Chapter Five

Sequences and Series

This chapter is devoted mainly to series representations of analytic functions. To begin with, we shall give some definitions and results concerning the sequences of complex numbers.

Definition:

The function $f(n)$ defined for every positive integer $n = 1,2,3, \ldots,$ is a sequence writing $z_n = f(n)$, the sequence z_0 is denoted by $\{z_n\} = \{z_1, z_2, \ldots, z_n\}$. For example, if $f(n) = n$, then the sequence is denoted by $\{n\} = \{1,2,3, \ldots, n, \ldots\}$.

If g is a function defined by $g(n) = i^n$, then the sequence is denoted by

$$\{i^n\} = \{i, -1, -i, 1, i, \ldots\}$$

The range R of a sequence $\{z_n\}$ is the set of distinct values of $\{z_n\}$. A sequence $\{z_n\}$ has a limit z and written as

$$\lim_{n\to\infty} z_n = z$$

If for every $\epsilon > 0$, there exists a positive integer N, such that $|z_n - z| < \epsilon$ whenever $n > N$. When the limit z exists, $\{z_n\}$ is called convergent, otherwise it is called divergent.

Theorem 1:

If $\{z_n\}$ is convergent to z, then z is unique.

Proof: let $z_n \to z$ and $z_n \to z^*$, to prove that $z = z^*$. Now,

$$|z - z^*| = |z - z_n + z_n - z^*|$$

$$\leq |z - z_n| + |z_n - z^*|$$

$$< \epsilon_1 + \epsilon_2 = \epsilon$$

Hence, the limit is unique. ■

Theorem 2:

Suppose that $z_n = x_n + iy_n,\ z = x + iy,$ then $\lim_{n\to\infty} z_n = z$ if and only if $x_n \to x$ and $y_n \to y$.

Proof: let $z_n \to z$, so $|z_n - z| < \epsilon$, whenever $n > N$, i.e.:

$$|x_n + iy_n, -x - iy| = |x_n - x + i(y_n - y)| < \epsilon, \text{ whenever } n > N$$

Now,

$$\begin{aligned} |x_n - x| &= \sqrt{(x_n - x)^2} \\ &\leq \sqrt{(x_n - x)^2 + (y_n - y)^2} \\ &= |x_n - x + i(y_n - y)| < \epsilon, \text{ whenever } n > N \end{aligned}$$

Similarly,

$|y_n - y| < \epsilon$, whenever $n > N$

Hence, $x_n \to x$ and $y_n \to y$. Conversely, if $x_n \to x$, i.e.:

$$|x_n - x| < \frac{\epsilon}{2}, \text{ whenever } n > N_1$$

And $y_n \to y$, i.e.:

$$|y_n - y| < \frac{\epsilon}{2}, \text{ whenever } n > N_2$$

Now,

$$\begin{aligned} |z_n - z| &= |x_n - x + i(y_n - y)| \\ &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

whenever $n > N$, where $N = \max\{N_1, N_2\}$, thus $z_n \to z$. ■

Definition:

The sequence $\{z_n\}$ diverges to infinity if there exists a positive integer μ, such that $|z_n| > \mu, \forall n > N$.

Example:

1. $\{1+ni\} = \{1+i, 1+2i, \dots, 1+ni, \dots\}$, is divergent to ∞.
2. $\{ni\} = \{i, 2i, 3i, \dots\}$, is divergent.
3. $\left\{\frac{i}{n}\right\} = \left\{i, \frac{i}{2}, \frac{i}{3}, \frac{i}{4}, \dots\right\}$, converges to 0.
4. $\sum_{n=1}^{\infty} \frac{3i}{2^n}$,

Note: $S_1 = \frac{3i}{2}, S_2 = \frac{3i}{2^2}, \dots, S_n = 3i\left(\frac{1}{2}, \frac{1}{2^2}, \dots, \frac{1}{2^n}\right)$ is geometric series, the first term is $\frac{1}{2}$, i.e.:

$$S = 3i\left(\frac{\frac{1}{2}}{1-\frac{1}{2}}\right) = 3i, \left(\text{Geometric series and its sum } S = \frac{a}{1-r}\right)$$

This series is convergent, since $\left|\frac{1}{2}\right| < 1$.

Note:

1. If $\sum z_n$ is convergent then $\lim_{n\to\infty} z_n = 0$, but the converse is not true. For example, $\sum \frac{1}{n}$ is divergent, but $\lim_{n\to\infty} \frac{1}{n} = 0$.
2. We say that $\sum z_n$ is convergent (absolute convergent) if $\sum |z_n|$ is convergent.

Note:

Every absolute convergent series is convergent, but the converse is not true.

Definition:

The series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots$$

is called a power series, where z_0 and a_n are complex constants and z may be any point in a stated region containing z_0. If $z_0 = 0$, then the series is called a Maclaurin series.

Note:

If the function is analytic somewhere then it can be represented by a power series and vice versa.

Example: Let $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

1. $z = i$
2. $z = 2$

Is convergent or divergent series?

Solution:

1. When $z = i$

$\therefore \sum_{n=1}^{\infty} \left|\frac{i^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}, \left(1, \frac{1}{4}, \frac{1}{9}, \ldots \to 0\right)$

Then the series is convergent.

