# Advanced Calculus

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- Taylor and Maclauring Series, Convergence of Taylor. Series.

### Text Book:

Maurice Weir, Joel Hass, Frank Giordano, Thomas calculus.

### References:


- G Stephenson, Mathematical Methods For Science student
- Anton Bivens Davis, Calculus

# Chapter One: Conic Section and Polor Cordinates

الفصل الأول: الاحداثيات القطبية والقطوع المخروطية

## 1. Conic Section and Quadratic Equations

This section show how to the Conic Sections that originated in Greek Geometry are discribed today as the graphs of quadratic equations in the Coordinate Plane. These Curves as the Curves formed by Cutting a double cone with Plane. These curved are called Conic sections

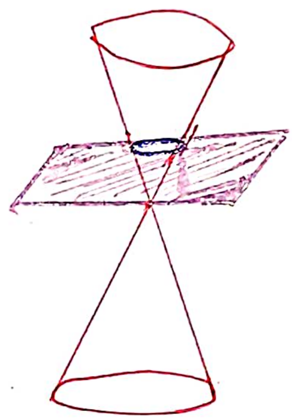


Fig. 1

In this Section we define and review Circle, Parabolas, ellipses and Hyperbolars, geometrically and drive their standard Cartesian

Conic Section

- Circle
  الدائرة
- Parabolas
  القطع المكافئ
- ellipses
  القطع الناقص
- Hyperbolars
  القطع الزائد

- Classifying Conic Section

1. Circle الدائرة

A circle is the set of points in a plane whose distance from a given fixed point in the plane as a constant.

The fixed point is the center of the circle. The constant distance is the radius of the circle.

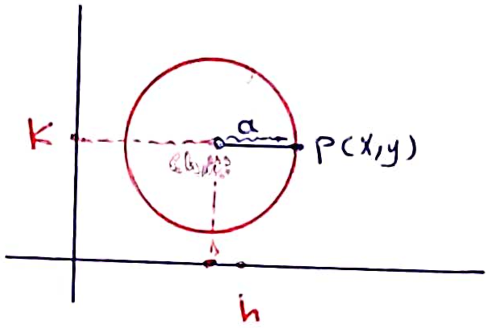



Fig. (2)

The standard equation for the circle of radius a centered at the origin, i.e. (h, k) = (0, 0)

$$x^2+y^2=a^2$$

The standard equation of the circle of radius a centered at the point (h, k)

$$(x-h)^2+(y-k)^2=a^2$$

or

$$\sqrt{(x-h)^2+(y-k)^2}=a$$ [the distance from (x, y) to (h, k) equals a

Ex.1: The standard equation for the circle of radius 2 sentered at the point $(3,4)$ is:

$$(x-3)^2+(y-4)^2=(2)^2$$

or

$$(x-3)^2+(y-4)^2=4$$

Ex.2: Find the center and radius of the circle $(x-1)^2+(y+5)^2=3$

Sol.

Comparing $(x-h)^2+(y-k)^2=a^2$

with $(x-1)^2+(y+5)^2=3$

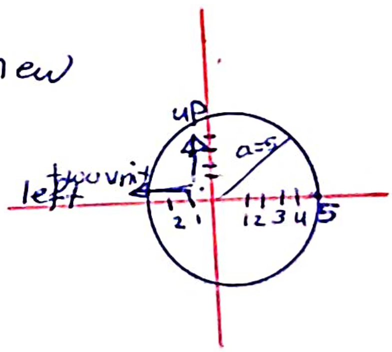
shows that: $h=1$ , $y=-5$ , $a=\sqrt{3}$

The center is the point $(1,-5)=(h,k)$, the radius is $a=\sqrt{3}$

Ex3: If the circle $x^2+y^2=25$ is shifted two units to the left and three units to the up. find the center and radius of the circle.

Sol

$\because$ the circle is shifted two units to the left and three units to the up its new equation is




$$(x-(-2))^2+(y-3)^2=25$$

or $(x+2)^2+(y-3)^2=25$

The radius of the circle is 5 and the center is $(h,k)=(-2,3)$

## 2. The Parabola القطع المكافئ

**Definition:**

Aset that Consists of all the Points in a plane equidistant from agiven fixed Point and agiven fixed line in the Plane is a Parabola. The fixed Point is the focus of the Parabola and the fixed line is the directrix.

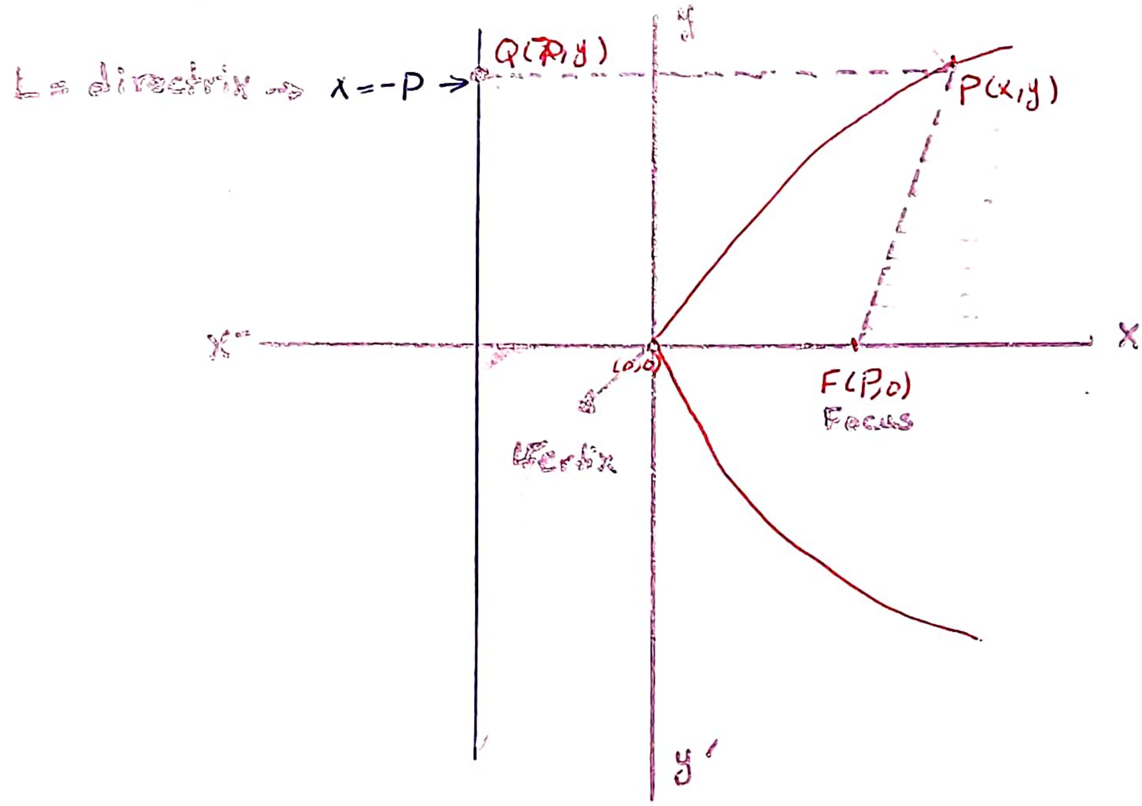



Fig 3: The standard Form of the Parabola

$y^2 = 4Px$ on $x^{+}$-axis

If the focus F Lie on the directrix L, F does not lie on L. For example, suppose that the the focus lie at the Point F(P,0) on the Positive x-axis and that the directrix is the line $x=-P$, in the notation from the figar3, a Point P(x,y) lies on the Parabola if and only if $PF = PQ$ From the distance formula

$PF = PQ$

$P=(x,y)$ , $F=(P,0)$ , $Q=(-P,y)$

$$\sqrt{(x-P)^2+(y-0)^2} = \sqrt{(x+P)^2+(\cancel{y}-\cancel{y})^2}$$

$$(x-P)^2+y^2=(x+P)^2$$

$$\cancel{x^2}-2xP+\cancel{P^2}+y^2=\cancel{x^2}+2xP+\cancel{P^2}$$

$\boxed{y^2=4Px}$ the standard form of the Parabola on x-axis (Positive)

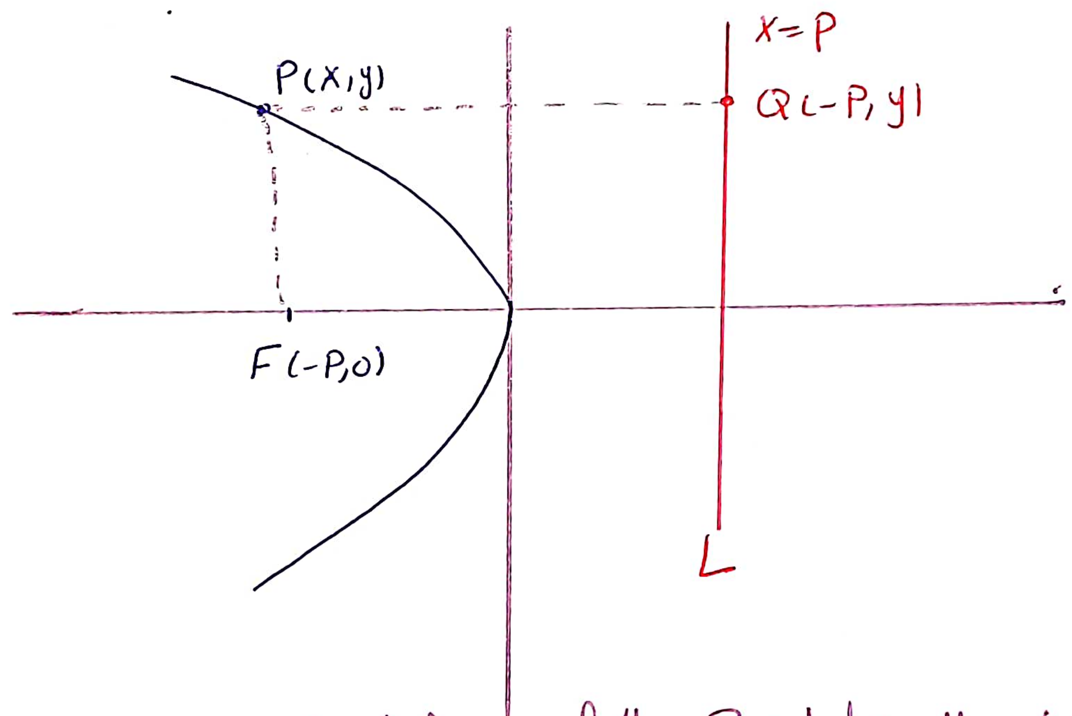



Fig 41 The standard form of the Parabla $y=-4Px$ on x-axis (negative)

$PF = PQ$

$P=(x,y)$ , $F=(-P,0)$ , $Q=(P,y)$

$$\sqrt{(x+P)^2+(y-0)^2} = \sqrt{(x-P)^2+(\cancel{y}-\cancel{y})^2}$$

$$(x+P)^2+y^2=(x-P)^2$$

$$\cancel{x^2}+2Px+\cancel{P^2}+y^2=\cancel{x^2}-2Px+\cancel{P^2}$$

$\boxed{y^2=-4Px}$ the standard form of Parabola on x-axis (negative)

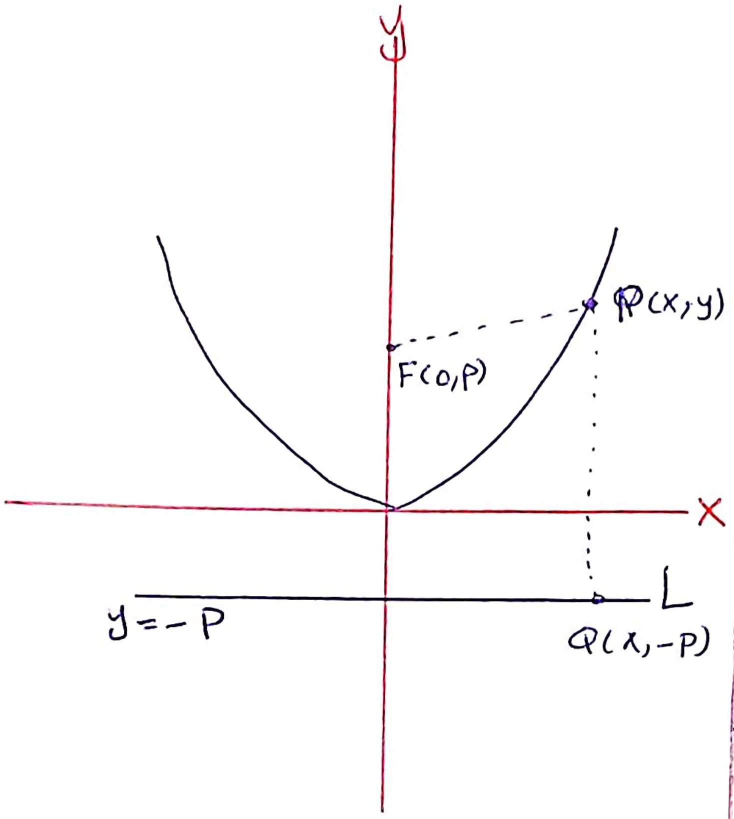



Fig 5: The standard form of the Parabola $x^2 = 4py$ on y-axis (positive)

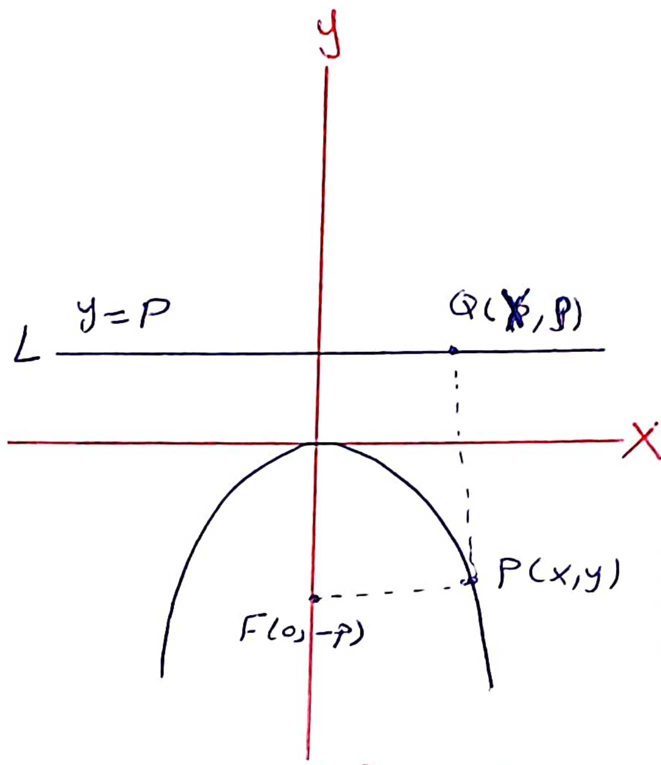



Fig 6: The standard form of the Parabola $x^2 = -4py$ on y-axis (negative)

$PF = PQ$

$$\sqrt{(x-0)^2+(y-p)^2} = \sqrt{\cancel{(x-x)^2}+(y+p)^2}$$

$$x^2+(y-p)^2 = (y+p)^2$$

$$x^2+\cancel{y^2}-2py+\cancel{p^2} = \cancel{y^2}+2py+\cancel{p^2}$$

$\boxed{x^2 = 4py}$ the standard form of the Parabola on y-axis (positive)

$PF = PQ$

$$\sqrt{(x-0)^2+(y+p)^2} = \sqrt{(x-\cancel{x})^2+(y-p)^2}$$

$$x^2+(y+p)^2 = (\cancel{y}-p)^2$$

$$\cancel{x^2}+y^2+2py+\cancel{p^2} = \cancel{x^2}-2px+\cancel{p^2}$$

$\boxed{y^2 = -4py}$ the standard form of the Parabda on y-axis (negative ?

## Remark

1. If vertix is $(0,0)$, then:

$y^2 = 4p(x)$ , $x^2 = 4py$

$y^2 = -4px$ , $x^2 = -4py$

2. If the vertix is $(h,k)$ then:

$(y-k)^2 = 4p(x-h)$ , $(x-h)^2 = 4p(y-k)$

$(y-k)^2 = -4p(x-h)$ , $(x-h)^2 = -4p(y-k)$

Ex:1 :- Find the focus and directrix of the Parabola

$y^2 = 10x$.

Sol: $y^2 = 4px$

$10\cancel{x} = 4p\cancel{x} \Rightarrow p = \frac{10}{4} = \frac{5}{2}$

The focus $(5/2, 0)$ and directrix $(-\frac{5}{2}, 0)$.

Ex:2: Find the equation of Parabola if the focus is $(0,2)$.

Sol: $\because$ The focus is $(0,2)$, then focus is y-axis, and the directrix is $(0,-2)$.

$x^2 = 4py$

$x^2 = 4 * -2y \Rightarrow x^2 = -8y$

Ex.3: The Point (+1,-2) is belong to the Parabola $x^2=4Py$ find the focus and directrix.

Sol. $x^2 = 4Py$ , $\because (x,y)=(1,-2)$

the Parabola lie in y-axis (Positive)

$\therefore 1^2 = 4P(-2)$

$1 = -8P \Rightarrow P = -1/8$

The focus $F(0,-1/8)$ , directrix is $y = 1/8$

Ex.4: Find the equation of Parabola if the Vertix is (-1,4) and Pass through (4,-2) directrix this Parabola is, the y-axis parallel

Sol: $\because$ The vertix is (-1,4) , from remark 2, to get:

$(y-k)^2 = 4P(x-h)$

$\therefore (-2-4)^2 = 4P(4+1)$

$(-6)^2 = 4P(5)$

$36 = 20P \Rightarrow P = \frac{36}{20}$

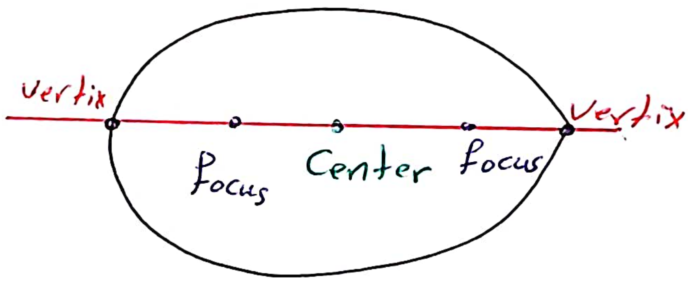
$\therefore (y-k)^2 = 4P(x-h)$

$(y-4)^2 = \cancel{4} \times \frac{36}{\cancel{20}_{5}} (x+1)$

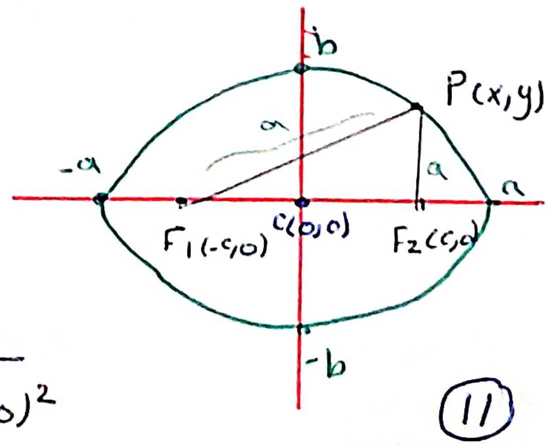
$(y-4)^2 = \frac{36}{5}(x+1)$ the Parabola eq. where vertix is (-1,4).

## 3- Ellipses: القطع الناقص

**Def:** An ellipse is the set of Point in a plane whose distances from two fixed Points in the Plane have a Constant. The two fixed Points are the foci of the ellipse and the line through the foci of an ellipse is the focal axis. The Point of this line have way between the foci is the ellipses Center, the Points where the focal axis and the ellipse cross are the ellipses Vertices




to find the equation of ellipse, suppose that, the two fixed Points are $F_1(c,0)$, $F_2(-c,0)$ and the sum of the distances $PF_1 + PF_2$ is denoted by $2a$, the Coordinates of a Point P on the ellipse satisfy:




$$PF_1 + PF_2 = 2a$$

$$\sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x-c)^2+(y-0)^2} = 2a$$

or

$$\sqrt{(x+c)^2+(y-0)^2} = 2a - \sqrt{(x-c)^2+(y-0)^2}$$

Sequare two side and simplified, to get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1 \quad , \quad \text{let } a^2-c^2 = b^2 \text{ , to get}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Remark :-

1. The standard form equation for ellipses Centered at the origin

A- Foci on the x-axis:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad a > b$$

- Center to focus distance $c = \sqrt{a^2 - b^2}$
- Foci = $(\mp c, 0)$
- Vertices = $(\mp a, 0)$

B- Foci on the y-axis:

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad , \quad a < b$$

- Center to focus distance $c = \sqrt{a^2 - b^2}$
- Foci = $(0, \mp c)$
- Vertices = $(0, \mp a)$

In each cases, $a$ is the Semimajor axis and $b$ is the Semiminor axis.

2. The standard form equation for ellipses centered at the Point $(h, k)$

A. Foci on the x-axis

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad , \quad a > b$$

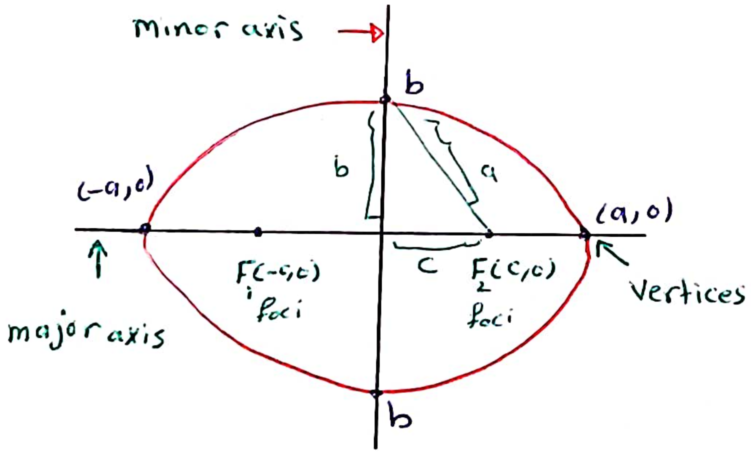
- the foci is: $(h+c, k)$, $(h-c, k)$

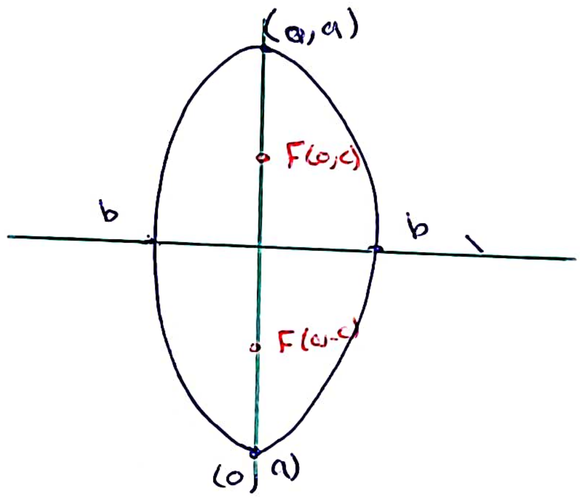
B. Foci on the y-axis

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \quad , \quad a < b$$

the foci is: $(h, k+c)$, $(h, k-c)$

The Length of major axis $2a$, the length of minor axis is $2b$.




the standard form of ellipses equation on x-axis

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

the standard form of of ellipses equation on y-axis

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$

Ex: Find the semimajor axis and semiminor axis, vertix and foci, and the center to the following ellipses equation.

1. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Sol

The semimajor axis: $a = \sqrt{16} = 4$ ,

The semi minor axis: $b = \sqrt{9} = 3$

The center of the focus: $c = \sqrt{a^2 - b^2} = \sqrt{16-9} = \sqrt{7}$

$\therefore$ the foci is: $(\sqrt{7}, 0), (-\sqrt{7}, 0)$

The length of major axis $= 2a = 8$

The length of minor axis $= 2b = 6$ , The vertices is $(4,0), (-4,0)$

2. $\frac{x^2}{9} + \frac{y^2}{25} = 1$

Sol

The semimajor axis: $a = \sqrt{25} = 5$

The semi ~~major~~ minor axis: $b = \sqrt{9} = 3$

The center of the focus: $c = \sqrt{a^2 - b^2} = \sqrt{25-9} = \sqrt{16} = 4$

$\therefore$ The foci is $(0,4), (0,-4)$

The vertices is $(0,5), (0,-5)$

The length of major axis $= 2a = 10$

the length of minor axis $= 2b = 6$

$(6,-4)$

**Ex:** find the equation of ellipse where the center is $(0,0)$ and $(0,3)$ is focus and the half length of the major axis is 5.

**Sol**

the focus is $(0,3)$, $c=3$

the half length of the major axis is 5 i.e. $a=5$

$\therefore$ vertix is $(0,5)$, $(0,-5)$

$$c=\sqrt{a^2-b^2} \Rightarrow c^2=a^2-b^2$$

$$9=25-b^2 \Rightarrow b^2=25-9=16$$

$\therefore b=4$ , $(4,0)$, $(-4,0)$

$$\therefore \frac{x^2}{16}+\frac{y^2}{25}=1$$

**Ex:** find the equation of ellipse

$$4x^2+9y^2-48x+72y+144=0$$

**Sol:**

$$(4x^2-48x)+(9y^2+72y)+144=0$$

$$4(x^2-12x)+9(y^2+8y)+144=0$$

$$4(x^2-12x+36-36)+9(y^2+8y+16-16)+144=0$$

$$4(x^2-12x+36)-144+9(y^2+8y+16)\cancel{-144}\cancel{+144}=0$$

$$4(x-6)^2+9(y+4)^2=144 \quad ]\div 144$$

$$\frac{(x-6)^2}{36}+\frac{(y+4)^2}{16}=1$$

$\therefore$ the center is $(h,k) = (6,-4)$

$a^2 = 36 \Rightarrow a = 6$ , $(6,0)$, $(-6,0)$ ,

$b^2 = 16 \Rightarrow b = 4$ $(0,4)$, $(0,-4)$

$c = \sqrt{a^2-b^2} = 2\sqrt{5}$ , foci is $(6+2\sqrt{5}, -4)$, $(6-2\sqrt{5}, -4)$

the length of simemajor is $2a = 12$

" " " simeminor is $2b = 8$

**Ex:** Find the equation of Ellipse, Vertices, foci, Semimajor seminminor, center, Length of major, length of minor

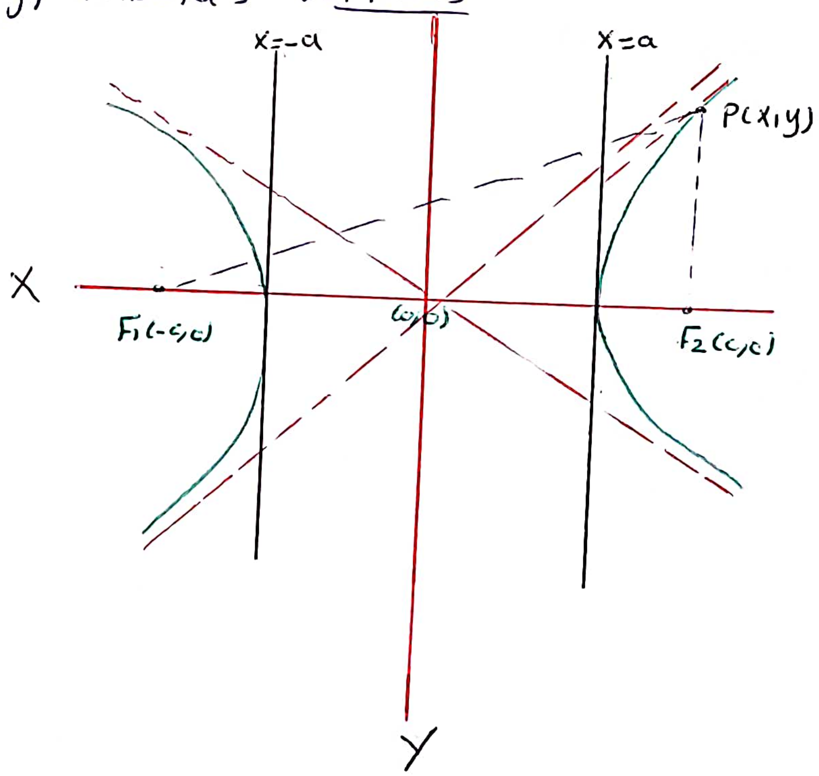
1- $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1$

2- $\dfrac{x^2}{81} + \dfrac{y^2}{36} = 1$

3- $4x^2 + 25y^2 = 100$

## 4. HyPerabola.

**Def:** A HyPerabola is a set of Points in a plane whose distances from two fixed Points in the Plane have a constant difference.
The Two fixed Points are the foci of the hyperabola.
The line through the foci of a HyPerabola is the focal axis.
The Point on this line halfway between the foci is the hyPerabola Center. The Points where the hyperabola and focal axis cross are the hyperabola's vertices.




If the foci are $F_1(-c,0)$, $F_2(c,0)$ and the constant difference is $2a$, then a point $(x,y)$ lies on the hyperabola if and only if

$$PF_1 - PF_2 = 2a$$

$$\frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1, \quad c > a, \quad \text{let } c^2 - a^2 = b^2$$

$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}$, the equation of hyperabola where the foci belong to the x-axis

Remark:

1- The standard equations for hyparabolas Centered at the origin.

A. Focus on the x-axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

- Center to focus distance $c = \sqrt{a^2+b^2}$
- Foci : $(c,0)$, $(-c,0)$
- Vertices : $(a,0)$, $(-a,0)$
- Asymptotes: we can find the asymptotes by the following :

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \Rightarrow \frac{y^2}{b^2} = \frac{x^2}{a^2} \Rightarrow \boxed{y = \mp \frac{b}{a} x}$$

the equation of asymptotes

B- focus on the y-axis: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

- Center to foci distance: $c = \sqrt{a^2+b^2}$
- Foci: $(0,c)$, $(0,-c)$
- Vertices: $(0,a)$, $(0,-a)$
- Asymptotes: we can find the asymptotes by the following

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 0 \Rightarrow \frac{y^2}{a^2} = \frac{x^2}{b^2} \Rightarrow \boxed{y = \mp \frac{a}{b} x}$$

2- The standard form equation for hyparabolas Centered at the Point $(h,k)$

A- Focus on the x-axis : $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

- Foci : $(h+c,k)$, $(h-c,k)$
- Asympetotes : $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 0 \Rightarrow \frac{(y-k)^2}{b^2} = \frac{(x-h)^2}{a^2}$

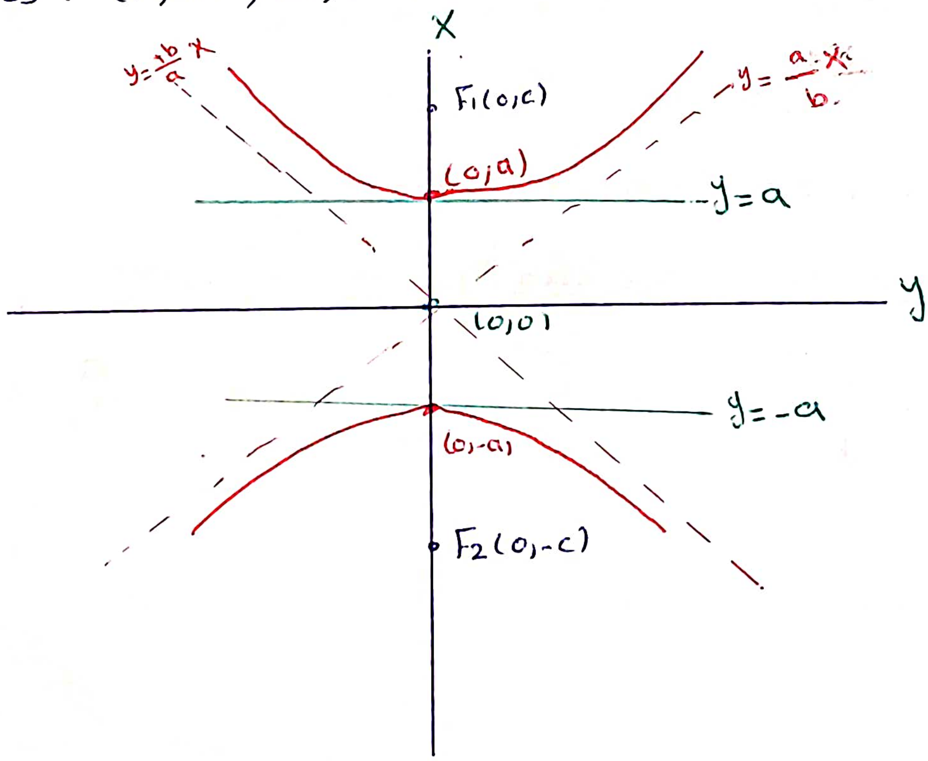
$$\Rightarrow \boxed{\frac{y-k}{b} = \mp \frac{x-h}{a}}$$

- Vertices : $(a,0)$, $(-a,0)$

B- Focus on the y-axis : $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$

- Foci : $(h,k+c)$, $(h,k-c)$
- Asympetotes : $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 0 \Rightarrow \boxed{\frac{y-k}{a} = \mp \frac{x-h}{b}}$

- Vertices : $(0,a)$, $(0,-a)$



Example : $\frac{x^2}{4} - \frac{y^2}{5} = 1$ , The equation of hyparabola where the foci in x-axis.

