every open set G in X is $\{b\}$which is not open in X^*but the image of every closed set F in X is $\{b\}$which is closed in X^*.

4.2.5 Example:

Give an example of a real function $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau)$ such that f is continuous and closed, but not open.

Solution:

Let $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau^*)$ be a constant function, $f(x) = 1$, $\forall x \in \mathbb{R}$. Then $f(A) = \{1\}$ for any $A \subseteq \mathbb{R}$. Hence if A is open then $f(A) = \{1\}$ is not open, so f is not open function and if if A is closed then $f(A) = \{1\}$ is closed, so f is closed function (since singleton sets are closed in the usual topology).Also by example 4.1.8, f is continuous on $\mathbb{R}$.

4.2.6 Example:

Let the real function $f: (\mathbb{R},\tau) \longrightarrow (\mathbb{R},\tau)$ be defined by $f(x) = x$, $\forall x \in \mathbb{R}$. Show that f is not open.

Solution:

Let $A = (-1,1)$ be an open set. Note that $f(A) = [0,1)$,which is not open hence f is not an open function.

4.2.7 Example:

Let $f:(X,\tau)\longrightarrow(X^*,\tau^*)$ be a function from any topological space (X,τ) to the discrete topology (X^*,τ^*) then f is open function.

Solution:

Let $G\in\tau$ then $f(G)\subseteq X^*$, since X^* discrete topology then $f(G)\in\tau^*$,i.e f is open function.

4.2.8 Remark:

1. Let (X,τ),(X^*,τ) be the discrete topologies then the function $f:(X,\tau)\longrightarrow(X^*,\tau)$ is continuous , open and closed function.
2. Let (X,τ) be the discrete topologies and (X,τ^*) be the indiscrete topology , X contain more than one point then the function $f:(X,\tau)\longrightarrow(X^*,\tau)$ is continuous function not open and not closed function.
3. Let (X,τ) be the indiscrete topologies and (X,τ^*) be the discrete topology , X contain more than one point then the function $f:(X,\tau)\longrightarrow(X^*,\tau)$ is open and closed function not continuous.

4.2.9 Example:

Let the functions $f:(\mathrm{X},\tau)\longrightarrow(\mathrm{X}^*,\tau^*)$ and $\mathrm{g}:(\mathrm{X}^*,\tau^*)\longrightarrow(\mathrm{X}^{**},\tau^{**})$ be open functions then the composition $g\circ f:(\mathrm{X},\tau)\longrightarrow(\mathrm{X}^{**},\tau^{**})$ is an open function.

Solution:

Let $G\in\tau$ then $f(G)\in\tau^*$since f is an open function and $g\big(f(G)\big)\in\tau^{**}$ since g is an open function then $g\circ f$ is an open function.

4.2.10 Theorem:

A function $\boldsymbol{f}:(\boldsymbol{X},\boldsymbol{\tau})\longrightarrow(\boldsymbol{X}^*,\boldsymbol{\tau}^*)$ ***is open iff*** $\boldsymbol{f}(\boldsymbol{E}^\circ)\subseteq\boldsymbol{f}(\boldsymbol{E})^\circ$ ***for every*** $\boldsymbol{E}\subseteq\boldsymbol{X}$.

Proof:

Suppose f is open and $E\subseteq X$.Since $E^\circ$is an open set and f is an open function, then $f(E^\circ)$ is an open set in X^*. Since $E^\circ\subseteq E$, $f(E^\circ)\subseteq f(E)$. Thus $f(E^\circ)$ is an open set contained in $f(E)$, and hence $f(E^\circ)\subseteq f(E)^\circ$.

Conversely, if G is an open set in X and $f(G^\circ)\subseteq f(G)^\circ$ for all $E\subseteq X$ then $f(G)=f(G^\circ)\ \subseteq f(G)^\circ$, and so $f(G)$ an open set in X^*.□

4.2.11 Theorem:

A function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ is closed iff $\overline{f(E)} \subseteq f(\bar{E})$ for every $E \subseteq X$.

Proof:

Suppose f is closed and $E \subseteq X$.Since $\bar{E}$ is closed set and f is closed function, then $f(\bar{E})$ is a closed set in X*. Since $E \subseteq \bar{E}$,$f(E) \subseteq f(\bar{E})$. Thus$f(\bar{E})$ is a closed set contain $f(E)$, and hence $\overline{f(E)} \subseteq f(\bar{E})$.

Conversely, if F is a closed set in X and $\overline{f(F)} \subseteq f(\bar{F})$ for all $F \subseteq X$ then $\overline{f(F)} \subseteq f(\bar{F}) = f(F)$, and so $f(F)$ closed set in X^*.□

4.2.12 Theorem:

Let $\mathcal{B}$ be a base for a topological space (X,τ).Show that if function $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ has the property that $f(B)$ is open for every $B \in \mathcal{B}$ then f is an open function.

Proof:

We want to show that the image of every open subset of X is open in X^*. Let $G \subseteq X$ be open. By definition of a base $G = \bigcup_i B_i$ where $B_i \in \mathcal{B}$.Now $f(G) = f(\bigcup_i B_i) = \bigcup_i f(B_i)$. By hypothesis, each $f(B_i)$ is open in $X^*$and so $f(G)$ a union of open sets in X^*, hence f is an open function.□

4.3 Homeomorphisms

4.3.1 Definition:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be function from a topological space (X,τ) to the topological space (X^*,τ^*), f is said to be a ***homeomorphism*** if it satisfy the following:

1. f is one to one.
2. f is onto.
3. f is an open function (i.e. f^{-1} is a continuous function)
4. f is a continuous function.

4.3.2 Remark:

If there exists a homeomorphism between (X,τ) and (X^*,τ^*),we say that X and $X^*$are ***homotopic*** or ***topologically equivalent*** denote by $X \cong X^*$.

4.3.3 Definition:

A property p of sets is called ***topological*** or a ***topological invariant*** if whenever a topological space (X,τ) has p then every space homeomorphic to (X,τ) also has p.

4.3.4 Example:

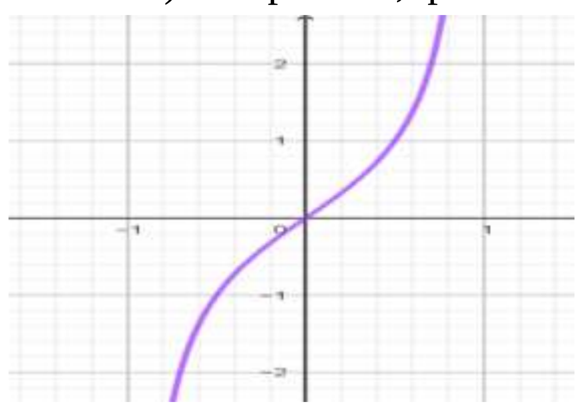
Let $X=\{a,b,c\}$,$X^*=\{1,2,3\}$,$\tau=\{X,\emptyset,\{a\},\{c\},\{a,c\}\}$ and $\tau^*=\{X^*,\emptyset,\{1\},\{3\},\{1,3\}\}$. Define $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ by $f(a)=1$,f(b)=2,$f(c)=3$. The function f is a homeomorphism since it is a bijection (1-1 and onto) on points ,open and continuous function.

4.3.5 Example:

Show that $X=(-1,1) \cong \mathbb{R}$.

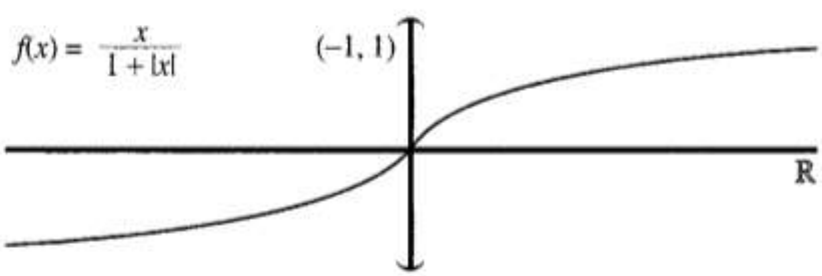
Solution:

Define $f:(-1,1) \longrightarrow \mathbb{R}$ by $f(x) = tan\frac{1}{2}\pi x$. f is one to one, onto, continuous function and open function. Hence $(-1,1) \cong \mathbb{R}$.



4.3.6 Remark:

1. We can use function $f:\mathbb{R} \longrightarrow (-1,1)$ by $f(x)=\frac{x}{1+|x|}$.From the graph of f is shown f is one to one, onto, continuous function and open function. Hence $(-1,1) \cong \mathbb{R}$.



2. Example 4.3.5 shows that the length and boundness is not homeomorphism since $\mathbb{R}$ is unbounded but $(-1,1)$ is bounded and its length is 2.

4.3.7 Remark:

Not only $(-1,1)$ is homomorphic to $\mathbb{R}$, but every nonempty open interval (a,b) is as well. Now consider the following collections of intervals with the usual topology (assume a and b are arbitrary real numbers with $a < b$):

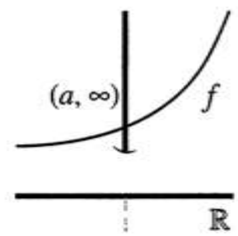
1) Open intervals (a,b),(-∞,a),(a,∞),$\mathbb{R}$.

2) Closed bounded intervals [a,b].

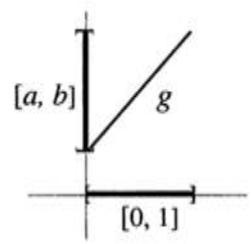
3) Half – open intervals and closed unbounded intervals $[a,b)$,(a,b],(-∞,a],[a,∞).

Each of the collections 1),2) and 3) all of the spaces are topologically equivalent.

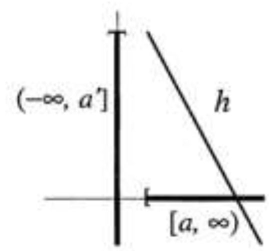
The function $f\colon \mathbb{R} \longrightarrow (a,\infty)$ defined by $f(x) = e^x + a$ is a homeomorphism. Thus $\mathbb{R}$ is homeomorphism to every interval (a,∞). Since topological equivalence is an equivalence relation, it also follow that every interval (a,∞) is homeomorphic to every other intervals in the form (a',∞).



The linear function $g : [0,1] \longrightarrow [a,b]$ given by $g(x) = (b-a)\,x + a$ is a homeomorphisms between $[0,1]$ and $[a,b]$. Therefore every interval $[a,b]$ is homeomorphic to $[0,1]$ and consequently every interval $[a,b]$ is homeomorphic to every other closed interval $[a',b']$ with $a' < b'$.



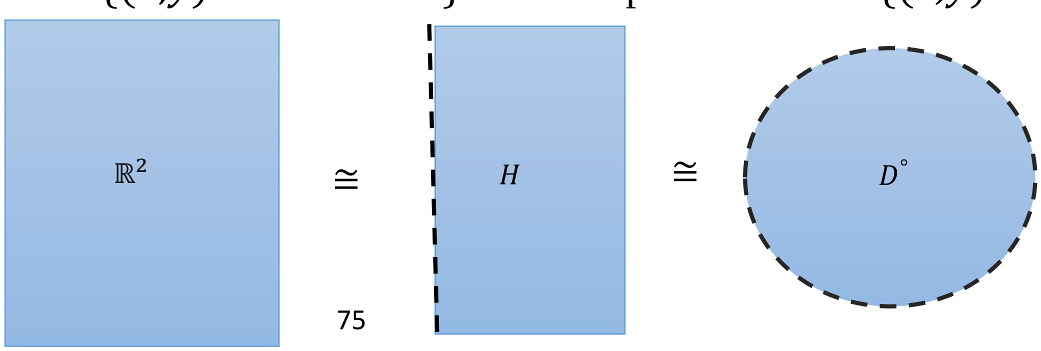
The function $h : [a,\infty) \longrightarrow (-\infty,a']$ given by $h(x) = -x + a' + a$ is a homeomorphism between intervals $[a,\infty)$ and $(-\infty,a']$. Thus if I_1 and I_2 are intervals of either form $[a,\infty)$ or $(-\infty,a']$.Then I_1 and I_2 are homotopic.



4.3.8 Example:

The usual topology on each ,the plane $\mathbb{R}^2$is topologically equivalent to the open right half plane $H = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and the open disk $D^\circ = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

Solution:



The function $f:\mathbb{R}^2 \longrightarrow H$, defined by f(x,y)=(e^x,y) is a homeomorphism between $\mathbb{R}^2$ and H. It maps $\mathbb{R}^2$ to H, sending vertical lines to vertical lines as followings:

1) The left half plane is mapped to the strip in H where $0 < x < 1$.

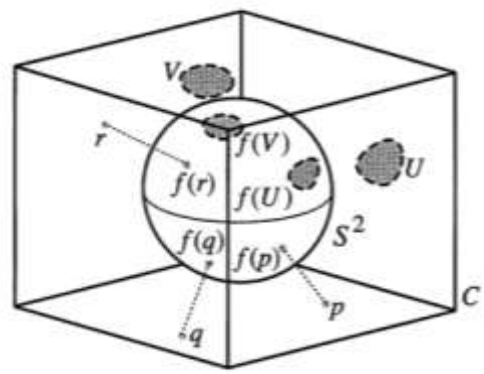
2) The y-axis is mapped to the line $x = 1$.

3) The right half plane is mapped to the region in H where $x > 1$.

The function $g:\mathbb{R}^2 \longrightarrow D^{\circ}$, defined by f(r,θ)=($\frac{r}{1+r}$,θ) is a homeomorphism between $\mathbb{R}^2$ and D°.It contracts the whole plane radially inwards to coincidence with the open disk D°.

<u>4.3.9 Example:</u>

The surface of cube C is homeomorphic to the sphere S^2. If we regard each as centered at the origin origin in 3- space the function $f: C \longrightarrow S^2$defined by $f(p) = \frac{p}{|p|}$ is a homeomorphism. f maps points in C bijectively to points in $S^2$and maps the collection of the open sets in C bijectively to the collection of open sets in S^2.