2. When $z = 2$

$\therefore \sum_{n=1}^{\infty} \frac{z^n}{n^2} = \sum_{n=1}^{\infty} \frac{2^n}{n^2}$, then by ratio test we get:

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to\infty} \frac{2^{n+1}/(n+1)^2}{2^n/n^2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \frac{2n^2}{n^2+2n+1}$$

$$= 2 > 1$$

The series is divergent.

[1] Taylor Series

Taylor's Theorem:

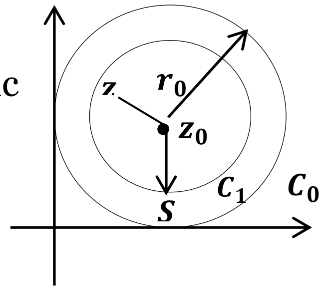
Let f be analytic everywhere inside a circle C_0 with center z_0 and radius r_0. Then at each point z inside C_0

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

that is, the power series converges to $f(z)$ when $|z - z_0| < r_0$.

Proof:

Let z be any point inside C_0 and $|z - z_0| = r$, where $r < r_0$. Let S be any point lying on a circle C_1 centered at z_0 and with radius r_1 where $r < r_1 < r_0$.



Thus $|S - z_0| = r_1$, since z inside C_1 and f is analytic within and on that circle, it follows that by (C.I.F):

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(S)dS}{S-z} \qquad \dots (1)$$

Now,

$$\frac{1}{S-z} = \frac{1}{S-z_0+z_0-z}$$

$$= \frac{1}{(S-z_0)-(z-z_0)}$$

$$= \frac{1}{(S-z_0)\left[1-\frac{z-z_0}{S-z_0}\right]}$$

$$= \frac{1}{S-z_0}\left[1 + \frac{z-z_0}{S-z_0} + \left(\frac{z-z_0}{S-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{S-z_0}\right)^{N-1} + \frac{\left(\frac{z-z_0}{S-z_0}\right)^N}{1-\left(\frac{z-z_0}{S-z_0}\right)}\right]$$

Since $\left[\frac{1}{1-c} = 1 + c + c^2 + \cdots + c^{N-1} + \frac{c^N}{1-c}\right]$

$$\rightarrow \frac{f(S)}{S-z} = \frac{f(S)(z-z_0)}{(S-z_0)^2} + \cdots + \frac{f(S)(z-z_0)^{N-1}}{(S-z_0)^N} + \frac{f(S)(z-z_0)^N}{(S-z)(S-z_0)^N}$$

Integrating around C_1 and dividing by $2\pi i$, we get:

$$\frac{1}{2\pi i}\oint_{C_1}\frac{f(S)dS}{S-z} = \frac{1}{2\pi i}\oint_{C_1}\frac{f(S)dS}{(S-z_0)^2}(z-z_0) + \cdots + \frac{1}{2\pi i}\oint_{C_1}\frac{f(S)dS}{(S-z_0)^N}(z-z_0)^{N-1}$$

$$+ \frac{1}{2\pi i}\oint_{C_1}\frac{f(S)dS}{(S-z)(S-z_0)^N}(z-z_0)^N$$

$$\rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(N-1)}(z_0)}{(N-1)!}(z-z_0)^{N-1} + R_N(z)$$

Where $R_N(z) = \frac{1}{2\pi i}\oint_{C_1}\frac{f(S)dS}{(S-z)(S-z_0)^N}(z-z_0)^N \qquad \dots (2)$

Note that:

$$|R_N(z)| \le \frac{r^N}{2\pi} \frac{2\mu r_1 \pi}{(r_1 - r)\, r_1^N}$$

$$= \frac{\mu r_1}{r_1 - r}\left(\frac{r}{r_1}\right)^N \ , \ \frac{r}{r_1} < 1$$

So, when $N \to \infty$, we have $R_N(z) \to 0$. Therefore, for each point z inside C_0, the limit of the sum for the first N terms on the right in Eq.(2) as $N \to \infty$, is $f(z)$. That is, if f is analytic inside a circle centered at z_0 with radius r_0, then $f(z)$ is represented by a Taylor series

$$f(z) = f(z_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \text{where } |z - z_0| < r_0. \blacksquare$$

Important Note:

The special case in which $z_0 = 0$; i.e.:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

$$= f(0) + f'(0)z + \frac{f''(0)z^2}{2!} + \cdots + \frac{f^{(n)}(0)z^n}{n!} + \cdots$$

is called a Maclaurin series.

Example: Find the Maclaurin series expansion for the following:

$$\sin z, \cos z, \sinh z, \cosh z \text{ and } e^z$$

Solution:

❖ Let $f(z) = \sin z$, then

$f(0) = \sin 0 = 0$

$f'(z) = \cos z \to f'(0) = 1$

$f''(z) = -\sin z \to f''(0) = 0$

$f^{(3)}(z) = -\cos z \to f^{(3)}(0) = -1$

$f^{(4)}(z) = \sin z \to f^{(4)}(0) = 0$

$\vdots$

$$f(z) = \sin z = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \cdots + \frac{f^{(n)}(0)}{n!}z^n + \cdots$$

$$= 0 + z + 0 - \frac{z^3}{3!} + 0 + \frac{z^5}{5!} + \cdots$$

$$= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

i.e.:

$$\sin z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}\ , \ |z| < \infty \qquad \dots (1)$$

❖ To find the series of $\cos z$:

Differentiating both sides of (1) with respect to z, we get:

$$\cos z = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}\ , \ |z| < \infty \qquad \dots (2)$$