Sol.

with $a^2 = 4$ , $b^2 = 5$ , and Center $(0,0)$, we have:

$a^2 = 4 \Rightarrow a = \mp 2$ , $(2,0), (-2,0)$

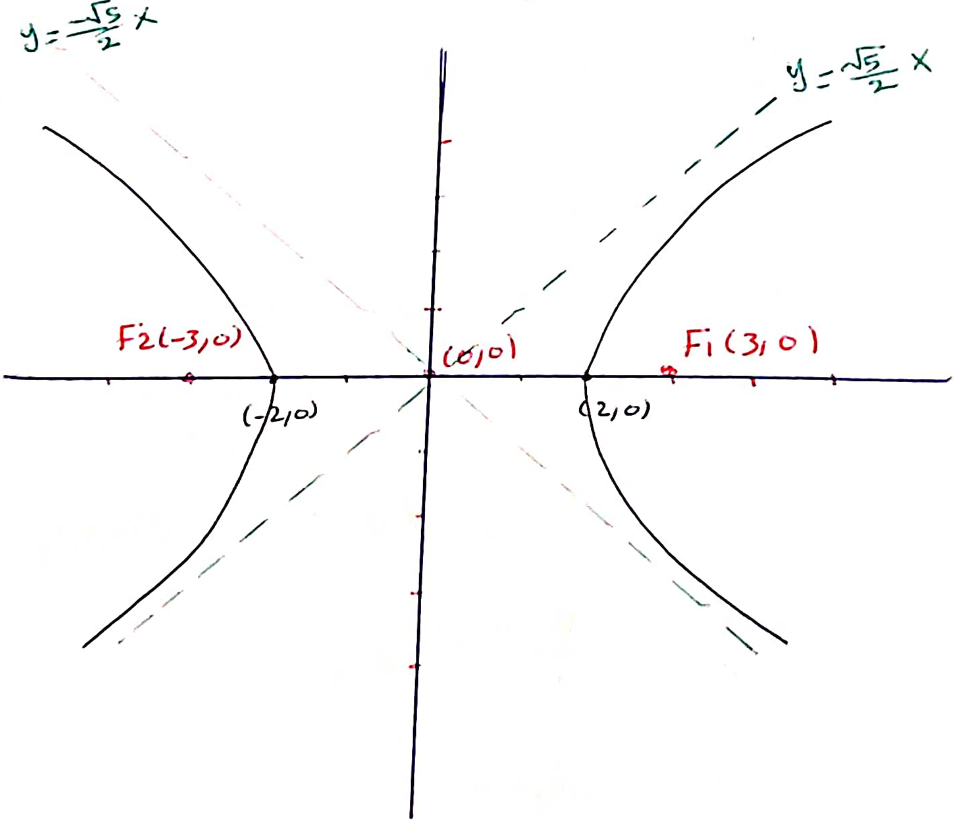
$b^2 = 5 \Rightarrow b = \mp \sqrt{5}$

$c = \sqrt{a^2 + b^2} = \sqrt{4+5} = \sqrt{9} = \mp 3$

The foci : $(3,0), (-3,0)$

The Asymptotes eq. $y = \mp \frac{b}{a} x$

$y = \mp \frac{\sqrt{5}}{2} x$



Ex: Show that the equation $x^2-4y^2+2x+8y-7=0$ represented a hyperabola, find the Center, foci, and asymptotes.

Sol: $x^2-4y^2+2x+8y-7=0$

$(x^2+2x)-(4y^2-8y)-7=0$

$(x^2+2x+1-1)-4(y^2-2y+1-1)-7=0$

$(x+1)^2-1-4(y-1)^2+4-7=0$

$(x+1)^2-4(y-1)^2-4=0$

$(x+1)^2-4(y-1)^2=4 \quad ]\div 4$

$$\boxed{\frac{(x+1)^2}{4}-\frac{(y-1)^2}{1}=1}$$ the hyperabola eq. with

the center $(\overset{h}{-1},\overset{k}{1})$

$a^2=4 \Rightarrow a=\mp 2$, vertices : $(2,0),(-2,0)$

$b^2=1 \Rightarrow b=\mp 1$

Foci: $c=\sqrt{4+1}=\sqrt{5}$, $(-1+\sqrt{5},1),(-1-\sqrt{5},1)$

Asymptotes: $\frac{(x+1)^2}{4}-\frac{(y-1)^2}{1}=0$

$y-1=\mp\frac{x+1}{2}$

$y=\frac{x+1}{2}+1 \Rightarrow \boxed{y=\frac{x+3}{2}}$

$y=-\frac{x+1}{2}+1 \Rightarrow \boxed{y=\frac{-x+1}{2}}$

Ex: Find the Center and vertices and foci, Asymptotes of the equation: $9y^2 - 16x^2 = 144$

Sol:

$$9y^2 - 16x^2 = 144 \quad ] \div 144$$

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$ , the eq. of hyperabola with origin

Center on y- axis.

Vertices: $a^2 = 16 \Rightarrow a = \mp 4$ , $(0,4)$, $(0,-4)$

$b^2 = 9 \Rightarrow b = \mp 3$

$c = \sqrt{16+9} = \sqrt{25} = \mp 5$ , foci is $(0,5)$, $(0,-5)$

asymptotes: $y = \mp \frac{a}{b} x$

$$y = \mp \frac{4}{3} x$$

H-W: Find the foci, Center, Vertees, and asymptotes of the following eq.

1. $\frac{x^2}{9} - \frac{y^2}{4} = 1$

2. $\frac{y^2}{16} - x^2 = 1$

3. $x^2 - y^2 = 1$

4. $\frac{(x+2)^2}{9} - \frac{(y-1)^2}{16} = 1$.

## 1-3. Classifying Conic Sections by Eccentricity.

**Def:**

1. The eccentricity of Parabola is $e = 1$.

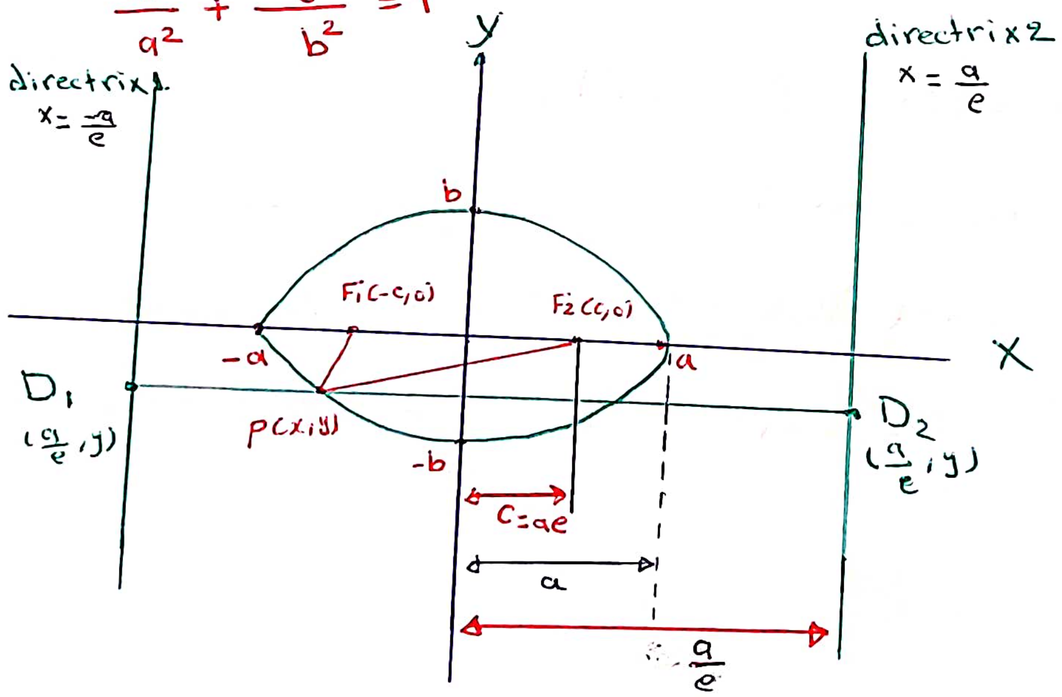
2. The eccentricity of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ is the number $e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} < 1$, $0 < e < 1$.

3. The eccentricity of the hyparabola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the number $e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}$, $e > 1$.

$$\text{Eccentricity} = \frac{\text{Distance between foci}}{\text{Distance between Vertices}} = \frac{2c}{2a} = \frac{c}{a}$$

- The eccentricity in ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



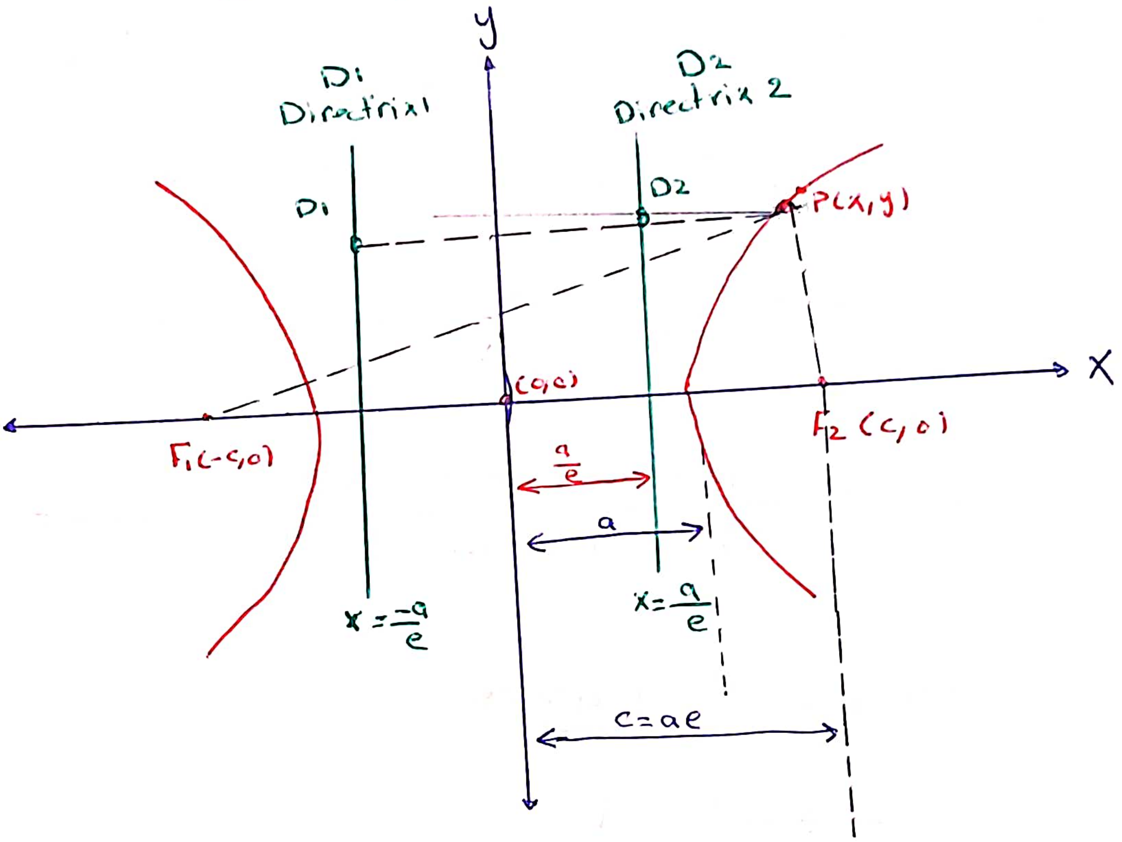

$PF_1 = e\,PD_1$ , $PF_2 = e\,PD_2$

where $F_1$, $F_2$ are Focus , $P$ is a Point belong to ellipse

$D_1$ is a point belong to directrix $x = \frac{-a}{e}$

$D_2$ is a point belong to directrix $x = \frac{a}{e}$

- The eccentricity in hyperabola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$




$PF_1 = e\,PD_1$ , $PF_2 = e\,PD_2$

where $F_1$, $F_2$ are Foci , $e = \frac{c}{a}$ is eccentricity ,

P is a point belong to hyperabola , $D_1$ is a point in directrix $x = \frac{-a}{e}$ , $D_2$ is a point belong to directrix $x = \frac{a}{e}$

Ex: Find the eccentricity of the hy perabola

$$9x^2 - 16y^2 = 144.$$

Sol:

$$9x^2 - 16y^2 = 144 \;]\div 144$$

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1 \Rightarrow \frac{x^2}{16} - \frac{y^2}{9} = 1$$

with $a^2 = 16 \Rightarrow a = 4$ , $(4,0), (-4,0)$ is the vertices

and $b^2 = 9 \Rightarrow b = \pm 3$

The foci $c = \sqrt{a^2+b^2} = \sqrt{16+9} = \sqrt{25} = 5$

$$e = \frac{c}{a} = \frac{5}{4}.$$

H.W: Find the eccentricity and directrix for:

1. $2x^2 - y^2 = 6$

2. $144y^2 - 25x^2 = 3600$

## 1.2. General Quadratic Equation and Rotations

المعادلة التربيعية العامة والدوران

**Theorem:** General Quadratic Equation

Consider asecond degree equation:

$$\boxed{Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0} \quad \text{--- } (*)$$

The general quadratic eq.

where A, B, C, D, E, F are Coefficients then the discriminant at eq. (*) is $\boxed{B^2 - 4AC}$

1. if $B^2 - 4AC = 0$ The eq (*) represented Perabola.
2. if $B^2 - 4AC < 0$ The eq (*) represented ellipse.
3. if $B^2 - 4AC > 0$ The eq (*) represented hyperabola.

**Ex:** show that the following equation represented Parabola, ellipse, hy Perabola.

1. $2x^2 + 3xy + 2y^2 - x - 6 = 0$

**Sol:**

Comparing with: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

then $A = 2$, $B = 3$, $C = 2$, $D = -1$, $E = -6$, $F = 0$

$$B^2 - 4AC = (3)^2 - 4 * 2 * 2$$

$$= 9 - 16 = -7 < 0$$

this equation represented ellipse eq.

2. $x^2+6xy+y^2-1=0$

Sol: Comparing with $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$A=1$, $B=6$, $C=1$, $D=0$, $E=0$, $F=-1$

discriminant is $B^2-4AC = 36-4*1*1 = 32 > 0$

This eq. represented by hy perabola eq.

3. $x^2-2xy+y^2+x-y+3=0$

Sol:

Comparing with $Ax^2+Bxy+Cy^2+Dx+Ey+F=0$

$A=1$, $B=-2$, $C=1$, $D=1$, $E=-1$, $F=3$

discriminant is $B^2-4AC=4-4=0$

This eq. represented by perabola eq.

H-W:

show that the following equation represented Parabola, ellipse, and hy perabola:

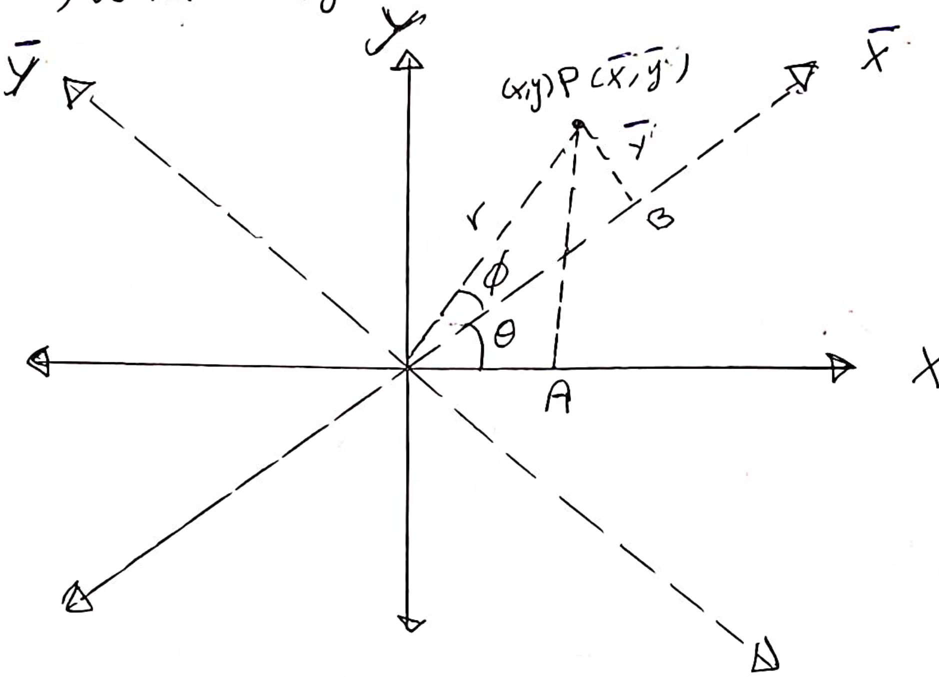
1. $3x^2-9xy+2y^2+2=0$

2. $4x^2-2xy+y^2-5x-1=0$

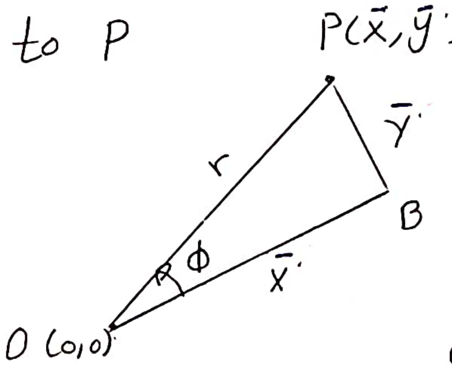
3. $x^2+2xy+y^2+x-y=0$

## - Rotation Quadratic Equation. دوران المعادلات التربيعية

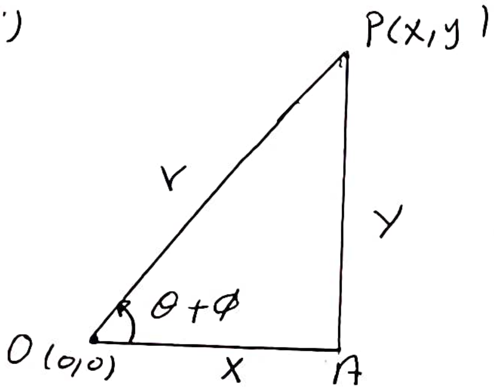
If arectangular x-y- Coordinate system is arotated through an angle $\theta$ to form an $\bar{x}\bar{y}$-Coordinate system, then a point $P(x,y)$ will have Coordinates $P(\bar{x},\bar{y})$ in the new system, where $(x,y)$ and $(\bar{x},\bar{y})$ related by:




First introduce anew pair of variables $r$ and $\phi$, represented respectively, the distance from P to origin and the angle fromed by the $\bar{x}$-axis and the line connecting the origin to P



(a)



(b)

From fig.(a) the triangle OBP, we see that:

$$\boxed{\bar{x} = r\cos\phi} \quad , \text{ and } \boxed{\bar{y} = r\sin\phi}$$

and From fig(b) the triangle OAP, we see that:

$x = r\cos(\phi + \theta)$ and $y = r\sin(\phi + \theta)$

$x = r\cos(\phi + \theta) = r\cos\phi\cos\theta - r\sin\phi\sin\theta$

$y = r\sin(\phi + \theta) = r\cos\phi\sin\theta + r\sin\phi\cos\theta$

Since $\bar{x} = r\cos\phi$ and $\bar{y} = r\sin\phi$, we have the following

result:

$x = \bar{x}\cos\theta - \bar{y}'\sin\theta$ and $y = \bar{x}\sin\theta + \bar{y}\cos\theta$

and

$\bar{x} = x\cos\theta + y\sin\theta$ and $\bar{y} = -x\sin\theta + y\cos\theta$

Ex: suppose that axis $xy$- Coordinate system are related through an angle of $\theta = 45^\circ$ to obtain an. $\bar{x}\bar{y}$- Coordinate system. Find the equation of the Curve $x^2 - xy + y^2 - 6 = 0$ in $\bar{x}\bar{y}$ coordinate.

Sol

$$x = \bar{x}\cos\frac{\pi}{4} - \bar{y}\sin\frac{\pi}{4} = \frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}$$

$$y = \bar{x}\sin\frac{\pi}{4} + \bar{y}\cos\frac{\pi}{4} = \bar{x}\cdot\frac{1}{\sqrt{2}} + \bar{y}\cdot\frac{1}{\sqrt{2}} = \frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}$$

substituting thes expressions in to the eq. $x^2 - xy + y^2 - 6 = 0$

$$\left(\frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}\right)^2 - \left(\frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}\right)\left(\frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}\right) + \left(\frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}\right)^2 - 6 = 0$$

$$\frac{\bar{x}^2}{2} - \cancel{2}\cdot\frac{\bar{x}\bar{y}}{\cancel{2}} + \frac{\bar{y}^2}{2} - \left(\frac{\bar{x}^2}{2} + \cancel{\frac{\bar{x}\bar{y}}{2}} - \cancel{\frac{\bar{x}\bar{y}}{2}} - \frac{\bar{y}^2}{2}\right) + \frac{\bar{x}^2}{2} + \cancel{2}\frac{\bar{x}\bar{y}}{\cancel{2}} + \frac{\bar{y}^2}{2} - 6 = 0$$

$$\cancel{\frac{\bar{x}^2}{2}} - \cancel{\bar{x}\bar{y}} + \frac{\bar{y}^2}{2} - \cancel{\frac{\bar{x}^2}{2}} + \frac{\bar{y}^2}{2} + \frac{\bar{x}^2}{2} + \cancel{\bar{x}\bar{y}} + \frac{\bar{y}^2}{2} - 6 = 0$$

$$\frac{\bar{x}^2}{2} + \frac{3\bar{y}^2}{2} = 6 \quad ] \div 6$$

$$\boxed{\frac{\bar{x}^2}{12} + \frac{\bar{y}^2}{4} = 1}$$ the ellipse equation

Ex: suppose that axis $xy$- Coordinate system are related through an angle of $\theta = 45^\circ$ to obtain an. $\bar{x}\bar{y}$- Coordinate system. Find the equation of the Curve $x^2 - xy + y^2 - 6 = 0$ in $\bar{x}\bar{y}$ coordinate.

Sol

$$x = \bar{x}\cos\frac{\pi}{4} - \bar{y}\sin\frac{\pi}{4} = \frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}$$

$$y = \bar{x}\sin\frac{\pi}{4} + \bar{y}\cos\frac{\pi}{4} = \bar{x}\cdot\frac{1}{\sqrt{2}} + \bar{y}\cdot\frac{1}{\sqrt{2}} = \frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}$$

substituting thes expressions into the eq. $x^2 - xy + y^2 - 6 = 0$

$$\left(\frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}\right)^2 - \left(\frac{\bar{x}}{\sqrt{2}} - \frac{\bar{y}}{\sqrt{2}}\right)\left(\frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}\right) + \left(\frac{\bar{x}}{\sqrt{2}} + \frac{\bar{y}}{\sqrt{2}}\right)^2 - 6 = 0$$

$$\frac{\bar{x}^2}{2} - \cancel{2}\cdot\frac{\bar{x}\bar{y}}{\cancel{2}} + \frac{\bar{y}^2}{2} - \left(\frac{\bar{x}^2}{2} + \cancel{\frac{\bar{x}\bar{y}}{2}} - \cancel{\frac{\bar{x}\bar{y}}{2}} - \frac{\bar{y}^2}{2}\right) + \frac{\bar{x}^2}{2} + \cancel{2}\frac{\bar{x}\bar{y}}{\cancel{2}} + \frac{\bar{y}^2}{2} - 6 = 0$$

$$\cancel{\frac{\bar{x}^2}{2}} - \cancel{\bar{x}\bar{y}} + \frac{\bar{y}^2}{2} - \cancel{\frac{\bar{x}^2}{2}} + \frac{\bar{y}^2}{2} + \frac{\bar{x}^2}{2} + \cancel{\bar{x}\bar{y}} + \frac{\bar{y}^2}{2} - 6 = 0$$

$$\frac{\bar{x}^2}{2} + \frac{3\bar{y}^2}{2} = 6 \quad ] \div 6$$

$$\boxed{\frac{\bar{x}^2}{12} + \frac{\bar{y}^2}{4} = 1}$$ the ellipse equation

Ex: Find the new coordinates of the point (2,4) if the coordinate axes are retated through the angle $\theta = 30^\circ$.

Sol.

$\bar{x} = x\cos\theta + y\sin\theta = 2\cos 30 + 4\sin 30$

$= \cancel{2} * \frac{\sqrt{3}}{\cancel{2}} + 4 * \frac{1}{2} = 2 + \sqrt{3}$

$\bar{y} = -x\sin\theta + y\cos\theta = -2 * \sin 30 + 4 * \cos 30$

$= \frac{-2}{2} + 4 * \frac{\sqrt{3}}{2} = -1 + 2\sqrt{2}$

$\therefore (2+\sqrt{3}, -1+2\sqrt{2})$ is a point in $\bar{x}\bar{y}$ - coordinate system

H.w: show that the graph of the equation $xy = 1$ is a Hyperabola by rolated the $xy$ - axes through an angle of $\pi/4$.

If we apply the rotation equations

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta$$

$$y = \bar{x}\sin\theta + \bar{y}\cos\theta$$

to the general quadratic equation (*), we obtain a new quadratic equation. $A\bar{x}^2 + B\bar{x}\bar{y} + C\bar{y}^2 + D\bar{x} + E\bar{y} + F = 0$ ...(A)

$$A(\bar{x}\cos\theta - \bar{y}\sin\theta)^2 + B(\bar{x}\cos\theta - \bar{y}\sin\theta)(\bar{x}\sin\theta + \bar{y}\cos\theta)$$

$$+ C(\bar{x}\sin\theta + \bar{y}\cos\theta)^2 + D(\bar{x}\cos\theta - \bar{y}\sin\theta) + E(\bar{x}\sin\theta +$$

$$\bar{y}(\cos\theta) + F = 0$$

then

$$\bar{x}^2\left(A(\cos\theta)^2 + B(\sin\theta\cos\theta) + C(\sin\theta)^2\right) +$$

$$\bar{x}\bar{y}\left[-2A\cos\theta\sin\theta + B((\cos\theta)^2 - (\sin\theta)^2) + 2C\sin\theta\cos\theta\right]$$

$$+ \bar{y}^2\left(A(\sin\theta)^2 - B(\sin\theta\cos\theta) + C(\cos\theta)^2\right) +$$

$$\bar{x}(D\cos\theta + E\sin\theta) + \bar{y}(-D\sin\theta + E\cos\theta) + F = 0$$

The new coefficients are related to the old ones by the equations:

$$\bar{A} = A\cos^2\theta + B\cos\theta\sin\theta + C\sin^2\theta$$

$$\bar{B} = B\cos 2\theta + (C - A)\sin 2\theta$$

$$\bar{C} = A\sin^2\theta - B\sin\theta\cos\theta + C\cos^2\theta$$

$$\bar{D} = D\cos\theta + E\sin\theta$$

$$\bar{E} = -D\sin\theta + E\cos\theta \qquad \ldots\ldots (7)$$
$$\bar{F} = F$$

To find $\theta$, Put $\bar{B}=0$ in the second equation in (7) and solve the resulting equation

$B\cos 2\theta + (C-A)\sin 2\theta = 0$ , Then find $\theta$ from:

$$\left.\begin{aligned} &\cot 2\theta = \frac{A-C}{B} \quad \text{or} \quad \tan 2\theta = \frac{B}{A-C} \\ &\text{or} \quad \frac{\cos 2\theta}{\sin 2\theta} = \frac{A-C}{B} \end{aligned}\right\} \ldots (**)$$

Remark

at is always Possible to satisfy $(**)$ with an angle $\theta$ in the range $0 < \theta < \frac{\pi}{2}$.