<u>4.3.10 Example:</u>

Let X be the set of positive real numbers ,i.e. $X = (0,\infty)$. The function $f: X \longrightarrow X$ defined by $f(x) = \frac{1}{x}$ is a homeomorphism from X to X.

<u>4.3.11 Remark:</u>

In example 4.3.10 if we take the cushy sequence $\langle a_n \rangle = \langle 1,\frac{1}{2},\frac{1}{3},\ldots \rangle$ then the corresponds$\langle f(a_n) \rangle = \langle f(1) = 1, f(\frac{1}{2})=2, f(\frac{1}{3})=3, \ldots \rangle$ under the homeomorphism is not a cushy sequence, hence the property of being a cushy sequence is not topological.

<u>4.3.12 Example:</u>

Show that area is not a topological property.

<u>Solution:</u>

1. The open disk $D = \{(r,\theta): r < 1\}$ with radius 1 is homeomorphism to the open $D^{\circ} = \{(r,\theta): r < 2\}$ with radius 2.The function $f: D \longrightarrow D^{\circ}$ defined by $f\big((r,\theta)\big)=(2r,\theta)$ is a homeomorphism. Here (r,θ) denotes the polar coordinates of a point in the plane $\mathbb{R}^2$ the area of D is $r^2\pi \neq 4r^2\pi$ the area of D°.

4.3.13 Remark:

1. From remarks 4.3.6 and 4.3.11and example 4.3.12 show that the length, boundness, area and cushy sequence are not homeomorphism.

2. Let (X,τ) and (X^*,τ^*) be discrete topological spaces then from examples 4.1.5 and 4.2.7 every bijective (one to one and onto) functions $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ are homeomorphism.

4.3.14 Example:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a one to one and open function , let $A \subset X$, and let $f(A) = B$.Show that the function $f_A:(A,\tau_A) \longrightarrow (B,\tau_B^*)$ is also one to one and open function. Here f_A denote the restriction of f to A and τ_A and τ_B^* are relative topologies.

Solution:

If f is one to one then every restriction of f is also one to one, hence we need only show that f_A is open.

Let $H \subset A$ be $\tau_A -$ open. Then by definition of the relative topology, $H^* = A \cap G$ where $G \in \tau$. Since f is one to one $f(A \cap G) = f(A) \cap f(G)$, and so

$$f_A(H) = f(H) = f(A \cap G) = f(A) \cap f(G) = B \cap f(G).$$

Since f is open and $G \in \tau$, $f(G) \in \tau_B^*$ then $B \cap f(G) = \tau_B^*$ and so $f_A$is open.

4.3.15 Example:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homomorphism and let (A,τ_A) be any subspace of (X,τ). Show that $f_A:(A,\tau_A) \longrightarrow (B,\tau_B^*)$ is also a homomorphism where f_A is the restriction of f to A ,$f(A) = B$, and τ_B^* is the relative topology on B.

Solution:

Since f is one to one and onto, $f_A:(A,\tau_A) \longrightarrow (B,\tau_B^*)$,where $f(A) = B$ is also one to one and onto. Hence we need only show that $f_A$is continuous and open function. By example 4.3.14 f_A is open and the restriction of any continuous function is also continuous hence f_A is a homeomorphism.

4.3.16 Theorem:

The perfect property is a topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a topological space (X,τ) to the topological space (X^*,τ^*) and $let\ E\ be\ a\ perfect$ (closed and dense in itself) subset of X, we want to prove that $f(E)$ is perfect subset of X^*.

By theorem 4.1.34 $f(E)$ is dense itself. Since E is closed subset of X then $E^c$is open in X. Since f is open function then $f(E^c)$ is open set in X*.Since f is bijective then $f(E^c) = f(E)^{\ c}$, so $f(E)$ is closed in X*, i.e. perfect set in X*.□

4.3.17 Theorem:

The locally compact set property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a topological space (X,τ) to the topological space (X*,τ*) and let E be a locally compact set in X we want to prove that $f(E)$ is a locally compact subset of X*.

Let $x^* \in f(E)$, since f is onto then $\exists x \in E,$ s.t. $f(x) = x^*$.Since E is locally compact set in X then there exists a compact neighborhood G for x. Since f is open function and G is compact then $f(G)$ is a compact neighborhood for x^* in $f(E)$, so $f(E)$ is a locally compact subset of X*.□

4.3.18 Definition:

A subset E of a topological space is ***isolated*** iff no point of E is a limit point of E that is, if $E \cap d(E) = \emptyset$.

4.3.19 Example:

Let $X = \{a$,b,c,d,$e\}$ and τ = {∅,X,{a},{a,b},{a,c,d},{a,b,c,d},{a,b,e}} then $E = \{c,e\}$ is isolated set since $d(E) = \{d\}$ and $E \cap d(E) = \emptyset$.

4.3.20 Theorem:

The isolated property is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be a homeomorphism from a topological space (X,τ) to the topological space (X*,τ*) and let E is isolated set in X we want to prove that $f(E)$ is isolated subset of X*.

Let $x^* \in f(E)$, since f is onto then $\exists x \in E$, s.t. $f(x) = x^*$.Since E is isolated then $x \notin d(E)$ then there exists an open set G containing x such that $G/\{x\} \cap E = \emptyset$. But f is a homeomorphism, and so f(G) is an open set in X*which contains $f(x) = x^*$. From the fact that f is one-to-one it follows that $f(E) \cap f(G)/\{x^*\} = \emptyset$, i.e $x^* \notin d(f(E))$.□

4.3.21 Theorem:

The countably compact property is a topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a topological space (X,τ) to the topological space (X^*,τ^*) and let E is countably compact set in X we want to prove that $f(E)$ is countably compact subset of X^*.

Assume that A^* be infinite subset of $f(E)$. Since f is bijective then there exists an infinite subset of E such that $f(A) = A^*$.Since A is countably compact set then it has a limit point x in E ($x \in E$,$x \in d(E)$).

Since f is open and one to one function then $x^* = f(x) \in f(E)$,$x^* \in d(f(A))$,so $A^* = f(A)$ has a limit point in $f(E)$, i.e. $f(E)$ is countably compact. □

4.3.22 Theorem:

The locally connected property is a topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a topological space (X,τ) to the topological space (X^*,τ^*) and let E is locally connected set in X we want to prove that $f(E)$ is locally connected subset of X^*.

Let $x^* \in f(E)$ and G^*open subset of $f(E)$ contain x^*.Since f is onto then $\exists x \in E$, s.t. $f(x) = x^*$, so $x \in f^{-1}(G^*)$. Since f is continuous then $f^{-1}(G^*)$ is open subset of E. Since $E = f^{-1}(f(E)) \subseteq f^{-1}(G^*)$, by theorem 4.1.4.

Since E is locally connected and $x \in f^{-1}(G^*) \subseteq E$ then there exists an open connected G such that $x \in G \subseteq f^{-1}(G^*)$, so by theorem 4.1.4 we get $f(x) \in f(G) \subseteq f(f^{-1}(G^*)) \subset G^*$.Since f is onto and $f(G)$ is connected by theorem 4.1.22,so $f(E)$ is locally connected. □

4.4 Hereditary Properties

4.4.1 Definition:

A property P of a topological space (X,τ) is said to be ***hereditary*** iff every subspace of X also possesses property P.

4.4.2 Example:

A property of being a topological space a discrete topological spaces is a hereditary property.

Solution:

Let (Y,τ_Y) be a subspace of a discrete topological space (X,τ) we want to show that (Y,τ_Y) is also a discrete topological space.

Let $A^* \subset Y \subset X$ and let $A^* = A \cap Y$. Now $A \subset X$ and X is a discrete topology then $A \in \tau$ .Since (Y,τ_Y) is a subspace of (X,τ) then $A^* \in \tau_Y$, i.e. (Y,τ_Y) is a discrete topological space.

4.4.3 Example:

A property of being a topological space an indiscrete topological spaces is a hereditary property.

Solution:

Let (Y,τ_Y) be a subspace of an indiscrete topological space (X,τ) then (Y,τ_Y) is also an indiscrete topological space, since the only open sets in X are $X,\emptyset$ and their intersect with Y are $Y,\emptyset$.

4.4.4 Definition:

A subset ***E*** of a topological space $(\boldsymbol{X},\boldsymbol{\tau})$ will be called ***dense*** in ***X*** iff $\bar{E} = X$.

4.4.5 Example:

Consider the topology $\tau = \{\emptyset, X, \{b,c,d,e\}, \{a,b,e\}, \{b,e\}, \{a\}, X\}$ on $X = \{a,b,c,d,e\}$ then $\{a,c\}$ is a dense subset of X ,since $\overline{\{a,c\}} = X$ but $\{b,d\}$ is not dense since $\overline{\{b,d\}} = \{b,c,d,e\}$.

4.4.6 Example:

The usual topology $(\mathbb{R},\tau)$ the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, since $\overline{\mathbb{Q}} = \mathbb{R}$.

4.4.7 Example:

Let (X,τ) be the discrete topology then X is the only dense set in X, Since every $A \subset X$, A is closed and $\bar{A} = A$.

4.4.8 Definition:

A topological space (X,τ) will be called ***separable*** iff it satisfies the following condition:

[S] There exists a countable dense subset of X**.**

4.4.9 Example:

In example 4.4.6 we show that ℚ is dense in the usual topology (ℝ,τ) and since ℚ is countable then ℝ is a separable space.

4.4.10 Example:

Let (X,τ) be the co-finite topology. Show that (X,τ) is separable, i.e. contains a countable dense subset.

Solution:

If X is countable then X is a countable dense subset of (X,τ). On the other hand, suppose X is not countable then X contains a non-finite countable subset A. Since the closed sets in X are the finite sets then the closure of the non-finite set A is the space X, i.e. $\bar{A} = X$. Gut A is countable hence (X,τ) is separable.

4.4.11 Example:

Let (ℝ,τ) be the discrete topology. Since every subset of ℝ is both open and closed so the only dense subset of ℝ is ℝ itself. But ℝ is not countable set, hence (ℝ,τ) is not a separable space.

4.4.12 Example:

A discrete topological space (X,τ) is separable iff X is countable.

Solution:

Since every subset of a discrete topological space (X,τ) is both open and closed then the only subset of X is X itself. Hence X contains a countable dense subset iff X is countable, i.e. X is separable iff X is countable.

4.4.13 Example:

Let τ be the topology on the real line $\mathbb{R}^2$ generated by the half- open rectangles, [a,b)×[c,d)={(x,y):a≤x<b,c≤y $< d$}. Show that ($\mathbb{R}^2$,τ) is seperable.

Solution:

Now there are always rational numbers x_0 and y_0 such that $a < x_0 < b$ and $c < y_0 < d$, so the above open rectangle contains the point $p = (x_0\ ,y_0)$ with rational coordinates. Hence the set $A = \mathbb{Q} \times \mathbb{Q}$ consisting of all points in $\mathbb{R}^2$with rational coordinates is dense in $\mathbb{R}^2$.But A is a countable set thus $(\mathbb{R}^2,\tau)$ is seperable.

<u>4.4.14 Theorem:</u>

The separable property is a topological property.

<u>Proof:</u>

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a separable topological space (X,τ) to the topological space (X^*,τ^*), we want to prove that X^* is separable space.

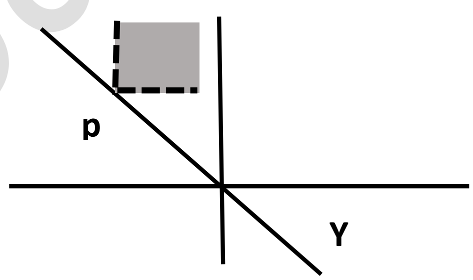
Since X is a separable then there exists a countable A subset of X such that $\bar{A} = X$. Now since f is a homeomorphism then $f(A)$ is countable subset of $X^*$and $X^* = f(X) = f(\bar{A}) \subseteq \overline{f(A)}$.So $X^* = \overline{f(A)}$, i.e. $f(A)$ is dense in X^*, i.e. (X^*,τ^*) is separable space.□

<u>4.4.15 Example:</u>

Show that by a counterexample that a subspace of a separable space need not be separable, i.e. reparability is not a hereditary property.

<u>Solution:</u>

Consider the separable topological space $(\mathbb{R}^2,\tau)$ in example 4.4.13 and let $Y = \{(x,y): x + y = 0\}$ be a subset of $(\mathbb{R}^2,\tau)$ then τ_Y the relative topology is the discrete topology since each singleton $\{p\}$ of Y is τ_Y- open. But an uncountable space is not separable. Thus the reparability of $(\mathbb{R}^2,\tau)$ is not inherited by the subspace (Y,τ_Y).



<u>4.4.16 Example:</u>

Show that by a counterexample that a subspace of compact space need not be compact, i.e. compactness is not a hereditary property.

<u>Solution:</u>

The closed interval $[0,1]$ is compact subset in the usual topology $(\mathbb{R},\tau)$ since its closed and bounded (by Hein Boral theorem) but the subset $(0,1)$ of $[0,1]$ is not compact. Thus compactness is not a hereditary property.

Chapter Five

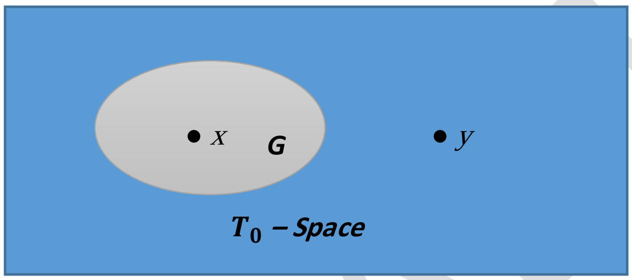
Separation Axioms

5.1 T_0 - Space

5.1.1 Definition:

A topological space (X,τ) is called $\boldsymbol{T_0}$ **– *Space*** iff it satisfies the following axiom of Kolomogorov:

[T_0] If x and y are two distinct points of X, then there exists an open set which contains one of them but not the other, $\forall x , y \in X, x \neq y$, $\exists G \in \tau$, s. t. $x \in G , y \notin G$.



5.1.2 Example:

Let $X = \{a,b\}$, $\tau = \{\{X,\emptyset,\{a\}\}$ then (X,τ) is T_0 – Space ,since a , b ∈ X , a ≠ b , $\exists\{a\} \in \tau$, s. t. $x \in \{a\} , y \notin \{a\}$.