❖ To find the series of $\sinh z$:

Since $\sinh z = -i \sin iz$, it follows from (1), that

$$\sinh z = -i \sum_{k=0}^{\infty}(-1)^k \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$= -i \sum_{k=0}^{\infty}(-1)^k (i)^{2k+1} \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k (-i)(i)(i^2)^k \frac{z^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{\infty}(-1)^k (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad [((-1)^2)^k = 1]$$

$$\sinh z = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!}\ , \ |z| < \infty \qquad \dots (3)$$

❖ To find the series of $\cosh z$:

Differentiating both sides of (3) with respect to z, we get:

$$\cosh z = \sum_{k=0}^{\infty}\frac{z^{2k}}{(2k)!}\ , \ |z| < \infty \qquad \dots (4)$$

❖ To find the series of e^z:

When $f(z) = e^z$, then $f^{(n)}(z) = e^z$

$f^{(n)}(0) = 1$, since e^z is analytic for all z, so:

$$e^z = e^0 + e^0 z + \frac{e^0}{2!} z^2 + \cdots + \frac{e^0}{n!} z^n + \cdots$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{z^k}{k!} \qquad \ldots (5)$$

<u>Example</u>: Expand $\cos z$ into a Taylor series about the point $z = \frac{\pi}{2}$.

<u>Solution</u>: let $f(z) = \cos z$, then

$$f(z) = \cos z = f\left(\frac{\pi}{2}\right) + \frac{f'\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)}{1!} + \frac{f''\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^2}{2!} + \cdots + \frac{f^{(n)}\left(\frac{\pi}{2}\right)\left(z-\frac{\pi}{2}\right)^n}{n!} + \cdots$$

Now,

$$f\left(\frac{\pi}{2}\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

$$f'(z) = -\sin z \to f'\left(\frac{\pi}{2}\right) = -1$$

$$f''(z) = -\cos z \to f''\left(\frac{\pi}{2}\right) = 0$$

$$f^{(3)}(z) = \sin z \to f^{(3)}\left(\frac{\pi}{2}\right) = 1$$

$$f^{(4)}(z) = \cos z \to f^{(4)}\left(\frac{\pi}{2}\right) = 0$$

$$\vdots$$

$$\to \cos z = 0 - \frac{\left(z-\frac{\pi}{2}\right)}{1!} + 0 + \frac{\left(z-\frac{\pi}{2}\right)^3}{3!} + 0 - \frac{\left(z-\frac{\pi}{2}\right)^5}{5!} + \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^{2n+1}}{(2n+1)!}$$

<u>Example</u>: Show that

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} (n+1)(z+1)^n$$

where $|z+1| < 1$.

Solution:

Since $|z+1|<1 \rightarrow z_0=-1$ and,

$f(z)=\frac{1}{z^2} \rightarrow f(-1)=1$

$f'(z)=\frac{-2}{z^3} \rightarrow f'(-1)=2$

$f''(z)=\frac{2.3}{z^4} \rightarrow f''(-1)=3!$

$f^{(3)}(z)=\frac{-2.3.4}{z^5} \rightarrow f^{(3)}(-1)=4!$

$\vdots$

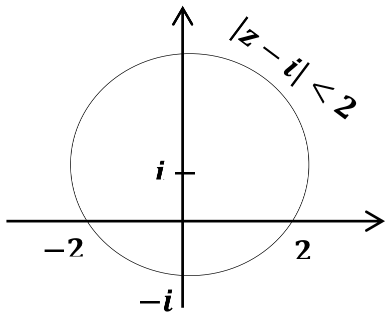
$f^{(n)}(z)=\frac{(-1)^n.2.3\ldots(n+1)}{z^{n+2}} \rightarrow f^{(n)}(-1)=(n+1)!$

$\frac{1}{z^2}=\sum_{n=0}^{\infty}\frac{(n+1)!}{n!}(z+1)^n$

$\quad =\sum_{n=0}^{\infty}(n+1)(z+1)^n$

Example: Expand $f(z)=\frac{3}{z+i}$ into a Taylor series about $|z-i|<2$.

Solution:



Note that $-i$ is a singular point located on the perimeter. The largest size circle that can be found is the one that the function is not analytic at it, which is $-i$. The distance between i and $-i$ represents the radius of convergence which is 2, and that's why we have the circle $|z-i|<2$. And if we have $|z-i|<3$ then the Taylor series cannot be applied, since the function will not be analytic and one of its conitions is that the function must be analytic inside C.

$\frac{3}{z+i}=\frac{3}{z+2i-i}$

$\quad =\frac{3}{2i+(z-i)}$

$= \frac{3}{2i\left(1+\frac{z-i}{2i}\right)}$

$= \frac{3}{2i}\left[\frac{1}{1+\frac{z-i}{2i}}\right]$ (Geometric series $a = 1, r = \frac{z-i}{2i}$)

$= \frac{3}{2i}\left[\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-i}{2i}\right)^n\right]$

Note: $\left|\frac{z-i}{2i}\right| < 1 \rightarrow |z-i| < 2.$

Example:

1. Expand $f(z) = \frac{1}{1+z}$ about $z = 0.$

Solution:

$f(z) = \frac{1}{1-(-z)}$

$= 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots \quad , |z| < 1$

$= \sum_{n=0}^{\infty}(-1)^n z^n$

2. Expand $f(z) = \frac{1}{1-z^2}$ about $z = 0.$

Solution:

$f(z) = \sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n} \ , \ |z| < 1$

Note: to find the radius of convergence $= \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|}$, such as in the previous example,

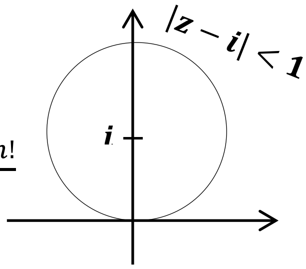
$$\left.\begin{array}{l} a_{n+1} = -1 \rightarrow |a_{n+1}| = 1 \\ \quad a_n = 1 \quad \rightarrow |a_n| = 1 \end{array}\right\} \rightarrow \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{1}{1} = 1$$

$\therefore r_0 = 1$

$|z-0| < r_0 \rightarrow |z| < 1.$

Example: Write $f(z) = \frac{1}{z}$ into a Taylor series about $z = i, r_0 = 1.$

Solution: from Taylor's theorem $|z-z_0| < r_0 \rightarrow |z-i| < 1$



$\frac{1}{z} = \sum_{n=0}^{\infty} a_n (z-i)^n \ , \ |z-i| < 1$

$f(i) = \frac{0!}{i}, f'(i) = \frac{-1!}{i^2}, f''(i) = \frac{2!}{i^3}, \dots, f^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$

$\therefore \frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$

Or:

$$\frac{1}{z} = \frac{1}{z-i+i} = \frac{1}{i+(z-i)}$$

$$= \frac{1}{i\left(1+\frac{z-i}{i}\right)}$$

$$= \frac{1}{i}\left[\frac{1}{1+\frac{z-i}{i}}\right], \text{since } |z-i| < 1$$

$$= \frac{1}{i}\sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{i^{n+1}}$$

Example: Write $f(z) = \frac{1}{z}$ into a power series for $(z-1)$.

Solution:

$$\frac{1}{z} = \frac{1}{z-1+1}$$

$$= \frac{1}{1+(z-1)} \quad \text{(Geometric series } a = 1, r = (z-1))$$

$$= \sum_{n=0}^{\infty} (-1)^n (z-1)^n \ , \ |z-1| < 1$$

Example: Represent the function

$$f(z) = \frac{z}{(z-3)(z-1)}$$

into a series of negative power of $(z-1)$, which converges to $f(z)$ where $0 < |z-1| < 2$

Solution:

$$f(z) = \frac{z}{(z-3)(z-1)} = \frac{A}{z-1} + \frac{B}{z-3}$$

$\rightarrow A = -\frac{1}{2}\ ,\ B = \frac{3}{2}$

$\therefore f(z) = \frac{-1}{2}(z-1)^{-1} + \frac{3/2}{z-3}$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3/2}{2-(z-1)}$$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3/2}{2\left[1-\frac{(z-1)}{2}\right]}$$

$$= \frac{-1}{2}(z-1)^{-1} - \frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^{n}$$

$$= \frac{-1}{2}(z-1)^{-1} - 3\sum_{n=0}^{\infty}\frac{(z-1)^{n}}{2^{n+2}}$$

Example: Represent the function

$$f(z) = \frac{1}{1+z}$$

into a series of negative power of z.

Solution:

$\frac{1}{1+z} = \frac{1}{z\left(1+\frac{1}{z}\right)}$

$$= \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right)$$

$$= \frac{1}{z}\sum_{n=0}^{\infty}(-1)^{n}\left(\frac{1}{z}\right)^{n}$$

$$= \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\ ,\ \left(\ \left|\frac{1}{z}\right| < 1 \rightarrow |z| > 1\right)$$

Example: Evaluate Taylor series of $f(z) = \log(1+z)$ about zero.

Solution: note that $f(z) = \log(1+z)$ is not analytic when

$$Im(1+z) = 0 \text{ and } Re(1+z) < 0$$

$\rightarrow y = 0$ and $x + 1 < 0 \rightarrow x < -1$

$f(0) = \log 1 = 0$

$f'(z) = \frac{1}{1+z} \to f'(0) = 1$

$f''(z) = \frac{-1}{(1+z)^2} \to f''(0) = -1$

$f^{(3)}(z) = \frac{2}{(1+z)^3} \to f^{(3)}(0) = 2$

$\vdots$

$\therefore f(z) = \log(1+z) = z - \frac{1}{2!}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \cdots, (|z| < 1)$

[2] Laurent Series

If a function f fails to be analytic at z_0, then we can apply Taylor's theorem at z_0. It is possible however, to find a series representation for $f(z)$ involving both positive and negative powers of $(z - z_0)$. Now, we represent the theory of such representation and begin with Laurent theorem.