Ex : Identify and sketch the curve $xy = 1$

sol :

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
$$B = 1 , \ A = C = D = E = 0 \ ; \ F = -1$$

$$\cot 2\theta = \frac{A-C}{B} , \ B \neq 0 \quad , \ \cot 2\theta = \frac{0-0}{1} = 0$$

$$2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$x = \bar{x}\cos\theta - \bar{y}\sin\theta$$

$$x = \frac{1}{\sqrt{2}}\bar{x} - \frac{1}{\sqrt{2}}\bar{y}$$

$$y = \bar{x}\sin\theta + \bar{y}\cos\theta$$

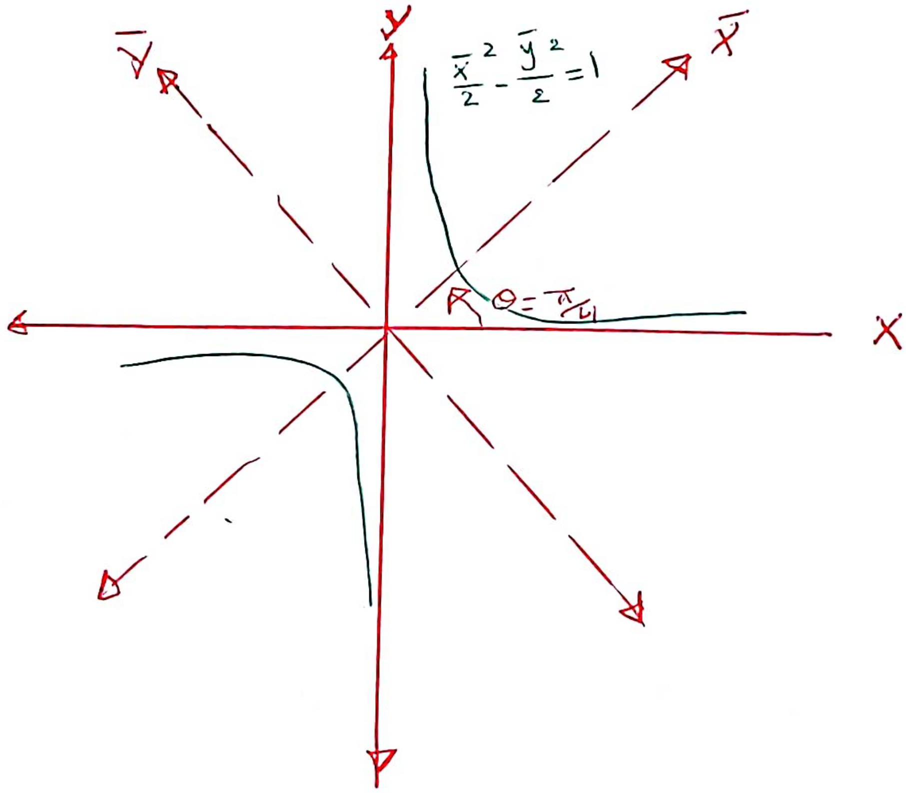
$$y = \frac{1}{\sqrt{2}}\bar{x} + \frac{1}{\sqrt{2}}\bar{y}$$

$$x \cdot y = 1$$

$$\left(\frac{1}{\sqrt{2}}\bar{x} - \frac{1}{\sqrt{2}}\bar{y}\right)\left(\frac{1}{\sqrt{2}}\bar{x} + \frac{1}{\sqrt{2}}\bar{y}\right) = 1$$

$$\frac{\bar{x}^2}{2} + \cancel{\frac{\bar{x}\bar{y}}{2}} - \cancel{\frac{\bar{x}\bar{y}}{2}} - \frac{\bar{y}^2}{2} = 1$$

$\frac{\bar{x}^2}{2} - \frac{\bar{y}^2}{2} = 1$ , the hyperabola eq. with Center $(0,0)$ and vertix $(\sqrt{2},0)$ $(-\sqrt{2},0)$



Ex: The coordinate axes are to be related through an angle $\theta$ to produce an equation for the curve

$$x^2+xy+y^2-6=0$$

Sol

$$A=1,\ B=1,\ C=1,\ D=E=0,\ F=-6$$

$$\cot 2\theta=\frac{A-C}{B}=\frac{1-1}{1}=0 \quad ,$$

$$2\theta=\frac{\pi}{2} \Rightarrow \theta=\frac{\pi}{4}$$

$$x=\bar{x}\cos\theta-\bar{y}\sin\theta$$

$$y=\bar{x}\sin\theta+\bar{y}\cos\theta$$

$$(\bar{x}\cos\theta-\bar{y}\sin\theta)^2+(\bar{x}\cos\theta-\bar{y}\sin\theta)(\bar{x}\sin\theta+\bar{y}\cos\theta)$$

$$+(\bar{x}\sin\theta+\bar{y}\cos\bar{\theta})^2-6=0$$

$$(\bar{x}\cos\frac{\pi}{4}-\bar{y}\sin\frac{\pi}{4})^2+(\bar{x}\cos\frac{\pi}{4}-\bar{y}\sin\frac{\pi}{4})(\bar{x}\sin\frac{\pi}{4}+\bar{y}\cos\frac{\pi}{4})$$

$$+(\bar{x}\sin\frac{\pi}{4}+\bar{y}\cos\frac{\pi}{4})^2-6=0$$

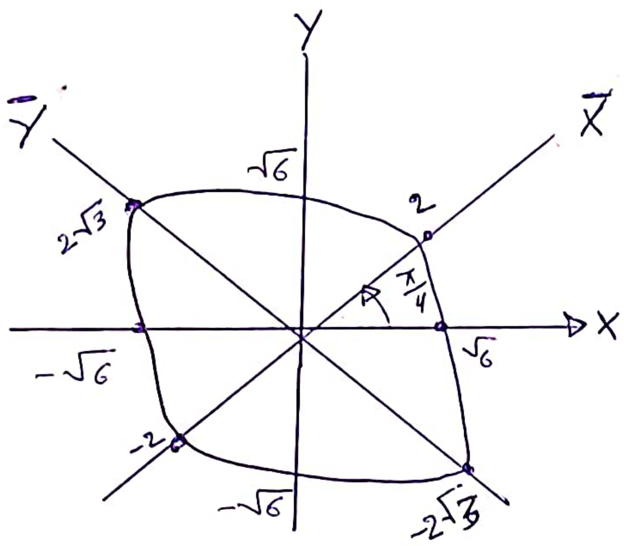
$$(\frac{\bar{x}}{\sqrt{2}}-\frac{\bar{y}}{\sqrt{2}})^2+(\frac{\bar{x}}{\sqrt{2}}-\frac{\bar{y}}{\sqrt{2}})(\frac{\bar{x}}{\sqrt{2}}+\frac{\bar{y}}{\sqrt{2}})+(\frac{\bar{x}}{\sqrt{2}}+\frac{\bar{y}}{\sqrt{2}})^2-6=0$$

$$\frac{1}{2}\bar{x}^2-\cancel{\bar{x}\bar{y}}+\cancel{\frac{1}{2}\bar{y}^2}+\frac{1}{2}\bar{x}^2+\cancel{\frac{1}{2}\bar{x}\bar{y}}-\cancel{\frac{1}{2}\bar{x}\bar{y}}-\cancel{\frac{1}{2}\bar{y}^2}+\frac{1}{2}\bar{x}^2$$

$$+\bar{x}\bar{y}+\frac{1}{2}\bar{y}^2-6=0$$

$$\frac{3}{2}\bar{x}^2+\frac{1}{2}\bar{y}^2-6=0 \quad \text{or} \Rightarrow \frac{\bar{x}^2}{4}+\frac{\bar{y}^2}{12}=1$$

This is ellipse eq. with center $(0,0)$ on $y$-axis




The ellipse eq. $\frac{\bar{x}^2}{4} + \frac{\bar{y}^2}{12} = 1$

**Ex:** Skech the graph of $4x^2 - 4xy + 7y^2 - 24 = 0$

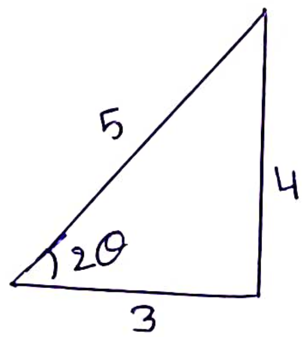
**Remark:** $\cot 2\theta = \frac{A-C}{B}$, $B \neq 0$, then

$$\cos\theta = \sqrt{\frac{1+\cos 2\theta}{2}}, \quad \sin\theta = \sqrt{\frac{1-\cos 2\theta}{2}}$$

Sol: rotate the Coordinate axes through the angle $\theta$

s.t

$$\cot 2\theta = \frac{A-C}{B} = \frac{4-7}{-4} = \frac{3}{4}$$




from the triangle, see that

$$\cos 2\theta = \frac{3}{5}$$

by remark above, to get

$$\cos\theta = \sqrt{\frac{1+\frac{3}{5}}{2}}, \quad \sin\theta = \sqrt{\frac{1-3/5}{2}}$$

$$\cos\theta = \frac{2\sqrt{5}}{5}, \quad \sin\theta = \frac{\sqrt{5}}{5}$$

$$x = \frac{2\sqrt{5}}{5}\bar{x} - \frac{\sqrt{5}}{5}\bar{y} \quad \& \quad y = \frac{\sqrt{5}}{5}\bar{x} + \frac{2\sqrt{5}}{5}\bar{y}$$

Substituting in $4x^2 - 4xy + 7y^2 - 24 = 0$ , to get:

$$4\left(\frac{2\sqrt{5}}{5}\bar{x} - \frac{\sqrt{5}}{5}\bar{y}\right)^2 - 4\left(\frac{2\sqrt{5}}{5}\bar{x} - \frac{\sqrt{5}}{5}\bar{y}\right)\left(\frac{\sqrt{5}}{5}\bar{x} + \frac{2\sqrt{5}}{5}\bar{y}\right) +$$

$$7\left(\frac{\sqrt{5}}{5}\bar{x} + \frac{2\sqrt{5}}{5}\bar{y}\right)^2 - 24 = 0$$

$$4\left(\frac{4*5}{25}\bar{x}^2 - \frac{4*5}{25}\bar{x}\bar{y} + \frac{5}{25}\bar{y}^2\right) - 4\left(\frac{10}{25}\bar{x}^2 + \frac{20}{25}\bar{x}\bar{y} - \frac{5}{25}\bar{x}\bar{y} - \frac{10}{25}\bar{y}^2\right)$$

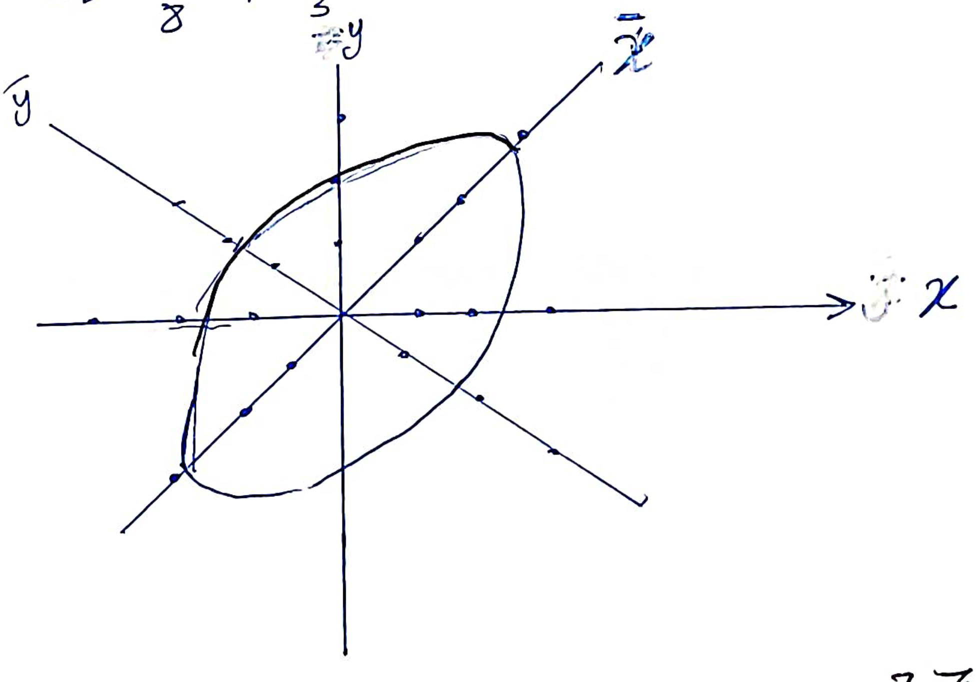
$$+ 7\left(\frac{5}{25}\bar{x}^2 + \frac{20}{25}\bar{x}\bar{y} + \frac{20}{25}\bar{y}^2\right) - 24 = 0$$

$$\frac{80}{25}\bar{x}^2 - \frac{80}{25}\bar{x}\bar{y} + \frac{20}{25}\bar{y}^2 - \frac{40}{25}\bar{x}^2 - \cancel{\frac{80}{25}\bar{x}\bar{y}} + \cancel{\frac{20}{25}\bar{x}\bar{y}} + \frac{40}{25}\bar{y}^2 + \frac{35}{24}\bar{x}^2$$

$$+ \cancel{\frac{140}{25}\bar{x}\bar{y}} + \frac{140}{25}\bar{y}^2 - 24 = 0$$

$$\frac{75}{25}\bar{x}^2 + \frac{200}{25}\bar{y}^2 = 24 \Rightarrow 3\bar{x}^2 + 8\bar{y}^2 = 24 \;]\div 24$$

$$\frac{3}{24}\bar{x}^2 + \frac{8}{24}\bar{y}^2 = 1 \Rightarrow \frac{\bar{x}^2}{8} + \frac{\bar{y}^2}{3} = 1$$



# Parametic Equations for Plane Curve

Def: If $x$ and $y$ are given as functions:

$$x = f(t), \qquad y = g(t) \qquad \text{....(1)}$$

over an interval $t$-values, then the set of Points $(x,y) = (f(t), g(t))$ defined by these equations is called a curve in the coordinate plane.

Remark:

1. The variable $t$ is the parameter of the curve.
2. The Domain $I$ is called is called the parameter interval, if $I$ is closed interval, $a \leq t \leq b$, $t \in [a,b]$
3. the Point $(f(a), g(a))$ is the initial point of the curve and $(f(b), g(b))$ is the terminal point of curve
4. when we write parametric equations for a curve in the Plane, we say that, we have Parameterize the curve.
5. The equation and interval togather constitute a parameterized of the curve

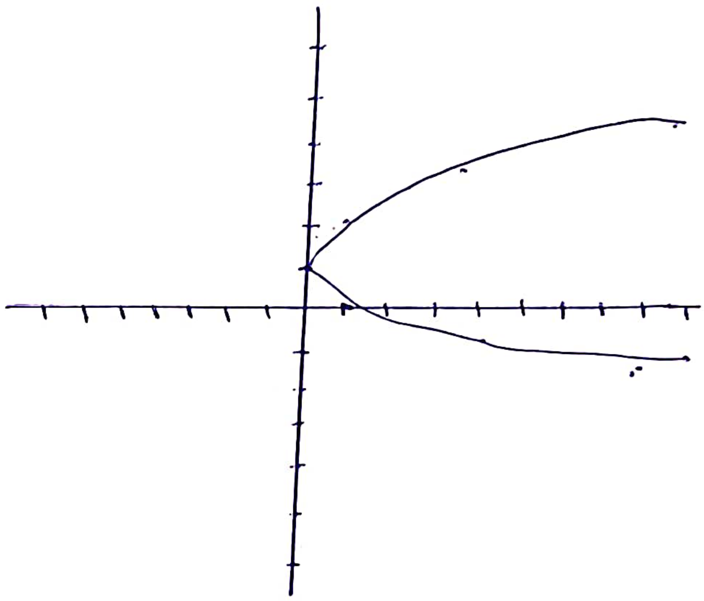
6- A given curve can be represented by different sets of parametic equations.

Ex: Sketch the curve defined by parametic eqs.

$$x = t^2 \quad , \quad y = t + 1 \quad , \quad -\infty < t < \infty$$

Sol

| $t$ | $x$ | $y$ | $(x, y)$ |
|---|---|---|---|
| $-3$ | $9$ | $-2$ | $(9, -2)$ |
| $-2$ | $4$ | $-1$ | $(4, -1)$ |
| $-1$ | $1$ | $0$ | $(1, 0)$ |
| $0$ | $0$ | $1$ | $(0, 1)$ |
| $1$ | $1$ | $2$ | $(1, 2)$ |
| $2$ | $4$ | $3$ | $(4, 3)$ |
| $3$ | $9$ | $4$ | $(9, 4)$ |



$t = y - 1 \Rightarrow x = (y-1)^2, \quad x = y^2 - 2y + 1$

$y^2 - 2y - x + 1 = 0$

Comparing with: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

$A = 0, \; B = 0, \; C = 1, \; D = -1, \; E = -2, \; F = 1,$

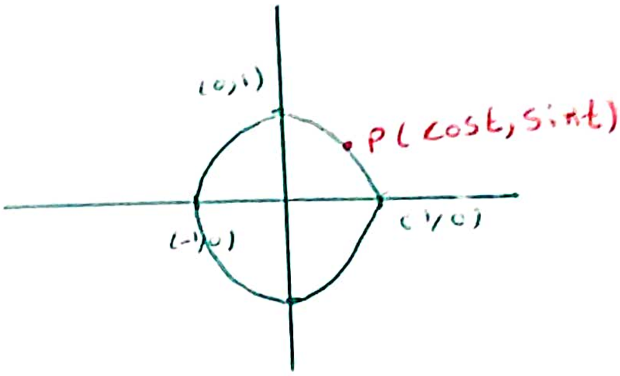
by discreminant: $B^2 - 4AC = 0 - 4 * 0 * 1 = 0$

$\therefore$ This equation is parabola.

**Ex:** graph the Parametric curve

$x = \cos t$ , $y = \sin t$ , $0 \le t \le 2\pi$

| $t$ | $x$ | $y$ | $(x,y)$ |
|---|---|---|---|
| $\pi$ | $-1$ | $0$ | $(-1,0)$ |
| $0$ | $1$ | $0$ | $(1,0)$ |
| $\frac{\pi}{2}$ | $0$ | $1$ | $(0,1)$ |




$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$

$\therefore (x^2 + y^2) = 1$

**Ex:** Find a parametrazation for the line through the point $(a,b)$ having slope $m$.

**Sol** $(y - b) = m(x - a) \quad \cdots (1)$ معادلة المستقيم

let $x - a = t \Rightarrow x = t + a$

subtituting $x$ in eq. (1), to get

$y - b = mt \Rightarrow y = mt + b$

$\therefore x = t + a$ , $y = mt + b$

**H-W:** graph the paramebic curve $x = \sqrt{t}$ , $y = t$ , $t \ge 0$

Ex: Find the Parametic eq. for the Circle $x^2+y^2=a^2$

Sol:

$$x = a\cos t \quad , \quad y = a\sin t \quad , \quad 0 \le t \le 2\pi$$

$$x^2+y^2 = a^2\cos^2 t + a^2\sin^2 t$$

$$\Rightarrow a^2(\underbrace{\cos^2 t + \sin^2 t}_{1}) = a^2$$

H.W: Find the Parametic equation for the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Remark:

General Parametic equation

1. Circle $x^2+y^2=a^2$ $\quad$ $x = a\cos t$, $y = a\sin t$ $\quad$ , $0 \le t \le 2\pi$

2. Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ , $\quad$ $x = a\cos t$, $y = b\sin t$ $\quad$ , $0 \le t \le 2\pi$

## The Derivative parametic equations:

**Def:** A Parametrized curve $x = f(t)$, $y = g(t)$ is said to be differentable at $t = t_0$ and $g$ is differentable at $t = t_0$. The curve is differentable if it is differentable at every paramet value.

At a point on a differentable parametrized curve where $y$ is also differentable function of $x$, the derivatives $dx/dt$ and $dy/dt$ and $dy/dx$ are related by chain rule.

* Formula for finding $dy/dx$ from $dy/dt$ and $dx/dt$ s.t $dx/dt \neq 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

* formula for finding $d^2y/dx^2$ from $dx/dt$ and $y' = dy/dx$ s.t $dx/dt \neq 0$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Ex: Find the tangent to the curve:

$$x = \sec t \quad , \quad y = \tan t \quad , \quad \frac{-\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$ where $t = \pi/4$

Sol:

$$(y - y_0) = m(x - x_0)$$

$$m = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sec^{\cancel{2}} t}{\cancel{\sec t} \tan t} = \frac{\sec t}{\tan t}$$

$\because t = \pi/4$ , then

$$\frac{dy}{dx} = \frac{\sec \pi/4}{\tan \pi/4} = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\therefore y - y_0 = m(x - x_0)$$

$$y - 1 = \sqrt{2}(x - \sqrt{2})$$

$$y = \sqrt{2}x - 2 + 1$$

$$y = \sqrt{2}x - 1$$

Ex: Find $d^2y/dx^2$ if $x = t - t^2$ and $y = t - t^3$

Sol: $y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-3t^2}{1-2t}$

$$\frac{dy'}{dt} = \frac{(1-2t)*(-6t) - (1-3t^2)*(-2)}{(1-2t)^2}$$

$$= \frac{-6t + 12t^2 + 2 - 6t^2}{(1-2t)^2} = \frac{6t^2 - 6t + 2}{(1-2t)^2}$$

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{\frac{6t^2-6t+2}{(1-2t)^2}}{1-2t} = \frac{6t^2-6t+2}{(1-2t)^3}$$

## The Length of a Parametic Curve:

Def: If the functions $x = f(t)$ and $y = g(t)$ have continuous first derivatives with respect to $t$, for $a \le t \le b$, and if the point $P(x, y)$ traces the curve defined by these equations exactly ones as $t$ moves from $t = a$ to $t = b$, then the length of the curve is define by:

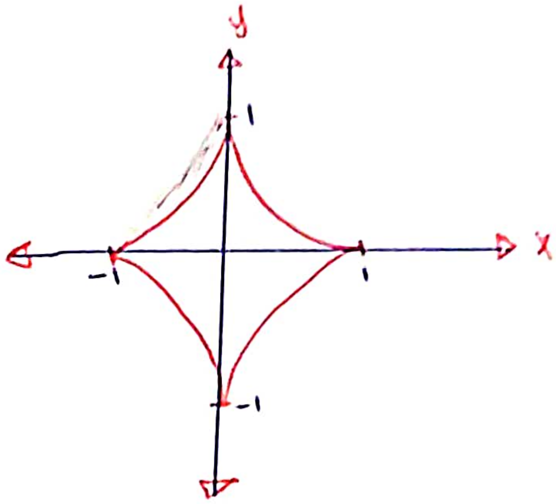
$$\text{Length} = L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: Find the Length of the Curve:

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$$

Sol: we find the length of the first-quadrant

Part, $0 \le t \le \pi/2$




$$\frac{dx}{dt} = -3\cos^2 t \sin t$$

$$\frac{dy}{dt} = 3\sin^2 t \cos t$$

$$L = \int_0^{\pi/2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t (\underbrace{\cos^2 t + \sin^2 t}_{1})}\, dt$$

$$= \int_0^{\pi/2} \sqrt{9\cos^2 t \sin^2 t}\, dt$$

# The Area of the Surface of Revolution

If the functions $x = f(t)$ and $y = g(t)$ have continuous First dirivatives with respect to $t$ for $a \leq t \leq b$, and if the point $P(x, y)$ traces the curve defined by these equations exactly ones as $t$ moves from $t = a$ to $t = b$, Then the areas of the surfaces generated by revolving the curve about the coordinate axes are:

## 1- Revolution about the x-axis ($y \geq 0$)

$$\text{Area} = A = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

## 2- Revolution about the y-axis ($x \geq 0$)

$$\text{Area} = A = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex: The standard Parametric equations for the Circle of radius 1 centered at the Point $(0,1)$ in the xy-plane are:

$x = \cos t, \quad y = 1+\sin t, \quad 0 \leq t \leq 2\pi$

use these equations to find the area of the surface about x-axis.

$$A = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{2\pi} 2\pi (1+\sin t) \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t) \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{= 1}}\, dt$$

$$= 2\pi \int_0^{2\pi} (1+\sin t)\, dt$$

$$= 2\pi \left[ t - \cos t \right]_0^{2\pi}$$

$$= 2\pi \left[ (2\pi - \cos(2\pi)) - (0 - \cos 0) \right]$$

$$= 2\pi \left[ (2\pi - 1) - (0 - 1) \right]$$

$$= 2\pi (2\pi \cancel{-1} \cancel{+1})$$

$$= 4\pi^2$$

## * Polar Coordinates

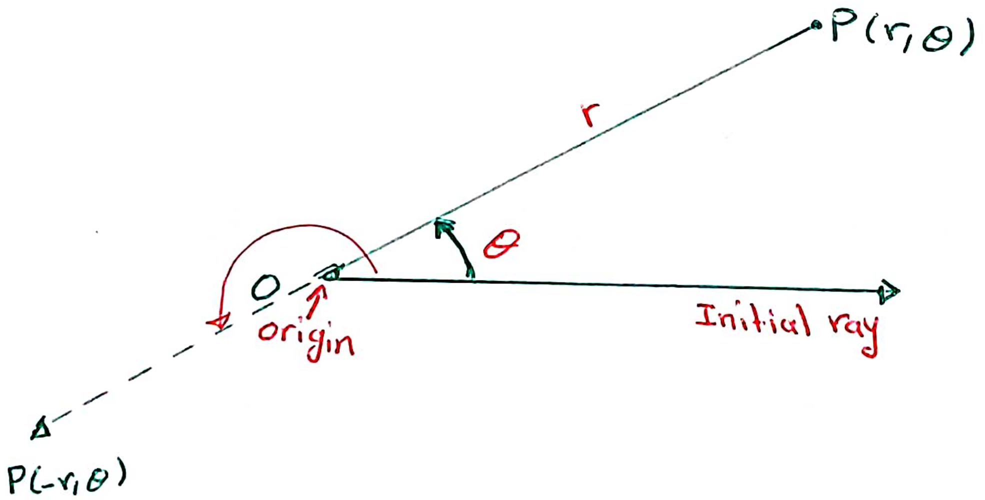
Def:

To define Polar Coordinates, we first fix an origin O and and initial ray from O. Then each Point P Can be located by assigning to it apolar Coordinate pair $(r, \theta)$. in which we first number, $r$, gives the directed distance from O to P and the second number, $\theta$, gives the directed angle from the initial ray to the segment OP.

$$P(r, \theta)$$

Directed distance from O to P

Directed angle from initial ray to OP



## Remark:

1- The angle $\theta$ is Positive when measured Counter clockwise and negative when measured clock wise.

2. $\theta$ is not unique with agiven point while apoint in the Plane has just one Pair of cartesiam Coodinated.

3- It has infinitely many pairs of Polor Coordinates, i.e.

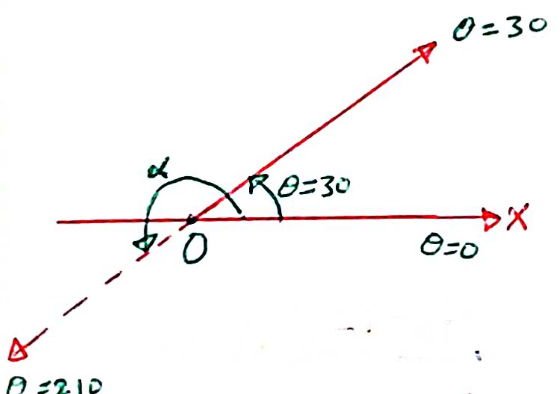
$(r, \theta + 2i\pi)$, $i = 1, 2, 3, \dots$

$r, \theta$ are positive or negative.

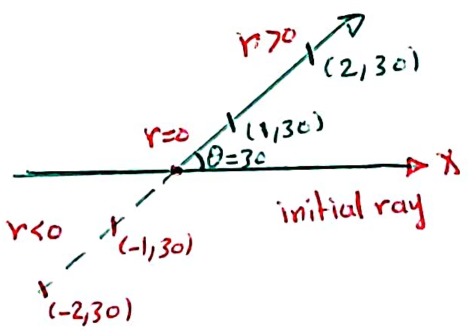
4. $r$ is Length $r = OP$, if $r$ is negative then $r = -|OP|$ and if $r$ Positive then $r = |OP|$

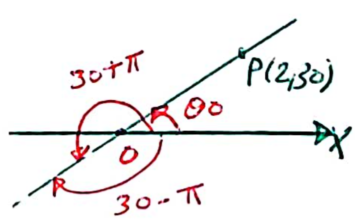
5- If $r = 0$, then P is lies in O and if $\theta = 0$, then P is lies in initial ray

6- If $r$ negativen then $r$ is exist back ward.




$P(r, 210) = p(-r, 30)$



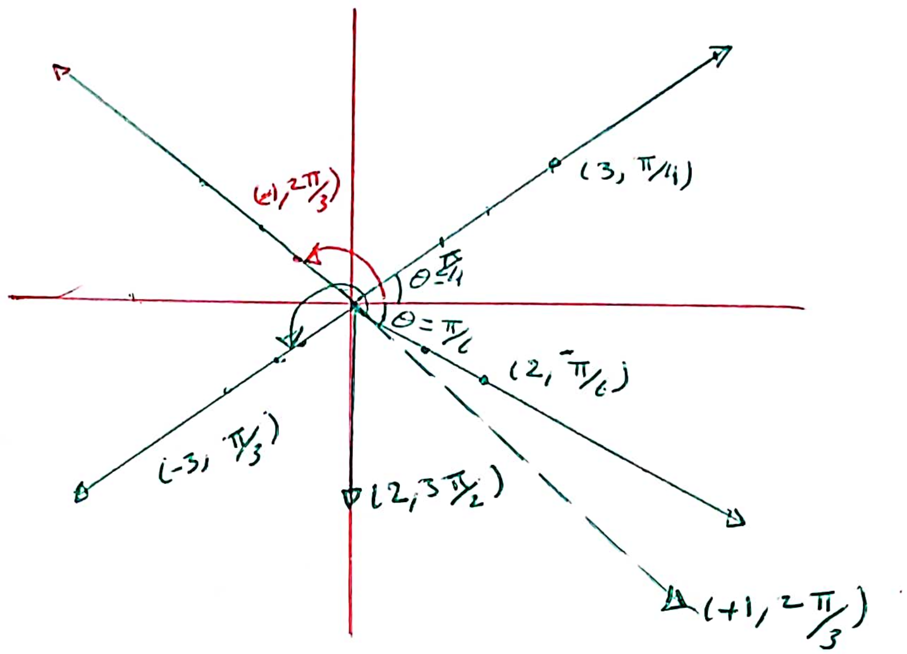




$(2,30) = (-2, 150) = (-2, 210) = $ etc

Ex: graph the Polar Coordinates:

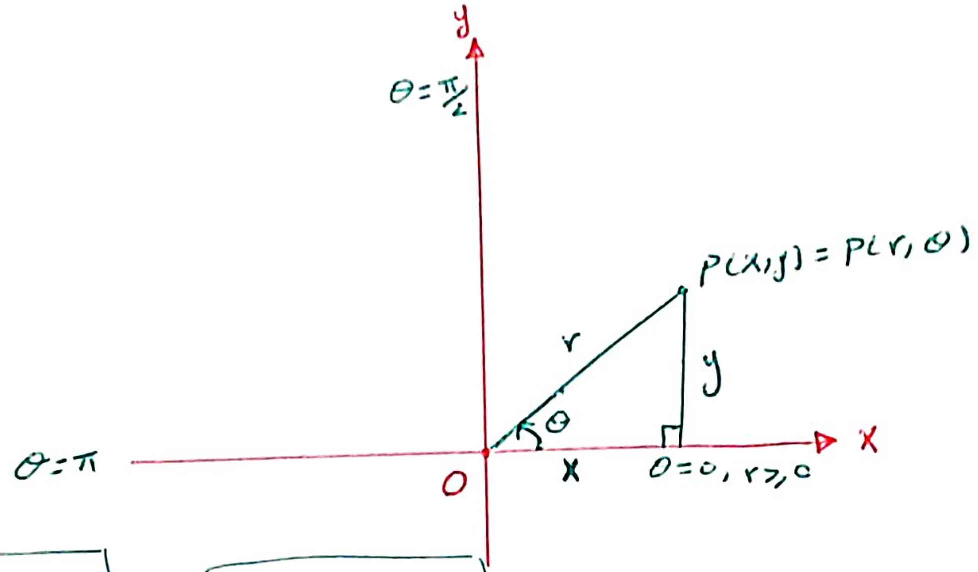
$(3, \pi/4)$, $(2, -\pi/6)$, $(-3, \pi/3)$, $(2, 3\pi/2)$, $(-1, 2\pi/3)$,

$(1, 2\pi/3)$




H.W: $(-3, \pi/6)$, $(5, \pi/6)$, $(-2, -\pi)$, $(1, \pi)$

# Equations Relating Polar and Cartesian Coordinates




$$\boxed{\cos\theta = \frac{x}{r}} \Rightarrow \boxed{x = r\cos\theta}$$

$$\boxed{\sin\theta = \frac{y}{r}} \Rightarrow \boxed{y = r\sin\theta}$$

Polar $\Rightarrow$ Cartesian

$$\boxed{r^2 = x^2 + y^2} \Rightarrow \boxed{r = \mp\sqrt{x^2+y^2}}$$

$$\boxed{\tan\theta = \frac{y}{x}} \Rightarrow \boxed{\theta = \tan^{-1}\frac{y}{x}}$$

Cartesian $\Rightarrow$ Polar

Ex: Replace $(3, \pi)$ to Cartesian Coordinates

Sol: $r = 3$, $\theta = \pi$

$x = r\cos\theta \Rightarrow x = 3\cos\pi = -3$

$y = r\sin\theta \Rightarrow y = 3\sin\pi = 0$

$\therefore$ $(3, \pi)$ equavelante $(-3, 0)$ in Cartesian Coordinates

Ex: find the Cartesian Coordinates to the Point $(-3, \pi/6)$

Sol: $r = -3$, $\theta = \pi/6$

$x = r\cos\theta \Rightarrow x = -3\cos\pi/6 = -3 * \frac{\sqrt{3}}{2} = \frac{-3\sqrt{3}}{2}$

$y = r\sin\theta \Rightarrow y = -3\sin\pi/6 = -3 * \frac{1}{2} = \frac{-3}{2}$

$\therefore$ Cartesian point is $\left(\frac{-3\sqrt{3}}{2}, \frac{-3}{2}\right)$

H.W. find the Cartesian Coordinate to the Points;

1- $(4, \frac{\pi}{2})$,

2- $(-5, \frac{\pi}{3})$

**Ex:** Replace the Point $(1/2, 0)$ to the Polar Coordinate

**Sol:** $x = 1/2$ , $y = 0$

$$r = \mp\sqrt{x^2+y^2} = \mp\sqrt{(1/2)^2+0^2} = \mp 1/2$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{1/2} = 0 \quad , \quad \theta = \mp n\pi \, , \, n = 0, 1, 2, \dots$$

**Ex:** Replace the following equations Cartesian Coordinates:

| Polar Coordinate | Cartesian equivalent |
|---|---|
| 1- $r\cos\theta = 2$ | $x = 2$ |
| 2- $r^2\cos\theta\sin\theta = 4$ | $\underbrace{r\cos\theta}_{x}\,\underbrace{r\sin\theta}_{y} = 4$ <br> $xy = 4$ |
| 3. $r^2\cos^2\theta - r^2\sin^2\theta = 1$ | $x^2 - y^2 = 1$ |
| 4. $r = 1 + 2r\cos\theta$ | $\sqrt{x^2+y^2} = 1 + 2x$ <br> $x^2+y^2 = (1+2x)^2$ <br> $x^2+y^2 = 1+4x+4x^2$ <br> $y^2 - 3x^2 - 4x - 1 = 0$ |

Ex: Replace the equation $x^2+(y-3)^2=9$ to Polar eq.