5.1.3 Example:

Let $X = \{a,b,c\}$, $\tau = \{\{X,\emptyset,\{a,b\}\}$ then (X,τ) is not T_0 – Space ,since a , b ∈ X , a ≠ b , every open set contain a contain b.

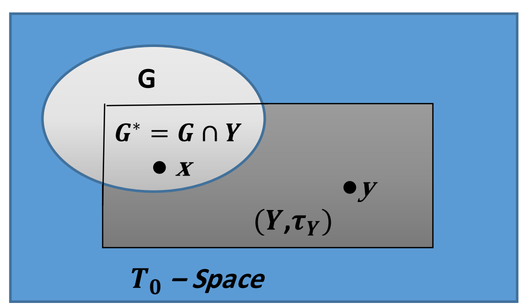
5.1.4 Theorem:

T_0 – Space is a hereditary property.

Proof:

Let (Y,τ_Y) be a subspace of a T_0 – Space(X,τ). We want to prove that (Y,τ_Y) is T_0 – Space.

Let $x, y \in Y, x \neq y$. Since $Y \subset X$ then $x, y \in X$ but X is T_0 – Space then $\exists G \in \tau$, s.t. $x \in G, y \notin G$. Let $G^* = G \cap Y$ then $x \in G^*$(since $x \in G$, $x \in Y$) But $y \notin G^*$(since $y \notin G$, $y \in Y$) , so (Y,τ_Y) is T_0 – Space. □



Exercise:

Prove that T_0 – Space is a topological property.

5.1.5 Theorem:

A topological space (X,τ) is called T_0 – Space iff the closures of distinct points are distinct.

Proof:

$\Longrightarrow$

Suppose that $x \neq y$ implies that $\overline{\{x\}} \neq \overline{\{y\}}$ and that x and y are distinct points of X. Since the sets $\overline{\{x\}}$ and $\overline{\{y\}}$ are not equal, there must exist some point $z \in X$ which is contained in one of them but not the other.

Suppose that $z \in \overline{\{x\}}$ but $z \notin \overline{\{y\}}$. If we had $x \in \overline{\{y\}}$, then we would have $\overline{\{x\}} \subseteq \overline{\overline{\{y\}}} = \overline{\{y\}}$ and so $z \in \overline{\{x\}} \subseteq \overline{\{y\}}$, which is a contradiction. Hence $x \notin \overline{\{y\}}$ and so $\overline{\{y\}}^c$ is an open set containing x but not y.

$\Longleftarrow$

Let us suppose that X is a T_0 – Space, and that x and y are two distinct points of X. By $[T_0]$, there exists an open set G containing one of them but not the other.

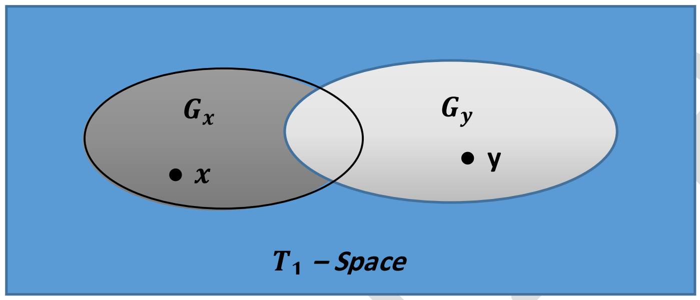
Suppose that $x \in G$ but $y \notin G$. Clearly, G^c is a closed set containing y but not x. From the definition of $\overline{\{y\}}$ as the intersection of all closed sets containing $\{y\}$ we see that $y \in \overline{\{y\}}$, but $x \notin \overline{\{y\}}$ because of G^c. Hence, $\overline{\{x\}} \neq \overline{\{y\}}$. □

5.2 T_1 - Space

5.2.1 Definition:

A topological space (X,τ) is called ***T_1 – Space*** iff it satisfies the following axiom of Fréchet:

[T_1] If x and y are two distinct points of X, then there exists two open sets one containing x not y, and the other containing y but not x, i.e. $\forall x , y \in X, x \neq y$, $\exists G_x , G_y \in \tau$, s. t. $x \in G_x , y \notin G_x$ and $y \in G_y , x \notin G_y$.



5.2.2 Example:

Let $X = \{a,b\}$, $\tau = \{\{X, \emptyset , \{a\} , \{b\}\}$ then (X,τ) is T_1 – Space ,since a , b ∈ X , a ≠ b , $\exists\{a\} , \{b\} \in \tau$, s. t. $a \in \{a\}$,$b \notin \{a\}$ and $b \in \{b\}$, $a \notin \{b\}$.

5.2.3 Remark:

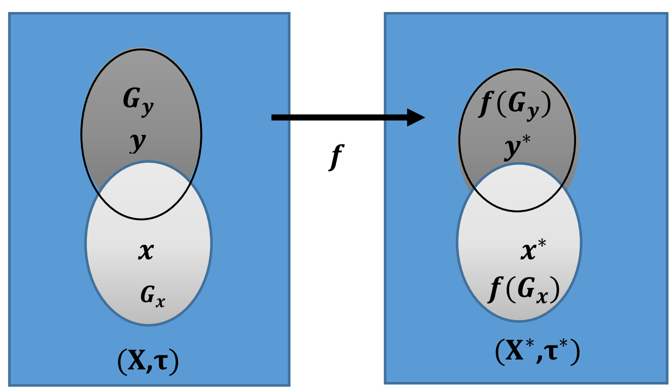
Every T_1 – Space is obviously a T_0 – Space, the converse is not true as the following example:

5.2.4 Example:

Let $X = \{a,b\}$, $\tau = \{\{X,\emptyset,\{a\}\}$ then (X,τ) is T_0 – Space not T_1 – Space, since X is the only open set contain a and b.

5.2.5 Theorem:

T_1 – Space is a topological property.



Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be A homeomorphism from a T_1 – Space (X,τ) to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) is T_1 – Space .

Let $x^*,y^* \in X^*$, $x^* \neq y^*$.Since f is onto then $\exists x,y \in X$, s. t. $f(x) = x^*, f(y) = y^*$. Since f is 1-1 and $x^* \neq y^*$then $x \neq y$. Since (X,τ) is T_1 – Space then $\exists G_x , G_y \in \tau$, s. t. $x \in G_x$, $y \notin G_x$ and $y \in G_y , x \notin G_y$, so $x^* \in f(G_x)$,$y^* \notin f(G_x)$ and $y^* \in f\left(G_y\right)$,$x^* \notin f\left(G_y\right)$. Since f is open function then $f(G_x)$,$f\left(G_y\right) \in \tau^*$, $x^* \in f(G_x)$,$y^* \in f\left(G_y\right)$. So (X^*,τ^*) is T_1 – Space. □

Exercise:

Prove that T_1 – Space is a hereditary property.

5.2.6 Theorem:

A topological space (X,τ) is called T_1 – Space iff every singleton is closed.

Proof:

$\Longrightarrow$

If x and y are distinct points of a space X in which subsets consisting of exactly one point are closed, then $\{x\}^c$ is an open set containing y but not x, while $\{y\}^c$ is an open set containing x but not y. Thus (X,τ) is a T_1 – Space.

$\Longleftarrow$

Suppose that (X,τ) is a T_1 – Space, and that x is a point of X. By **[T_1]** if $y \neq x$, there exists an open set G_y containing y but not x, that is, $y \in G_y \subseteq \{x\}^c$. But then $\{x\}^c = \bigcup\{G_y : y \neq x\}$ and so $\{x\}^c$ is the union of open sets, and hence is itself open. Thus $\{x\}$ is a closed set for every $x \in X$.□

5.2.7 Example:

Let $X = \mathbb{N}$ the set of positive integers, and let τ be the family consisting of $\emptyset$, X and all subsets of the form $\{1 , 2 , ... , n\}$ then $(\mathbb{N},\tau)$ is not a T_1 – Space, since $\forall n \in \mathbb{N}, \{n\}$ is not a closed set (Note that $(\mathbb{N},\tau)$ is a T_0 – Space).

5.2.8 Example:

Let $X = \mathbb{R}$ the set of real numbers, and let τ be the family consisting of $\emptyset$ and all subsets of $\mathbb{R}$ whose complement is finite then $(\mathbb{R},\tau)$ is a T_1 – Space, since $\forall p \in \mathbb{R}, \{p\}$ is a closed set.

5.2.9 Theorem:

In a T_1 – Space (X,τ), a point x is a limit point of a set E iff every open set containing x contains an infinite number of distinct points of E.

Proof:

$\Longrightarrow$

The sufficiency of the condition is obvious, since if G is an open set containing x and $G \cap E$ contains an infinite number of distinct points of E, i.e. $G \cap E/\{x\} \neq \emptyset$. So that $x \in d(E)$.

$\Longleftarrow$

To prove the necessity, suppose there were an open set G containing x for which $G \cap E$ was finite. If we let $G \cap E/\{x\} = \bigcup_{i=1}^{n}\{x_i\}$, then each set $\{x_i\}$ would be closed by the above theorem, and the finite union $\bigcup_{i=1}^{n}\{x_i\}$ would also be a closed set. But then $(\bigcup_{i=1}^{n}\{x_i\})^c \cap G$ would be an open set containing x with $((\bigcup_{i=1}^{n}\{x_i\})^c \cap G) \cap E/\{x\} = ((\bigcup_{i=1}^{n}\{x_i\})^c \cap \bigcup_{i=1}^{n}\{x_i\}) = \emptyset$.Thus x would not be a limit point of E.□

5.2.10 Corollary:

The finite subset of T_1 – Space (X,τ) has no limit point.

Proof:

Suppose A be a finite subset of X. If A has a limit point $x \in X$ (i.e. $x \in d(E)$) then by theorem 5.2.9 every open set G containing x contains infinite number of A but A is finite set and this contradiction, so A has no limit points. □

5.2.11 Remark:

Countably compact spaces are more useful in T_1 – Spaces, since we may then characterize them in a way that is exactly analogous to that for compact spaces. The following theorem, in fact, explains why we chose the name "countably compact."

5.2.12 Theorem:

A T_1 – Space (X,τ) is countably compact iff every countable open covering of X is reducible to a finite subcover.

Proof:

$\Longrightarrow$

Suppose $\{G_n\}_{n\in\mathbb{N}}$ is a countable open covering of the countably compact space X which has no finite subcover. This means that $\bigcup_{i=1}^{n} G_i$ does not contain X for any $n \in \mathbb{N}$. If we let $F_n = (\bigcup_{i=1}^{n} G_i)^c$, then each F_n is a nonempty closed set contained in the preceding one. From each F_n let us choose a point x_n , and let $E=\bigcup_{n\in\mathbb{N}}\{x_n\}$. The set E cannot be finite because there would then be some point in an infinite number, and hence all of the sets F_n , and this would contradict the fact that the family $\{G_n\}_{n\in\mathbb{N}}$ is a covering of X. Since E must be infinite, we may use the countable compactness of X to obtain a limit point x of E.

By theorem 5.2.9, every open set containing x contains an infinite number of points of E. and so x must be a limit point of each of the sets $E_n = \bigcup_{i>n}\{x_i\}$. For each n, however, E_n is contained in the closed set F_n , and so x must belong to F_n for every $n \in \mathbb{N}$. This again contradicts the fact that the family $\{G_n\}_{n\in\mathbb{N}}$ is a covering of $\boldsymbol{X}$. Hence the condition is necessary.

$\Leftarrow$

Now let us suppose that E is an infinite subset of X and that E has no limit points. Since E is infinite, we may choose an infinite sequence of distinct points x_n from E. The set $A=\bigcup_{n\in\mathbb{N}}\{x_n\}$ has no limit points since it is a subset of E, and so, in particular, each point x_n is not a limit point of A**.** This means that for every $n \in \mathbb{N}$ there exists an open set G_n containing x_n such that $A \cap G_n/\{x_n\} = \emptyset$ **.** From the definition of A we see that $A \cap G_n = \{x_n\}$ for every $n \in \mathbb{N}$**.** Since A has no limit points, it is a closed set, and hence A^c is open. The collection $A^c \cup \{G_n\}_{n\in\mathbb{N}}$ is then a countable open covering of X which has no finite subcover, since the set G_n is needed to cover the point x_n for every $n \in \mathbb{N}$. Thus, the condition is sufficient. □

5.2.13 Corollary:

A T_1 – Space (X,τ) is countably compact iff every countable family of closed sets having the finite intersection property has a nonempty intersection.

5.2.14 Example:

Every finite T_1 – *Space* has the discrete topology.

Solution:

Let (X,τ) be a finite T_1 – Space, so every subset of X is finite, i.e. equal a union of finite numbers of singleton and therefore closed. Hence every subset of X is also open, i.e. X is a discrete topology.

5.2.15 Remark:

Although countable compactness is a topological property, we noted from remark 4.1.32 that it may not be preserved by continuous mappings. With the aid of one-to-oneness, we may show that it is preserved by continuous mappings of T_1 – Spaces .

5.2.16 Theorem:

If f is a continuous mapping of the T_1 – Space (X,τ) into the topological space (X^*,τ^*), then f maps every countably compact subset of X onto a countably compact subset of X^*.

Proof:

Suppose E is a countably compact subset of X and $\{G_n^*\}_{n\in\mathbb{N}}$ is a countable open covering of $f(E)$. We need only show that there is a finite subcovering of $f(E)$, since we noted above that the condition of theorem 5.2.12 is always sufficient. Since f is continuous, $\{f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ is a countable open covering of E**.** In the induced topology, $\{E \cap f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ is a countable open covering of the countably compact T_1 – Space ***E.*** By theorem 5.2.12, there exists some finite subcovering $\left\{E \cap f^{-1}(G_{n_i}^*)\right\}_{i=1}^{k}$, and clearly the family $\left\{G_{n_i}^*\right\}_{i=1}^{k}$ is the desired finite subcovering of $f(E)$**.** □

<u>5.2.17 Example:</u>

Let (X,τ) be a T_1 – Space and let $\mathcal{B}_p$ be a local base at $p \in X$. Show that if $q \in X$ distinct from p then some member of $\mathcal{B}$ does not contain q.