<u>Laurent's Theorem</u>:

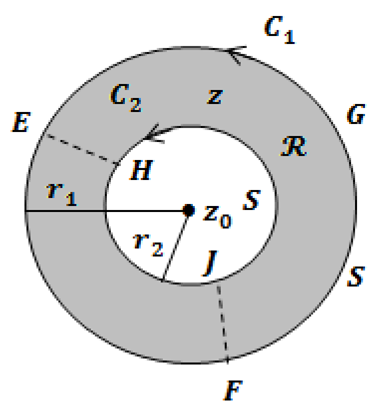
If $f(z)$ is analytic inside and on the boundary of the ring $\mathcal{R}$ bounded by two concentric C_1 andC_2 with center z_0 and respective radii r_1 and r_2, then for all z in $\mathcal{R}$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

Such that:

$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)dz}{(z - z_0)^{n+1}}, \ n = 0,1,2, \ldots$

$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(z)dz}{(z - z_0)^{-n+1}}, \ n = 1,2, \ldots$



Note:

1. If f is analytic on and inside C_2, then $b_n = 0$, i.e.:

$$b_n = \int f(z)(z - z_0)^{n-1}\, dz\,, n = 1,2, \ldots$$

2. Laurent's theorem can be reform as:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Such that: $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(z)dz}{(z-z_0)^{n+1}}$, $n = 0,1,2, \ldots$, where C is any simple closed curve lies between C_1 and C_2.

3. If f is analytic on and inside C_1 then the Laurent series turns to Taylor series.
4. The Laurent series expansion contains negative powers and usually begins from $-\infty$.
5. The Taylor series expansion about z_0 is a special case of Laurent expansion, that is when calculating Laurent coefficients in this case all the negative power coefficients appear as zeros and the Taylor series remains.

   For example, if $n = -1$ then $(z - z_0)^0 = 1$ and the function is analytic i.e. ($\oint_C = 0$), and it is the same when $n = -2, -3, \ldots$

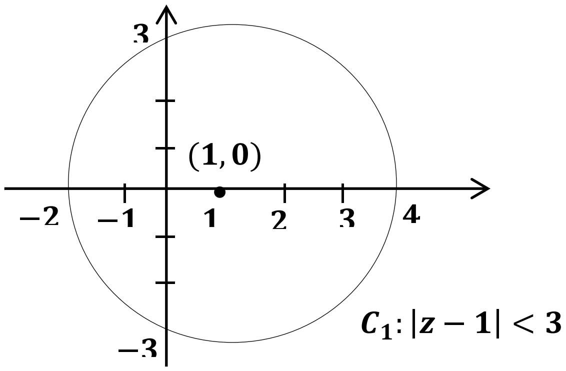
6. z_0 might be the only singular point of f on C_1 and in this case $0 < r_2 < r_1$, so the series will be at the region $0 < |z - z_0| < r_1$.
7. Taylor series can be written for a point inside the circle.
8. Laurent series can be written for a point outside the circle.

Example: Represent $f(z) = \frac{1}{(z-1)(z+2)}$ into a Laurent series about $0 < |z - 1| < 3$.

Solution:

$$f(z) = \frac{1}{(z-1)(z+2)}$$

$$= \frac{1}{3}\,\frac{1}{(z-1)} - \frac{1}{3}\,\frac{1}{(z+2)}$$



We don't need to make it $(z-1)$, the singular points are $1, -2$. Note that the singular point $0 < |z-1| < 3 \not\ni -2$.

$$f(z) = \frac{1}{3(z-1)} - \frac{1}{3}\left(\frac{1}{3+z-1}\right)$$

$$= \frac{1}{3(z-1)} - \frac{1}{3}\cdot\frac{1}{3}\left[\frac{1}{1+\frac{z-1}{3}}\right]$$

$\rightarrow \frac{1}{1+\frac{z-1}{3}}$ is a geometric series and $a = 1\,, r = \frac{z-1}{3}$ with alternative sign $(-1)^n$.

$$f(z) = \frac{1}{3(z-1)} - \frac{1}{3^2}\sum_{n=0}^{\infty}(-1)^n\,\frac{(z-1)^n}{3^n}$$

$$\left|\frac{z-1}{3}\right| < 1 \rightarrow |z-1| < 3$$

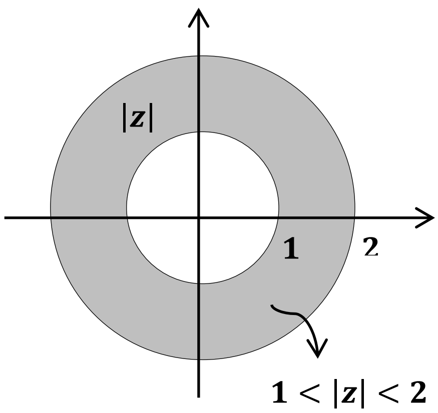
The other part of the region $|z-1| > 0$ we avoid that $|z-1| \neq 0$ in the term $\frac{1}{3(z-1)}$.

In the same example about $1 < |z| < 2$

$$f(z) = \frac{1}{3}\,\frac{1}{(z-1)} - \frac{1}{3}\,\frac{1}{(z+2)}$$

$$= \frac{1}{3}\cdot\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{3}\cdot\frac{1}{2}\left(\frac{1}{1+\frac{z}{2}}\right)$$

$\lvert 1/z\rvert < 1$ $\rightarrow 1 < \lvert z\rvert$	$\lvert z/2\rvert < 1$ $\rightarrow \lvert z\rvert < 2$



$$\therefore f(z) = \frac{1}{3z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n$$

Example: Expand $f(z) = \frac{1}{z^2(z-1)(z-2)}$ in a Laurent series about $z_0 = 0$.

Solution: the possibilities depend on the singular points $0, 1, 2$:

Case 1: $0 < |z| < 1$

Case 2: $1 < |z| < 2$

Case 3: $2 < |z| < \infty$

♦ Case 1: if $0 < |z| < 1$

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-\frac{z}{2}}\right]$$

$$= \frac{1}{z^2}\left[\sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right]$$

$|z| < 1$ & $\left|\frac{z}{2}\right| < 1 \to |z| < 1$ & $|z| < 2$, note the connection between the intervals and the solution, then $0 < |z| < 1$.