Sol $x^2+(y-3)^2=9$

$x^2+y^2-6y+\cancel{9}=\cancel{9}$

$\underbrace{x^2+y^2}_{r^2}-6y=0$

$r^2-6r\sin\theta=0$

$r(r-6\sin\theta)=0$

$\therefore r=0$ or $r-6\sin\theta=0 \Rightarrow \sin\theta=\frac{r}{6}$

H-W: Replace the following equations to the Cartesian equations:

a- $r\cos\theta=-4$

b- $r^2=4r\cos\theta$

c. $r=\dfrac{4}{2\cos\theta-\sin\theta}$

Remark:

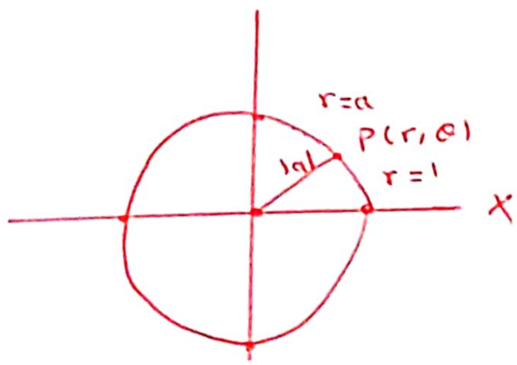
1. The graph of $r=a$ is the circle of radius $|a|$ and center at the origine

2- The graph $\theta=\theta_0$ is the line through the origin

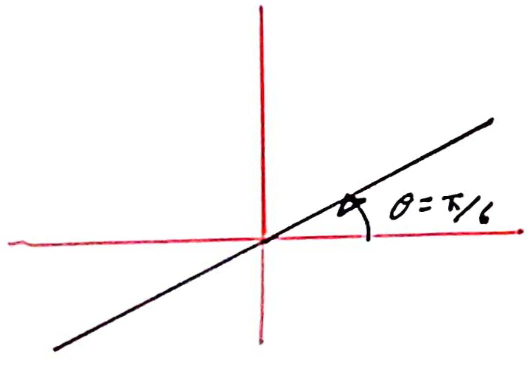
Ex: graph the set of Points whose Polar Coordinates satisfy the following Conditions.

1. $r = 1$ , $r = -1$

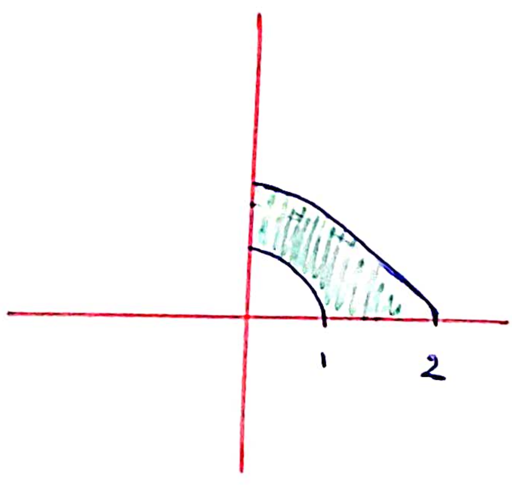
Sol: The equations $r=1$, $r=-1$ are an equation for the circle of radius 1 and center at the origin.



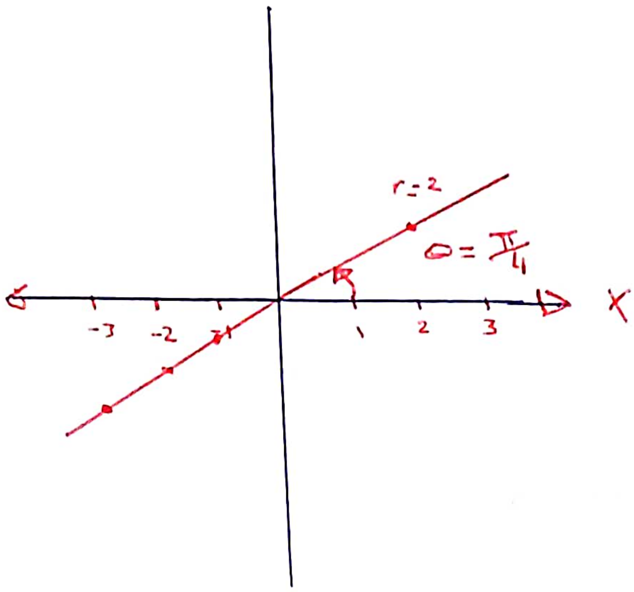

2. $\theta = \pi/6$



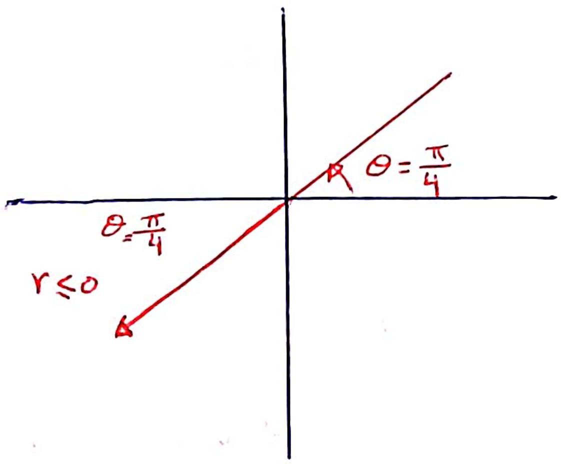

3. $1 \le r \le 2$ , $0 \le \theta \le \pi/2$



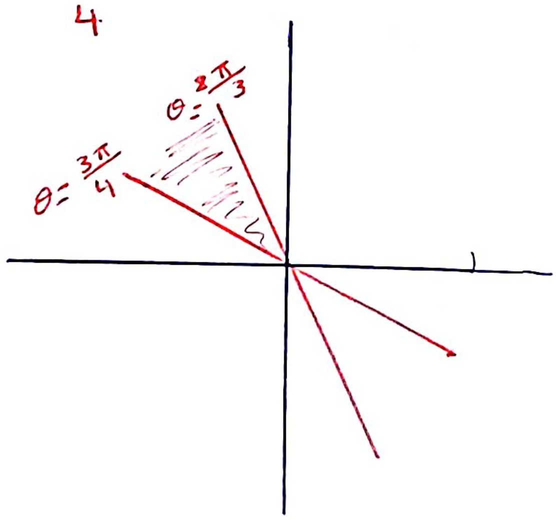
4. $-3 \leq r \leq 2$, $\theta = \frac{\pi}{4}$




4. $r \leq 0$, $\theta = \pi/4$



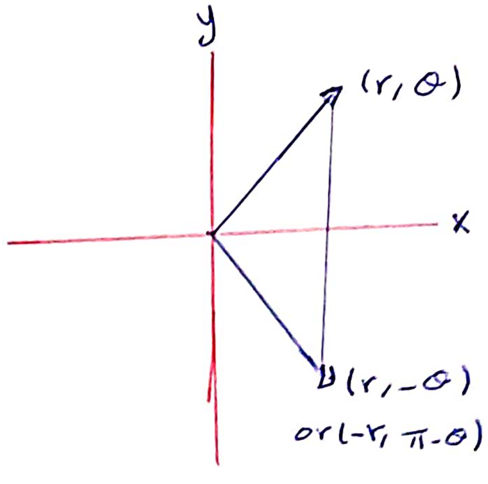

5. $\frac{2\pi}{3} \leq \theta \leq \frac{3\pi}{4}$



# Graphing in Polar Coordinate

## Symmetry Test for Graphs

1- Symmetry about the $x$-axis: If the Point $(r,\theta)$ Lies on the graph, the Point $(r,-\theta)$ Lies on the graph.




about $x$-axis

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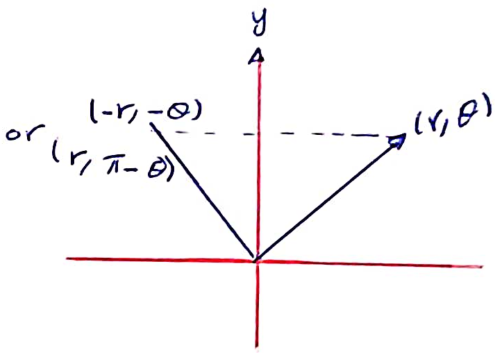
2- Symmetry about the $y$-axis:
If the Point $(r,\theta)$ Lies on the graph, the Point $(-r,-\theta)$ or $(r,\pi-\theta)$ Lies on the graph.




about $y$-axis

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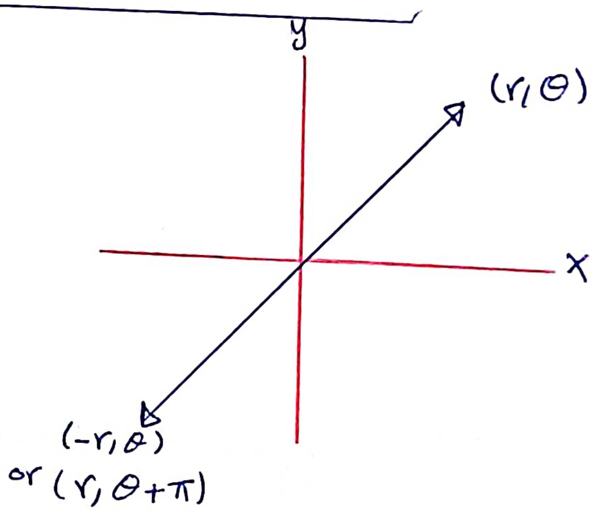
3- Symmetry about the origin:
If the Point $(r,\theta)$ Lies on the graph, the Point $(-r,\theta)$ or $(r,\theta+\pi)$ Lies on the graph.



## The slope of the Polar Curve $r=f(\theta)$

The slope of apolar curve $r=f(\theta)$ is not given by the derivative $r'=f'(\theta)=df/d\theta$, but by a different formula. the formula of graph of f as the graph of the Parametic equations:

$$x = r\cos\theta = f(\theta)\cos\theta$$

$$y = r\sin\theta = f(\theta)\sin\theta$$

If f, x, and y are differentable function of $\theta$, when $dx/d\theta \neq 0$, we calculate $dy/dx$ from the Parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\frac{d}{d\theta}(f(\theta)\cdot\sin\theta)}{\frac{d}{d\theta}(f(\theta)\cdot\cos\theta)}$$

$$= \frac{\frac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\frac{df}{d\theta}\cos\theta - f(\theta)\sin\theta}$$ [حاصل ضرب دالتين]

$$= \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$ $[r=f,\ r'=df/d\theta$

Remark:

1. Slope of a Polar Curve:

If $r = f(\theta)$ is diffirentable and $dx/d\theta \neq 0$, then the slope $dy/dx$ at the Point $(r, \theta)$ of $f$ is given by the formula:

$$\text{Slope at}(r,\theta) = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} \quad \text{-----} \quad (1)$$

2. If $r=0$ when $\theta = \theta_0$, then eq.(1) reduces to

$$\therefore \text{ slope at } (0, \theta_0) = \frac{r'\sin\theta_0}{r'\cos\theta_0} = \frac{\sin\theta_0}{\cos\theta_0} = \tan\theta_0 \quad \text{---} \quad (2)$$

3. Slopes at the origin

If the graph of the $r = f(\theta)$ Passes through the origin at the value , then the slope of the Curve there is

$$\text{Slope at } (0, \theta_0) = \tan\theta_0$$

Ex: Graph the curve $r = 1 - \cos\theta$

Sol:

1. Symmetry

① symmetric about the origin

$(r,\theta) \stackrel{?}{=} (-r,\theta)$

$r = 1 - \cos\theta$

$-r = 1 - \cos\theta$

∴ no symmetricon the origin

② Symmetric about the $x$-axis

$(r,\theta) \stackrel{?}{=} (r,-\theta)$

$r = 1 - \cos\theta$

$r = 1 - \cos(-\theta)$

$r = 1 - \cos\theta$

∴ there is symmetric about the $x$-axis

③ symmetric about the $y$-axis

$(r,\theta) \stackrel{?}{=} (-r,-\theta)$

$r = 1 - \cos\theta$

$-r = 1 - \cos(-\theta)$

∴ $-r = 1 - \cos\theta$

∴ no symmetric about $y$-axis

② $r = 0 \Rightarrow 1 - \cos\theta = 0 \Rightarrow \cos\theta = 1 \Rightarrow \boxed{\theta = 0}$

∴ slope is $\theta = 0$

∴ $r = 1 - \cos 0 \Rightarrow r = 1 - 1 = 0$

∴ $r = 0$

③

| $\theta$ | $r$ | $(r,\theta)$ |
|---|---|---|
| $0$ | $0$ | $(0,0)$ |
| $\pi/3$ | $1/2$ | $(1/2, \pi/3)$ |
| $\pi$ | $2$ | $(2,\pi)$ |
| $3\pi/2$ | $1$ | $(1, 3\pi/2)$ |
| $2\pi$ | $0$ | $(0, 2\pi)$ |

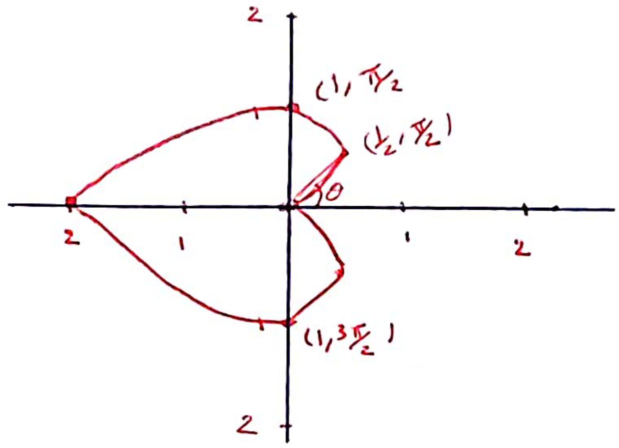
$\Rightarrow r = 1-\cos 0 \Rightarrow r = 1-1 = 0$

$, \; r = 1-\cos \pi/3 \Rightarrow r = 1-1/2 = 1/2$

$, \; r = 1-\cos \pi \Rightarrow r = 1-(-1) = 2$

$, \; r = 1-\cos(3\pi/2) \Rightarrow r = 1-0 = 1$

$, \; r = 1-\cos(2\pi) \Rightarrow r = 1-1 = 0$




**Ex:** Graph the curve $r^2 = 4\cos^2\theta$

**Sol:** ⓑ symmetric

① symmetric about $y$-axis

$(r,\theta) \overset{?}{=} (-r,-\theta)$

$r^2 = 4\cos^2\theta$

$(-r)^2 = 4\cos^2(-\theta)$

$r^2 = 4\cos^2\theta$

$\therefore$ There is symmetric about $y$-axis.

② symmetric about $x$-axis

$(r,\theta) \overset{?}{=} (r,-\theta)$

$r^2 = 4\cos^2\theta$

$(-r)^2 = 4\cos^2(-\theta)$

$r^2 = 4\cos^2\theta$

$\therefore$ There is symmetric about $x$-axis

③ symmetric about origin

$(r,\theta) \overset{?}{=} (-r,\theta)$

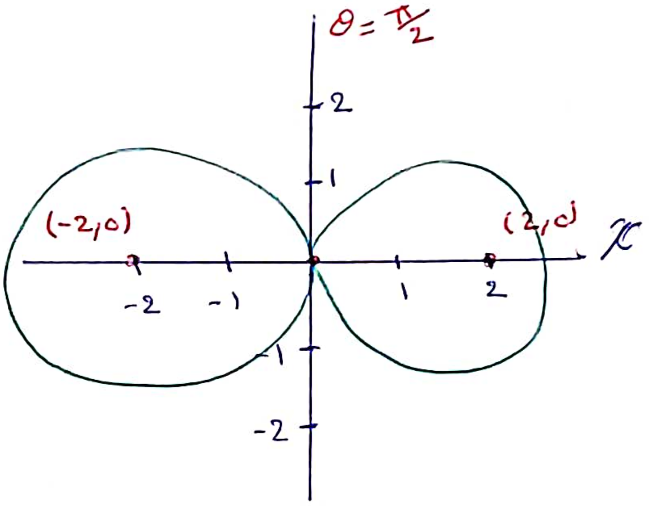
$r^2 = 4\cos^2\theta$

$(-r)^2 = 4\cos^2\theta$

$r^2 = 4\cos^2\theta$

$\therefore$ There is symmetric about origin

② $r = 0 \Rightarrow 0 = 4\cos^2\theta \Rightarrow \cos\theta = 0 \quad \therefore \theta = \pi/2$




③

| $\theta$ | $r$ | $(r,\theta)$ |
|---|---|---|
| 0 | $\mp 2$ | $(2,0), (-2,0)$ |
| $\pi/2$ | 0 | $(0, \pi/2)$ |
| $\pi$ | $\mp 2$ | $(2,\pi), (-2,\pi)$ |
| $3\frac{\pi}{2}$ | 0 | $(0, \frac{3\pi}{2})$ |
| $2\pi$ | $\mp 2$ | $(2,2\pi), (-2,2\pi)$ |

, $r^2 = 4(\cos 0)^2 = 4(1)^2 = 4 \Rightarrow r = \mp 2$

, $r^2 = 4(\cos \pi/2)^2 = 4(0) = 0 \Rightarrow r = 0$

, $r^2 = 4(\cos\pi)^2 = 4(-1)^2 = 4 \Rightarrow r = \mp 2$

, $r^2 = 4(\cos\frac{3\pi}{2})^2 = 4 \cdot 0 = 0 \Rightarrow r = 0$

, $r^2 = 4(\cos 2\pi)^2 = 4 \cdot 1 = 4 \Rightarrow r = \mp 2$

H.W. Graph the curve:

1- $r^2 = 4\cos 2\theta$

2- $r = 4\cos\theta + 2$

3- $r = 4\sin 2\theta$

4. $r = 4(1+\cos\theta)$

## Area and Lengths in polar Coordinates:

**Def:** Area of the region between the origin and the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$

$$\boxed{A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta}$$

This is the integral of the area differential

$$dA = \frac{1}{2} r^2 \, d\theta$$

**Ex:** find the area of the origin in the plane enclosed by the curve:

$$r = 2(1+\cos\theta), \quad 0 \leq \theta \leq 2\pi$$

Sol: $A = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$

$$A = \frac{1}{2}\int_{0}^{2\pi} 4(1+\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 4(1+2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \frac{4}{2}\int_0^{2\pi} (1 + 2\cos\theta + \frac{1+\cos 2\theta}{2})\, d\theta$$

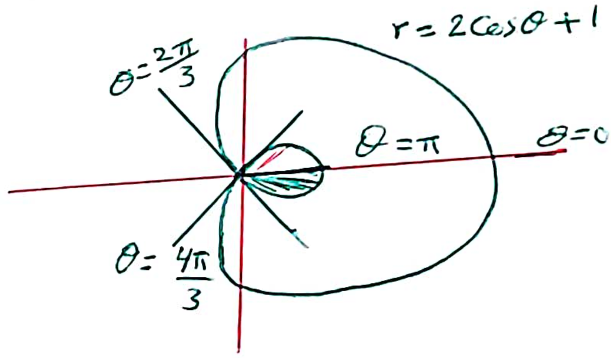
$$= 2\left[\theta + 2\sin\theta + \frac{1}{2}(\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= 2[2\pi + 2\cdot 0 + \frac{1}{2}(2\pi + \frac{1}{2}*0))] - 2[0]$$

$$= 2[2\pi + \pi] = 2*3\pi = 6\pi$$

Ex: Find the area inside the smaller loop of the equation

$r = 2\cos\theta + 1$




sol:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2\, d\theta$$

$$A = \int_{2\pi/3}^{\pi} r^2\, d\theta$$

$$= \int_{2\pi/3}^{\pi} (2\cos\theta + 1)^2\, d\theta$$

$$= \int_{2\pi/3}^{\pi} (4\cos^2\theta + 4\cos\theta + 1)\, d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(4\left(\frac{1+\cos 2\theta}{2}\right) + 4\cos\theta + 1\right) d\theta$$

$$= \int_{2\pi/3}^{\pi} \left(4 * \frac{1}{2} + 4 \frac{\cos 2\theta}{2} + 4\cos\theta + 1\right) d\theta$$

$$= \int_{2\pi/3}^{\pi} (2 + 2\cos 2\theta + 4\cos\theta + 1)\, d\theta$$

$$= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta$$

$$= \left[3\theta + \sin 2\theta + 4\sin\theta\right]_{2\pi/3}^{\pi}$$

$$= [3\pi + 0 + 4*0] - \left[3 \cdot \frac{2\pi}{3} + \sin\frac{4\pi}{3} + 4\sin\frac{2\pi}{3}\right]$$

$$= 3\pi - \left[2\pi - \frac{\sqrt{3}}{2} + 4 \cdot \frac{\sqrt{3}}{2}\right.$$

$$= \pi - \frac{3\sqrt{3}}{2}$$

H-W: Find the area between $r = 4\cos\theta$, $\theta = \pi/3$

find the smallest area

# Area of the Region $r_1(\theta) \le r \le r_2(\theta)$ , $\alpha \le \theta \le \beta$

The area of the region $r_1(\theta) \le r \le r_2(\theta)$ , $\alpha \le \theta \le \beta$ is:

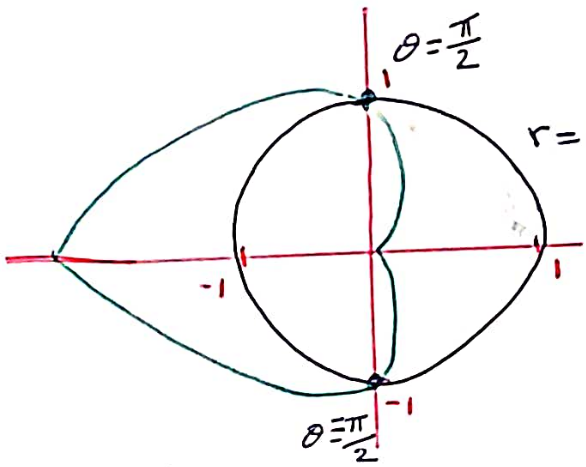
$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta$$

Ex: Find The area of the region that lies inside the Circle $r=1$ and outside the Cardioid $r = 1 - \cos\theta$

Sol: $r_2 = 1$ , $r_1 = 1 - \cos\theta$

and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$




$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta$$

$$= 2 \int_{0}^{\pi/2} \frac{1}{2} (r_2^2 - r_1^2) \, d\theta \quad \text{(symmetry)}$$

$$= \int_{0}^{\pi/2} (r_2^2 - r_1^2) \, d\theta$$

$$=\int_0^{\pi/2}\left((1)^2-(1-\cos\theta)^2\right)d\theta$$

$$=\int_0^{\pi/2}\left(1-(1-2\cos\theta+\cos^2\theta)\right)d\theta$$

$$=\int_0^{\pi/2}\left(\cancel{1}-\cancel{1}+2\cos\theta-\cos^2\theta\right)d\theta$$

$$=\int_0^{\pi/2}\left(+2\cos\theta-\frac{\cos 2\theta+1}{2}\right)d\theta$$

$$=\int_0^{\pi/2}\left(2\cos\theta-\frac{\cos 2\theta}{2}-\frac{1}{2}\right)d\theta$$

$$=\int_0^{\pi/2}2\cos\theta-\frac{2}{2}\cdot\frac{\cos 2\theta}{2}-\frac{1}{2}\Big)\,d\theta$$

$$=\left[2\sin\theta-\frac{1}{4}\sin 2\theta-\frac{1}{2}\theta\right]_0^{\pi/2}$$

$$=\left[2\sin\pi/2-\frac{1}{4}\sin 2\frac{\pi}{2}-\frac{1}{2}\cdot\frac{\pi}{2}\right]-\left[2\sin 0-\frac{1}{4}\sin 0-0\right]$$

$$\left[2\cdot 1-0-\frac{\pi}{4}\right]-0 \quad = 2-\frac{\pi}{4}$$

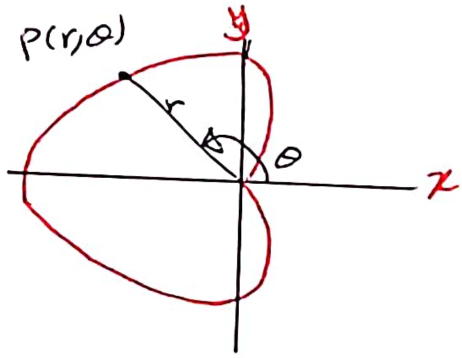
# Length of a Curve

**Def:** If $r=f(\theta)$ has a continous first derivative for $\alpha \leq \theta \leq \beta$ and of the point $P(r,\theta)$ traces the curve $r=f(\theta)$ exactly once; as $\theta$ runs from $\alpha$ to $\beta$, then the length of the curve is given by the formula:

$$L = \text{Length} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

**EX!** Find the Length of the Cardioid $r = 1 - \cos\theta$

**Sol** $0 \leq \theta \leq 2\pi$




$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

$r = 1 - \cos\theta \quad , \quad \frac{dr}{d\theta} = \sin\theta$

$$\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right) = (1-\cos\theta)^2 + (\sin\theta)^2$$

$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1}$$

$$= 1 - 2\cos\theta + 1$$

$$= 2 - 2\cos\theta$$

$$L = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$

$$= \int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta \qquad [1 - \cos\theta = 2\sin^2\frac{\theta}{2}]$$

$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta$$

$$= \int_0^{2\pi} \left|2\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad [\sin\theta \geq 0, \quad 0 \leq \theta \leq 2\pi]$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} \qquad [\int \sin\frac{\theta}{2}\, d\theta = \int \frac{2}{2}\sin\frac{\theta}{2} = 2\sin\frac{\theta}{2}]$$

$$= \left[-4\cos\frac{2\pi}{2}\right] - \left[-4\cos 0\right]$$

$$= [-4 * -1] - [-4 * 1] = 4 + 4 = 8$$

# Conic Sections in Polar Coordinats

To find Polar equations for elipses, Parabolas, and hyparabola we first assume that the conic has one focus at the origin (for the parabola, it's only focus) and the Corresponding the directrix is the vertical line $x=h$ Lying to the right of the origin (Fig a). This makes

$$PF=r$$

$$PD=h-FB=K-r\cos\theta$$

The conic focus-directrix equation $PF=e\,PD$ then becomes:

$$r=e(K-r\cos\theta)$$

we can solved for r to obtain

$$\boxed{r=\frac{Ke}{1+e\cos\theta}}$$

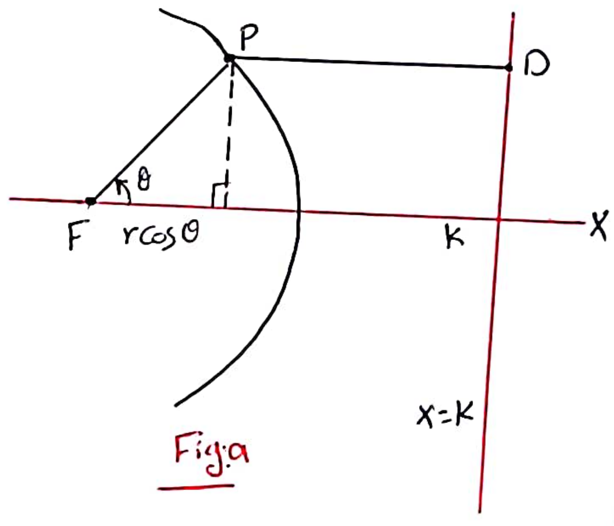
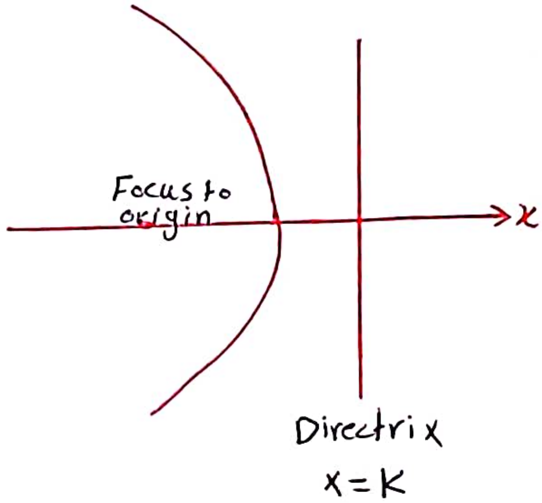



Fig a

This equation represent an elipse if $0<e<1$, a Parabola if $e=1$, and a hyparabola if $e>1$. and there we have it ellipse, Parabolas, and hyperabolas all with the same basic equation

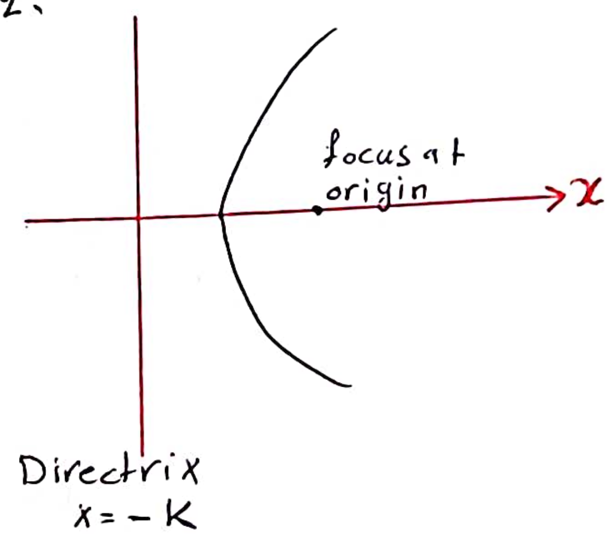
## Remark: Equations for Conic Sections $(e > 0)$

1.