<u>Solution:</u>

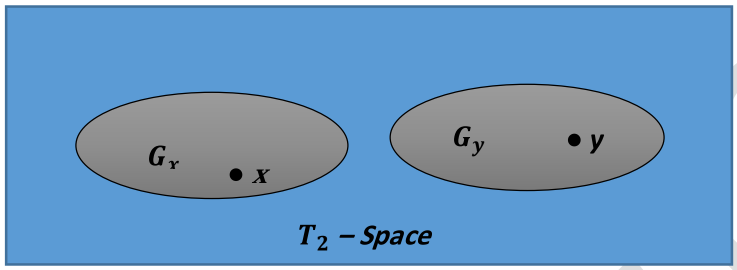
Since $p \neq q$ and X satisfies $[T_1]$,∃ an open set $G \subset X$ consisting p but not q. Now $\mathcal{B}_p$ is a local base at p, so G is contain of some $B \in \mathcal{B}_p$ and Balso does not contain q.

5.3 T_2 - Space

5.3.1 Definition:

A topological space (X,τ) is called $\boldsymbol{T_2}$ **– *Space* or Hausdorff space** iff it satisfies the following axiom of Hausdorff:

[T_2] If x and y are two distinct points of X, then there exists two disjoint open sets one containing x and the other containing y . $\forall x\ ,y \in X, x \neq y$, $\exists G_x$, $G_y \in$ τ, s. t. $x \in G_x$ and y$\in G_y$,$G_x \cap G_y = \emptyset$.



5.3.2 Example:

Let $X = \{a,b\}$, $\tau = \{\{X,\emptyset,\{a\},\{b\}\}$ then (X,τ) is T_2 – Space, a,b∈X,a $\neq b$,∃{a},{b}∈τ and {a}∩{b}=∅, s. t. a∈{a} , b∈{b}.

5.3.3 Remark:

From definition of T_2 – Space we get

$$T_2 - \text{Space} \;\underset{\nLeftarrow}{\Rightarrow}\; T_1 - \text{Space} \;\underset{\nLeftarrow}{\Rightarrow}\; T_0 - \text{Space}$$

5.3.4 Example:

Let (X,τ) be the co-finite topology then (X,τ) is T_1 – Space not T_2 – Space.

Solution:

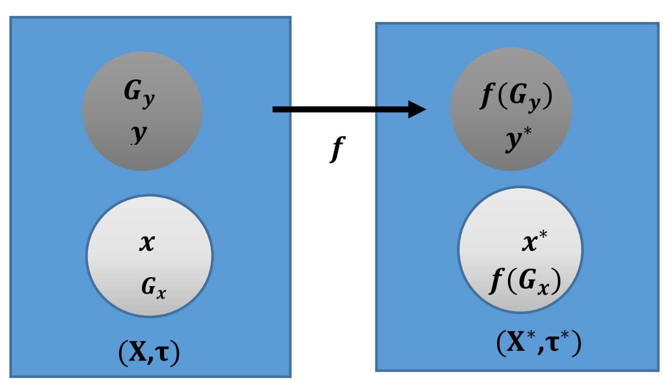
If $G,H \in \tau$ then G^c,$H^c$are finite sets. If $H \cap G = \emptyset$ then $G \subseteq H^c$and this is contradiction ,since $H^c$is finite set and G is infinite set. Then $H \cap G \neq \emptyset$.So (X,τ) is not T_2 – Space.

5.3.5 Theorem:

T_2 – Space is a topological property.

Proof:

Let f: (X,τ) ⟶ (X*,τ*) be A homeomorphism from a T_2 – Space (X,τ) to the topological space (X*,τ*),we want to show

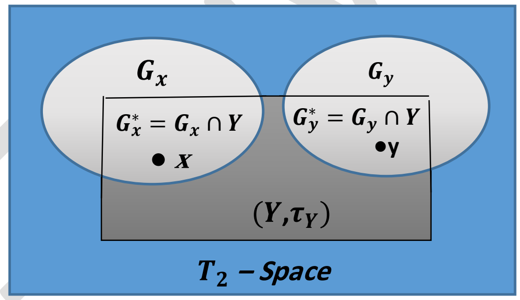


that (X^*,τ^*) is T_2 – Space .

Let $x^*,y^* \in X^*$, $x^* \neq y^*$.Since f is onto then $\exists x$,y∈X, s.t. $f(x)=x^*$,$f(y)=y^*$. Since f is 1-1 and $x^* \neq y^*$then $x \neq y$. Since (X,τ) is T_2 – Space then $\exists G_x$, $G_y \in \tau$,$G_x \cap G_y = \emptyset$, s. t. $x \in G_x$, y∈ G_y . Since f is open function then $f(G_x)$,$f(G_y) \in \tau^*$. Since f is 1-1 and $G_x \cap G_y = \emptyset$ then $f(G_x) \cap f(G_y) = \emptyset$.Since $x \in G_x$, y∈ G_y then $x^* \in f(G_x)$,$y^* \in f(G_y)$. So (X^*,τ^*) is T_2 – Space.□

5.3.6 Theorem:

T_2 – Space is a hereditary property.



Proof:

Let (Y,τ_Y) be a subspace of a T_2 – Space(X,τ). We want to prove that (Y,τ_Y) is T_2 – Space.

Let $x, y \in Y$, $x \neq y$.Since $Y \subset X$ then $x, y \in X$ but X is T_2 – Space then $\exists$ G_x , $G_y \in \tau$, $G_x \cap G_y = \emptyset$, s. t. $x \in G_x$, y∈ G_y.By definition of subspace let $G_x^* = G_x \cap Y$, $G_y^* = G_y \cap Y$ are τ_Y – open sets. Furthermore $x \in G_x^*$(since $x \in G_x$, $x \in Y$), $y \notin G_y^*$(since y $\notin G_y$, y $\in Y$) and and $G_x \cap G_y = \emptyset$ then$(G_x \cap Y) \cap (G_y \cap Y) = (G_x \cap G_y) \cap Y = \emptyset \cap Y = \emptyset$.So (Y,τ_Y) is T_2 – Space. □

5.3.7 Remark:

Compact sets are more useful in T_2 – Spaces since we may prove a part of the Heine-Borel Theorem which does not hold in general topological spaces.

5.3.8 Theorem:

Every compact subset E of a Hausdorff space X is closed.

Proof:

Let x be a fixed point in E^c. By $[T_2]$, for each point $y \in E$,there exist two disjoint open sets G_x and G_y such that $x \in G_x$ and y∈ G_y.The family of sets $\{G_y : y \in E\}$ is an open covering of E. Since E is compact, there must be some finite subcovering $\{G_{y_i}\}_{i=1}^{n}$. Let $\{G_{y_i}\}_{i=1}^{n}$be the corresponding open sets containing x, and let $G = \bigcap_{i=1}^{n} G_{x_i}$.Then G is an open set containing x since it is the intersection of a finite number of open sets containing x. Furthermore, we see that $G = \bigcap_{i=1}^{n} G_{x_i} \subseteq \bigcap_{i=1}^{n} {G_{y_i}}^{c} = \left(\bigcup_{i=1}^{n} G_{y_i}\right)^{c} \subseteq E^c$. Thus each point in E^c is contained in an open set which is itself contained in E^c.Hence E^c is an open set, and so E must be closed.□

5.3.9 Corollary:

If f is a one-to-one continuous mapping of the compact topological space (X,τ) onto the T_2 – Space (X^*,τ^*), then f is also open, and so f is a homeomorphism.

Proof:

Let G be open in X, so that G^c is closed. By theorem 3.5.6, G^c is compact. By theorem 4.1.23 $f(G^c)$ is compact. By theorem 5.3.8, $f(G^c)$ is closed. Thus $\left(f(G^c)\right)^c$ is open. Since f is one-to-one and onto, $\left(f(G^c)\right)^c = f(G)$ which is open.□

5.3.10 Theorem:

Every metric space is T_2 – Space (Hausdorff space).

Proof:

Let $a,b \in X$ be distinct points $d(a,b) = \varepsilon > 0$. Consider the open spheres $G = B_{\frac{1}{3}\varepsilon}(a)$ and $H = B_{\frac{1}{3}\varepsilon}(b)$ centered at a and b respectively.

We claim that $G \cap H = \emptyset$ if not then $\exists x \in G \cap H$ s.t. $d(a,x) = \frac{1}{3}\varepsilon$ and $d(x,b) = \frac{1}{3}\varepsilon$ hence by Triangle Inequality, $d(a,b) \le d(a,x) + d(x,b) < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \frac{2}{3}\varepsilon$ but this is contradicts the fact that $d(a,b) = \varepsilon$. Hence G and H are disjoint, i.e. a and b belong respectively to the disjoint open spheres G and H. So X is Hausdorff space. □

5.3.11 Remark:

The following theorem shows in T_2 – Space we can separate a point from compact set by using open sets.

5.3.12 Theorem:

In T_2 – Space we can separate any point and compact subset not contain the point by disjoint open sets.

Proof:

Let (X,τ) be a T_2 – Space ,F compact subset of X ,$x \in X$and $x \notin F$.Let $y \in F$ then $y \neq x$. Since (X,τ) is T_2 – Space then $\exists G_x$, $H_y \in \tau$, s. t. $x \in G_x$ and y$\in H_y$,$G_x \cap H_y = \emptyset$.

The family $\{H_y : y \in F\}$ is an open cover for F. Since F is compact then there exist $\left\{H_{y_i}\right\}_{i=1}^{n}$finite subcover for F corresponding $\{G_i\}_{i=1}^{n}$family of finite open sets contain x.Let $H = \bigcup_{i=1}^{n} H_{y_i}$,$G = \bigcap_{i=1}^{n} G_i$, i.e. $x \in G$,$F \subseteq H$ and $G \cap H = \emptyset$.□

5.3.13 Remark:

Since the notion of a convergent sequence of real numbers plays such a basic role in the study of the real number system, we might expect that the equivalent notion for topological spaces would be as primitive a concept as the closure. Although convergence has been used as the primitive notion for abstract spaces, we will see below that some of the natural properties fail to hold in more general spaces than Hausdorff spaces.

5.3.14 Definition:

Let (X,τ) be a topological space and let $\langle x_n \rangle$ be a sequence in X. We say that $\langle x_n \rangle$ converge in X if $\exists x \in X$ (denote by $x_n \to x$) such that

for every open set G contain x , $\exists k \in \mathbb{N}$, s.t. $x_n \in G, \forall n > k$.

5.3.15 Example:

Let $\langle a_1, a_2, \ldots \rangle$ be a sequence of points in an indiscrete topological space (X,τ). Since X is only open set containing any point $b \in X$ and X contains every term of the sequence $\langle a_n \rangle$, so the sequence $\langle a_1, a_2, \ldots \rangle$ converge to every point of $b \in X$.

5.3.16 Example:

Let $\langle a_1, a_2, \ldots \rangle$ be a sequence of points in a discrete topological space (X,τ).Since $\forall b \in X$ the singleton set $\{b\}$ is an open set contain b , so if $a_n \to b$ then the set $\{b\}$ must contain almost all of the terms of the sequence. In other words the sequence $\langle a_n \rangle$ converges to a point $b \in X$ iff the sequence is of the form $\langle a_1, a_2, \ldots, a_{n_0}, b, b, b, \ldots \rangle$.

5.3.17 Example:

Let τ be the topology on an infinite set X which consists of $\emptyset$ and the complements of countable sets . A sequence $\langle a_1, a_2, \ldots \rangle$ in X converges to $b \in X$ iff the sequence is also of the form $\langle a_1, a_2, \ldots, a_{n_0}, b, b, b, \ldots \rangle$, i.e. the set A consisting of the terms of $\langle a_n \rangle$ different from b is finite .Now A is countable and so A^c is an open set containing b. Hence if $a_n \to b$ then A^c contains all except a finite number of the terms of the sequence ,so A is finite

5.3.18 Remark:

It is the failure of limits of sequences to be unique that makes this concept unsatisfactory in general topological spaces. The following example shows that a T_0 – Space in which limits of sequences need not be unique.

5.3.19 Example:

Let $X = \mathbb{N}$, and let τ be the family consisting of $\emptyset$, X, and all subsets of the form $\{n, n+1, n+2, \ldots\}$ then $(\mathbb{N},\tau)$ is $T_0 - Space$ not $T_2 - Space$,(since if $n_1, n_2 \in \mathbb{N}$. $n_1 \neq n_2$ with $n_2 < n_1$ then there exists $\{n_1, n_1+1, \ldots\}$ contain n_1 not n_2 if $n_1 < n_2$ then there exists $\{n_2, n_2+1, \ldots\}$ contain n_2 not n_1) but the sequence $< a_n = n >$ for which converges to every point of that space, i.e. $< n >$ converge to ,2,3,… .

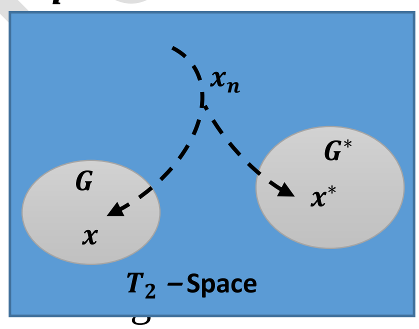
5.3.20 Remark:

The following theorem shows that this anomalous behavior cannot occur in a Hausdorff space.

5.3.21 Theorem:

In a Hausdorff space, a convergent sequence has a unique limit.

Proof:



Suppose a sequence $\langle x_n \rangle$ converged to two distinct points x and x^* in a Hausdorff space X. By $[T_2]$, there exist two disjoint open sets G and G^* such that $x \in G$ and $x^* \in G^*$. Since $x_n \to x$, there exists an integer k such that $x_n \in G$ whenever $n > k$. Since $x_n \to x^*$ there exists $x_n \in G^*$ whenever $n > k^*$. If m is any integer greater than both k and k^* , then x_m must be in both G and G^*,which contradicts the fact that G and $G^*$are disjoint.□

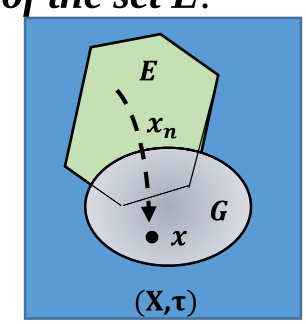
5.3.22 Remark:

1. The converse of theorem 5.3.21 is not true. An example of a non-Hausdorff space in which every convergent sequence has not unique limit was given in example 5.3.19 .
2. A relationship between the limit points of sets and the limit points of sequences of points is given in the following theorem.