♦ Case 2: if $1 < |z| < 2$

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{-1}{2}\frac{1}{1-\frac{z}{2}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right]$$

$\left|\frac{z}{2}\right| < 1 \to |z| < 2$ & $\left|\frac{1}{z}\right| < 1 \to |z| > 1$, note the connection between the intervals and the solution, then $1 < |z| < 2$.

$$\therefore f(z) = \frac{1}{z^2}\left[\frac{-1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

$$= -\left[\sum_{n=0}^{\infty}\frac{z^{n-2}}{2^{n+1}} + \sum_{n=0}^{\infty}\frac{1}{z^{n+3}}\right]$$

♦ Case 3: if $2 < |z| < \infty$ ($|2/z| < 1$)

$$f(z) = \frac{1}{z^2}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{z}\frac{1}{1-\frac{2}{z}} - \frac{1}{z}\frac{1}{1-\frac{1}{z}}\right]$$

We need $|z| > 2$, so

$\left|\frac{2}{z}\right| < 1 \to |z| > 2$ & $\left|\frac{1}{z}\right| < 1 \to |z| > 1$, note the connection between the two intervals and the solution, then $2 < |z|$.

$$\therefore f(z) = \frac{1}{z^3}\left[\sum_{n=0}^{\infty}\frac{2^n}{z^n} - \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n\right]$$

<u>Example</u>: Expand $f(z) = \frac{e^z}{z^2}$ in a Laurent series.

<u>Solution</u>: $f(z)$ is analytic everywhere except the origin. We take C_1 big and C_2 a little smaller.

$$f(z) = \frac{e^z}{z^2} = \frac{1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}{z^2}$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots\ ,\ 0 < |z| < \infty$$

<u>Example</u>: Write $f(z) = \frac{\sin z}{z^2-1}$ into a Laurent series in powers of $z-1$.

<u>Solution</u>:

1. Locate the singular points which are $-1,1$.
2. Leave every factor of the form $z-1$ in the denominator and otherwise is considered a part of the numerator.

$$f(z) = \frac{\sin z}{z^2-1} = \frac{\sin z}{(z-1)(z+1)}$$

$$= \frac{\sin z/(z+1)}{z-1}$$

3. Write Taylor expansion for the new numerator about $z_0 = 1$ and then simplify to get Laurent series,

$$\frac{\sin z}{(z+1)} = \frac{\sin 1}{1+1} + \frac{(\sin z/(z+1))'\big|_{z=1}}{1!}(z-1) + \frac{(\sin z/(z+1))''\big|_{z=1}}{2!}(z-1)^2 + \cdots$$

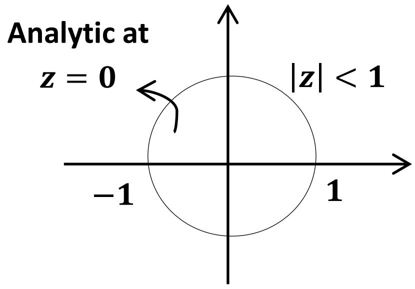
$$f(z) = \frac{\sin z/(z+1)}{z-1}$$

$$f(z) = \frac{\sin 1/2}{z-1} + \frac{(\sin z/(z+1))'\big|_{z=1}/1!\,(z-1)}{z-1} + \frac{(\sin z/(z+1))''\big|_{z=1}/2!\,(z-1)^2}{z-1} + \cdots$$

Example: Let $f(z) = \frac{z+1}{z-1}$, find:

1. Maclaurin series (Taylor about $z_0 = 0$).
2. Laurent series about $z_0 = 0$.

Solution:



1. $f(z) = \frac{z+1}{z-1} = \frac{z-1+2}{z-1}$

$$= 1 - \frac{1}{1-z}$$

$$= 1 - 2\left(\frac{1}{1-z}\right)$$

$$= 1 - 2(1 + z + z^2 + \cdots),\ \ |z| < 1$$

$$= -1 - 2z - 2z^2 - \cdots$$

2. $f(z) = \frac{z+1}{z-1} = 1 + \frac{2}{z-1}$

$$= 1 + \frac{2}{z\left(1-\frac{1}{z}\right)}$$

$$= 1 + \frac{2}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right]$$

$$= 1 + \frac{1}{z} + \frac{2}{z^2} + \frac{2}{z^3} + \cdots$$

Example: Let $f(z) = \frac{z-1}{z^2}$, calculate:

1. Taylor series expansion about $z = 1$.
2. Laurent series expansion about $z = 1$.

Solution:

Since $z = 1$ then the series is of power $(z - 1)$:

"Inside the circle Taylor means positive powers for $(z-1)$"

"Outside the circle Laurent means negative powers for $(z-1)$"