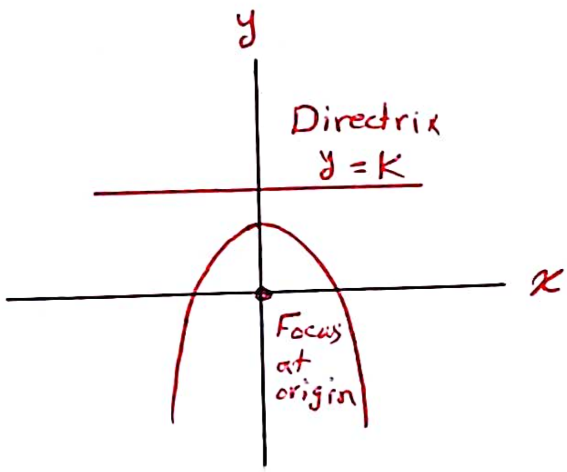

$$r = \frac{ke}{1 + e\cos\theta}$$

2.



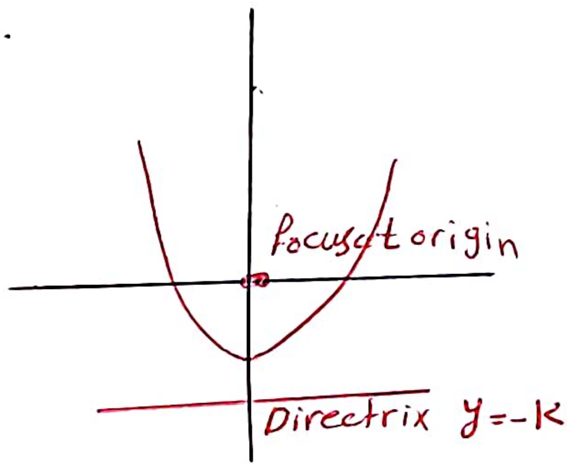

$$r = \frac{ke}{1 - e\cos\theta}$$

3.




$$r = \frac{ke}{1 + e\sin\theta}$$

4.




$$r = \frac{ke}{1 - e\sin\theta}$$

Fig (b)

Example: Typical Conics from equation:

1. $e = 1/2 \Rightarrow r = \dfrac{\frac{1}{2} K}{1 + \frac{1}{2}\cos\theta} = \dfrac{K}{2 + \cos\theta}$ ellipse

2. $e = 1 \Rightarrow r = \dfrac{K}{1 + \cos\theta}$ , parabola

3. $e = 2 \Rightarrow r = \dfrac{2K}{1 + 2\cos\theta}$ , hyperabola

Ex: Find the equation for the hyperabola with eccentricity $3/2$ and directrix $x = 2$

Sol: we use fig(b) eq.(1) s.t $K = 2$ , $e = 3/2$

$$r = \frac{eK}{1 + e\cos\theta}$$

$$= \frac{\frac{3}{2} * 2}{1 + \frac{3}{2}\cos\theta} = \frac{3}{1 + \frac{3}{2}\cos\theta} \Big] * \frac{2}{2}$$

$$= \frac{6}{2 + 3\cos\theta}$$

Ex: find the directrix for the parabola:

$$r = \frac{25}{10 + 10\cos\theta}$$

Sol: $r = \dfrac{25}{10 + 10\cos\theta} \Big] \div 10$

$$= \frac{25/10}{\frac{10}{10} + \frac{10}{10}\cos\theta} \Rightarrow r = \frac{5/2}{1 + \cos\theta}$$

Comparing with the eq. $r = \dfrac{ke}{1 + e\cos\theta}$

to get, $e = 1$ and $k = 5/2$

H.w: find the directrix and eccentricity of the equations:

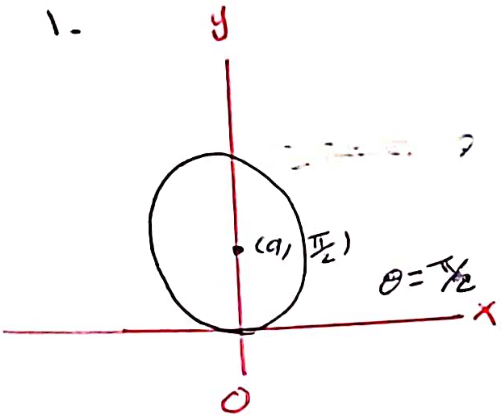
1- $r = \dfrac{6}{2 + \cos\theta}$

2. $r = \dfrac{12}{3 + 3\sin\theta}$

3. $r = \dfrac{4}{2 - \sin\theta}$

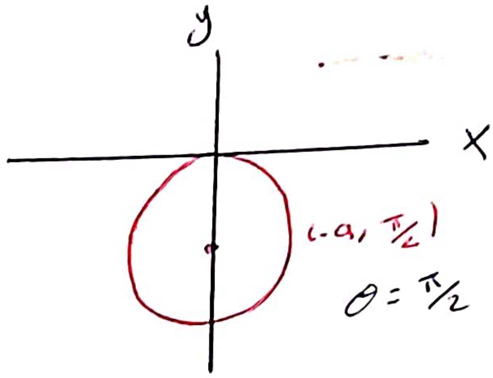
**Remark:** Polar Equations for circles Through the origin Centered on the $x$-axis and $y$-axis, Radius $a$

1.



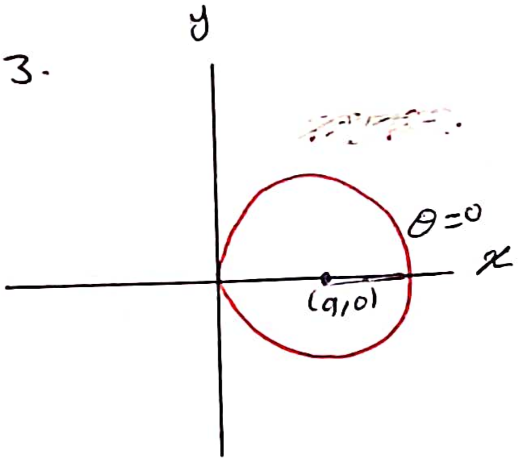

$r = 2a \sin\theta$

2.



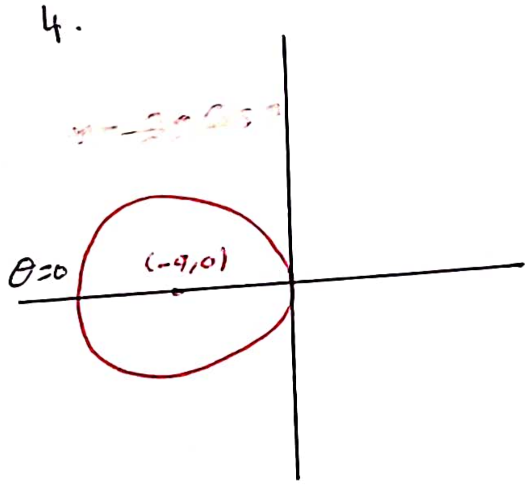

$r = -2a \sin\theta$

3.




$r = 2a \cos\theta$

4.




$r = -2a \cos\theta$

Fig (c)

Ex: Find the polar coordinate and radius of the circle equations:

1- $r = 6\cos\theta$

Sol: Comparing with fig(c),(3)

$r = 2a\cos\theta$ with eq. $r = 6\cos\theta$, to get:

$2a = 6 \Rightarrow a = 3$ the radius

$(3, 0)$ the polar coordinate.

2. $r = 4\sin\theta$

Sol: Comparing with fig(c),(1)

$r = 2a\sin\theta$ with eq. $r = 4\sin\theta$, to get

$2a = 4 \Rightarrow a = 2$ the radius

$(2, \pi/2)$ the polar coordinate

3. $r = -\cos\theta$ H.W

4. $r = -2\sin\theta$ H.W

$$= \int_0^{\pi/2} 3 \underbrace{\cos t}_{\text{مشتقة}} \underbrace{\sin t}_{\text{دالة}} \, dt$$

$u = \sin t$

$du = \cos t \, dt$

$u(0) = 0, \; u(\pi/2) = 1$

$$= \int_0^{\pi/2} 3u \, du = \frac{3}{2} \left[ u^2 \right]_0^1 = \frac{3}{2} [1 - 0] = \frac{3}{2}$$

the length of the complete curve is:

$$L = 4 * \frac{3}{2} = 6$$

<u>Ex</u> find the length of the circle with radius $r$ defined parametrically by:

$x = r\cos t \;, \; y = r \sin t \;, \qquad 0 \leq t \leq 2\pi$

<u>Sol:</u>
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$= \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2} \, dt$$

$$= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t} \, dt$$

$$= \int_0^{2\pi} \sqrt{(r^2 (\underbrace{\sin^2 t + \cos^2 t}_{=1})} \, dt$$

$$= \int_0^{2\pi} r \cdot dt = r\, t \Big]_0^{2\pi} = 2r\pi$$

# Chapter two
# Partial Derivatives

## Function of Several Variables

**Def:** Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A real Valued Function $f$ on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's domain. The set of $w$-values taken on by $f$ is the function's range. The symbol $w$ is the dependent variable of $f$, and $f$ is said to be a function of the n independent variables $x_1$ to $x_n$. we also call the $x_j$'s the function's input variables, and call $w$ the function's output variable.

For example: $z = f(x, y)$

$F$ is function of two variable

$x, y$ is the independent variables,

$z$ is the dependent variable.

The domain of $f$ as a region in $xy$-plane.

**Ex:** find the value of $f(x,y,z)=\sqrt{x^2+y^2+z^2}$ at the Point $(3,0,4)$

$$f(3,0,4)=\sqrt{(3)^2+(0)^2+(4)^2}=\sqrt{9+16}=\sqrt{25}=5$$

**Ex:** Find the domain and range for the following function

| | Function | Domain | Range |
|---|---|---|---|
| 1. | $z=\sqrt{y-x^2}$ | $y \geq x^2$ | $[0,\infty)$ |
| 2. | $z=\dfrac{1}{xy}$ | $xy \neq 0$ | $(-\infty,0)\cup(0,\infty)$ |
| 3. | $z=\sin xy$ | $x,y \in R$ | $[-1,1]$ |
| 4. | $w=\sqrt{x^2+y^2+z^2}$ | $x,y,z \in R$ | $[0,\infty)$ |
| 5. | $w=\dfrac{1}{x^2+y^2+z^2}$ | $(x,y,z)\neq 0$ | $(0,\infty)$ |
| 6. | $w=xy\ln z$ | $x,y \in R,\ z>0$ | $(-\infty,\infty)$ |

**H.W.** find the domain and range for the following function

1. $w=\dfrac{1}{x+y+z}$
2. $z=2xy$
3. $z=x-y$

# Limits in Higher Dimensions:

## Limits For Functions of Two Variables

**Def:** we say that a function $f(x,y)$ approachs the limit $L$ as $(x,y)$ approaches $(x_0,y_0)$, and write

$$\boxed{\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L}$$

if for every number $\epsilon>0$, there exists a corresponding number $\delta>0$ such that for all $(x,y)$ in the domain of $f$.

$$\boxed{\begin{array}{l} \forall \epsilon>0,\ \exists \delta>0,\ \text{s.t. } \forall (x,y) \text{ in the domain of } f \\ |f(x,y)-L|<\epsilon \text{ whenever } 0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta \end{array}}$$

**Remark:**

If the function of a single variable, then

1. $\lim_{(x,y)\to(x_0,y_0)} x = x_0$

2. $\lim_{(x,y)\to(x_0,y_0)} y = y_0$

3. $\lim_{(x,y)\to(x_0,y_0)} k = k$ (any number $k$).

Theorem 1: "Properties of limits of Function of Two Variables"

let $F, g$ are functions and $L, M$, and $k$ are real numbers

and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M$$

Then,

1- Sum Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)+g(x,y)) = L+M$

2- Difference Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y)-g(x,y)) = L-M$

3- Constant Multiple Rule: $\lim_{(x,y)\to(x_0,y_0)} k f(x,y) = K.L$ (any number $k$)

4- Product Rule: $\lim_{(x,y)\to(x_0,y_0)} (f(x,y).g(x,y)) = L.M$

5- Qoutient Rule: $\lim_{(x,y)\to(x_0,y_0)} \frac{F(x,y)}{g(x,y)} = \frac{L}{M}$, $M \neq 0$

6- Power Rule: $\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, $n$ is a positive integer

7- Root Rule: $\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, $n$ is positive integer, and if $n$ is even, we assume that $L > 0$.

Ex: Find the limit values of the following

1- $$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3}$$

sol

$$\lim_{(x,y)\to(0,1)} \frac{x-xy+3}{x^2y+5xy-y^3} = \frac{(0)-(0)(1)+3}{(0)^2(1)+5(0)(1)-(1)} = \frac{3}{-1} = -3$$

2- $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2}$$

sol

$$\lim_{(x,y)\to(3,-4)} \sqrt{x^2+y^2} = \sqrt{(3)^2+(-4)^2} = \sqrt{25} = 5$$

3- $$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$

sol

$$\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} ] * \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}(\sqrt{x}+\sqrt{y})}{\cancel{(x-y)}}$$

$$= \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0(\sqrt{0}+\sqrt{0}) = 0$$

H.W: Find the limit values of the following

1- $\lim\limits_{(x,y)\to(1,-3)} (x+5y)$

2- $\lim\limits_{(x,y)\to(1,2)} \dfrac{x^2-y^2}{x^2+y^2}$

3- $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x-y}$

4- $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x-y}$

5- $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$

Def: Continuity

A function $f(x,y)$ is continuous at the point $(x_0,y_0)$ if

1- $f$ is defined at $(x_0,y_0)$

2. $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists

3- $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

A function is continuous if it is continuous at every point its domain.

Ex: Show that the function $f(x,y)=\frac{x}{\sqrt{y}}$ is Continuous at the Point $(1,4)$

Sol

① $f$ is defined at $(1,4)$ s.t

$f(1,4)=\frac{1}{\sqrt{4}}=\frac{1}{2}$

② $\lim_{(x,y)\to(1,4)} \frac{x}{\sqrt{y}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

③ $\lim_{(x,y)\to(1,4)} \frac{x}{\sqrt{y}} = f(1,4) = \frac{1}{2}$

$\therefore$ The function $f(x,y)$ is Continuous at $(1,4)$.

Ex: Show that the function $f(x,y)=\sqrt{x^2+y^2-1}$ is Continuous at $(3,4)$

Sol:

① $f$ is defined at $(3,4)$ s.t $f(3,4)=\sqrt{9+16-1}=\sqrt{24}$

② $\lim_{(x,y)\to(3,4)} \sqrt{x^2+y^2-1} = \sqrt{9+16-1} = \sqrt{24}$

③ $\lim_{(x,y)\to(3,4)} = f(3,4)=\sqrt{24}$.

$\therefore$ The function $f(x,y)=\sqrt{x^2+y^2-1}$ is Continuous at $(3,4)$

# Partial Derivatives of a function of Two Varibles:

**Def:** ① The Partial derivative of $f(x,y)$ with respect to $x$ at the Point $(x_0, y_0)$ is :

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h}$$

Provided the limit exists.

An equivalent for the Partial derivative is $\boxed{\left.\frac{d}{dx} f(x,y_0)\right|_{x=x_0}}$

② The Partial derivative of $f(x,y)$ with respect to $y$ at the Point $(x_0, y_0)$ is:

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{d}{dy} f(x_0,y)\right|_{y=y_0}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$$

Provided the limit exists.

* $\frac{\partial f}{\partial x} = f_x$ , $\frac{\partial f}{\partial y} = f_y$

* if $z = f(x,y)$

$$\frac{\partial z}{\partial x} = z_x \quad , \quad \frac{\partial z}{\partial y} = z_y$$

Ex: Find the Value of $\partial f/\partial x$ and $\partial f/\partial y$ at the Point $(4,-5)$ if $f(x,y) = x^2+3xy+y-1$

Sol:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2+3xy+y-1) = 2x+3y+0+0 = 2x+3y$$

The value of $\partial f/\partial x$ at $(4,-5)$ is $2(4)+3(-5) = 8-15 = -7$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2+3xy+y-1) = 0+3x+1-0 = 3x+1$$

The Value of $\partial f/\partial y$ at $(4,-5)$ is $3(4)+1 = 13$

Ex: find $\partial f/\partial y$ as a function $f(x,y) = y\sin xy$

sol:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y\sin xy) = y\frac{\partial}{\partial y}\sin xy + \sin xy\frac{\partial}{\partial y}y$$

$$= y(\cos xy)\frac{\partial}{\partial y}(xy) + 1\cdot\sin xy$$

$$= xy\cos xy + \sin xy$$

Ex: find $\partial z/\partial x$ of the Function $yz - \ln z = x+y$

Sol:

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1+0$$

$$\frac{\partial z}{\partial x}\left(y-\frac{1}{z}\right) = 1 \Rightarrow \frac{\partial z}{\partial x} = \frac{1}{y-\frac{1}{z}} = \frac{z}{yz-1}$$

Ex: find $\frac{\partial f}{\partial z}$ where $f(x,y,z) = x\sin(y+3z)$

Sol: $$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(x\sin(y+3z))$$

$$= x\cos(y+3z) * 3 = 3x\cos(y+3z).$$

H.W ① The plane $x=1$ intersects the curve $z = x^2 + y^2$ find the slope of the tangent to the curve at the Point $(1,2,5)$.

② find $\partial f/\partial x$ and $\partial f/\partial y$ of $f(x,y) = x^2 - xy + y^2$

③ find $\partial f/\partial x$ and $\partial f/\partial y$ of $f(x,y) = 2x^2 - 3y - 4$

## Second Order Partial Derivatives

when we differente a function $f(x,y)$ tiwce, we produce its second-order derivatives. This derivatives are usually denoted by:

1- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = f_{xx}$

2. $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = f_{yy}$

3. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = f_{xy}$

4. $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = f_{yx}$

Ex: If $f(x,y) = x\cos y + ye^x$ find the second order derivatives ① $\frac{\partial^2 f}{\partial x^2}$, ② $\frac{\partial^2 f}{\partial y \partial x}$, ③ $\frac{\partial^2 f}{\partial y^2}$, ④ $\frac{\partial^2 f}{\partial x \partial y}$

Sol: $f_x = \frac{\partial f}{\partial x} = \cos y + ye^x$

$f_y = \frac{\partial f}{\partial y} = -x\sin y + e^x$

① $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(\cos y + ye^x) = ye^x$

② $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(\cos y + ye^x) = -\sin y + e^x$

③ $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(-x\sin y + e^x) = -x\cos y$

④ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(-x\sin y + e^x) = -\sin y + e^x$

Ex: If $f(x,y) = x + y + xy$ find the all second-order Partial derivatives.

Sol: $f_x = 1 + y$ , $f_y = 1 + x$

① $\frac{\partial^2 f}{\partial y^2} = 0$

② $\frac{\partial^2 f}{\partial x^2} = 0$

③ $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}(1+y) = 1$

④ $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{x}(1+x) = 1$

Theorem2: "The Mixed Derivative Theorem"

If $f(x,y)$ and its Partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined throughout an open region containing a Point $(a,b)$ and are all continuous at $(a,b)$, then:

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Ex: find $\partial^2 w / \partial y \partial x$ if $w = xy + \frac{e^y}{y^2+1}$

Sol: $\frac{\partial w}{\partial x} = y$ , $\frac{\partial^2 w}{\partial y \partial x} = 1$

The Partial Derivatives of Higher Order

1- $\frac{\partial^3 f}{\partial x \partial y^2} = f_{yyx}$ is third partial derivatives also,

$f_{xxy}$, $f_{yxy}$, $f_{yyx}$.

2. $f_{xxyy}$, $f_{xyxy}$, $f_{yxyx}$, $f_{yyxx}$ is the fourth Partial derivatives.

Ex: find $f_{yxyz}$ if $f(x,y,z) = 1-2xy^2z + x^2y$

Sol $f_y = -4xyz + x^2$

$f_{yx} = -4yz + 2x$

$f_{yxy} = -4z$

$f_{yxyz} = -4$

H.W: Verify that $w_{xy} = w_{y}$

$W = xy^2 + x^2y^3 + x^3y^4$

The chain Rulue for the Function of Two Variables

If $w = f(x, y)$ is differentable function and $x = x(t)$ $y = y(t)$ are differentable functions of $t$, then the Composite $w = f(x(t), y(t))$ is adifferentable functions of $t$, and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

or $\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t)$

Ex: If $w = xy$ find $\frac{dw}{dt}$ where $x = \cos t$ and $y = \sin t$. and find the derivative value where $t = \pi/2$ ?

Sol: we apply the chain rule to find $\frac{dw}{dt}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= \frac{\partial (xy)}{\partial x} \cdot \frac{d}{dt}(\cos t) + \frac{\partial (xy)}{\partial y} \cdot \frac{d(\sin t)}{dt}$$

$= (y)(-\sin t) + (x)(\cos t)$

$= (\sin t)(-\sin t) + (\cos t)(\cos t)$

$= -\sin^2 t + \cos^2 t$

$= \cos 2t$

$$\left(\frac{dw}{dt}\right)_{t=\pi/2} = \cos(2 * \frac{\pi}{2}) = \cos \pi = -1$$

<u>Theorem3</u>: "Chain Rule for Functions of Three Independent Variables"

If $w = f(x, y, z)$ is differentiable and $x, y,$ and $z$ are differentiable functions of $t$, then $w$ is differentiable functions of $t$ and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

<u>Ex:</u> Find $\frac{dw}{dt}$ if $w = xy + z$, $x = \cos t$, $y = \sin t$ and $z = t$. and find derivative Value at $t = 0$ ?

<u>Sol:</u> Using the chain rule of three independent Variable to find $dw/dt$

Theorem6: "A formula for implicit Differentiation"

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (y)(-\sin t) + (x)(\cos t) + (1)(1)$$

$$= (\sin t)(\sin t) + (\cos t)(\cos t) + 1$$

$$= -\sin^2 t + \cos^2 t + 1$$

$$= 1 + \cos 2t$$

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 1+1=2$$

Theorem 4: Chain Rule for Two Independent Variables and three Intermediate Variables"

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, $z = k(r, s)$

If all four functions are differentable, then $w$ has Partial derivatives with respect to $r$ and $s$, given by the formulas:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial s}.$$

Ex: find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ where $w = x + 2y + z^2$

and $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

Sol:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= (1)\left(\frac{1}{s}\right) + (2)\cdot(2r) + (2z)(2)$$

$$= \frac{1}{s} + 4r + 4z = \frac{1}{s} + 4r + 8r = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

$$= (1)\left(\frac{-r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0)$$

$$= \frac{2}{s} - \frac{r}{s^2}$$

Ex: find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ where $w = x^2 + y^2$, and

$x = r - s$, $y = r + s$

Sol: $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= (2x)(1) + (2y)(1) = 2x + 2y$$

$$= 2(r - s) + 2(r + s)$$

$$= 2r - \cancel{2s} + 2r + \cancel{2s}$$

$$= 4r$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x)(-1) + (2y)(1)$$

$$= -2x + 2y = -2(r-s) + 2(r+s)$$

$$= -\cancel{2r} + 2s + \cancel{2r} + 2s = 4s$$

Remarks:

① If $w = f(x)$ and $x = g(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} \quad , \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s}$$

② If $w = f(x,y)$ and $x = g(r,s)$ , $y = h(r,s)$ then

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

③ If $w = f(u,v)$ and $u = u(x_1, x_2, \dots, x_n)$

$v = v(x_1, x_2, \dots, x_n)$ then

$$\frac{\partial w}{\partial x_1} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x_1} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x_1}$$

$$\frac{\partial w}{\partial x_2} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x_2} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x_2}$$

$\vdots$

$$\frac{\partial w}{\partial x_n} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial x_n} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial x_n}$$

H.W:

1. find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ where $w = xy + yz + xz$

and $x = u+v$, $y = u-v$, $z = uv$.

2. find $\frac{dw}{dt}$ where $w = x^2+y^2$, $x = \cos t$, $y = \sin t$, $t = \pi$

3- Find $\partial w/\partial u$ and $\partial w/\partial v$ where $w = xy + yz + xz$

and $x = u+v$, $y = u-v$, $z = uv$.

4. find $\frac{\partial w}{\partial r}$ when $r=1$, $s=1$, if $w = (x+y+z)^2$, and

$x = r-s$, $y = \cos(r+s)$, $z = \sin(r+s)$

Theorem 6: "A formula for implicit Differentiation"

Suppose that $F(x,y)$ is differentiable and the equation $F(x,y)=0$ defines $y$ as a differentiable function of $x$. Then at any point where $F_y \neq 0$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Ex: Use Theorem 6 to find $dy/dx$ if $y^2 - x^2 - \sin xy = 0$

Sol: take $F(x,y) = y^2 - x^2 - \sin xy$, then by theorem 6:

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-2x - y\cos xy}{2y - x\cos xy} = \frac{2x + y\cos xy}{2y - x\cos xy}$$

Ex: Use theorem 6 to find $dy/dx$ if $\cos xy + y^2 x = 5$

Sol: $F(x,y) = \cos xy + y^2 x - 5$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-y\sin xy + y^2}{-x\sin xy + 2yx} = \frac{y\sin xy - y^2}{-x\sin xy + 2yx}$$

**Remark:** If $F(x,y,z)=0$ where $z=f(x,y)$, then the function $F(x,y,f(x,y))=0$ and $f(x,y)$ are differentable functions and defined by:

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}$$

**Ex:** Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0,0,0)$ if $x^3+z^2+ye^{xz}+z\cos y=0$

**Sol:**

Let $F(x,y,z)=x^3+z^2+ye^{xz}+z\cos y$, then

$$F_x=3x^2+yze^{xz}$$

$$F_y=e^{xz}-z\sin y$$

$$F_z=2z+xye^{xz}+\cos y$$

$$\frac{\partial z}{\partial x}=\frac{-F_x}{F_z}=-\frac{3x^2+yze^{xz}}{2z+xye^{xz}+\cos y}=\frac{-3x^2-yze^{xz}}{2z+xye^{xz}+\cos y}$$

$$\frac{\partial z}{\partial y}=\frac{-F_y}{F_z}=-\frac{e^{xz}-z\sin y}{2z+xye^{xz}+\cos y}=\frac{-e^{xz}+z\sin y}{2z+xye^{xz}+\cos y}$$

At $(0,0,0)$, to find

$$\frac{\partial z}{\partial x}=\frac{-0-0}{0+0+\cos 0}=\frac{0}{1}=0$$

$$\frac{\partial z}{\partial y}=\frac{-e^0+0}{0+0+\cos 0}=\frac{-1}{1}=-1$$

**Ex:** find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $F(x,y,z) = \sin xz + y^2 x$

Sol:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{-z\cos xz - y^2}{x\cos xz}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{-2y\cancel{x}}{\cancel{x}\cos xz} = \frac{-2y}{\cos xz}$$

## The Directional derivatives and Gradient Vectors

الاشتقاق الاتجاهي والانحدار

**Def:** The Directional Derivatives

The derivative of $f$ at $P_0(x_0, y_0)$ in the direction of unit vector $u = u_1 i + u_2 j$ is the number

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

Provided the limit exists. and it is also denoted by :

$(D_u f)_{P_0}$ : the derivative of $f$ at $P_0$ in the direction of $u$.

Ex: Find the derivative of $f(x,y) = x^2 + xy$ at $P_0(1,2)$ in the direction of unit vector $u = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

Sol:

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{f(1 + s\frac{1}{\sqrt{2}}, 2 + s\frac{1}{\sqrt{2}}) - f(1,2)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + s\frac{1}{\sqrt{2}})^2 + (1 + \frac{s}{\sqrt{2}})(2 + \frac{s}{\sqrt{2}}) - (1^2 + 2 * 1)}{s}$$

$$= \lim_{s \to 0} \frac{(1 + \frac{2s}{\sqrt{2}} + \frac{s^2}{2}) + (2 + \frac{3s}{\sqrt{2}} + \frac{s^2}{2}) - 3}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}} + s^2}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{2}}}{s} + \lim_{s \to 0} \frac{s^2}{s}$$

$$= \lim_{s \to 0} \frac{5\cancel{s}}{\sqrt{2}} \frac{1}{\cancel{s}} + \lim_{s \to 0} s =$$

$$= \lim_{s \to 0} \frac{5}{\sqrt{2}} + 0 = \frac{5}{\sqrt{2}}$$

**Ex:** Find the derivative of $f(x,y) = x+y$ at $P_0(1,1)$ in the direction of the unit vector $u = \frac{2}{\sqrt{13}} i + \frac{3}{\sqrt{13}} j$

**Sol:**

$$\left(\frac{df}{ds}\right)_{u,P_0} = \lim_{s \to 0} \frac{f(x_0 + s u_1, y_0 + s u_2) - f(x_0, y_0)}{s}$$

$$= \lim_{s \to 0} \frac{f(1 + s\frac{2}{\sqrt{13}}, 1 + s\frac{3}{\sqrt{13}}) - f(1,1)}{s}$$

$$= \lim_{s \to 0} \frac{\cancel{1} + \frac{2s}{\sqrt{13}} + \cancel{1} + \frac{3s}{\sqrt{13}} - \cancel{2}}{s}$$

$$= \lim_{s \to 0} \frac{\frac{5s}{\sqrt{13}}}{s} = \lim_{s \to 0} \frac{5\cancel{s}}{\sqrt{13}} \cdot \frac{1}{\cancel{s}} =$$

$$= \lim_{s \to 0} \frac{5}{\sqrt{13}} = \frac{5}{\sqrt{13}}$$

**Def:** The Gradient Vector " مهمة لإختبار

The gradient vector of $f(x,y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

obtained by evaluating the Partial derivatives of $f$ at $P_0$.

the symbol $\nabla$ by itself is read "del"

the notation $\nabla f$ is read "grad f" or "gradien of f"

**Theorem:** "The Directional Derivative Is a dot Product"

If $f(x,y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot u$$

the dot Product of the gradient $\nabla f$ at the $P_0$ and $u$.