5.3.23 Theorem:

If $\langle x_n \rangle$ is a sequence of distinct points of a subset E of a topological space (X,τ) which converges to a point $x \in X$ then x is a limit point of the set E.

Proof:



If x belongs to an open set G,then there exists an integer k such that $x_n \in G$ for all $n > k$. Since the points x_n are distinct, at most one of them equals x and so $E \cap G / \{x\} \neq \emptyset$.□

5.3.24 Remark:

The converse of theorem 5.3.23 is not true, even in a Hausdorff space .as the following example

5.3.25 Example:

Let $X = \{a,b,c\}$,$\tau=\{\emptyset,\{a,b\},\{c\},X\}$. Let $x_1=a.x_2=b,x_n=c$, $\forall n\geq 3$, i.e. $\langle x_n\rangle = \langle a,b,c,c,\ldots\rangle$. It's clear $x_n \to c$ but $c \notin d(\{a,b,c\})$ since $c \in \{c\} \in \tau$, $\{a,b,c\} \cap \{c\}/\{c\} = \emptyset$. Also $a,b \in d(\{a,b,c\})$ but $x_n \nrightarrow a$ and $x_n \nrightarrow b$, since $a,b \in \{a,b,c\}$ and $x_n \notin \{a,b\}, \forall n \geq 3$.

5.3.26 Remark:

A relationship between continuity of functions and convergent sequences of points is given in the following theorem.

5.3.27 Theorem:

If f is a continuous mapping of the topological space (X,τ) into the topological space (X^*,τ^*) and $\langle x_n\rangle$ is a sequence of points of X which converges to the point $x \in X$ then the sequence $\langle f(x_n)\rangle$converges to the point $f(x) \in X^*$.

Proof:

If $f(x)$ belongs to the open set G^* in X^*,then $f^{-1}(G^*)$ is an open set in X containing x since f is continuous. There must then exist an integer k such that $x_n \in f^{-1}(G^*)$ whenever $n > k$. Thus we have $f(x_n) \in G^*$whenever $n > k$, and so $f(x_n) \to f(x)$.□

5.3.28 Remark:

The converse of theorem is also not true, even in a Hausdorff space. That is, a mapping f for which $x_n \to x$ implies $f(x_n) \to f(x)$ may not be continuous as the following example:

5.3.29 Example:

Let $\mathbb{R}$ be the set of real numbers and $\tau = \{\emptyset\} \cup \{G \subseteq X: G^c \text{ is countable}\}$.Let $X^* = [0,1],\tau^* = \{G \cap [0,1]: G \in \tau\}$ be the relative topology and let $f: (\mathbb{R},\tau) \longrightarrow (X^*,\tau^*)$ be a function defined by

$$f(x) = \begin{cases} x & x\in[0,1] \\ 0 & x\notin[0,1] \end{cases}.$$

Then f is not continuous since $(0,1) \in \tau^*$but $f^{-1}\big((0,1)\big)=(0,1) \notin \tau$,where $\mathbb{R}/(0,1)$ is not countable. If $x_n \to x$ in X and iff $x_n = x$, $\forall n \in k$, k is positive integers iff $f(x_n) = f(x)$, $\forall n \in k$ iff $f(x_n) \to f(x)$.

5.3.30 Remark:

The failure of the converses of the preceding three theorems 5.3.21,5.3.23 and 5.3.27 to hold shows that the notion of limit for sequences of points is not completely satisfactory, even if the space satisfies the axiom [T_2].The Axioms of Countability we will introduce another axiom for the open sets of a topological space with which we may prove these converses.

5.4 Axioms of Countability

5.4.1 Definition:

A topological space (X,τ) is a ***first axiom space*** iff it satisfies the following ***first axiom of countability***:

$[\boldsymbol{C_I}]$ For every point $x \in X$, there exists a countable family $\{B_n(x)\}$ of open sets containing x such that whenever x belongs to an open set G, $B_n(x) \subseteq G$ for some n.

5.4.2 Example:

Let (X,d) be a metric space and $p \in X$ then the countable class of open balls $\{\mathrm{B}_1(\mathrm{p}),\mathrm{B}_{\frac{1}{2}}(\mathrm{p}), \dots\}$ with center p is a local base at p.Hence every metric space satisfies the first axiom of countability.

5.4.3 Example:

Let $(\mathbb{R},\tau)$ be the usual topology and $p \in \mathbb{R}$ then the countable class of open sets $\left\{\mathrm{B}_\mathrm{n}(\mathrm{p}) = \left(p-\frac{1}{n}, p+\frac{1}{n}\right): n \in \mathbb{N}\right\}$ is a local base at p.Hence the usual topology satisfies the first axiom of countability.

5.4.4 Example:

Let (X,τ) be any discrete topology. The singleton set {p} is open and is contained in every open set G containing $p \in X$.Hence every discrete space satisfies $[\boldsymbol{C_I}]$.

5.4.5 Example:

Let $(\mathbb{R},\tau)$ be the co-finite topology dose not satisfy the first axiom of countability.

Solution:

Suppose that $(\mathbb{R},\tau)$ satisfy $[C_I]$ then $1 \in \mathbb{R}$ possesses a countable open local base $\mathcal{B}_1 = \{B_n: n \in \mathbb{N}\}$.Since each B_n is open then B_n^c is closed and hence is finite , the set $A = \cup \{B_n^c: n \in \mathbb{N}\}$ is the countable union of finite sets and is therefore countable. But $\mathbb{R}$ is not countable then there exists a point $p \in \mathbb{R}$ different from 1 which does not belong to A ,i.e. $p \in A^c = (\cup \{B_n^c: n \in \mathbb{N}\})^c = \cap \{B_n^{cc}: n \in \mathbb{N}\} = \cap \{B_n: n \in \mathbb{N}\}$, hence $p \in B_n, \forall n \in \mathbb{N}$.On the other hand $\{p\}^c$ is open set since it is the complement of a finite set, and $\{p\}^c$contains 1 since p is different from 1. Since $\mathcal{B}_1$is a local base there exists a member $B_{n_0} \in \mathcal{B}_1$ such that $B_{n_0} \subset \{p\}^c$.Hence $p \notin B_{n_0}$.But this is contradicts the statement that $p \in B_n, \forall n \in \mathbb{N}$. So

$(\mathbb{R},\tau)$ does not satisfy the first axiom of countability.

5.4.6 Remark:

If (X,τ) is a topological space satisfy $[C_I]$,i.e. for every $x \in X$ $\exists\{B_n(x)\}$ countable base at x then we arranged the base in decreasing order as following

$$B_1^*(x) = B_1(x)$$
$$B_2^*(x) = B_1^*(x) \cap B_2(x)$$
$$B_3^*(x) = B_2^*(x) \cap B_3(x)$$
$$\vdots$$
$$B_n^*(x) = B_{n-1}^*(x) \cap B_n(x).$$

We get $\{B_n^*(x)\}$ a countable base s.t. $B_n^*(x) = \cap\{B_k(x): k \le n\}$.Also we can arrange the base as increasing order by replace the intersection with union.

Exercise:

Prove that $[C_I]$ is a hereditary property.

5.4.7 Theorem:

$[C_I]$ is a topological property.

Proof:

Let $f: (X,\tau) \longrightarrow (X^*,\tau^*)$ be A homeomorphism from a topological space (X,τ) which satisfy $[C_I]$ to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) satisfy $[C_I]$.

Let $x^* \in X^*$.Since f is onto $\exists x \in X$, s.t. $f(x) = x^*$.Since X satisfy $[C_I]$ then $\exists\{B_n(x)\}$ countable base at x ,so the family $\{f(B_n(x))\}$ is a base since f is open function and countable since f is one to one ,so (X^*,τ^*) satisfy $[C_I]$.□

5.4.8 Remark:

In the next three important theorems, we will show the converse of theorems 5.3.21,5.3.23 and 5.3.27 is true in spaces which satisfy the first axiom of countability.

5.4.9 Theorem:

A topological space (X,τ) satisfying the first axiom of countability is a Hausdorff space iff every convergent sequence has a unique limit.

Proof:

$\Longrightarrow$

In theorem 5.3.21 in T_2 –Space every convergent sequence has a unique limit.

$\Longleftarrow$

Assume that every convergent sequence has a unique limit, we want to prove that (X,τ) is T_2 –Space.

If not $\exists x,y \in X. x \neq y$ such that every open set containing x has a nonempty intersection with every open set containing *y*. Since X satisfy $[C_I]$ then $\exists\{B_n(x)\}$ and $\{B_n(y)\}$ are monotone decreasing countable open bases at x and *y* respectively with , $B_n(x) \cap B_n(y) \neq \emptyset, \forall n$, so we choose a point $x_n \in B_n(x) \cap B_n(y), \forall n$. If G_x and G_y are arbitrary open sets containing x and *y* respectively, there must exist some integer k such that $B_n(x) \subseteq G_x$ and $B_n(y) \subseteq G_y$ for all $n > k$ by the definition of a monotone decreasing base. Hence $x_n \to x$ and $x_n \to y$, so that we have a convergent sequence without a unique limit and this is contradiction .so (X,τ) is T_2 –Space.□

5.4.10 Theorem:

If x is a point and E a subset of a T_1 –Space (X,τ)satisfying the first axiom of countability , then x is a limit point of E iff there exists a sequence of distinct points in E converging to x.

Proof:

$\Longrightarrow$

In theorem 5.3.23 we proved the limit point of convergent sequence in E is a limit point of E.

$\Longleftarrow$

Let (X,τ) is T_1 *–Space* and satisfy $[C_I]$ .Let E be a subset of X and $x \in X$ s.t. $x \in d(E)$.Since X satisfy $[C_1]$ then $\exists\{B_n(x)\}$ a monotone decreasing countable open base at *x*. Since x belongs to the open set $B_n(x)$, the set $B_n(x) \cap E/\{x\}$ must be infinite by theorem 5.2.9. By induction we may choose a point x_n in this set different from each previously chosen x_n with *k* < *n*. Clearly, $x_n \to x$ since the sets $\{B_n(x)\}$ form a monotone decreasing base at x.□

5.4.11 Theorem:

If f is a mapping of the first axiom space (X,τ) into the topological space (X^*,τ^*) , then f is continuous at $x \in X$ iff for every sequence $\langle x_n\rangle$ of points in X converging to x we have the sequence $\langle f(x_n)\rangle$ converges to the point $f(x) \in X^*$.

Proof:

$\Longrightarrow$

In theorem 5.3.27 we proved if *f* is continuous and $x_n \to x$ then $f(x_n) \to f(x)$.

$\Longleftarrow$

We want to prove that f is continuous at $x \in X$, if not then $\exists G^* \in \tau^*, f(x) \in G^*$,s.t. $f(G) \nsubseteq G^*$,i.e. $f(G) \cap G^{*c} \neq \emptyset$ for any open set G containing x. Let $\{B_n(x)\}$ be a monotone decreasing countable open base at x (since (X,τ) satisfy $[C_I]$).Then $f(B_n(x)) \cap G^{*c}) \neq \emptyset, \forall n$ and we may pick $x_n^* \in f(B_n(x)) \cap G^{*c}$. Since $x_n^* \in f(B_n(x))$ we may choose a point $x_n \in B_n(x)$ such that $f(x_n) = x_n^*$. We now have $x_n \to x$ since the sets $\{B_n(x)\}$ form a monotone decreasing base at x. The sequence $\langle f(x_n) \rangle = \langle x_n^* \rangle$ cannot converge to $f(x)$, however, since $x_n^* \in G^{*C}, \forall n$. □

5.4.12 Definition:

A topological space(X,τ) is a ***second axiom space*** iff it satisfies the following ***second axiom of countability***:

$[\boldsymbol{C_{II}}]$ There exists a countable base for the topology τ.

5.4.13 Remark:

1. The property $[C_I]$ is local (i.e. there exist a base at each point) but $[C_{II}]$ is global (i.e. there exist a base for every points in a space X).

2. Every topological space satisfy $[C_{II}]$ satisfy $[C_I]$ but the converse is not true as the following examples:

5.4.14 Example:

The discrete topology on any uncountable set, has no countable base (i.e. not satisfy $[C_{II}]$) .Since each set consisting of exactly one point must belong to any base, even though there is a countable open base at each point x obtained by letting $\{B_n(x)\} = \{x\}$, i.e. satisfy $[C_I]$.

5.4.15 Example:

Let (ℝ,τ) be the discrete topology on ℝ .A class $\mathcal{B}$ is a base for a discrete topology iff it contains all singleton $\{p\}$ subset of ℝ, but ℝ is non- countable ,so the discrete topology does not satisfy $[C_{II}]$ but satisfy $[C_I]$.

5.4.15 Example:

The class $\mathcal{B}$ of open intervals (a,b) with rational endpoints ,i.e. $a,b \in \mathbb{Q}$ is countable and is a base for the usual topology on the real line ℝ.Thus (ℝ,τ) satisfies $[C_{II}]$.

Exercise:

Prove that $[C_{II}]$ is a topological property.

5.4.17 Theorem:

$[\boldsymbol{C_{II}}]$ ***is a hereditary property.***

Proof:

Let (Y,τ_Y) be a subspace of a topological space (X,τ) which satisfy $[C_{II}]$. We want to prove that (Y,τ_Y) satisfy$[C_{II}]$.

Since (X,τ) satisfy$[C_{II}]$ then $\exists\{B_n\}$ countable base for X then family $\{B_n^* = B_n \cap Y\}$ is a countable base for Y,so (Y,τ_Y) satisfy$[C_{II}]$.□

5.4.18 Remark:

The relationship between compact and countably compact sets is made clearer by application of the following theorem due to Lindelöf. Indeed, it shows that the two notions are equivalent in second axiom
T_1 – Spaces**.**

5.4.19 Theorem:

In a second axiom space, every open covering of a subset is reducible to a countable subcovering.