**Ex:** find the derivative of $f(x,y) = xe^y + \cos xy$ at the Point $(2,0)$ in the direction of $v = 3\hat{i} - 4\hat{j}$

**Sol:**

$$\left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot u$$

$$(\nabla f)_{P_0} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

$f_x = e^y - y \sin xy$ , $f_x(2,0) = e^0 - 0 \sin 0 = 1 - 0 = 1$

$f_y = xe^y - x \sin xy$ , $f_y(2,0) = 2e^0 - 2\sin 0 = 2 - 0 = 2$

The direction of $v$ is $\frac{v}{|v|}$

$$u = \frac{v}{|v|} = \frac{3i - 4j}{\sqrt{9+16}} = \frac{3}{5} i - \frac{4}{5} j$$

$$\therefore (\nabla f)_{P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0} i + \left(\frac{\partial f}{\partial y}\right)_{P_0} j = 1i + 2j$$

$$\left(\frac{df}{ds}\right)_{u,P_0} = (i + 2j) \cdot \left(\frac{3}{5} i - \frac{4}{5} j\right) = 1 * \frac{3}{5} + 2 * \frac{-4}{5} = \frac{3}{5} - \frac{8}{5}$$

$$= \frac{-5}{5} = -1$$

Remark:

Properties of the Directional Derivative $D_u f = \nabla f \cdot u = |\nabla f| \cos\theta$

1. If $\theta = 0$ or $\cos\theta = \cos 0 = 1$ then $D_u f = |\nabla f|$
the function $f$ is increasing.

2. If $\theta = \pi$ or $\cos\theta = \cos\pi = -1$ then $D_u f = -|\nabla f|$
the function $f$ is decreasing

3. Any direction $u$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in $f$ because $\theta = \pi/2$ and

$$D_u f = |\nabla f| \cos \pi/2 = |\nabla f| \cdot 0 = 0$$

Ex: Find the directions in which $f(x,y) = x^2/2 + y^2/2$

① increasing at the point $(1,1)$

② decreases at the point $(1,1)$

③ what are the directions of zero change in $f$ at $(1,1)$

sol:
① The function $f$ is increasing at the point $(1,1)$ i.e.

$$D_u f_{P_0} = |\nabla f|_{P_0}$$

$$|\nabla f|_{P_0} = \frac{\partial f}{\partial x} i \Big|_{P_0} + \frac{\partial f}{\partial y} j \Big|_{P_0}$$

$$f_x = \frac{\partial f}{\partial x} = \frac{2}{2} x = x \Big|_{(1,1)} = 1$$

$$f_y = \frac{\partial f}{\partial y} = \frac{2}{2} y = y \Big|_{(1,1)} = 1$$

$$|\nabla f|_{(1,1)} = i + j$$

The Direction is $u = \frac{i+j}{|i+j|} = \frac{i+j}{\sqrt{(1)^2+(1)^2}} = \frac{i+j}{\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j$

(2) The function $f$ is decreasing at the Point $(1,1)$ i.e.

$$D_u f_{P_0} = -|\nabla f|_{P_0} = -\frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

(3) The direction of zero change at $(1,1)$ are ~~the~~ directional is orthogonal to $\nabla f$

$$n = -\frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} j \quad , \quad -n = \frac{1}{\sqrt{2}} i - \frac{1}{\sqrt{2}} j$$

**Remark: Algebra Rules for Gradients**

1. Sume Rule: $\nabla(f+g) = \nabla f + \nabla g$
2. Difference Rule: $\nabla(f-g) = \nabla f - \nabla g$
3. Constant Multiple Rule: $\nabla(kf) = k(\nabla f)$ (any number $k$)
4. Product Rule: $\nabla(fg) = f\nabla g + g\nabla f$
5. Quotient Rule: $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

**Ex:** Find algebra rule for gradients with

$f(x,y) = x-y, \quad g(x,y) = 3y$

**Sol:**

$\nabla f = f_x i + f_y j = i - j$

$\nabla g = g_x i + g_y j = +3j$

① $\nabla(f+g) \stackrel{?}{=} \nabla(f) + \nabla(g)$

$\nabla(f) + \nabla(g) = i - j + 3j = i + 2j$

$\nabla(f+g) = \nabla(x - y + 3y) = \nabla(x + 2y)$

$= i + 2j = \nabla f + \nabla g$

② $\nabla(f-g) \stackrel{?}{=} \nabla f - \nabla g$

$\nabla f - \nabla g = i - j - 3j = i - 4j$

$\nabla(f-g) = \nabla(x + y - 3y) = \nabla(x - 4y) = i - 4j = \nabla f - \nabla g$

③ $\nabla(fg) \stackrel{?}{=} f\nabla g + g\nabla f$

$\nabla(fg) = \nabla((x-y)3y) = \nabla(3xy - 3y^2) = 3y\, i + (3x - 6y) j$

$f\nabla g + g\nabla f = (x-y)3j + 3y(i-j)$

$= 3xj - 3yj + 3yi - 3yj$

$= 3yi + (3x - 6y)j = \nabla(fg)$

④ $= \nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

$= \frac{3y(i-j) - (x-y)(3j)}{9y^2} = \frac{3yi - \cancel{3yj} - 3xj + \cancel{3yj}}{9y^2}$

$= \frac{3yi - 3xj}{9y^2} = \frac{\cancel{3}(yi - xj)}{\cancel{9}_3 y^2} = \frac{yi - xj}{3y^2}$

$\underline{\underline{Ex}}$: find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $v = 2i - 3j + 6K$

$\underline{\underline{Sol}}$

The derivative of $f$ at $P_0$ in the direction of $v$

$$(D_u f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot u \quad (f_x i + f_y j + f_z K) \cdot u$$

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 3 - 1 = 2$$

$$f_y = (-2xy)_{(1,1,0)} = -2$$

$$f_z = (-1)_{(1,1,0)} = -1 \quad , \quad (\nabla f)_{(1,1,0)} = 2i - 2j - K$$

the direction of $v$ is $\frac{v}{|v|}$

$$\frac{v}{|v|} = \frac{2i - 3j + 6K}{\sqrt{4+9+36}} = \frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}K$$

$$(D_u f)_{(1,1,0)} = (\nabla f)_{(1,1,0)} \cdot u$$

$$= (2i - 2j - K)\left(\frac{2}{7}i - \frac{3}{7}j + \frac{6}{7}K\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}$$

Linearazation of function of Two Variables

Def: The Linearazation of a function $f(x,y)$ at a Point $(x_0,y_0)$ where $f$ is differentiable is the function

$$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

The approximation $f(x,y) \approx L(x,y)$

Ex: find the linearazation of $f(x,y) = x^2 - xy + \frac{1}{2}y^2 + 3$ at the Point $(3,2)$

Sol: $f(3,2) = (3)^2 - (3)(2) + \frac{1}{2}(2)^2 + 3$

$= 9 - 6 + 2 + 3 = 8$

$f_x = (2x - y)_{(3,2)} = 2*3 - 2 = 4$

$f_y = (-x + y)_{(3,2)} = -3 + 2 = -1$

$L(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$

$= 8 + 4(x - x_0) - 1(y - y_0)$

$= 8 + 4x - 12 - y + 2 = 4x - y - 2$

$L(x,y) = 4x - y - 2$

**Ex**: find the linearization of $f(x,y) = x^2 - y^2 + 2xy$ at the Point $(2,1)$

**Sol:**

$$f(2,1) = 4 - 1 + 4 = 7$$

$$f_x = (2x + 2y)_{(2,1)} = 4 + 2 = 6$$

$$f_y = (-2y + 2x)_{(2,1)} = -2 + 4 = 2$$

$$L(x,y) = f(2,1) + f_x(2,1)(x - x_0) + f_y(2,1)(y - y_0)$$

$$= 7 + 6(x-2) + 2(y-1)$$

$$= 7 + 6x - 12 + 2y - 2 = 6x + 2y - 7$$

**Remark:**

The linearization of $f(x,y,z)$ at a Point $P_0(x_0, y_0, z_0)$ is

$$L(x,y,z) = f(P_0) + f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0)$$

**Ex:** find the linearization of $f(x,y,z) = x^2 - xy + 3\sin z$

at the Point $P_0(x_0, y_0, z_0) = (2,1,0)$

**Sol:**

$$f(2,1,0) = 4 - 2 - 3\sin 0 = 2$$

$$f_x = (2x - y)_{(2,1,0)} = 3$$

$$f_y = (-x)_{(2,1,0)} = -2$$

$f_z = (3\cos z)_{(2,1,0)} = 3\cos 0 = 3$

$L(x,y,z) = f(P_0) + f_x(P_0)(x-x_0) + f_y(P_0)(y-y_0) + f_z(P_0)(z-z_0)$

$= 2 + 3(x-2) + (-2)(y-1) + 3(z-0)$

$= 2 + 3x - 6 - 2y + 2 + 3z \quad = 3x - 2y + 3z - 2$

H.w: ① find Linearazation $L(x,y)$ of the function

$f(x,y) = x^2 + y^2 + 1$ at ① $(0,0)$, ② $(1,1)$

② Find linearazation of $f(x,y) = x^2 + y^2 + z^2$ at the

ⓐ $(1,1,1)$ , ⓑ $(0,1,0)$

③ find the gradient of the function $f(x,y) = y - x$ at the Point $(2,1)$

④ find the gradient of the function $f(x,y,z) = x^2 + y^2 - 2z^2 + z\ln x$ at the Point $(1,1,1)$

⑤ find the derivative of the function $f(x,y) = 2xy - 3y^2$ at the Point $(5,5)$ in the direction of $u = 4i + 3j$

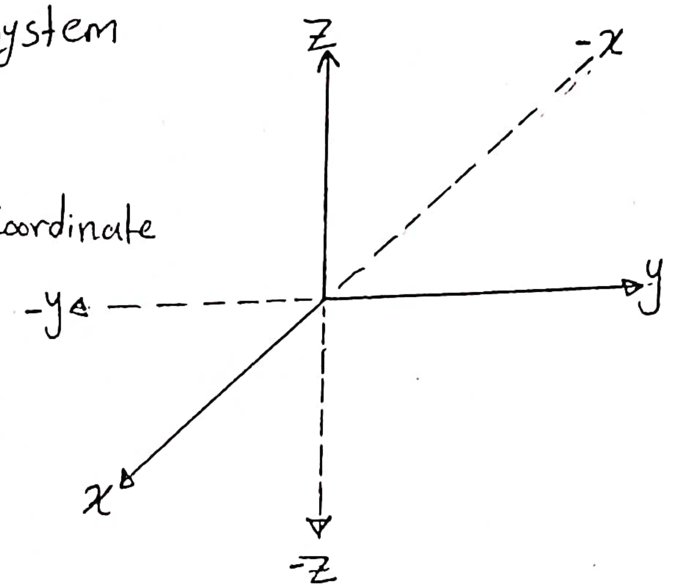
# Chapter Two
## Vectors and Geometry of Space

### - Three Dimensional Coordinate System.

The Point $XY$ Coordinate system represented by $(x, y)$.

In three dimensional $XYZ$-Coordinate system represented by $(x, y, z)$.

we use three prependicular Coordinate axes $x, y, z$ they



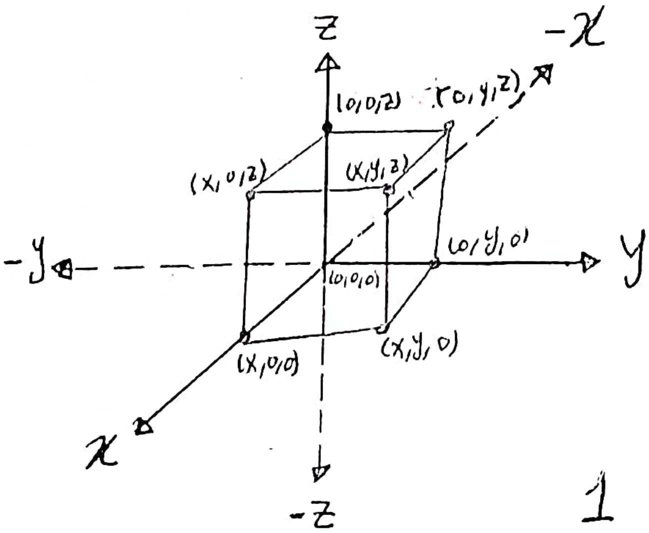

meet at the origin $O(0,0,0)$.

we have three Planes $xy$-plane and $yz$-Plane and $zx$-Plane

The Point $(x,0,0)$ Lies in $x$-axis

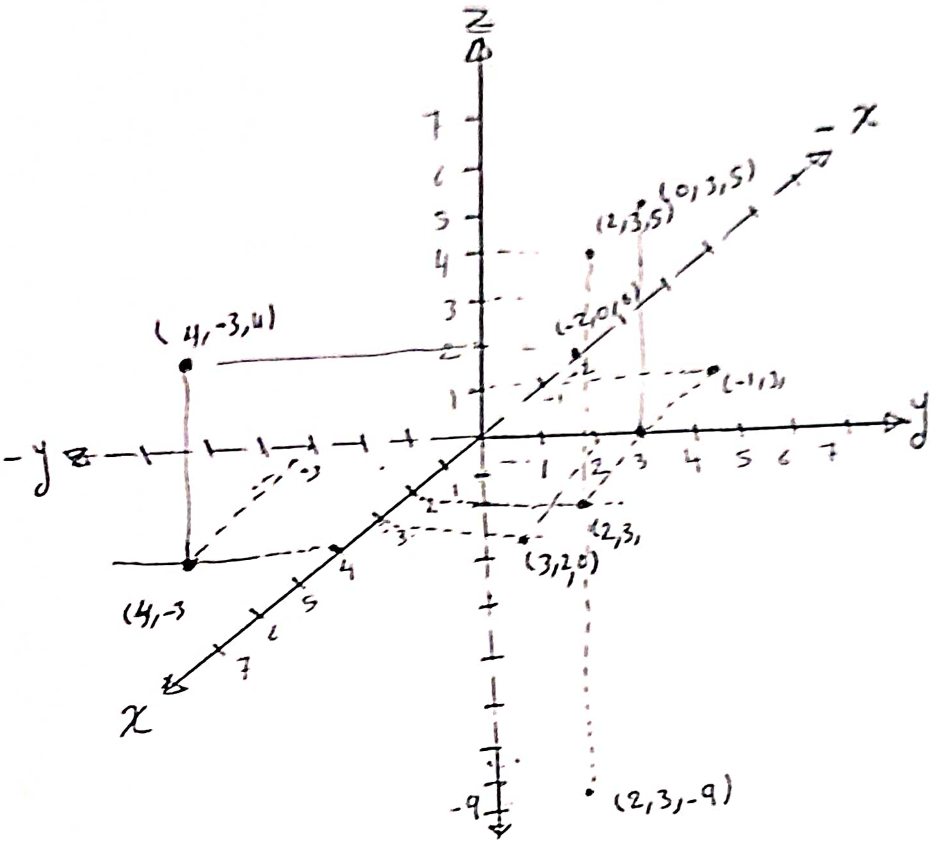
The Point $(0,y,0)$ lies in $y$-axis

The Point $(0,0,z)$ lies in $z$-axis



Ex: Graph the points:

(2, 3, 5), (3, 2, 0), (-2, 0, 0), (-1, 3, -9), (4, -3, 4), (0, 3, 5)




Ex: Graph the Points:

(3, 2, 0), (-1, 3, -5)

(3,2,0) =

(3,0,0)

(0,2,0)

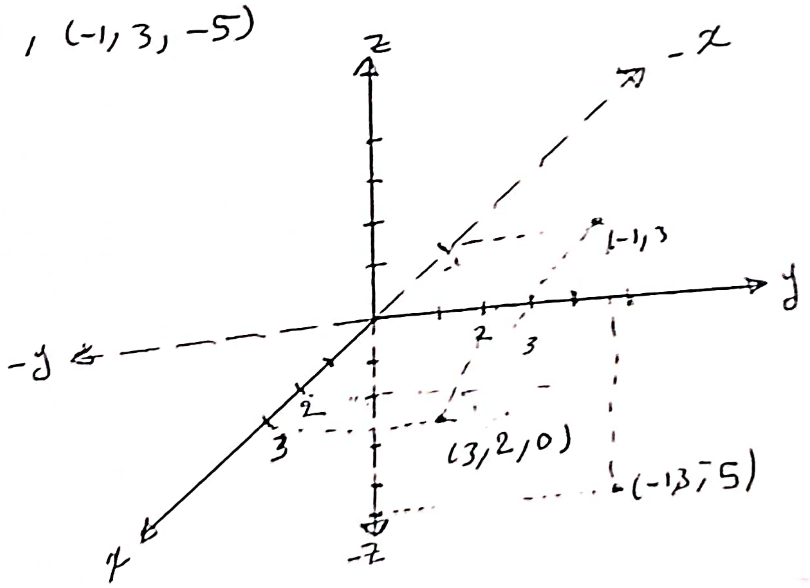
(0,0,0)

(-1, 3,5) =

(-1, 0,0)

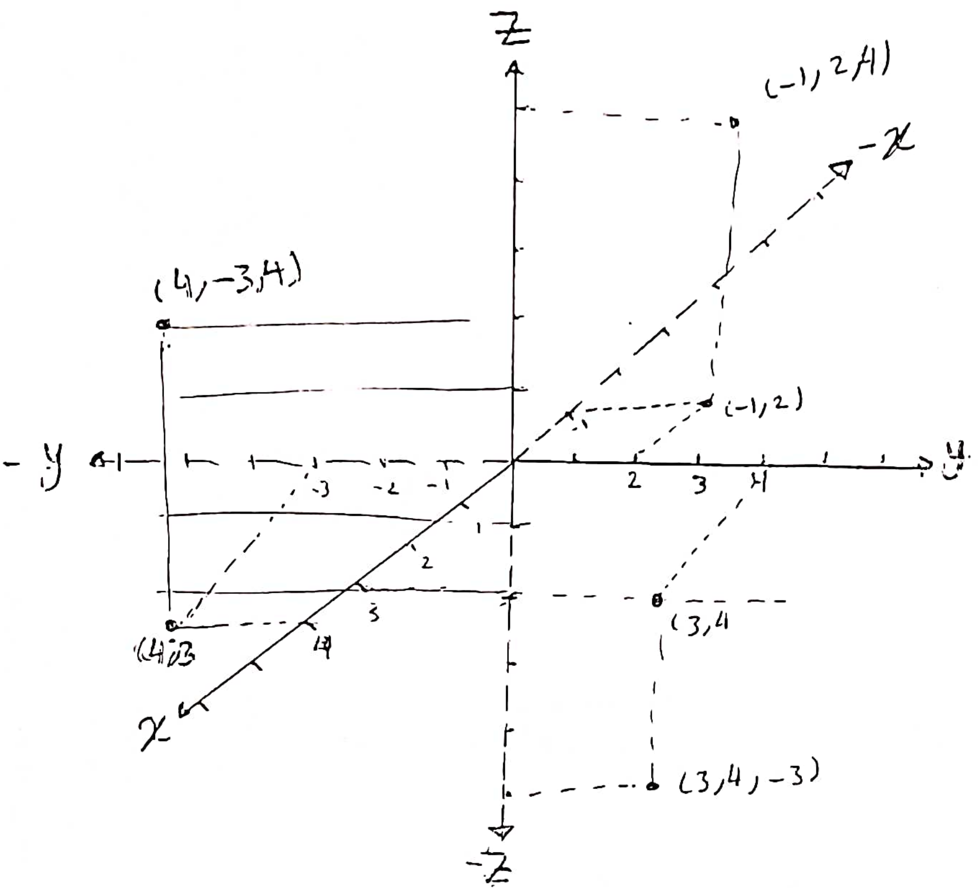
(0,3, 0)

(0,0,5)



Ex: Graph the points:

(4,-3,4), (-1,2,4) (3,4,-3)




(4,-3,4) =

(4,0,0)

(0,-3,4)

(0,0,4)

H-W: Graph the Points:

(-1,2,4),(3,0,-1),(2,-2,1),(0,5,-1),(-1,-1,-1)

Def: The distance between two Points in the xyz-Plane where $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is:

$$\boxed{|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1^2)+(z_2-z_1)^2}}$$

Ex: Find the distance between the Points $P_1(2,1,5)$ and $P_2(-2,3,0)$.

sol:

$$|P_1P_2| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

$$= \sqrt{(-2-2)^2+(3-1)^2+(0-5)^2}$$

$$= \sqrt{(-4)^2+(2^2)+(-5)^2} = \sqrt{16+4+25} = \sqrt{45}$$

Ex: Find the distance between the Points $P_1(-1,2,-3)$ and $P_2(2,0,-3)$

Sol:

$$|P_1P_2| = \sqrt{(2+1)^2+(0-2)^2+(-3+3)^2}$$

$$= \sqrt{(3)^2+(-2)^2+(0)^2} = \sqrt{9+4} = \sqrt{13}$$

H-W: Find the distance the Points $P_1(2,3,5)$ and $P_2(-1,2,9)$

**Def:** The standard equation for the sphere of radius $a$ and center $(x_0, y_0, z_0)$ is:

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=a^2}$$

**Ex:** Find the center and radius of the sphere

$$x^2+y^2+z^2+3x-4z+1=0$$

**Sol:** Complete the sequares on the $x$-; $y$- and $z$- terms

$$(x^2+3x)+(y^2+0)+(z^2-4z)=-1$$

$$\left(x^2+3x+\left(\tfrac{3}{2}\right)^2\right)+(y^2+0)+\left(z^2-4z+\left(\tfrac{-4}{2}\right)^2\right)=-1+\left(\tfrac{3}{2}\right)^2+\left(\tfrac{-4}{2}\right)^2$$

$$\left(x+\tfrac{3}{2}\right)^2+(y^2+0)+(z-2)^2=-1+\tfrac{9}{4}+\tfrac{16}{4}$$

$$\left(x+\tfrac{3}{2}\right)^2+y^2+(z-2)^2=\tfrac{21}{4}$$

from the standard form, to get:

The center $(-3/2, 0, +2)$ and radius $\sqrt{21/4}=\frac{\sqrt{21}}{2}=a$

H.W:

1. Find equation for sphere whose center $(1, 2, 3)$ and radius $\sqrt{14}$
2. Find the center and raduis of sphere
   $x^2+y^2+z^2+4x-4z=0$

# Vectors المتجهات

**Def:** The vecter represented by the directed line segment $\vec{AB}$ has initial Point $A$ and terminal Point $B$ and the length denoted by $|\vec{AB}|$.

Two Vectors are equal if thy have the same length and direction.

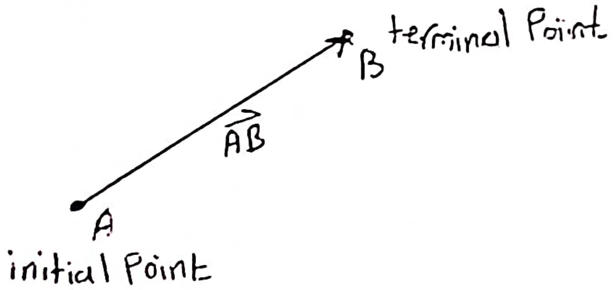
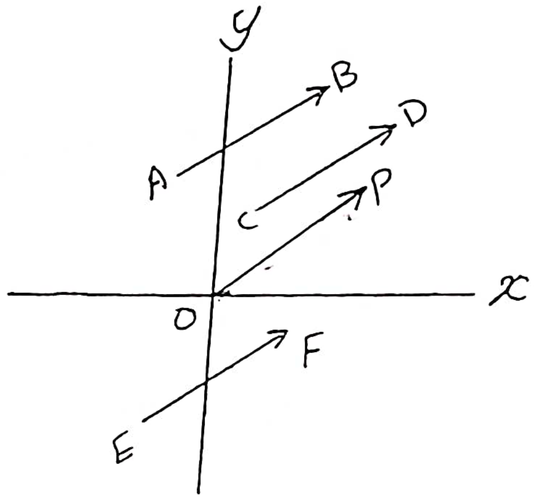



Fig (a): The directed Line Segment $\overrightarrow{AB}$ is Called a vector.




Fig(b): The four directed Line segment in a plane have a same length and direction.

$\vec{AB} = \vec{CD} = \vec{OP} = \vec{EF}$

Def:

1- If $\underline{\underline{v}}$ is a two-dimensional vector in the Plane equal to the vector with initial Point at the origin and terminal Point $(v_1, v_2)$, then the Component form of $v$ is

$$v = (v_1, v_2)$$

2. If $\underline{\underline{v}}$ is a Three-dimensional vector in the Plane equal to the vector with initial Point at the origin and terminal Point $(v_1, v_2, v_3)$, then the Component form of $v$ is

$$v = (v_1, v_2, v_3)$$

Remark:

1- If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two give Points then the standard Position vector $v = (v_1, v_2, v_3)$ equal to $\overrightarrow{PQ}$ is

$$\boxed{v = \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)}$$

2- If $v$ is two-dimensional with $P(x_1, y_1)$ and $Q(x_2, y_2)$ as Points in a Plane, then

$$\boxed{v = \overrightarrow{PQ} = (x_2 - x_1, y_2 - y_1)}$$

3- The length of the vector $v=\overrightarrow{PQ}$ is the non-negative number

- Two dimensional

$$\boxed{|v| = \sqrt{v_1^2 + v_2^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}}$$

- Three dimensional

$$|v| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1^2)}$$

4- The Zero Vector is the vector with length 0, $0=(0,0)$, $0=(0,0,0)$, this Vector also with no specific direction.

Ex: Find the (a) Component form and
(b) length of the vector
with initial Point $P(-3,4,1)$ and terminal Point $Q(-5,2,2)$

Sol:
(a). The component form $v=\overrightarrow{PQ}$ is the standard Position vector $v$,

$$v = (v_1, v_2, v_3) = (x_2-x_1,\ y_2-y_1,\ z_2-z_1)$$
$$= (-5+3,\ 2-4,\ 2-1)$$
$$= (-2,-2,1)$$

b. the length of $v$ is

$$|v| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$
$$= \sqrt{(-2)^2 + (-2)^2 + (1)^2} = \sqrt{9} = 3$$

w: Find the Component form and length of the vector:

$w = \overrightarrow{PQ}$, where $P = (1,3)$ and $Q = (2,-1)$

## Vector Algebra Operations العمليات الجبرية للمتجهات

**Def:** Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ be vectors and let $k$ is a Scalar then,

1- Addition: الجمع

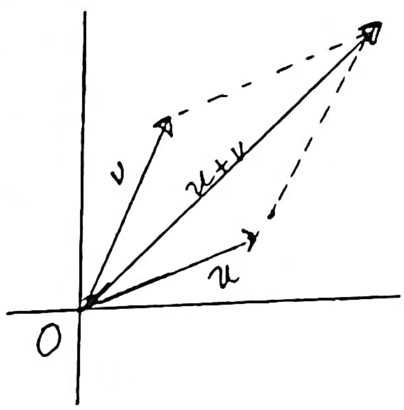
$$u + v = (u_1 + v_1 , u_2 + v_2 , u_3 + v_3)$$

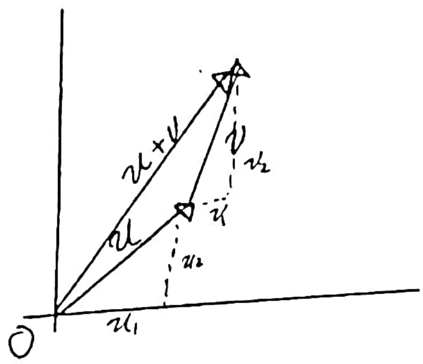
2- Scalar multiplication الضرب بثابت

$$ku = (ku_1, ku_2, ku_3)$$

3- Difference الطرح

$$u - v = u + (-v) = (u_1 - v_1 , v_2 - v_2 , u_3 - v_3)$$



Remark:

1- If $k>0$, then $k \cdot u$ has the same direction as $u$, and if $k<0$ then the direction of $ku$ is opposite to that of $u$.

2- $|ku| = \sqrt{(ku_1)^2 + (ku_2)^2 + (ku_3)^2}$

$$= \sqrt{k^2(u_1^2+u_2^2+u_3^2)} = \sqrt{k^2}\sqrt{u_1^2+u_2^2+u_3^2}$$

$$= |k||u|$$

Properties of Vector Operations

Let $u, v, w$, be vectors and $a, b$ be scalars

1. $u+v = v+u$
2. $(u+v)+w = u+(v+w)$
3. $u+0 = u$
4. $u+(-u) = 0$
5. $0u = 0$
6. $1u = u$
7. $a(bu) = (ab)u$
8. $a(u+v) = au + av$
9. $(a+b)u = au + bu$

Ex: Let $u=(-1,3,1)$ and $v=(4,7,0)$ Find the component of (a). $2u+3v$ (b). $u-v$ (c). $|\frac{1}{2}u|$

Sol:

(a). $2u+3v = 2(-1,3,1)+3(4,7,0)$

$= (-2,6,2)+(12,21,0)$

$= (10,27,2)$

(b). $u-v=(-1,3,1)-(4,7,0)=(-5,-4,1)$

(c). $|\frac{1}{2}u| = |\frac{1}{2}(-1,3,1)| = |(\frac{-1}{2},\frac{3}{2},\frac{1}{2})|$

$= \sqrt{(\frac{-1}{2})^2+(\frac{3}{2})^2+(\frac{1}{2})^2}$

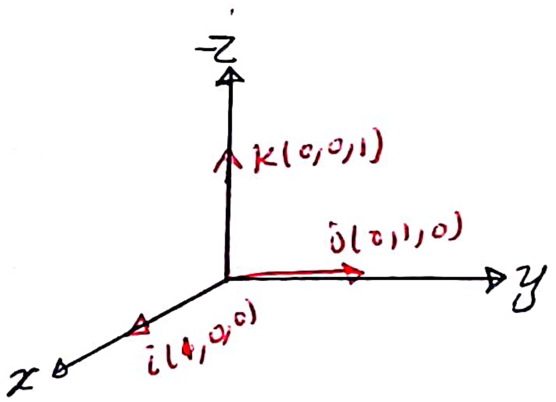
$= \sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}} = \sqrt{\frac{11}{4}} = \frac{1}{2}\sqrt{11}$

Unit Vectors: متجه الوحدة

The vector $v$ of length (1) is called a unit vector.

The standard unit vectors are

$i=(1,0,0)$, $j=(0,1,0)$, $k=(0,0,1)$.



Remark:

Any vector $v=(v_1, v_2, v_3)$ can be written as Linear combination of the standard unit vectors as follows:

$$v=(v_1, v_2, v_3)=(v_1,0,0)+(0,v_2,0)+(0,0,v_3)$$

$$=v_1\underbrace{(1,0,0)}_{i}+v_2\underbrace{(0,1,0)}_{j}+v_3\underbrace{(0,0,1)}_{k}$$

$$\boxed{=v_1 i+v_2 j+v_3 k}$$

We call the scalar (or number)

$v_1$ is the i-Component of vector $v$.

$v_2$ is the j-component of vector $v$.

$v_3$ is the k-Component of vector $v$.

2. The vector from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is:

$$\overrightarrow{P_1P_2}=(x_2-x_1,\ y_2-y_1,\ z_2-z_1)$$

$$\boxed{\overrightarrow{P_1P_2}=(x_2-x_1)i+(y_2-y_1)j+(z_2-z_1)k}$$

unit vector

3. If $v \neq 0$, $|v| \neq 0$, then

$$\left|\frac{1}{|v|}\cdot v\right|=\frac{1}{|v|}\cdot|v|=1$$

That is $\frac{v}{|v|}$ is a unit vector in the direction of $v$, is called the direction of the non zero vector $v$

**Ex:** Find a unit vector $u$ in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$

**Sol:**

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$
$$= (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)K$$
$$= (3-1)i + (2-0)j + (0-1)K$$
$$= 2i + 2j - K$$

$$|\overrightarrow{P_1P_2}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

$$u = \frac{\overrightarrow{P_1P_2}}{|P_1P_2|} = \frac{2i+2j-K}{3} = \frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}K$$

the unit vector $u$ is the direction of $\overrightarrow{P_1P_2}$

**Remark:**

If $v \neq 0$, then

1. $\boxed{\dfrac{v}{|v|}}$ is a unit vector in the direction of $v$.

2. The equation $\boxed{v = |v| \cdot \dfrac{v}{|v|}}$ expresses as a length times its direction

**Ex:** If $v = 3i - 4j$ is a velocity vector, express $v$ as a product of its speed times a unit vector in the direction of motion.

**Sol:**

The unit vector $\frac{v}{|v|}$ has the same direction of $v$.

$$\frac{v}{|v|} = \frac{3i-4j}{\sqrt{(3)^2+(-4)^2}} = \frac{3i-4j}{\sqrt{9+16}} = \frac{3i-4j}{5} = \frac{3}{5}i - \frac{4}{5}j$$

$v$ is speed times a unit vector, then

$v = 3i - 5j$

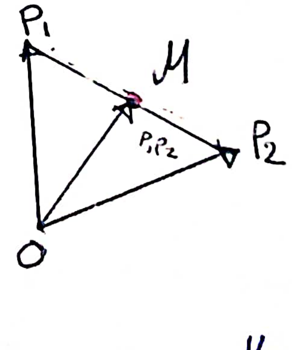
$$v = |v| \cdot \frac{v}{|v|} = 5\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

**H.W:** A force of 6 newtons is applied in the direction of the vector $v = 2i + 2j - k$. Express the force $F$ as a product of its Length and direction.

**Def:** MidPoint of a Line Segment نقطة المنتصف لقطعة مستقيم

The midPoint $M$ of the line segment joining Points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the Point

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$



Ex: find mid Point of the $P_1P_2$ where $P_1(3,-2,0)$ and $P_2(7,4,4)$

Sol:

$$M=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$$

$$=\left(\frac{3+7}{2}, \frac{-2+4}{2}, \frac{0+4}{2}\right)=(5,1,2)$$

H-W. find the direction of $\overrightarrow{P_1P_2}$ and the mid Point of Line segment $\overrightarrow{P_1P_2}$, where $P_1(-1,1,5)$, $P_2(2,5,0)$.

# The Dot Product الضرب العددي

**Theorem 1:** "Angle Between two Vectors"

The angle $\theta$ between two nonzero vectors $u=(u_1,u_2,u_3)$ and $v=(v_1, v_2, v_3)$ is given by:

$$\boxed{\theta = \cos^{-1}\left(\frac{u_1v_1+u_2v_2+u_3v_3}{|u|\cdot|v|}\right)}$$

angle between two vector

**Def:** The Dot Product

The dot Product $u\cdot v$ ($u$ dot $v$) of two vectors $u=(u_1,u_2,u_3)$ and $v=(v_1, v_2, v_3)$ is

$$\boxed{u\cdot v = u_1v_1+u_2v_2+u_3v_3}$$

**Remark:** The angle between two Vectors $u=(u_1,u_2,u_3)$ and $v=(v_1, v_2, v_3)$ is

$$\boxed{\theta = \cos^{-1}\left(\frac{u\cdot v}{|u|\cdot|v|}\right)}$$

Ex: If $u(1,-2,-1)$ and $v(-6,2,-3)$ are two vectors find $u \cdot v$?

Sol:

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$
$$= (1)(-6) + (-2)(2) + (-1)(-3)$$
$$= -6 + -4 + 3 = -7$$

Ex: find the dot Product of $P_1 = \frac{1}{2} i + 3j + k$ and $P_2 = 4i - j + 2k$

Sol:

$$P_1 \cdot P_2 = (\tfrac{1}{2})(4) + (3)(-1) + (1)(2)$$
$$= 2 - 3 + 2 = 1$$

Ex: Find the $\theta$ between $u = i - 2j - 2k$ and $v = 6i + 3j + 2k$

Sol:

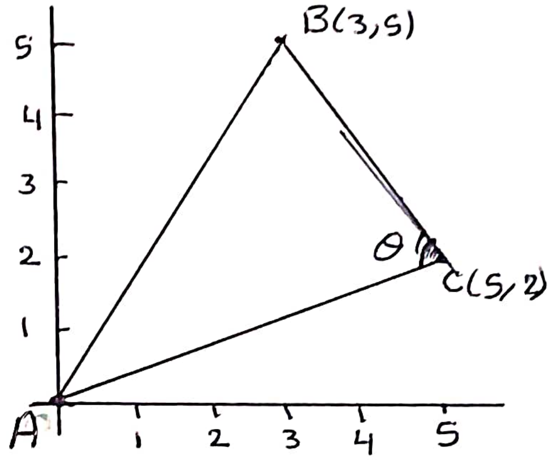
$$\theta = \cos^{-1}\left(\frac{u \cdot v}{|u| \cdot |v|}\right)$$

$$u \cdot v = (1)(6) + (-2)(3) + (-2)(2) = \cancel{6} - \cancel{6} - 4 = -4$$

$$|u| \cdot |v| = \sqrt{1+4+4} \cdot \sqrt{36+9+4} = \sqrt{9} \cdot \sqrt{49} = 21$$

$$\therefore \theta = \cos^{-1} \frac{-4}{21} \simeq 1.76 \text{ radians} \approx 100^\circ$$

**Ex:** Find the angle $\theta$ in the triangle $ABC$ determined by the vertices $A=(0,0)$, $B=(3,5)$, $C=(5,2)$




**Sol:**

$$\overrightarrow{CA} = (x_2-x_1,\, y_2-y_1)$$
$$= (0-5,\ 0-2) = (-5,-2)$$

$$\overrightarrow{CB} = (3-5,\ 5-2) = (-2,3)$$

$$|\overrightarrow{CA}| = \sqrt{(-5)^2+(-2)^2} = \sqrt{29}$$

$$|\overrightarrow{CB}| = \sqrt{(-2)^2+(3)^2} = \sqrt{4+9} = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{\overrightarrow{CA}\cdot\overrightarrow{CB}}{|\overrightarrow{CA}|\cdot|\overrightarrow{CB}|}\right)$$

$$= \cos^{-1}\left(\frac{(-5)(-2)+(-2)(3)}{(\sqrt{29})(\sqrt{13})}\right)$$

$$= \cos^{-1}\left(\frac{4}{(\sqrt{29})(\sqrt{13})}\right)$$

$$\approx 78.1^\circ \text{ or } 1.36 \text{ radians}$$

Def: Vectors u and v are orthogonal (or prependicular) if and only if $\boxed{u \cdot v = 0}$

EX: Determine if two vectors are orthogonal

a. $u = (3,-2)$, $v = (4,6)$

b. $u = 3i - 2j + k$, $v = 2j + 4k$

Sol:

a) $u \cdot v = (3)(4) + (-2)(6) = 12 - 12 = 0$

$\therefore$ u and v are orthogonal vectors.

b). $u \cdot v = (3)(0) + (-2)(2) + (1)(4)$

$= 0 - 4 + 4 = 0$

$\therefore$ u and v are orthogonal vectors.

Properties the Dot Product

If u, v, and w any vectors and c is a scalar, then

1- $u \cdot v = v \cdot u$

2- $(cu) \cdot v = u \cdot (cv)$

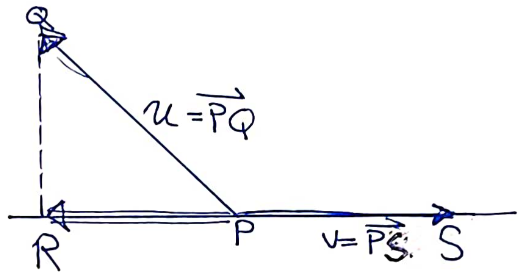
3. $u \cdot (v + w) = u \cdot v + u \cdot w$

4. $u \cdot u = |u|^2$

5. $0 \cdot u = 0$

**Def:** The vector Projection of $u$ on to non zero vector $v$ is the vector $\overrightarrow{PR}$ determined by the dropping a prependicular from $Q$ to the line $PS$.

$$\boxed{Proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) \cdot v}$$

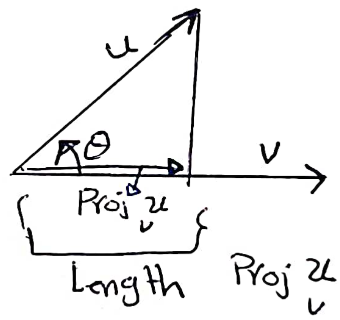



The scalar Component of $u$ in the direction of $v$ is the Scalar

$$\boxed{|u| \cos\theta = \frac{u \cdot v}{|v|} = u \cdot \frac{v}{|v|}}$$

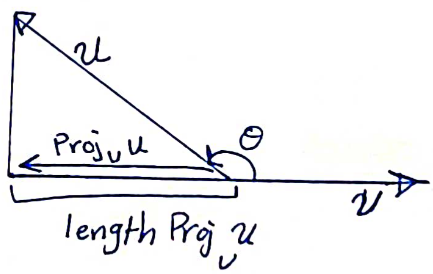
\- The length of $Proj_v u$ where $\cos\theta > 0$

$$\boxed{\text{Length } Proj_v u = |u| \cos\theta}$$




\- The length of $Proj_v u$ where $\cos\theta < 0$

$$\boxed{\text{Length } Proj_v u = -|u| \cos\theta}$$



Ex: Find the vector Projection of $u = 6i + 3j + 2k$ onto $v = i - 2j - 2k$ and the scalar component of $u$ in the direction of $v$.

Sol

$$Proj_v u = \left(\frac{u \cdot v}{|v|^2}\right) \cdot v$$

$$= \left(\frac{(6)(1) + (3)(-2) + (2)(-2)}{\sqrt{1+4+4} \cdot \sqrt{1+4+4}}\right) i - 2j - 2k$$

$$= \left(\frac{\cancel{6} + \cancel{-6} - 4}{(3)(3)}\right) i - 2j - 2k$$

$$= \frac{-4}{9}(i - 2j - 2k) = \frac{-4}{9} i + \frac{8}{9} j + \frac{8}{9} k$$

Scalar component

$$|u| \cos\theta = u \cdot \frac{v}{|v|} = \frac{u \cdot v}{|v|}$$

$$= (6i + 3j + 2k) \cdot \left(\frac{i - 2j - 2k}{3}\right)$$

$$= (6i + 3j + 2k)\left(\frac{i}{3} - \frac{2j}{3} - \frac{2k}{3}\right)$$

$$= (6)\left(\frac{1}{3}\right) + (3)\left(\frac{-2}{3}\right) + (2)\left(\frac{-2}{3}\right) = \cancel{2} - \cancel{2} - \frac{4}{3} = \frac{-4}{3}$$

**Ex** : Find the vector Projection of a force $F = 5i + 2j$ onto $U = i - 3j$ and the scalar Component of F in the direction of U.

**Sol:**

$$\text{Proj}_U F = \left(\frac{F \cdot U}{|U|^2}\right) \cdot U$$

$$= \left(\frac{5-6}{\sqrt{10}\,\sqrt{10}}\right)(i - 3j) = \frac{-1}{10}(i - 3j)$$

$$= \frac{-1}{10} i + \frac{3}{10} j$$

The scalar component of F in the direction of U.

$$|F| \cos\theta = \frac{F \cdot U}{|U|} = \frac{5-6}{\sqrt{1+9}} = \frac{-1}{\sqrt{10}}$$

**H.W:** Find the vector Projection of the vector $u = 2i + 2j + k$ onto $U = 2i + 10j - 11k$ and the scalar Component of $u$ in the direction of U.

# The Cross Product    الضرب الاتجاهي

**Def:** The cross Product of two vectors $u$ and $v$ is:

$$u \times v = (|u||v| \sin\theta)\, n$$

where $u$ and $v$ are not zero vectors, $n$ is a unit vector Prependicular to plane

**Remark:**

1- The dot Product is unlike the cross Product. The cross Product is vector but the dot Product is number

2- If one or both of $u$ and $v$ are zero then $u \times v$ is zero.

3- The cross Product of two vectors $u$ and $v$ is zero iff $u$ and $v$ are parallel or one of them are zero.

4- Nonzero vectors $u$ and $v$ are Parallel if and only if $u \times v = 0$

## The Properties of Cross Product

If $u, v$, and $w$ are any vectors and $r, s$ are scalars, then

1- $(ru) \times (sv) = (rs)(u \times v)$

2- $u \times (v + w) = u \times v + u \times w$

3- $v \times u = -(u \times v)$

4- $(v + w) \times u = v \times u + w \times u$

5- $0 \times u = 0$

6- $u \times (v \times w) = (u \cdot w)v - (u \cdot v)w$

The cross Product multiplication is not associative so $(u \times v) \times w$ does not generally equal $u \times (v \times w) \neq (u \times v) \times w$

## Remark:

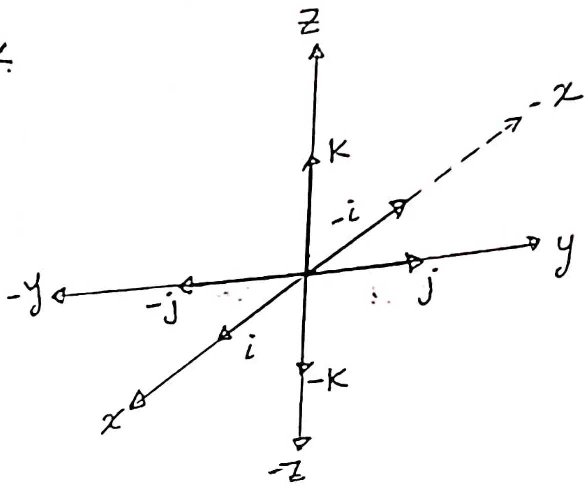
when we apply the definition to calculate the Pairwise cross Product of $i$, $j$, and $k$

$$i \times j = -(j \times i) = k$$

$$j \times k = -(k \times j) = i$$

$$k \times i = -(i \times k) = j$$

$$i \times i = j \times j = k \times k = 0$$



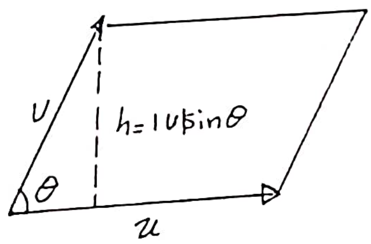

## The Area of Parallelogram

مساحة متوازي الاضلاع

Area = base · hight

$= |u| \cdot |v| \sin\theta$

$$\boxed{\text{Area} = |u \times v|}$$ area of Parallelogram




The area of Parallelogram where the vector $|u|$ is the base and $|v|\sin\theta$ is the hight

$\text{Area} = |u \times v| \neq (|u| \cdot |v| \cdot \sin\theta)n$, $n$ is unit vecter, $n = 1$

$= |u| \cdot |v| \sin\theta$

Def: If $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$, then

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$u \times v = (u_1 i + u_2 j + u_3 k) \times (v_1 i + v_2 j + v_3 k)$$

$$= u_1 v_1 \underbrace{(i \times i)}_{0} + u_1 v_2 \underbrace{(i \times j)}_{k} + u_1 v_3 \underbrace{(i \times k)}_{-j} + u_2 v_1 \underbrace{(j \times i)}_{-k} +$$
$$u_2 v_2 \underbrace{(j \times j)}_{0} + u_2 v_3 \underbrace{(j \times k)}_{i} + u_3 v_1 \underbrace{(k \times i)}_{j} + u_3 v_2 \underbrace{(k \times j)}_{-i} + u_3 v_3 \underbrace{(k \times k)}_{0}$$

$$= (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k$$

$$= \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Ex: Find $u \times v$ and $v \times u$ if $u = 2i + j + k$ & $v = -4i + 3j + k$

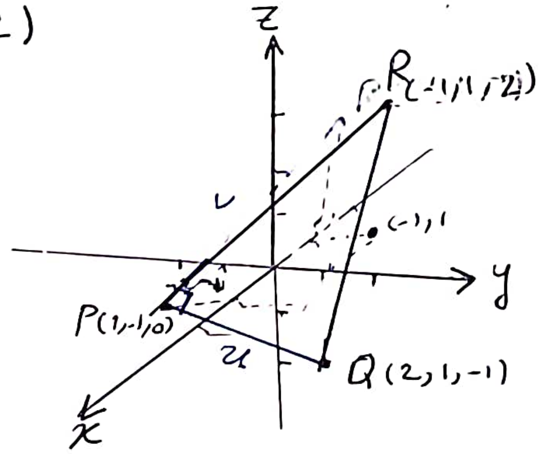
Sol:

$$u \times v = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= -2i - 6j + 10k$$

$$u \times v = -(v \times u) = -(-2i - 6j + 10k)$$
$$= 2i + 6j - 10k$$

**Ex:** Find a vector Prependicular to the Plane of $P(1,-1,0)$, $Q(2,1,-1)$, and $R(-1,1,2)$




**Sol:**

$$\overrightarrow{PQ} = (2-1)i + (1+1)j + (-1-0)K$$

$$= 1i + 2j - K$$

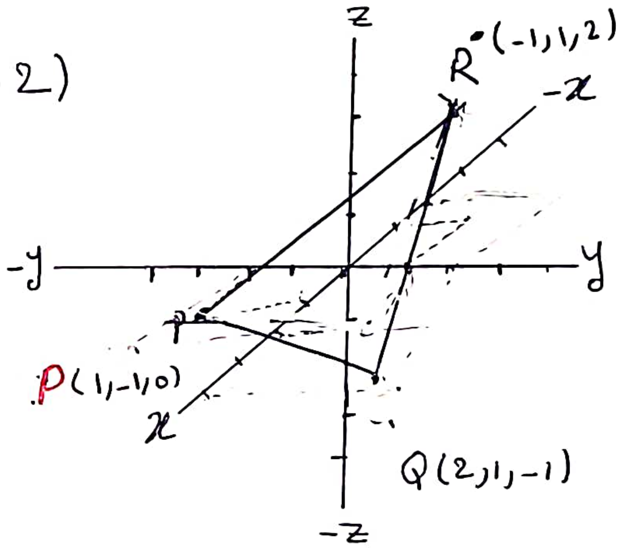
$$\overrightarrow{PR} = (-1-1)i + (1+1)j + (2-0)K$$

$$= -2i + 2j + 2K$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & K \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} i - \begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} j + \begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} K$$

$$= 6i + 6K$$

**Ex:** Find the area of traingle with vertices $P(1,-1,0)$, $Q(2,1,-1)$, $R(-1,1,2)$




**Sol:**

The area of Parellelogram

$$|\overrightarrow{PQ} \times \overrightarrow{PR}| = |6i + 6K|$$

$$= \sqrt{(6)^2 + (6)^2} = \sqrt{36+36}$$

$$= \sqrt{2 * 36} = 6\sqrt{2}$$

The area of triangle is half of this $= \dfrac{6\sqrt{2}}{2} = 3\sqrt{2}$

Ex: Find aunit vector Prependicular to the plane of

$P(1,-1,0)$, $Q(2,1,-1)$, $R(-1,1,2)$

Sol:

$$n = \frac{\overrightarrow{PQ} \times \overrightarrow{PR}}{|\overrightarrow{PQ} \times \overrightarrow{PR}|} = \frac{6i + 6K}{6\sqrt{2}} = \frac{1}{\sqrt{2}} i + \frac{1}{\sqrt{2}} K$$

## Triple Scalar or Box Product

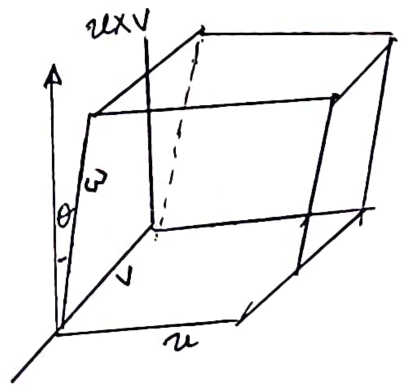
The Product $(u \times v) \cdot w$ is called triple scalar Product of $u, v$, and $w$.

$$|(u \times v) \cdot w| = |u \times v| |w| \, |\cos(\theta)|$$

the absolute value of this Product of the Parallelepiped determined by $u$, $v$, and $w$. the number $|u \times v|$ is the area of the base Parallelogram, the number $|w| \, |\cos\theta|$ is Parallelepiped's hight.

Calculating the trible scalar Product or Box Product as a determinant:

$$(u \times v) \cdot w = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



**Ex:** Find the valume of the Box (parallele piped) determined by $u = i + 2j - k$, $v = -2i + 3k$, and $w = 7j - 4k$

**Sol:**

$$(u \times v)\cdot w = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = (1)\begin{vmatrix} 0 & 3 \\ 7 & -4 \end{vmatrix} - 2\begin{vmatrix} -2 & 3 \\ 0 & -4 \end{vmatrix}$$

$$+ (-1)\begin{vmatrix} -2 & 0 \\ 0 & 7 \end{vmatrix} = -21 - 16 + 14 = -23 \text{ units cubed.}$$

## Lines and planes in space: المستقيمات والمستويات في الفضاء

**Def:** Vector Equation for a Line

A vector equation for the line $L$ through $P_0(x_0, y_0, z_0)$ Parallel to $v$ is

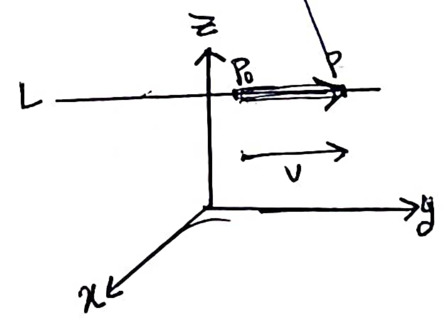
$$r(t) = r_0 + tv, \qquad -\infty < t < \infty$$

where $r$ is a position vector of a point $P(x, y, z)$ on $L$, and $r_0$ is the Position vector $P_0(x_0, y_0, z_0)$

**Def:** Parametric Equations for a Line

The standard Parametrization of the line throug $P_0(x_0, y_0, z_0)$ Parallel to $v = v_1 i + v_2 j + v_3 k$ is:

$$\boxed{\left.\begin{aligned} x &= x_0 + tv_1 \\ y &= y_0 + tv_2 \\ z &= z_0 + tv_3 \end{aligned}\right\} -\infty < t < \infty}$$



Ex: Find the Parametic equations for the line through $(-2,0,4)$ Parallel to ... $V = 2i + 4j - 2K$

Sol: the line through $(\underline{-2}_{x_0}, \underline{0}_{y_0}, \underline{4}_{z_0})$ , Parallel $v = \underline{2}_{v_1} i + \underline{4}_{v_2} j - \underline{2}_{v_3} K$

$x = x_0 + tv_1$

$x = -2 + 2t$

$y = y_0 + tv_2 \Rightarrow y = 0 + 4t = 4t$

$z = z_0 + tv_3 \Rightarrow z = 4 - 2t$.

Ex: Find the Parametic equations for the line through $P(-3,2,-3)$ and $Q(1,-1,4)$

Sol

$V = \overrightarrow{PQ} = (1+3)i + (-1-2)j + (4+3)K$

$= 4i - 3j + 7K$

$v = 4i - 3j + 7K$ , $P(-3,2,-3)$

$x = x_0 + tv_1 \Rightarrow x = -3 + 4t$

$y = y_0 + tv_2 \Rightarrow y = 2 - 3t$

$z = z_0 + tv_3 \Rightarrow z = -3 + 7t$

or $v = 4i - 3j + 7K$ , $Q(1,-1,4)$

$x = 1 - 4t$

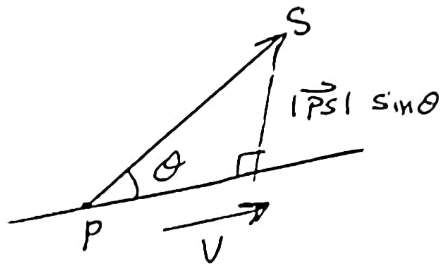
$y = -1 - 3t$

$z = 4 + 7t$

## The Distance from a Point to a line in space:

To find the distance from a point S to a line that passes through a Point P Parallel to a vector v, by:

$$\boxed{d = \frac{|\vec{PS} \times v|}{|v|}}$$




**Ex:** find the distance from the Point $S(1,1,5)$ to the line

$L: \quad x = 1+t, \quad y = 3-t, \quad z = 2t$

**Sol** From the equation of L, the Point $P(\overset{x_0}{1}, \overset{y_0}{3}, \overset{z_0}{0})$

and $v = i - j + 2k$

$$\therefore d = \frac{|\vec{PS} \times v|}{|v|}$$

$\vec{PS} = (1-1)i + (1-3)j + (5-0)k = -2j + 5k$

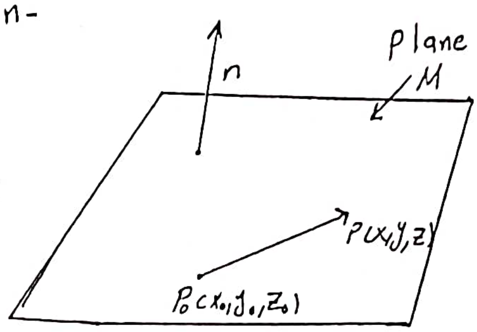
$$|\vec{PS} \times v| = \begin{vmatrix} i & j & k \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = i\begin{vmatrix} -2 & 5 \\ -1 & 2 \end{vmatrix} - j\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix} + k\begin{vmatrix} 0 & -2 \\ 1 & -1 \end{vmatrix}$$

$$= i + 5j + 2k$$

$$d = \frac{|\vec{PS} \times v|}{|v|} = \frac{\sqrt{1+25+4}}{\sqrt{1+1+4}} = \frac{\sqrt{30}}{\sqrt{6}} = \frac{\sqrt{5}\cancel{\sqrt{6}}}{\cancel{\sqrt{6}}} = \sqrt{5}$$

## Def:- The Equation For a plane in Space:

Suppose that plane $M$ passes through a point $P_0(x_0, y_0, z_0)$ and is normal to non-zero vector $n = Ai + Bj + Ck$. Then $M$ is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $n$.




Thus the dot Product $n \cdot \overrightarrow{P_0P} = 0$

i.e.

$$n \cdot \overrightarrow{P_0P} = (Ai + Bj + ck) \cdot [(x - x_0)i + (y - y_0)j + (z - z_0)k] = 0$$

$$\boxed{A(x - x_0) + B(y - y_0) + c(z - z_0) = 0}$$

the eq. of plane Pass ~~that~~ through $P_0(x_0, y_0, z_0)$ and orthogonal to $n$

Equation for a plane:

The Plane through $P_0(x_0, y_0, z_0)$ normal to $n = Ai + Bj + Ck$ has

Vector equation: $\quad n \cdot \overrightarrow{P_0P} = 0$

and

Component equation: $\quad A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $\quad Ax + By + Cz = D$

where $\quad D = Ax_0 + By_0 + Cz_0$

Ex: Find the equation for the Plane through $P_0(-3,0,7)$ Prependicular to $n = 5i + 2j - k$

Sol:

$$A(X - X_0) + B(y - y_0) + C(Z - Z_0) = 0$$

$$5(X - (-3)) + 2(y - 0) + (-1)(Z - 7) = 0$$

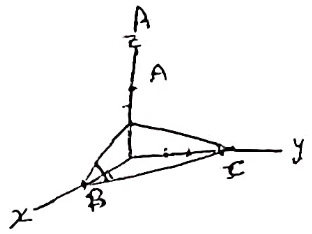
$$5X + 15 + 2y - Z + 7 = 0$$

$$5X + 2y - Z = -22$$

Ex: Find an equation for the Plane through $A(0,0,1)$, $B(2,0,0)$, and $C(0,3,0)$

Sol: first find the vector normal to aplane from the cross Product

$\vec{AB} \times \vec{AC}$, where




$$\vec{AB} = (2-0)i + (0-0)j + (0-1)K = 2i - K$$

$$\vec{AC} = (0-0)i + (3-0)j + (0-1)K = 3j - K$$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & K \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 3 & -1 \end{vmatrix} i - \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} j + \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} K$$

$$= 3i + 2j + 6K$$

$n = 3i + 2j + 6K$ , $A(0,0,1)$

The equation for the Plane is:

$$A(X - X_0) + B(y - y_0) + C(Z - Z_0) = 0$$

$$3(X - 0) + 2(y - 0) + 6(Z - 1) = 0$$

$$3X + 2y + 6Z = 6$$

**Remark:** Lines of Intersection

1. The equation of $xy$-plane is $z=0$ ; $(0,0,a) \cdot (0,0,0)$
   The equation of $yz$-plane is $x=0$ , $(a,0,0)=(0,0,0)$
   The equation of $xz$-plane is $y=0$ , $(0,a,0)=(0,0,0)$
2. The lines are Parallel iff they have the same direction.
3. Two plane are parallel iff their Prependicular Vactors (normal vectors) are Parallel. or

   $n_1 = K n_2$ , $n_1, n_2$ are prependicular vector, $K$ is scalar

Two plane that are not Parallel intersection in line.

**Ex:** Find avector Parallel to the line of intersection of the Planes $3x-6y-2Z=15$ and $2x+y-2Z=5$

**Sol:**

$n_1 = 3x-6y-2Z$ , $n_2 = 2x+y-2Z$

$n_1 = 3i-6j-2K$ , $n_2 = 2i+j-2K$

$$n_1 \times n_2 = \begin{vmatrix} i & j & K \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \begin{vmatrix} -6 & -2 \\ 1 & -2 \end{vmatrix} i - \begin{vmatrix} 3 & -2 \\ 2 & -2 \end{vmatrix} j + \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix} k$$

$= 14i + 2j + 15k$

Ex: Find the Point where the line

$x = \frac{8}{3} + 2t, \quad y = -2t, \quad z = 1 + t$

intersects the Plane $3x + 2y + 6z = 6$

Sol:

subsitituting the Point $(x, y, z) = (\frac{8}{3} + 2t, -2t, 1+t)$

in equation of the plane

$3(\frac{8}{3} + 2t) + 2(-2t) + 6(1+t) = 6$

$\therefore 8 + 6t - 4t + 6 + 6t = 6 \Rightarrow 8t + 14 = 6$

$8t = 6 - 14 \Rightarrow 8t = -8 \Rightarrow \boxed{t = -1}$

substituting $t = -1$ in the Point

$(x, y, z)_{t=-1} = (\frac{8}{3} + 2t, -2t, 1+t)_{t=-1}$

$(x, y, z)_{t=-1} = (\frac{8}{3} - 2, -2 \times -1, 1 - 1) = (\frac{2}{3}, 2, 0)$ the Point

of intersection.

c. Replac

**Def:** The Distance from a Point to a plane:

If $P$ is a Point on a plane with vector $n$, then the distance from any Point $S$ to the Plane is

$$\boxed{d = \left| \vec{PS} \cdot \frac{n}{|n|} \right|}$$

the length of the vector Projection of $\vec{PS}$ onto $n$

where $n = Ai + Bj + Ck$

**Ex:** find the distance from $S(1,1,3)$ to the Plane

$3x + 2y + 6z = 6$

**Sol:**

take $P(0, 3, 0)$ , $S(1,1,3)$

$$\boxed{\begin{array}{l} 3x + 2y + 6z \stackrel{?}{=} 6 \\ 3*0 + 2*3 + 6*0 = 6 \end{array}}$$

$\vec{PS} = (1-0)i + (1-3)j + (3-0)k$

$\vec{PS} = i - 2j + 3k$

$|n| = \sqrt{(3)^2 + (2)^2 + (6)^2} = \sqrt{9+4+36} = \sqrt{49} = 7$

The distance from $S$ to a plane is

$$d = \left| \vec{PS} \cdot \frac{n}{|n|} \right|$$

$$= \left| (i - 2j + 3k) \cdot \frac{3i + 2j + 6k}{7} \right|$$

$$= \left| (i - 2j + 3k) \cdot \left(\frac{3}{7}i + \frac{2}{7}j + \frac{6}{7}k\right) \right|$$

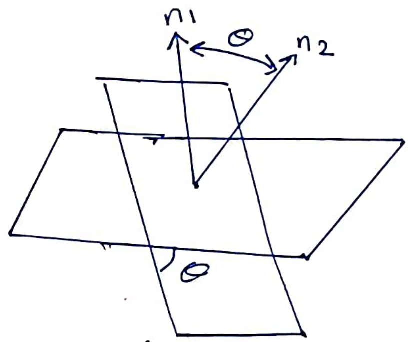
$=$

$$= \left| \frac{3}{7} - \frac{4}{7} + \frac{18}{7} \right| = \frac{17}{7}$$

**Def:** Angles Between Planes

The angle between to intersecting plane is defined to be the acute angle between their vector.

$$\boxed{\theta = \cos^{-1} \frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|}}$$




**Ex:** Find the angle between the planes

$3x - 6y - 2z = 15$ and $2x + y - 2z = 5$

**Sol:**

$3x - 6y - 2z = 15 \xRightarrow{\text{vector}} n_1 = 3i - 6j - 2k$

$2x + y - 2z = 5 \xRightarrow{\text{vector}} n_2 = 2i + j - 2k$

$$\theta = \cos^{-1}\left( \frac{n_1 \cdot n_2}{|n_1| \cdot |n_2|} \right)$$

$n_1 \cdot n_2 = 6 - 6 + 4 = 4$

$|n_1| \cdot |n_2| = \sqrt{9+36+4} \cdot \sqrt{4+1+4} = \sqrt{49} \cdot \sqrt{9} = 7 * 3 = 21$

$\theta = \cos^{-1} \frac{4}{21} \approx 1.38$ radians or $79^\circ$

H.W

1. Find the Parametic equation for the line through the point $P(3,-4,-1)$ Parallel to the Vector $i + i + k$

2. Find the Parametic equation for the line through $P(1,2,-1)$, and $Q(-1,0,1)$

3. Fined the distance from the Point $(0,0,12)$ to the line $x = 4t$, $y = -2t$, $z = 2t$

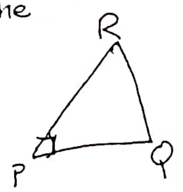
4. Find the distance from the Point $(2,-3,4)$ to the Plane $x + 2y + 2z = 13$

5. Find the angle between the planes $x + y = 1$ and $2x + y - 2z = 2$.

6. Find the acute angle between the planes $2x + 2y + 2z = 3$ and $2x - 2y - z = 5$

7. Find $u \times v$ and $v \times u$ where $u = 2i - 2j - k$ and $v = i - k$

8. Find the area of the triangle determined by the $P(1,-1,2)$, $Q(2,0,-1)$, $R(0,2,1)$ and find aunit vector prependicular to the plane $PQR$




9. Find the Valume of Parallelepiped where $u = i - j + k$, $v = 2i + j - 2k$, and $w = -i + 2j - k$.

...signs a unique (single) ...