Proof:

Suppose $\mathcal{A}$ *is an open covering of the subset* E of the second axiom space X which has $\mathcal{B}$ as a countable base.

Since $\mathcal{A}$ is an open covering of E then $E = \cup\{G: G \in \mathcal{A}\}$, i.e. $\forall p \in E, \exists G_p \in \mathcal{A}$ such that $p \in G_p$.

Since $\mathcal{B}$ is an open a countable base for X then $\forall p \in E, \exists B_p \in \mathcal{B}$ such that $p \in B_p \subset G_p$.

Hence $E = \cup\{B_p: p \in E\}$. But $\{B_p: p \in E\} \subset \mathcal{B}$,so it is countable ,hence $\{B_p: p \in E\} = \{B_n: n \in N\}$,where N is a countable index set. For each $n \in N$ choose one set $G_n \in \mathcal{A}$ such that $B_n \subset G_n$.Then $E \subset \{B_n: n \in N\} \subset \{G_n: n \in N\}$ and so $\{G_n: n \in N\}$ is a countable subcover of $\mathcal{A}$.□

5.4.20 Theorem:

In a second axiom space, we can find a countable subbase foe every base.

Proof:

Let $\mathcal{A}$ be a base for X. Since (X,τ) satisfy$[C_{II}]$ then X has a countable base $\mathcal{B} = \{B_n: n \in N\}$.Since $\mathcal{A}$ is also a base for X then for each $n \in \mathbb{N}$, $B_n = \cup\{G, G \in \mathcal{A}_n\}$ with $\mathcal{A}_n \subset \mathcal{A}$. So $\mathcal{A}_n$ is an open cover of B_n and by theorem 5.4.19 , $\mathcal{A}_n$ reducible to a countable over $\mathcal{A}_n^*$,i.e. for each $n \in \mathbb{N}$, $B_n = \cup\{G, G \in \mathcal{A}_n^*\}$ with $\mathcal{A}_n^* \subset \mathcal{A}$ and $\mathcal{A}_n^*$ countable. But $\mathcal{A}^* = \{G, G \in \mathcal{A}_n^*, n \in \mathbb{N}\}$ is a base for X since $\mathcal{B}$ is. Furthermore $\mathcal{A}^* \subset \mathcal{A}$, $\mathcal{A}^*$ is countable.□

5.4.21 Definition:

A topological space (X,τ) is called a ***Lindelöf space*** iff every open cover of X is reducible to a countable subcover.

5.4.22 Remark:

1. From definition of Lindelöf we get every compact space is a Lindelöf space (since every finite subcover is countable).

2. Every second countable space is a Lindelöf space.

5.4.23 Theorem:

The Lindelöf space is a topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a Lindelöf space (X,τ) to the topological space (X^*,τ^*),we want to prove that (X^*,τ^*) is a Lindelöf space.

Let $\{G_\lambda^*\}$ be an open cover for X^*. Since f is continuous then $\{f^{-1}(G_\lambda^*)\}$ is an open cover for X. Since (X,τ) is a Lindelöf space then there exists a countable subcover $\{f^{-1}(G_n^*)\}_{n\in\mathbb{N}}$ foe X, i.e. $X = \bigcup_{n\in\mathbb{N}} f^{-1}(G_n^*)$, so $X^* = f(X) = f(\bigcup_{n\in\mathbb{N}} f^{-1}(G_n^*)) = \bigcup_{n\in\mathbb{N}} f f^{-1}(G_n^*) = \bigcup_{n\in\mathbb{N}} G_n^*$,(since f is 1-1 and onto).Then (X^*,τ^*) is a Lindelöf space.□

5.4.24 Remark:

The following example show that the Lindelöf space is not a hereditary property.

5.4.25 Example:

Let $X = \mathbb{R}$ the set of real number and let $\tau = \{G: G \subseteq \mathbb{R}, 0 \notin G \text{ or } R/\{1,2\} \subseteq G\}$ then every open cover for X there exists a finite subcover for X, i.e. X is compact, so X is Lindelöf space. Let $X^* = \mathbb{R}/\{0\}$, τ^* the relative topology on X^*. We have the cover $\{\{r\}: r \in \mathbb{R}/\{0\}\}$ is an open cover for X^*but not have a countable subcover for X^*,i.e. $X^*$is not a Lindelöf space. So the Lindelöf property is not a hereditary property.

5.4.26 Theorem:

Every topological space satisfy $[C_{II}]$ is separable.

Proof:

Let (X,τ) be a topological space satisfy $[C_{II}]$ then there exists a countable base $\mathcal{B} = \{B_n: n \in \mathbb{N}\}$ for X. Let $x_n \in B_n, \forall n \in \mathbb{N}$ then the set $D = \{x_n: n \in \mathbb{N}\} \subseteq X$ is also countable. We shall prove that D is dense.

Let $x \in D^c$ and let G be an open set contain x then $\exists B_n \in \mathcal{B}$ s.t. $x \in B_n \subseteq G$. Since $D \cap B_n \neq \emptyset$ then $D \cap G/\{x\} \neq \emptyset$, so $x \in d(D)$, i.e. $\overline{D} = X$ so (X,τ) is separable.□

5.4.27 Remark:

1. The converse of theorem 5.4.26 is not true in general , since the lower limit topology on $\mathbb{R}$ is separable topological space which does not satisfy the second axiom of countability.

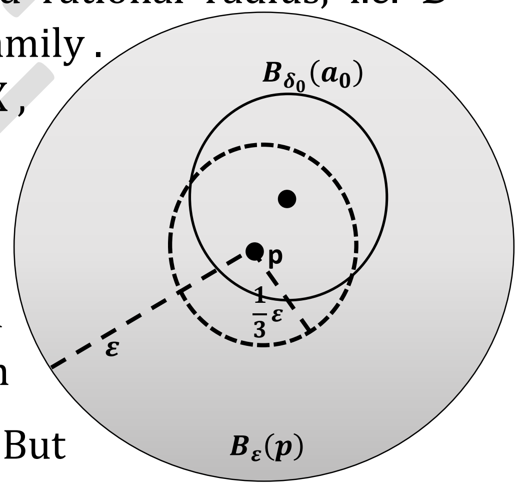
2. In metric space the converse of theorem 5.4.26 is true as the following theorem:

5.4.28 Theorem:

Every seperable metric space is second countable **($[C_{II}]$).**

Proof:

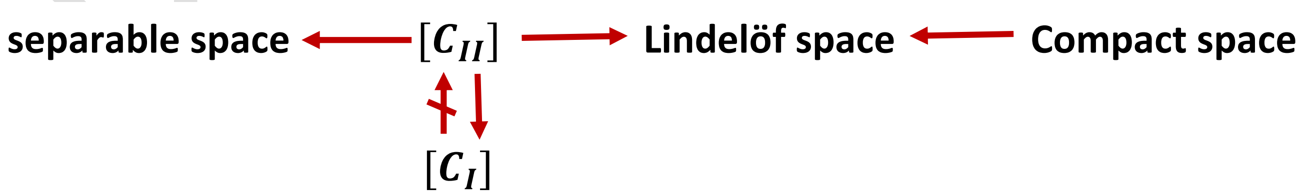
Since X is separable then X contain a countable dense subset A. Let $\mathcal{B}$ be a class of all open balls with centers in A and rational radius, i.e. $\mathcal{B} = \{B_\delta(a): a \in A, \delta \in \mathbb{Q}\}$. Note that $\mathcal{B}$ is a countable family .



We claim that $\mathcal{B}$ is a base for the topology on X , i.e. for every open set $G \subset X$ and every $p \in G$, $\exists B_\delta(a) \in \mathcal{B}$ s.t. $p \in B_\delta(a) \subset G$. Since $p \in$G there exists an open ball $B_\varepsilon(p)$ with center p such that $p \in B_\varepsilon(p) \subset G$. Since A is dense in X, $\exists a_0 \in A$ such that $d(p, a_0) < \frac{1}{3}\varepsilon$. Let δ_0 be a rational number such that $\frac{1}{3}\varepsilon < \delta_0 < \frac{2}{3}\varepsilon$. Then $p \in B_{\delta_0}(a_0) \subset B_\varepsilon(p) \subset G$.But $B_{\delta_0}(a_0) \in \mathcal{B}$,and so $\mathcal{B}$ is a countable base for the topology on X.□

5.4.29 Remark:

In the following diagram we denote by arrows the implications which hold in any topological space, while no other implications hold, even in a Hausdorff space.

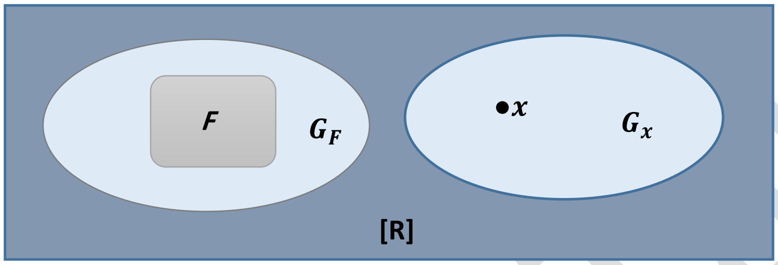


5.5 Regular and Normal Spaces

5.5.1 Definition:

A topological space X is regular iff it satisfies the following axiom of Vietoris:

[R] If F is a closed subset of X and x is a point of X not in F, then there exist two disjoint open sets G_F,G_x , one containing F and the other containing x.



5.5.2 Example:

Let $X = \{a,b,c\}$,$\tau=\{\emptyset,\{a,b\},\{c\},X\}$ then (X,τ) is regular space.

Solution:

The closed sets X,{c},{a,b},$\emptyset$, so if we take {c} closed set and $a \notin \{c\}$ then $\exists\{c\},\{a,b\} \in \tau$, s.t. $\{c\} \subset \{c\}$, $a \in \{a,b\}$.

5.5.3 Remark:

1. The above example is not T_2 – Space .Since $a,b \in X. a \neq b$ but we can't find disjoint open sets contain a and b.

2. The above example is not T_1 – Space. Since $\{a\}$,$\{b\}$ is not closed sets.

3. So regular space not necessary T_2 – Space and not T_1 – Space. Also T_2 – Space is not regular as the following example:

5.5.4 Example:

Let $X = \mathbb{R}$ the set of real numbers and let $U_x = \{(a,b):x\in(a,b)\}$ and let $U_0 = \{(-p,p)/\{\frac{1}{n}:n\in\mathbb{N}\}:p>0\}$ the family of all open sets form a base for a topology τ on $\mathbb{R}$ then $(\mathbb{R},\tau)$ is T_2 – Space , since if $a,b \in \mathbb{R}. a \neq b$, $a,b \neq 0$ then there exists two open intervals one of them contain a and the other contain b.Since every open interval is an element in U_x and all elements in $U_x$is in τ then it satisfy $[T_2]$.

If $b \neq 0$,$a = 0$, so it's clear if $b > 0$ the interval $(\frac{1}{b},b + 1)$ is a neighborhood of b and $\left(-\frac{b}{2},\frac{b}{2}\right)/\{\frac{1}{n}:n\in\mathbb{N}\}$ is a neighborhood of $a = 0$, then the first interval is an element in $U_x$and the second interval is an element in $U_0$and these intervals are disjoint then it satisfy $[T_2]$.

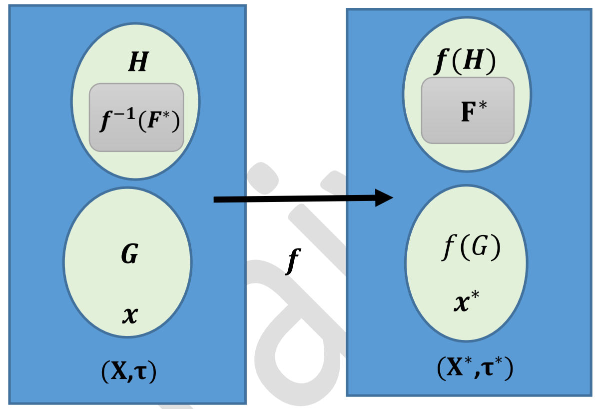
Now if $F=\{\frac{1}{n}:n\in\mathbb{N}\}$, x=0 then $0 \notin F$ and any neighborhood of F intersect with any neighborhood of x=0, so $(\mathbb{R},\tau)$ is not regular.

5.5.5 Remark:

The following theorems shows that the regularity is a topological and hereditary property:

5.5.6 Theorem:

The regularity is a topological property.



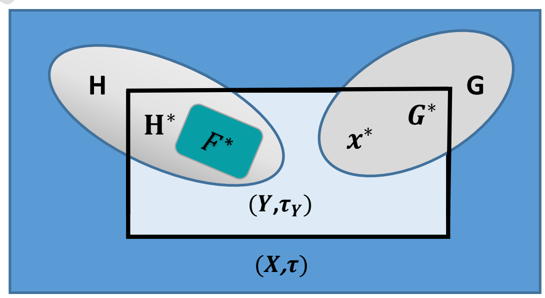
Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a regular space (X,τ) to the topological space (X^*,τ^*), we want to show that (X^*,τ^*) is a regular space.

Let F^* be a closed set in X^*, $x^* \in X^*, x^* \notin F^*$. Since f is onto then $\exists x \in X$ s.t. $f(x) = x^*$.Since f is continuous then $f^{-1}(F^*)$ is closed X .Since f is onto,1-1 and $x^* \notin F^*$then $x \notin f^{-1}(F^*)$, but (X,τ) is a regular space then $\exists G$,H$\in\tau$,$G \cap H = \emptyset$ with $x \in G, f^{-1}(F^*) \subseteq H$.Since f is open function then $f(x) \in f(G), F^* \subseteq f(H)$ with $f(G) \cap f(H) = \emptyset$, so (X^*,τ^*) is a regular space.□

5.5.7 Theorem:

The regularity is a hereditary property.



Proof:

Let (Y,τ_Y) be a subspace of a regular space (X,τ) topological space, we want to prove that (Y,τ_Y) is a regular space.