**Def:** The equation of sphere:

The equation of sphere where $P_0(x_0, y_0, z_0)$ is a Center and $P(x, y, z)$ is a Point, and $r$ is a distance from $P$ to $P_0$ (radius) is:

$$\boxed{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2}$$ , the eq. of sphere

where $P_0(x_0, y_0, z_0)$ is center & $r$ is radius

$$\boxed{x^2+y^2+z^2=r^2}$$ , the eq. of sphere where $(0,0,0)$ is Center and $r$ is radius.

**Ex:** Find the Center and radius for the sphere

$$x^2+y^2+z^2-10x-8y-12z+68=0$$

**Sol** $(x^2-10x)+(y^2-8y)+(z^2-12z)=-68$

$$(x^2-10x+(\tfrac{10}{2})^2)+(y^2-8y+(\tfrac{8}{2})^2)+(z^2-12z+(\tfrac{12}{2})^2)=-68+(\tfrac{10}{2})^2+(\tfrac{8}{2})^2+(\tfrac{12}{2})^2$$

$$(x-5)^2+(y-4)^2+(z-6)^2=-68+25+16+36$$

$$(x-5)^2+(y-4)^2+(z-6)^2=9$$

The Center of sphere is $(5, 4, 6)$, and radius is $3$

**Ex:** Find the equation of sphere with (1, 2, 3) is Center and radius is $r=4$, and if the points (5, 2, 3), (3, 0, 6) is inside or out side the sphere?

**Sol:** $(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$

$\because$ (1, 2, 3) is center & $r=4$, then

$(x-1)^2+(y-2)^2+(z-3)^2=16$ this eq. of sphere.

Substituting the point (5, 2, 3) in the eq. of sphere

$(5-1)^2+(2-2)^2+(3-3)^2 \overset{?}{=} 16$

$(4)^2+(0)^2=16 \Rightarrow$ The point (5, 2, 3) is inside the sphere

substituting the point (3, 0, 6) in the eq. of sphere

$(3-1)^2+(0-2)^2+(6-3)^2 \overset{?}{=} 16$

$(2)^2+(-2)^2+(3)^2=4+4+9$

$=17>16 \Rightarrow$ The point (3, 0, 6) is out-side the sphere.

**Ex:** Find the equation of sphere with center (3, 0, 4) and radius 5

# Chapter Four
## Infinite Sequences and Series
## المتتابعات اللانهائية والمتسلسلات

A Sequence is a list of number $a_1, a_2, a_3, \ldots, a_n, \ldots$ in a given order. Each of $a_1, a_2, a_3, \ldots$ is represented a number These are the terms of the Sequence.

For example, the sequence

$$2, 4, 6, 8, 10, \ldots, 2n, \ldots$$

the first term $a_1 = 2$

second term $a_2 = 4$

and nth term $a_n = 2n$

The integer $n$ is called the index of $a_n$. and denoted by $\{\ \}$

**Def:** The infinite sequance of numbers is the function $f: N \longrightarrow S$, whose domain is the set of Positive intigers, $N = \{1, 2, 3, \ldots\}$, $S$ is the set, $S \neq \emptyset$

ex: $a_1, a_2, a_3, \ldots, a_n, \ldots$

$f(1) = a_1 \rightarrow$ the first term

$f(2) = a_2 \rightarrow$ the second term

$f(3) = a_3$

$\vdots$

$f(n) = a_n \rightarrow$ nth term

$\vdots$

Ex:

① $a_n = \sqrt{n}$

$a_1 = \sqrt{1}, \; a_2 = \sqrt{2}, \; a_3 = \sqrt{3}, \ldots, a_n = \sqrt{n}, \ldots$

$\{a_n\} = \{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$

② $b_n = (-1)^{n+1} \cdot \frac{1}{n}$

$b_1 = (-1)^{1+1} \cdot \frac{1}{1}, \; b_2 = (-1)^{2+1} \frac{1}{2}, \; b_3 = (-1)^{3+1} \frac{1}{3}, \ldots, b_n = (-1)^{n+1} \frac{1}{n}, \ldots$

$\{b_n\} = \{1, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \ldots, (-1)^{n+1} \frac{1}{n}, \ldots\}$

③ $C_n = \frac{n-1}{n}$

$c_1 = \frac{1-1}{1}, \; C_2 = \frac{2-1}{2}, \; C_3 = \frac{3-1}{3}, \ldots, C_n = \frac{n-1}{n}, \ldots$

$\{C_n\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n-1}{n}, \ldots\}$

④ $d_n = (-1)^{n+1}$

$d_1 = (-1)^{1+1}, \; d_2 = (-1)^{2+1}, \; d_3 = (-1)^{3+1}, \ldots, d_n = (-1)^{n+1}, \ldots$

$\{d_n\} = \{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$

**Def:** The Sequence $\{a_n\}$ converges to the number $L$ for every positive number $\epsilon$ there corresponds an integer $N$ such that for all $n$,

$$n > N \Rightarrow |a_n - L| < \epsilon .$$

If no such number $L$ exists, we say that $\{a_n\}$ diverges.

If $\{a_n\}$ converges to $L$, we write $\lim_{n\to\infty} a_n = L$, or $a_n \to L$, and call $L$ the limit of the sequence.

**Ex:** show that $\lim_{n\to\infty} \frac{1}{n} = 0$

**Sol:** $\forall \epsilon > 0$, we must find an integer $N$ s.t

$$\forall n > N \Rightarrow |a_n - L| < \epsilon$$

$$\left|\frac{1}{n} - 0\right| < \epsilon$$

$$\left|\frac{1}{n}\right| < \epsilon \quad \text{or} \quad n > \frac{1}{\epsilon}$$

$$\therefore \lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$$

Ex: show that $\lim_{n\to\infty} k = k$, (any constant $k$)

Sol: $\forall \epsilon > 0$, we must find an integer $N$ s.t

$$\forall n > N \Rightarrow |a_n - L| < \epsilon$$

$$|k - k| < \epsilon$$

$$0 < \epsilon \Rightarrow \epsilon > 0$$

$\therefore \lim_{n\to\infty} k = k$ for any constant $k$.

Ex: The sequence $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$ diverges

$\lim_{n\to\infty} (-1)^{n+1}$ diverges

Ex: the sequence $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$ is

diverges

$$\lim_{n\to\infty} \sqrt{n} = \infty$$

<u>Theorem 1</u>: Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let $A$ and $B$ real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

① Sum Rule: $\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A+B$

② Difference Rule: $\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$

③ Constant Multiple Rule: $\lim_{n\to\infty} (k\, b_n) = k \lim_{n\to\infty} b_n = k \cdot B$

④ Product Rule: $\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = A \cdot B$

⑤ Quotient Rule: $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$, if $B \neq 0$

<u>Ex</u>: find the limits by Combining Theorem 1.

① $\lim_{n\to\infty} \left(\frac{-1}{n}\right)$

<u>Sol</u>: $\lim_{n\to\infty} \left(\frac{-1}{n}\right) = -1 \lim_{n\to\infty} \left(\frac{1}{n}\right) = -1 \cdot 0 = 0$

② $\lim_{n\to\infty} \left(\frac{n-1}{n}\right) = \lim_{n\to\infty} \left(\frac{n}{n} - \frac{1}{n}\right) = \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)$

$= \lim_{n\to\infty}(1) - \lim_{n\to\infty} \frac{1}{n} = 1 - 0 = 1$

③ $\lim_{n\to\infty} \frac{5}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n^2} = 5 \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n}$

$= 5 \cdot 0 \cdot 0 = 0$

④ $\lim_{n\to\infty} \frac{4-7n^6}{n^6+3} = \lim_{n\to\infty} \frac{4/n^6 - 7}{1 + 3/n^6}$

$= \frac{\lim_{n\to\infty} 4/n^6 - \lim_{n\to\infty} 7}{\lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{3}{n^6}} = \frac{0-7}{1+0} = -7$

Theorem 2: "The Sandwich Theorem for Sequences"

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be Sequences for real numbers

If $a_n \leq b_n \leq c_n$ holds forall $n$ beyond some index $N$

and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

Ex: show that the sequences is Converges to Zero.

1. $\frac{\cos n}{n} \overset{?}{\rightarrow} 0$, $\because \underbrace{\frac{-1}{n}}_{\searrow 0} \leq \frac{\cos n}{n} \leq \underbrace{\frac{1}{n}}_{\searrow 0}$

$\lim_{n\to 0} \frac{-1}{n} = 0$ $\lim_{n\to 0} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{\cos n}{n} = 0$ (by sandwich theorem)

2. $\frac{1}{2^n} \overset{?}{\to} 0$ , $\because$ $0 \leq \frac{1}{2^n} \leq \frac{1}{n}$

$\lim_{n\to\infty} 0 = 0$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\lim_{n\to\infty} \frac{1}{2^n} = 0$.

③ $(-1)^n \frac{1}{n} \overset{?}{\longrightarrow} 0$ , $\because$ $\frac{-1}{n} \leq (-1)^n \frac{1}{n} \leq \frac{1}{n}$

$\lim_{n\to\infty} \frac{-1}{n} = 0$ $\lim_{n\to\infty} \frac{1}{n} = 0$

$\therefore \lim_{n\to 0} (-1)^n \frac{1}{n} = 0$ (by sandwich theorem)

<u>Theorem 3:</u> The following sequences converges to the limits

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\lim_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim_{n\to\infty} x^{1/n} = 1$ , $x > 0$

4. $\lim_{n\to\infty} x^n = 0$ $(|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$ (any $x$)

6. $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ (any $x$)

Ex: find the limit of the following Sequence

① $\frac{\ln(n)^2}{n}$

Sol: $\lim_{n\to\infty} \frac{\ln(n)^2}{n} = \lim_{n\to\infty} \frac{2\ln(n)}{n} = 2\lim_{n\to\infty} \frac{\ln(n)}{n} = 2.0 = 0$

② $\sqrt[n]{n^2}$

Sol $\lim_{n\to\infty} \sqrt[n]{n^2} = \lim_{n\to\infty} n^{2/n} = \lim_{n\to\infty} (n^{1/n})^2 = (1)^2 = 1$

③ $\sqrt[n]{3n} =$

Sol: $\lim_{n\to\infty} \sqrt[n]{3n} = \lim_{n\to\infty} (3n)^{1/n} = \lim_{n\to\infty} 3^{1/n} \cdot n^{1/n}$

$= \underbrace{\lim_{n\to\infty} 3^{1/n}}_{} \lim_{n\to\infty} n^{1/n} = 1 . 1 = 1$

by ③ Theorem ③
where $x = 3$

④ $(\frac{-1}{2})^n$

Sol: $\lim_{n\to\infty} (\frac{-1}{2})^n = 0$ (by ④ theorem 3 where $x = \frac{-1}{2}$)

H.W: Is the following sequence is converges or diverges?

1. $\left\{\left(\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{\infty}$

2. $\left\{(-1)^n\right\}_{n=1}^{\infty}$

3. $\left\{(2n)\right\}_{n=1}^{\infty}$

4. $\left\{\frac{1}{n} - \frac{1}{n+1}\right\}_{n=1}^{\infty}$

5. $\left\{\frac{3n^2+2}{n^2+4n}\right\}_{n=1}^{\infty}$

6. $\left\{\frac{n+3}{n^2+5n+6}\right\}_{n=1}^{\infty}$

7. $\left\{\frac{n^2-1}{n}\right\}_{n=1}^{\infty}$

## Infinite Series:

**Def:** Given a sequance of numbers $\{a_n\}$, an expression of the form $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ is an infinite series. The number $a_n$ is the nth term of the series. The sequence $\{S_n\}$ defined by

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$S_3 = a_1 + a_2 + a_3$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$
$$\vdots$$

is the sequence of Partial sums of the series, the number $S_n$ being the nth Partial sum. If the sequence of Partial sums converges to a limit $L$, we say that the Series converges and that its sum is $L$. In this case we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L$$

If the sequence of Partial sums of the series does not convergent, we say that the series diverges.

Ex: The series $\underbrace{1}_{a_1} + \underbrace{\frac{1}{2}}_{a_2} + \underbrace{\frac{1}{4}}_{a_3} + \underbrace{\frac{1}{8}}_{a_4} + \underbrace{\frac{1}{16}}_{a_5} + \underbrace{\frac{1}{32}}_{a_6} + \underbrace{\frac{1}{64}}_{a_7} + \cdots$

$S_1 = a_1 = 1$ First term

$S_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2}$ second term

$S_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$ Third term

$\vdots$

$S_n = a_1 + a_2 + a_3 + \cdots + a_n = 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}$

$= \frac{2^n - 1}{2^{n-1}}$ nth term

Def: Geometric Series

The geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which $a$ and $r$ are fixed real numbers and $a \neq 0$.

Ex: The Series can also be written as $\sum_{n=0}^{\infty} ar^n$

- The ratio can be positive, as in

$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \left(\frac{1}{2}\right)^{n-1} + \cdots$ ,

$(1).(1) + (1)\left(\frac{1}{2}\right) + (1)\left(\frac{1}{4}\right) + \cdots + (1)\left(\frac{1}{2}\right)^{n-1}$

This geometric series s.t $a = 1$, $r = \frac{1}{2}$

//

- The ratio $r$ can be negative, as in

$$1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\cdots\cdots+\left(\frac{-1}{3}\right)^{n-1}+\cdots$$

This geometric series s.t $r=\frac{-1}{3}$ & $a=1$

Remark:

If $|r|<1$, the geometric series $a+ar+ar^2+\cdots+ar^{n-1}+\cdots$

Converges to $\frac{a}{1-r}$ s.t

$$\sum_{n=1}^{\infty} ar^{n-1}=\frac{a}{1-r}, \quad |r|<1.$$

If $|r|\geq 1$, the series diverges.

Ex: show that the geometric series is converges

$$\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\cdots+\left(\frac{1}{9}\right)^{n-1}+\cdots$$

Sol:

The geometric series with $a=\frac{1}{9}$ and $r=\frac{1}{3}$ s.t

$$\underbrace{\frac{1}{9}}_{a}+\underbrace{\left(\frac{1}{9}\right)}_{a}\underbrace{\left(\frac{1}{3}\right)}_{r}+\underbrace{\left(\frac{1}{9}\right)}_{a}\underbrace{\left(\frac{1}{9}\right)}_{r^2}+\cdots+\left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n-1}+\cdots$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n-1}$$, is converges to

$$\frac{a}{1-r}=\frac{1/9}{1-(1/3)}=1/6$$

**Ex:** Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n}$ is Converg

**Sol:**

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \frac{(-1)^0 5}{4^0} + \frac{(-1)^1 5}{4^1} + \frac{(-1)^2 5}{4^2} + \frac{(-1)^3 5}{4^3} + \cdots$$

$$= 5 - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots\cdots$$

$$= 5 + (5)\left(\frac{-1}{4}\right) + 5\left(\frac{-1}{4}\right)^2 + 5\left(\frac{-1}{4}\right)^3 + \cdots$$

This geometric series with $a = 5$ & $r = \frac{-1}{4}$

$\therefore |r| = \left|\frac{-1}{4}\right| = \frac{1}{4} < 1$

$\therefore$ The geometric Series Converges to $\frac{a}{1-r} = \frac{5}{1-(-1/4)}$

$$= \frac{5}{1-(-1/4)} = \frac{5}{1+1/4} = 4$$

**Theorem:** If $\sum_{n=1}^{\infty} a_n$ Converges, then $a_n \to 0$.

**Remark:** "The nth-Term test for divergence"

$\sum_{n=0}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from Zero.

Ex: show that the following series is divergent.

1. $\sum_{n=1}^{\infty} n^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + \cdots + (n-1)^2 + \cdots$

$= 1 + 4 + 9 + 16 + 25 + \cdots$

$\boxed{\begin{array}{l} S_1 = a_1 = 1 \\ S_2 = a_1 + a_2 = 1+4 = 5 \\ S_3 = a_1 + a_2 + a_3 = 14 \end{array}}$

is diverges: because $n^2 \to \infty$, $\lim_{n\to\infty} n^2 = \infty$

2. $\sum_{n=1}^{\infty} \frac{n+1}{n} = \frac{1+1}{1} + \frac{2+1}{2} + \frac{3+1}{3} + \frac{4+1}{4} + \cdots$

$= 2 + \frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \frac{6}{5} + \cdots$

is diverges: because $\frac{n+1}{n} \to 1$, $\lim_{n\to\infty} \frac{n+1}{n} \neq 0$

3. $\sum_{n=1}^{\infty} (-1)^{n+1} = (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 + \cdots$

$= +1 - 1 + 1 - 1 + 1 - 1 + \cdots$

is diverges because $\lim_{n\to\infty} (-1)^{n+1}$ does not exist.

4. $\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ H.W.

**Theorem**: If $\sum a_n = A$ and $\sum b_n = B$ are Convergent series, then

1. Sum Rule: $\sum (a_n + b_n) = \sum a_n + \sum b_n = A + B$

2. Difference Rule: $\sum (a_n - b_n) = \sum a_n - \sum b_n = A - B$

3. Constant Multiple Rule: $\sum K a_n = K \sum a_n = K \cdot A$

**Remark:**

1. Every non zero Constant multiple of a divergent series diverges.

2. If $\sum a_n$ Converges and $\sum b_n$ diverges, then $\sum (a_n + b_n)$ and $\sum (a_n - b_n)$ both diverges.

3. $\sum (a_n + b_n)$ Can Converge when $\sum a_n$ and $\sum b_n$ both diverge.

**Ex:** $\sum a_n = 1 + 1 + 1 + 1 + \cdots$ is diverge

$\sum b_n = (-1) + (-1) + (-1) + \cdots$ is diverge

$\sum a_n + b_n = 0 + 0 + 0 + \cdots$ is Converge to Zero

Ex: find the sum of the Series $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}}$

Sol: $\sum_{n=1}^{\infty} \frac{3^{n-1}-1}{6^{n-1}} = \sum_{n=1}^{\infty} \left( \frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right)$

$$= \sum_{n=1}^{\infty} \frac{3^{n-1}}{6^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{3^{n-1}}{(2*3)^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} - \sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$$

$$= 2 - 6/5 = 4/5$$

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \frac{1}{2^{1-1}} + \frac{1}{2^{2-1}} + \frac{1}{2^{3-1}} + \cdots = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ the geometric series with $a=1$ & $r=\frac{1}{2}$

$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is Converges to $\frac{a}{1-r} = \frac{1}{1-(1/2)} = 2$

$\sum_{n=1}^{\infty} \frac{1}{6^{n-1}} = \frac{1}{6^0} + \frac{1}{6^1} + \frac{1}{6^2} + \frac{1}{6^3} + \cdots$

$= 1 + \frac{1}{6} + \frac{1}{6^2} + \frac{1}{6^3} + \frac{1}{6^4} + \cdots$

$\sum_{n=1}^{\infty} \frac{1}{6^{n-1}}$ is geometric Series with $a=1$ & $r=\frac{1}{6}$

Converges to $\frac{a}{1-r} = \frac{1}{1-1/6} = \frac{6}{5}$

Ex: find the sum of the series $\sum_{n=0}^{\infty} \frac{4}{2^n}$

Sol: $\sum_{n=0}^{\infty} \frac{4}{2^n} = 4 \sum_{n=0}^{\infty} \frac{1}{2^n}$ (constant Multiple Rule)

$$= 4\left(\frac{1}{1-\frac{1}{2}}\right) = 8$$

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

$= 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$ this geometric series with $a=1$ & $r=\frac{1}{2}$ Converges to $\frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$

H.w: find the sum of the series $\sum_{n=1}^{\infty} \frac{7}{4^n}$

Theorem: "The Comparison test"

let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer $N$

$$d_n \leq a_n \leq c_n \quad \forall n > N$$

1- If $\sum c_n$ Converges, then $\sum a_n$ also Converges.

2. If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

Ex: Show that the series $\sum_{n=1}^{\infty} \frac{5}{5n-1}$ diverges ?

Sol: we apply theorem The Comparison test

$$\sum_{n=1}^{\infty} \frac{5}{5n-1} = \sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{5}}$$

s.t $\frac{1}{n-\frac{1}{5}} > \frac{1}{n}$ i.e. $\frac{1}{n} < \frac{1}{n-\frac{1}{5}}$

$n=1$

$\frac{1}{n} < \frac{1}{1-\frac{1}{5}}$

$1 < \frac{1}{\frac{4}{5}}$

$1 < \frac{5}{4}$

$\because \sum_{n=1}^{\infty} \frac{1}{n}$ is diverges [using nth term]

$\therefore \sum_{n=1}^{\infty} \frac{1}{n-\frac{1}{5}}$ diverges [the Comparison test]

Ex: Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ Converges ?

Sol: $\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

The series $\sum_{n=0}^{\infty} \frac{1}{n!}$ is converges because all terms are Positive and $\sum_{n=0}^{\infty} \frac{1}{n!} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$

$\because \sum_{n=0}^{\infty} \frac{1}{2^n}$ is geometric series Converges to 2

$\therefore 1 + \sum_{n=0}^{\infty} \frac{1}{2^n}$ : Converges to 3

**Theorem: "limit Comparison test"**

Suppose that $a_n > 0$ and $b_n > 0$ for all $n \geq N$ ($N$ an integer)

1. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both Converge or both diverge.

2. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ Converges, then $\sum a_n$ Converges.

3. If $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

**Ex:** Which of the following series Converge, and which diverge?

1. $\sum\limits_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1}$

2. $\sum\limits_{n=1}^{\infty} \frac{1}{2^n - 1}$

3. $\sum\limits_{n=2}^{\infty} \frac{1+n \ln n}{n^2+5}$

Sol:

1. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+2n+1} = \sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2} = \frac{3}{4} + \frac{5}{9} + \frac{7}{16} + \frac{9}{25} + \cdots$

let $a_n = \frac{2n+1}{n^2+2n+1}$, for large $n$ take $2n/n^2 = \underbrace{2/n}_{b_n}$, then let

$b_n = \frac{1}{n} \Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

and

$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{2n+1}{n^2+2n+1}}{1/n} = \lim_{n\to\infty} \frac{2n+1}{n^2+2n+1} \cdot \frac{n}{1} =$

$= \lim_{n\to\infty} \frac{2n^2+n}{n^2+2n+1} = 2 > 0$

$\therefore \sum a_n$ diverges by Theorem of limit Comparison (i).

2. $\sum_{n=1}^{\infty} \frac{1}{2^n-1} = \frac{1}{1} + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \frac{1}{31} + \cdots$

Let $a_n = \frac{1}{2^n-1}$, for large $n$ take $1/2^n$, then let

$b_n = \frac{1}{2^n} \Rightarrow \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$ Converges

and

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{2^n-1} \cdot \frac{2^n}{1} = \lim_{n\to\infty} \frac{2^n}{2^n-1}$

Converges to 1.

3. H.W

**Theorem: " The Ratio Test "**

Let $\sum a_n$ be a series with positive terms and suppose

that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = P$. Then

1. The Series Converges if $P<1$.
2. The series diverges if $P>1$ or $P=\infty$.
3. The test is inconclusive if $P=1$.

**Ex:** Investigate the convergence series

1. $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$

**Sol:** we apply the Ratio test for the Series $\sum_{n=1}^{\infty} \frac{2^n+5}{3^n}$.

$$\frac{a_{n+1}}{a_n} = \frac{(2^{n+1}+5)/3^{n+1}}{(2^n+5)/3^n} = \frac{2^{n+1}+5}{3^{\cancel{n}+1}} \cdot \frac{\cancel{3^n}}{2^n+5}$$

$$= \frac{1}{3} \cdot \frac{2^{n+1}+5}{2^n+5} \Big] * 2^{-n} = \left(\frac{1}{3} \cdot \frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}}\right) \longrightarrow \frac{1}{3}\cdot\frac{2}{1} = \frac{2}{3}$$

$\therefore$ The series Converges because $P=\frac{2}{3}<1$

$$* \quad \frac{2+5\cdot 2^{-n}}{1+5\cdot 2^{-n}} = \sum_{n=1}^{\infty} \frac{2+5*2^{-n}}{1+5*2^{-n}} \Rightarrow 2^{-n} = \frac{1}{2^n} = \frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$$

$$\longrightarrow 0$$

## Theorem: "The Root Test"

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$, and suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = P$. Then

1. the Series Converges if $P<1$.
2. The series diverges if $P>1$ or $P=\infty$
3. The test is inconclusive if $P=1$

**Ex:** Investigate The series is Converge or diverge?

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

**Sol:** we apply the root test

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \sqrt[n]{\frac{n^2}{2^n}} = \frac{\sqrt[n]{n^2}}{\sqrt[n]{2^n}} = \left(\frac{(\sqrt[n]{n})^2}{2}\right) \to \frac{1^2}{2} < 1$$

$\therefore$ The Series $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ is Converge to $\frac{1}{2}$

**Ex:** $\sum_{n=1}^{\infty} \frac{2^n}{n^3}$ is Converge or diverge? H.W.

Def: "The Power Series"

A Power Series about $x=0$ is a series of the form

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots + C_n x^n + \cdots \quad \cdots (1)$$

A Power Series about $x=a$ is the series of the form

$$\sum_{n=0}^{\infty} C_n (x-a)^n = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \cdots + C_n(x-a)^n + \cdots \quad \cdots (2)$$

in which the center $a$ and Coefficients $C_0, C_1, C_2, \cdots C_n, \cdots$

Ex: Taking the coefficient in eq. (1) to be 1

sol:

$$\sum_{n=0}^{\infty} C_n x^n = \sum_{n=0}^{\infty} x^n \quad , \text{taking } C_n = 1$$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

This a geometric series with $a=1$ and $r=x$

is Converges to $\frac{1}{1-x}$ for $|x| < 1$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad , \quad -1 < x < 1$$

Ex: The Power series

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\cdots+\left(\frac{-1}{2}\right)^n(x-2)^n+\cdots$$

find the converges of the series

Sol:

$$1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\cdots+\left(\frac{-1}{2}\right)^n(x-2)^n$$

The Power series with $a=2$, $C_0=1$, $C_1=-1/2$

$C_2=1/4, \ldots, C_n=\left(\frac{-1}{2}\right)^n$

This geometric series with the first term 1 and

$r=-\frac{x-2}{2}$ then $\frac{1}{1-r}=\frac{1}{1+\frac{x-2}{2}}=\frac{2}{x}$

$$\therefore \frac{2}{x}=1-\frac{1}{2}(x-2)+\frac{1}{4}(x-2)^2+\cdots+\left(\frac{-1}{2}\right)^n(x-2)^n+\cdots$$

Def: "Taylor Series" and "Maclaurian Series"

let $f$ be afunction with derivatives of all orders throughout some interval Containing $a$ as an interior Point. Then the Taylor Series generated by $f$ as $x=a$ is

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$$
$$+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{n}(a)}{n!}(x-a)^n+\cdots$$

The Maclaurian Series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{n}(0)}{n!}x^n + \cdots$$

The Taylor Series generated by f at $x=0$

Ex: find the Taylor series generated by $f(x) = 1/x$ at $a=2$, does the series converge to $1/x$ ?

Sol:
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(2)}{k!}(x-2) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots + \frac{f^{n}(2)}{n!}(x-2)^n + \cdots$$

$f(x) = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2!\,x^{-3}$, $f'''(x) = -6x^{-4}$

$f'''(x) = (-1)3!\,x^{-4}$, $f''''(x) = 24x^{-5} = (-1)4!\,x^{-5}$, ...)

$f^{n}(x) = (-1)n!\,x^{-(n+1)}$

$f(2) = 2^{-1} = 1/2$, $f'(2) = \frac{-1}{2^2}$, $f''(2) = 2!(2^{-3}) \Rightarrow \frac{f''(2)}{2!} = 2^{-3} = \frac{1}{2^3}$
$\vdots$

The Taylor series : $\frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}}$

$+ \cdots$

This geometric series with $a = 1/2$ and $r = -(x-2)/2$

Converge to $\frac{a}{1-r} = \frac{1/2}{1+(x-2)/2} \Big] \times 2 = \frac{1}{\cancel{2} + (x - \cancel{2})} = \frac{1}{x}$.

Ex: find the Tylor Series generated by $f(x) = \cos x$, at $x=0$

Sol $f(x) = \cos x$, $\quad f'(x) = -\sin x$

$f''(x) = -\cos x$, $\quad f'''(x) = \sin x$

$f''''(x) = \cos x$, $\quad f^{5}(x) = -\sin x$

$\vdots \quad\quad \vdots$

$f^{2n}(x) = (-1)^n \cos x \quad\quad f^{2n+1}(x) = (-1)^{n+1} \sin x$

at $x=0$, $f^{2n}(0) = (-1)^n \cos 0 = (-1)^n$

$f^{2n+1}(0) = (-1)^{n+1}(0) = 0$

$\therefore$ The Taylor series:

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{4}(0)}{4!}x^4 + \dots + \frac{f^{n}(0)}{n!}x^n + \dots$$

$$= 1 + 0\cdot x - \frac{x^2}{2!} + 0\cdot x^3 + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

H.W: find the Taylor Series generated by $f(x) = e^x$

at $x=0$