Let F^* be a closed set in Y, $x^* \in Y, x^* \notin F^*$then $F^* = F \cap Y$,were F is a closed set in X. Since $x^* \in Y \subset X$, $x^* \notin F^*$ then $x^* \notin F$. Since (X,τ) is a regular space then $\exists G$, H $\in\tau$,$G \cap H = \emptyset$ s.t. $x^* \in G, F \subseteq H$. Now $G^* = G \cap Y, x^* \in G^*$(since $x^* \in G, x^* \in Y$) , $H^* = H \cap Y, F^* \subseteq H^*$(since $F \subseteq H$) and $G^* \cap H^*$=$(G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset \cap Y = \emptyset$.So (Y,τ_Y) is a regular space.□

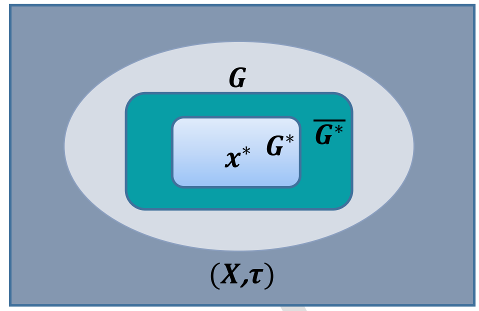
5.5.8 Theorem:

A topological space (X,τ) is regular iff for every point $x \in X$ and open set G containing x there exists an open set G^* such that $x^* \in G^*$and $\overline{G^*} \subseteq G$.

Proof:

$\Longrightarrow$

Suppose (X,τ) is regular, and the point x belongs to the open set G. Then $F = X/G$ is a closed set which does not contain x. By [R], there exist two open sets G_F and G_x such that $F \subseteq G_F$, $x \in G_x$, and $G_F \cap G_x = \emptyset$. Since $G_x \subseteq G_F^c$,$\overline{G_x} \subseteq \overline{G_F^c} = G_F^c \subseteq F^c = G$. Thus, $x \in G_x$ and $\overline{G_x} \subseteq G$ and G_x is the desired set.



$\Leftarrow$

Now suppose the condition holds and x is a point not in the closed set F. Then x belongs to the open set F^c, and by hypothesis there must exist an open set G^*such that $x \in G^*$ and $\overline{G^*} \subseteq F^c$. Clearly G^* and $\overline{G^*}^c$are disjoint open sets containing x and F , respectively.□

<u>5.5.9 Definition:</u>

A topological space (X,τ) is **T_3 – Space** if it regular and T_1 – Space, i.e.

$$T_3 \equiv [R]\&[T_1] .$$

<u>5.5.10 Remark:</u>

The following theorem shows that every T_3 – Space is T_2 – Space but the converse is not true as example 5.5.4.

<u>5.5.11 Theorem:</u>

Every T_3 – Space is Hausdorff space (T_2 – Space).

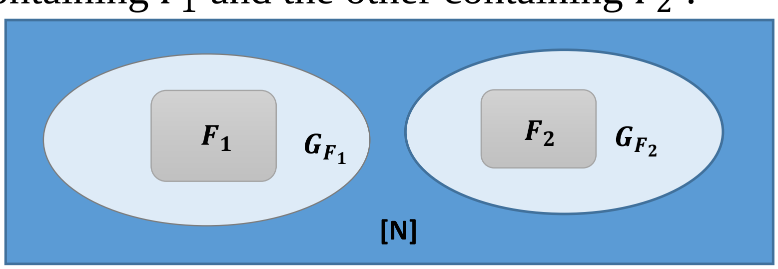
<u>Proof:</u>

Let (X,τ) be a T_3 – Space, we want to prove that (X,τ) is Hausdorff space. Let x,y∈X, $x \neq y$,since X is T_1 – Space then $\{x\}$ is closed set and since $x \neq y$, $y \notin \{x\}$ then by $[R]$, $\exists G$,H∈τ , $G \cap H = \emptyset$ and $\{x\} \subseteq G$,$y \in H$. Hence x and y belong respectively to disjoint open sets G and H.

<u>5.5.12 Definition:</u>

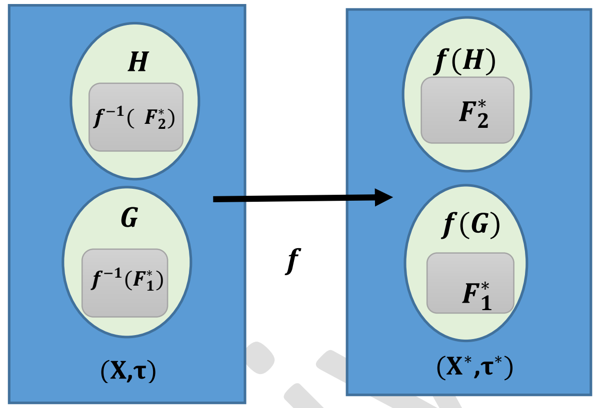
A topological space (X,τ) is ***normal*** iff it satisfies the following axiom of Urysohn:

[N] If F_1 and F_2 are two disjoint closed subsets of X, then there exist two disjoint open sets, one containing F_1 and the other containing F_2 .



5.5.13 Theorem:

The normality is a topological property.



Proof:

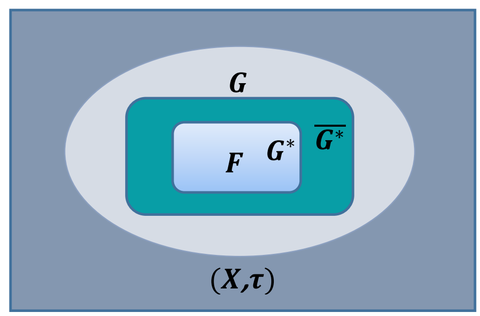
Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a normal space (X,τ) to the topological space (X^*,τ^*), we want to show that (X^*,τ^*) is a normal space.

Let F_1^* , F_2^* be a disjoint closed sets in X^*. Since f is continuous then $f^{-1}(F_1^*)$,$f^{-1}(F_2^*)$ are closed in X. Since f is onto,1-1 and $F_1^* \cap F_2^* = \emptyset$ then $f^{-1}(F_1^*) \cap f^{-1}(F_2^*) = \emptyset$, Since (X,τ) is normal then $\exists G,H \in \tau$ s.t. $f^{-1}(F_1^*) \subseteq G$, $f^{-1}(F_2^*) \subseteq H$ and $G \cap H = \emptyset$. Since f is an open function then $F_1^* \subseteq f(G)$, $F_2^* \subseteq f(H)$ and $f(G) \cap f(H) = \emptyset$.So (X^*,τ^*) is a normal space.

5.5.14 Theorem:

A topological space (X,τ) is normal iff for any closed set F and open set G containing F, there exists an open set G^* such that $F \in G^*$ and $\overline{G^*} \subseteq G$.

Proof:



$\Longrightarrow$

Suppose (X,τ) is normal and the closed set F is contained in the open set G. Then $K = X/G$ is a closed set which is disjoint from F. By [N], there exist two disjoint open sets G_F and G_K such that $F \subseteq G_F$ and $K \subseteq G_K$.Since $G_F \subseteq G_K^c$, we have $\overline{G_F} \subseteq \overline{G_K^c} = G_K^c \subseteq K^c = G$.Thus G_F is the desired set.

$\Longleftarrow$

Now suppose the condition holds, and let F_1 and F_2 be disjoint closed subsets of X. Then F_1 is contained in the open set $F_2^* = X/F_2$, and, by hypothesis, there exists an open set G^* such that $F_1 \subseteq G^*$ and $\overline{G^*} \subseteq F_2^*$.Clearly, G^* and $X/\overline{G^*}$ are the desired disjoint open sets containing F_1 and F_2, respectively.□

5.5.15 Definition:

A topological space (X,τ) is **T_4 – Space** if it normal and T_1 – Space, i.e.

$$\mathbf{T_4} \equiv [\boldsymbol{N}]\&[\mathbf{T_1}].$$

5.5.16 Example:

Let $X = \{a,b.c\}$,$\tau = \{\{a\},\{b\},\{a,b\},X,\emptyset\}$ then (X,τ) is normal space.

Solution:

Since the closed sets are {b,c},{a,c},{{c},∅,X are non-empty intersection ,i.e. if F_1 , F_2 are closed disjoint then $F_1 = \emptyset$,$F_2 = X$, so $\exists\emptyset, X \in \tau$, s. t. $F_1 \subseteq \emptyset$,$F_2 \subseteq X$, then (X,τ) is normal space. Also (X,τ) is not regular, since if F={a,c} is closed set and $x = b \notin F$ then every open set contain F intersect with every open set contain x. Also (X,τ) is not T_2 – Space.

5.5.17 Remark:

Example 5.5.16 show that the normal space need not be regular space .The following theorem 5.5.18 show that the T_4 – Space is T_3 – Space.

5.5.18 Theorem:

***Every* T_4 – Space is T_3 – Space**.

Proof:

Let (X,τ) be a T_4 – Space , let F be closed set , $x \in X$, $x \notin F$. Since (X,τ) is T_1 – Space then $F_1 = \{x\}$ is closed set. Since (X,τ) is T_4 – Space then ∃G , H ∈τ , F⊆G , F_1⊆H , G∩H=∅, i. e. x ∈H, F ∈ G , so (X,τ) is T_3 – Space.□

5.5.19 Remark:

The following theorem 5.5.20 gives a relation between normal and T_2 – Space. Also theorems 5.5.20, 5.5.21 give two sufficient conditions for a topological space to be normal.

5.5.20 Theorem:

Every compact Hausdorff space is normal.

Proof:

Let (X,τ) be a compact Hausdorff space and let F , F^* be two disjoint, closed subsets of the compact Hausdorff space X**.** F and F^* are compact since they are closed subsets of a compact space X.

By $[T_2]$,$\forall x \in F$,$\forall y \in F^*$, $\exists G_x, G_y^* \in \tau$,$G_x \cap G_y^* = \emptyset$, s.t. $x \in G_x$ & $y \in G_y^*$. For each fixed point $x \in F$ the collection $\{G_y^* : y \in F^*\}$ forms an open covering of the compact set F^*. There must be a finite subcovering, which we denote by $\{G_{y_i}^* : i = 1,2,..,n\}$. If we let $G_x^* = \bigcup_{i=1}^{n} G_{y_i}^*$ and the finite intersection $G_x = \bigcap_{i=1}^{n} G_x^i$ then G_x and G_x^* are disjoint open sets containing x and F^*, respectively. Now the collection $\{G_x : x \in F\}$ forms an open covering of the compact set F. There must be a finite subcovering, which we denote by $\{G_{x_i} : i = 1,2,..,m\}$. If we let $G =$

$\bigcup_{i=1}^{m} G_{x_i}$ and the finite intersection $G^* = \bigcap_{i=1}^{m} G_{x_i}^*$ then G and G^* are two disjoint open sets containing F and F^* respectively.□

5.5.21 Theorem:

Every regular Lindelöf space is normal.

Proof:

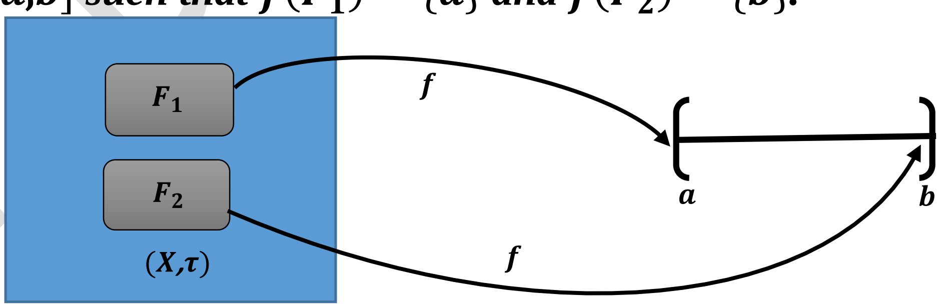
Let F and F^* be two disjoint closed subsets of the regular Lindelöf space (X,τ). Then F and F^* are Lindelöf since every closed subset of a Lindelöf space is Lindelöf space. By [R] , $\forall x \in F, \exists G_x \in \tau,$ s.t. $x \in G_x \subseteq \overline{G_x} \subseteq F^{*\,c}$. The collection $\{G_x : x \in F\}$ forms an open covering of the Lindelöf set F.There must be a countable subcovering, which we denote by $\{G_i\}_{i=1}^{n}$. Similarly, for each point $x \in F^*$ there must exist an open set $\exists G_x^* \in \tau,$ s.t. $x \in G_x^* \subseteq \overline{G_x^*} \subseteq F^{\,c}$.The collection $\{G_x^* : x \in F^*\}$ forms an open covering of the Lindelöf set F^*. There must be a countable subcovering, which we denote by $\{G_i^*\}_{i=1}^{n}$. The reader may show that the sets $G = \bigcup_{n\in\mathbb{N}}[G_n / \bigcup_{i\leq n} \overline{G_i^*}]$ and $G^* = \bigcup_{n\in\mathbb{N}}[G_n^* / \bigcup_{i\leq n} \overline{G_i}]$ are disjoint open sets containing F and F^*, respectively.□

5.5.22 Remark:

Another characterization of normality relates that concept to the number of real-valued continuous functions defined on the space.

5.5.23 Lemma (Urysohn's Lemma):

A topological space (X,τ) is normal iff for every two disjoint closed subsets F_1 and F_2 of X and closed interval $[a, b]$ of reals, there exists a continuous mapping $f: X \to [a,b]$ such that $f(F_1) = \{a\}$ and $f(F_2) = \{b\}$.



5.5.24 Definition:

A topological space (X,τ) is ***completely normal*** iff it satisfies the following axiom of Tietze:

[CN] If *A* and *B* are two separated subsets of *X*, then there exist two disjoint open sets, one containing *A* and the other containing *B*.

5.5.25 Definition:

A topological space (X,τ) is $\mathbf{T_5}$ **– Space** if it completely normal space and also T_1 – Space, i.e.

$$\mathbf{T_5 \equiv [CN]\&[T_1].}$$

5.5.26 Example:

Let $X = \{a,b,c\}$,τ={∅,{a,b},{c},X} then (X,τ) is completely normal.

Solution:

Since every set in τ is open and closed set, so if $A,B \in \tau$ then $\bar{A} \cap B = A \cap \bar{B} = A \cap B = \emptyset$ then A and B are separable and $A \subseteq A$,B⊆B so (X,τ) is completely normal .Also in example 5.5.2 we show that (X,τ) is regular space not T_1 – Space and not T_2 – Space.

5.5.27 Remark:

Since disjoint closed sets are separated, then every completely normal space is normal, and hence every T_5 – Space is a T_4 – Space but the converse is not true. Also the following example show that T_5 – Space does not transfer by continuity.

5.5.28 Example:

Let $X = X^* = \{a,b,c\}$ and let τ be the discrete topology and $\tau^*=\{\emptyset,\{a\},\{b,c\},X^*\}$ and let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be the identity function , i.e. $f(x) = x, \forall x \in X$.

Since (X,τ) is the discrete topology then f is continuous function and since the discrete topology is T_1 – Space and normal then (X,τ) is T_5 – Space. Since (X^*,τ^*) is not T_1 – Space then it's not T_5 – Space.

5.5.29 Theorem:

The completely normal space ([CN]) is topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a topological space (X,τ) satisfy [CN] to the topological space (X^*,τ^*),we want to show that (X^*,τ^*) satisfy [CN].

Let A^*,B^*be a separable sets in X^*. Since f is continuous and 1-1 then $f^{-1}(A^*),f^{-1}(B^*)$ are separated subset of X. Since (X,τ) satisfy [CN] then $\exists G,H \in \tau$, $G \cap H = \emptyset$, s.t. $f^{-1}(A^*) \subseteq G, f^{-1}(B^*) \subseteq H$.Since f is open ,1-1 and $G,H \in \tau$ then $A^* \subseteq f(G), B^* \subseteq f(H), f(G) \cap f(H) = \emptyset, f(G), f(H) \in \tau^*$,so (X^*,τ^*) satisfy [CN].◻

5.5.30 Theorem:

A topological space(X,τ) is completely normal iff every subspace of X is normal.

Proof:

$\Longrightarrow$

Suppose (X,τ) is completely normal and let (X^*,τ^*) be a subspace of (X,τ), we want to prove that (X^*,τ^*) is normal space.

Let F_1^* and F_2^* be disjoint (relatively) closed subsets of X^***,** so $F_1^* = \overline{F_1^*}$,$F_2^* = \overline{F_2^*}$. Since F_1^* and F_2^* are closed subsets of X^* then $\exists F_1$,F_2 closed subset of X such that $\overline{F_1^*}$=$\overline{F}_1 \cap X^*$, $\overline{F_2^*} = \overline{F_2} \cap X^*$.Now

$F_1^* \cap \overline{F_2} = \overline{F_1^*} \cap \overline{F_2} = \overline{F}_1 \cap X^* \cap \overline{F_2} = \overline{F}_1 \cap X^* \cap X^* \cap \overline{F_2} = \overline{F_1^*} \cap \overline{F_2^*}$=$F_1^* \cap F_2^* = \emptyset$. And similarly, $\overline{F_1} \cap F_2^* = \emptyset$. Hence F_1^* and F_2^* are separated subsets of X**.** By[CN], there exist disjoint open sets G_1 and G_2 containing F_1^* and F_2^* respectively. Then the sets $X^* \cap G_1$ and $X^* \cap G_2$ are disjoint (relatively) open subsets of X^* which contain F_1^* and F_2^*, respectively, so X^* is normal.

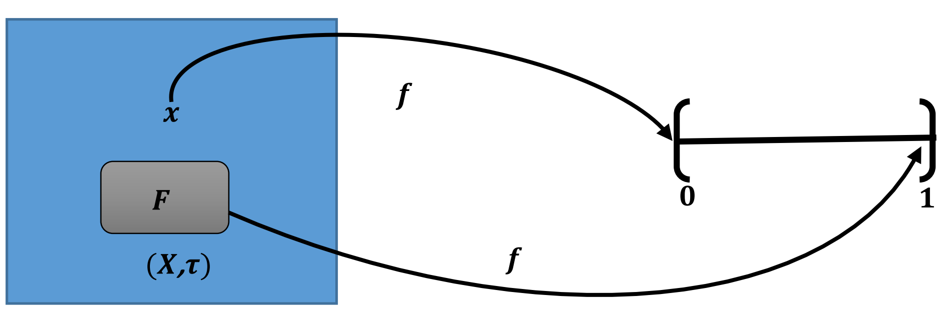
$\Longleftarrow$

Now let us suppose that every subspace of X is normal, and let A and B be separated subsets of X. Consider the open set $[\bar{A} \cap \bar{B}]^c = X^*$ as a subspace of X. By hypothesis, X^* is normal. The sets $X^* \cap \bar{A}$ and $X^* \cap \bar{B}$ will be disjoint, relatively closed subsets of X^* and so there must exist two disjoint relatively open sets G_A and G_B containing $X^* \cap \bar{A}$ and $X^* \cap \bar{B}$ respectively. Since X^* is an open subset of X, G_A and G_B are actually open subsets of X .Thus we have $A \subseteq X^* \cap \bar{A} \subseteq G_A$ and $B \subseteq X^* \cap \bar{B} \subseteq G_B$, so that X is completely normal.□

5.5.31 Definition:

A topological space (X,τ) is ***completely regular*** iff it satisfies the following axiom:

[CR] If F is a closed subset of X**,** and x is a point of X not in F**,** then there exists a continuous mapping $f: X \to [0,1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.



5.5.32 Definition:

A topological space (X,τ) is A ***Tichonov Space*** if it completely regular space and also T_1 – Space, i.e.

$$\mathbf{T_{3\frac{1}{3}}} \equiv [\mathbf{CR}]\&[\mathbf{T_1}].$$

5.5.33 Theorem:

The completely regular space is a topological property.

Proof:

Let $f:(X,\tau) \longrightarrow (X^*,\tau^*)$ be a homeomorphism from a completely regular space (X,τ) to the topological space (X^*,τ^*), we want to show that (X^*,τ^*) is compeletly regular space.

Let F^* be a closed subset of $X$and $x \in X$, $x \notin F^*$.Since f is continuous then $F = f^{-1}(F^*)$. Since f is onto then $\exists x \in X$, s.t. $f(x)=x^*$.Since f is 1-1 and $x^* \notin F^*$ then $x \notin F$. Since (X,τ) is completely regular then $\exists\, g : X \rightarrow [0,1]$, s. t. $g(x) = 0$ and $g(F) = \{1\}$ then the composition $g \circ f^{-1}$ is continuous (since g and f^{-1} are continuous functions).So $g \circ f^{-1}: X^* \rightarrow [0,1]$ and $(g \circ f^{-1})(F^*) = g(f^{-1}(F^*)) = g(F) = \{1\}$ and $(g \circ f^{-1})(x) = g(f^{-1}(x^*)) = g(x) = 0$. So (X^*,τ^*) is compeletly regular space.□

5.5.34 Theorem:

The completely regular space is a hereditary property.

Proof:

Let (Y,τ_Y) be a subspace of a regular space (X,τ) topological space, we want to prove that (Y,τ_Y) is a regular space.

Let F^* be a closed set in Y, $x^* \in Y$,$x^* \notin F^*$then $F^* = F \cap Y$,were F is a closed set in X. Since $x^* \in Y \subset X$, $x^* \notin F^*$ then $x^* \notin F$. Since (X,τ) is completely regular space then $\exists\, f : X \rightarrow [0,1]$, $s.\,t.$ $f(x) = 0$ and $f(F) = \{1\}$. Let $\exists\, f^*\, Y \rightarrow [0,1]$ defined as $f^*(x) = f(x)$,$\forall x \in Y$, i. e. $f^* = f|_Y$ is continuous and satisfy $f^*(x) = o$, since $x \in Y$ and $f^*(F^*) = \{1\}$, since $F^* = F \cap Y$,so (Y,τ_Y) is a regular space.□

5.5.35 Theorem:

Every completely regular space is regular.

Proof:

Let (X,τ) be a completely regular space. Let F be a closed subset of $X$and $x \in X$, $x \notin F$ then $\exists\, f : X \to [0,1]$, continuous function such that $f(x) = 0$ and $f(F) = \{1\}$. Since $\mathbb{R}$ is a T_2 – Space and $[0,1] \subseteq \mathbb{R}$ is also a T_2 – Space then $\exists G,H \in \tau$, $G \cap H = \emptyset$ and $0 \in G$, $1 \in H$. Since f is continuous function then $f^{-1}(G)$,$g^{-1}(H)$ are disjoint open subset of X and $x \in f^{-1}(0) \in f^{-1}(G)$, $F \subseteq f^{-1}(1) \subseteq f^{-1}(G)$.So (X,τ) is regular space.□

5.5.36 Remark:

Theorem 5.5.35 every [CR] is [R] ,every Tichonov space is a T_3 – Space, and every T_4 – Space is a Tichonov space by Urysohn's Lemma. Because of these facts, we might be inclined to call a Tichonov space a $T_{3\frac{1}{3}}$-space.

T_4 –Space ⟶ Tichonov space ⟶ $T_{3\frac{1}{3}}$-space

On the other hand, since a normal space need not be regular, it also need not be completely regular. The following implication does hold, however

5.5.37 Theorem:

A normal space is completely regular iff it is regular.

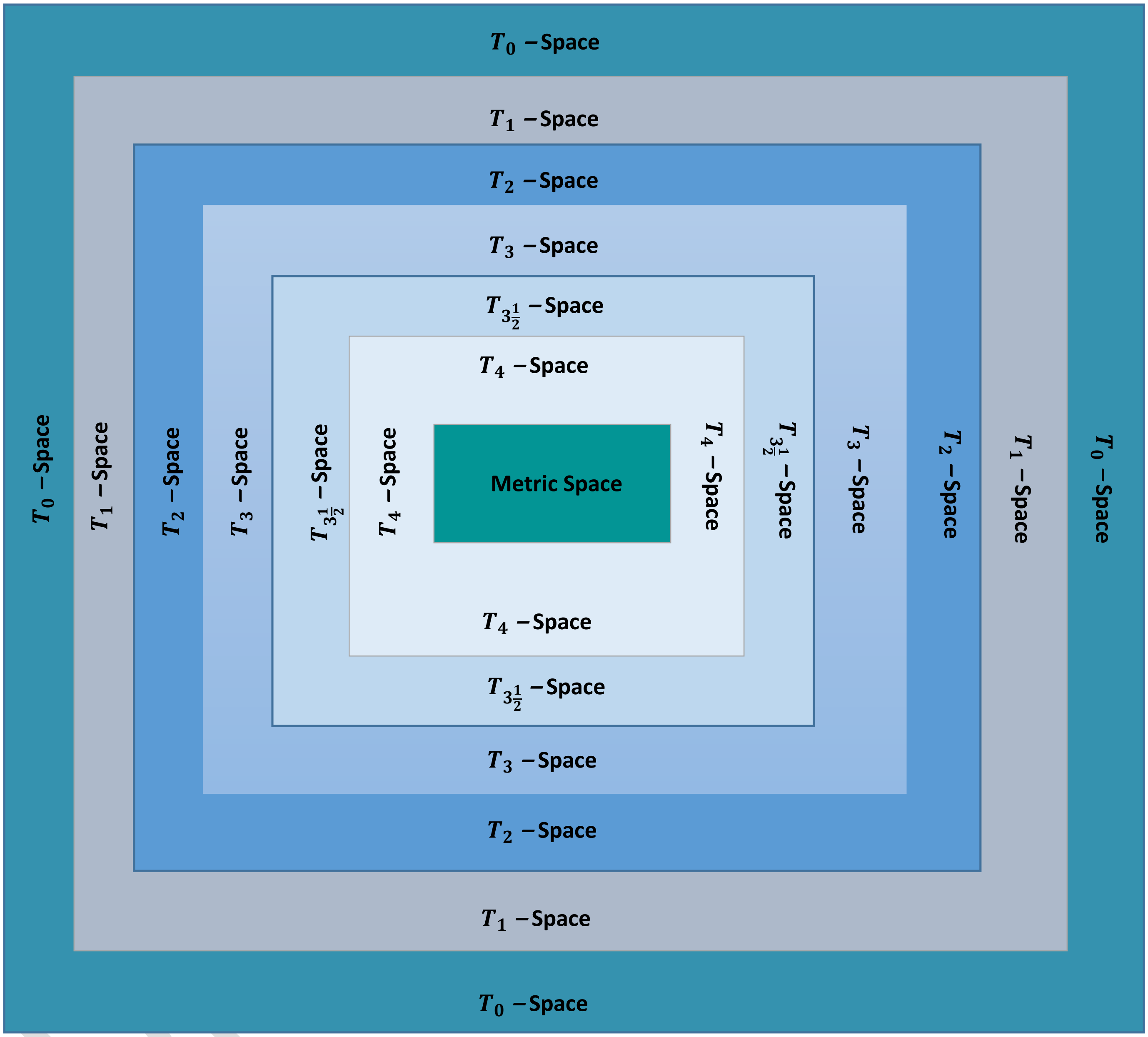
Proof:

$\Longrightarrow$

By theorem 5.5.18 a norm space is regular if it is completely regular.

$\Longleftarrow$

We need to show that any normal, regular space (X,τ) is completely regular. Suppose F is a closed subset of X not containing the point x**,** so that x belongs to the open set F^c**.** By theorem 5.5.14, there exists an open set G such that $x \in G$ and $\bar{G} \subseteq F^c$**.**Since F and $\bar{G}$ are disjoint closed sets in the normal space X, by Urysohn's Lemma there exists a continuous mapping $f : X \to [0,1]$ such that $f(F) = \{1\}$ and $f(\bar{G}) = \{0\}$.. Since $x \in G$, $f(x) = 0$, and so (X,τ) is completely regular.□



T_0 −Space
T_1 −Space
T_2 −Space
T_3 −Space
$T_{3\frac{1}{2}}$ −Space
T_4 −Space
Metric Space
T_4 −Space
$T_{3\frac{1}{2}}$ −Space
T_3 −Space
T_2 −Space
T_1 −Space
T_0 −Space