1. $f(z) = \frac{z-1}{z^2} = (z-1)\frac{1}{z^2}$

$$= (z-1)\left(\frac{1}{(z-1)+1}\right)^2$$

$$= (z-1)\left(\frac{1}{1+(z-1)}\right)^2$$

$$= (z-1)\left(\sum_{n=0}^{\infty}(-1)^n(z-1)^n\right)^2$$

$$= (z-1)(1-(z-1)+(z-1)^2-\cdots)^2$$

$$= (z-1)[1-2(z-1)+3(z-1)^2-\cdots],\ |z-1|<1$$

$$= (z-1)-2(z-1)^2+3(z-1)^3-\cdots$$

2. To find Laurent series of $f(z)$:

$$f(z) = \frac{z-1}{z^2} = (z-1)\left[\frac{1}{(z-1)\left(1+\frac{1}{z-1}\right)}\right]^2$$

$$= (z-1)\frac{1}{(z-1)^2}\left[\frac{1}{1+\frac{1}{z-1}}\right]^2$$

$$= \frac{1}{z-1}\left[1-\frac{1}{z-1}+\frac{1}{(z-1)^2}-\cdots\right]^2$$

$$= \frac{1}{z-1}\left[1-\frac{2}{z-1}+\frac{3}{(z-1)^2}-\cdots\right]$$

$$= \frac{1}{z-1}-\frac{2}{(z-1)^2}+\frac{3}{(z-1)^3}-\cdots\ ,\ \left(\left|\frac{1}{z-1}\right|<1 \to |z-1|>1\right)$$

[3] Integration and Differentiation of Power Series

Theorem:

Let C be any contour interior to the circle of convergence of $S(z)=\sum_{n=0}^{\infty}a_n z^n$ and let $g(z)$ be any continuous function on C, then

$$\int g(z)S(z)\,dz = \sum_{n=0}^{\infty}a_n\int_C g(z)z^n\,dz$$

Example: Expand the function $f(z) = \frac{1}{z}$ into a power series of $z-1$; then obtain by differentiation the expansion of $\frac{1}{z^2}$ in powers of $z-1$.

Solution:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$\rightarrow \frac{d}{dz}\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty}(-1)^n \frac{d}{dz}(z-1)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\, n(z-1)^{n-1}$$

$$\rightarrow \frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n\, n(z-1)^{n-1}$$

$$\rightarrow \frac{1}{z^2} = \sum_{n=0}^{\infty} n(-1)^{n+1}(z-1)^{n-1}$$

$$= \sum_{n=1}^{\infty} n(-1)^{n+1}(z-1)^{n-1}$$

Example: Expand the function $f(z) = \frac{1}{z}$ in a Laurent series in powers of $z-1$; then obtain by differentiation the Laurent series of $\frac{z-1}{z^2}$ in powers of $z-1$.

Solution:

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \frac{1}{(1-z)\left(\frac{1}{1-z}-1\right)}$$

$$= \frac{1}{(z-1)\left(1-\frac{1}{1-z}\right)}$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}\left(\frac{1}{1-z}\right)^n, \left|\frac{1}{1-z}\right| < 1 \rightarrow |1-z| > 1$$

$$= \frac{1}{z-1}\sum_{n=0}^{\infty}(-1)^n \frac{1}{(z-1)^n}$$

$$\therefore \frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(z-1)^{n+1}}$$

Now, differentiating both sides with respect to z, we get:

$\frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n - (n+1)(z-1)^{-(n+2)}$

$\frac{-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{-(n+1)}{(z-1)^{n+2}}$

Or $\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)}{(z-1)^{n+2}}$

$\rightarrow \frac{z-1}{z^2} = \sum_{n=0}^{\infty}(-1)^n \frac{(n+1)}{(z-1)^{n+1}}$

$= \sum_{n=1}^{\infty}(-1)^{n-1} \frac{n}{(z-1)^n}$

Which is a Laurent series for $f(z) = \frac{z-1}{z^2}$ in powers of $z-1$.

Example: Suppose that f and g are analytic functions at z_0 and $f(z_0) = g(z_0)$, while $g(z_0) \neq 0$, prove that

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$$

Solution:

$\lim_{z\to z_0} \frac{f(z)}{z-z_0} = f'(z_0)$, and

$\lim_{z\to z_0} \frac{g(z)}{z-z_0} = g'(z_0)$

Then,

$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)/(z-z_0)}{g(z)/(z-z_0)}$

$= \frac{\lim_{z\to z_0} f(z)/(z-z_0)}{\lim_{z\to z_0} g(z)/(z-z_0)}$

$= \frac{f'(z_0)}{g'(z_0)}$

Chapter Six

Residues and Poles

Definition 1:

A point z_0 is called a singular of f if the function f fails to be analytic at z_0 but it is analytic at some point in every neighborhood of z_0.

Definition 2:

A singular point z_0 is said to be isolated, if in addition, there is some neighborhood of z_0 for which f is analytic except at z_0.

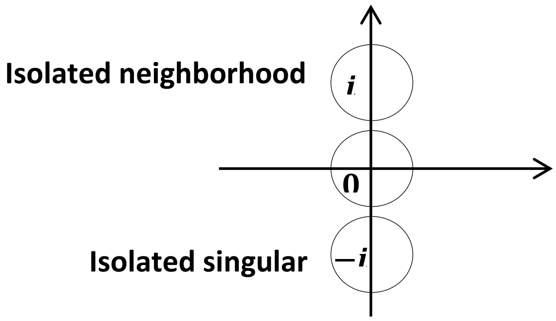
Example:

1. $f(z) = \frac{1}{z}$, this function has a singular point at $z = 0$, which is an isolated singular point of f.
2. $f(z) = \frac{1}{z^2(z-1)(z^2+1)}$, this function has four isolated singular points $z = 0, 1, \pm i$.
3. $f(z) = \operatorname{Log} z$, this function has a singular point at $z = 0$, but this point is not isolated, because each neighborhood of $z = 0$ contains points on the negative real axis and $\operatorname{Log} z$ fails to be analytic at each of these points.
4. $f(z) = e^z$, has no singular points.
5. $f(z) = \frac{1}{\sin\frac{\pi}{z}}$, has the singular points $z = 0$ and $z = \frac{1}{n}$, $n = \pm 1, \pm 2, \ldots$, each singular point $z = \frac{1}{n}$ is isolated but $z = 0$ is not isolated singular of f, since when $z = 0$ every neighborhood of $z = 0$ contains other singular points of f. For example, take $z = \frac{1}{N}$, N large enough, then

$$\frac{1}{N} \to 0 \Longrightarrow \sin\frac{\pi}{z} = \sin\frac{\pi}{\frac{1}{1/N}} = \sin N\pi = 0$$

<u>Note</u>: not every singular point is isolated, as in example 3, 5.

6. $f(z) = \frac{1}{z(z^2+1)}$, has singular and isolated points at $z = 0, i, -i$.



Let z_0 be any isolated singular point of f, then f is analytic at each point z, when $0 < |z - z_0| < R$, so $f(z)$ can be represented by a Laurent series

$$f(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \qquad \dots (1)$$

where $a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$

and $\quad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$

Hence:

$$b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz \qquad \dots (2)$$

Or

$$\int_C f(z)\, dz = 2\pi i b_1$$

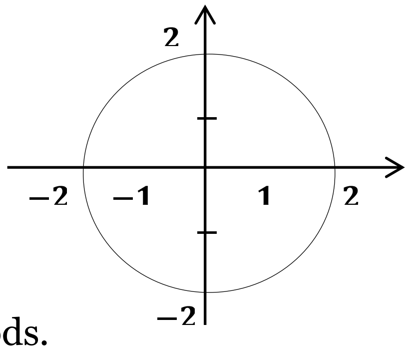
where C is any simple closed contour around z_0 described in positive sense. The coefficient $b_1$of $\frac{1}{z - z_0}$ in expansion (1) is called the residue of f at the isolated singular point z_0. Formula (2) gives us a powerful method for evaluating certain integrals around simple closed contours and it is denoted by

$$b_1 = Res[f, z_0]$$

Example: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz$$

such that $C : |z| = 2$.



Solution:

Note: we can solve this integral by two methods.

i. By Cauchy integral formula

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i f'(z_0)$$

$$f(z) = e^{-z} \to f'(z) = -e^{-z} \to f'(1) = -e^{-1}$$

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i f'(1)$$

$$= 2\pi i(-e^{-1})$$

$$= -\frac{2\pi i}{e}$$

ii. Note that $f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic over C except $z_0 = 1$, so by Laurent theorem

$$\frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}e^{-(z-1)}}{(z-1)^2}$$

$$= \frac{e^{-1}}{(z-1)^2} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^n}{n!} \text{ , } |z-1| < \infty$$

$$= \frac{1}{e} \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n-2}}{n!}$$

$$= \frac{1}{e}\left[\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{2!} - \frac{z-1}{3!} + \cdots\right]$$

where the coefficient of $(z-z_0)^{-1} = (z-1)^{-1}$ is $\frac{-1}{e} = b_1$, so:

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i(b_1)$$

$$= \frac{-2\pi i}{e}$$

Note: if z_0 is an isolated point, the we can find the integral by Laurent and then we find the residue of the function at $-z$.

Example: Evaluate

$$\oint_C \frac{e^z}{z} dz$$

such that $C : |z| = 1$.

Solution: Note $z_0 = 0$ is a singular point of f.

$\frac{1}{z} e^z = \frac{1}{z}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right]$

$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$

Note that $b_1$is the coefficient of $\frac{1}{z}$, then $b_1 = 1$ and

$\oint_C \frac{e^z}{z} dz = 2\pi i b_1$

$= 2\pi i$

Example: Evaluate

$$\oint_C e^{1/z^2}\, dz$$

such that $C : |z| = 2$.

Solution: Note that there is no fraction so we cannot solve by the two previous methods that is Cauchy integral formula cannot be applied here, so we will solve by residue.

$z_0 = 0$ is a singular and isolated point of f.

$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $|z| < \infty$, so

$e^{1/z^2} = \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!}$, $\left|\frac{1}{z^2}\right| < \infty$

$= \sum_{n=0}^{\infty} \frac{1}{n!\, z^{2n}} \to \left|\frac{1}{z}\right| < \infty$

$= 1 + \frac{1}{z^2} + \frac{1}{2!\, z^4} + \cdots , 0 < |z| < \infty$

The coefficient of $(z-z_0)^{-1} = (z-0)^{-1}$ is 0, then $b_1 = 0$ so that:

$\oint_C e^{1/z^2}\, dz = 2\pi i b_1 = 0$

And this is clear, since f is analytic on C and so by Cauchy $\oint_C f(z)\, dz = 0$.

Example: Evaluate the following integral by using residues:

$$\oint_C z^3 \cos\left(\frac{1}{z}\right) dz \; ; \; C : |z+1+i\,| = 4$$

Solution:

The point $z_0 = 0$ is an isolated singularity of $\cos\left(\frac{1}{z}\right)$ and lies in the given contour of integration; we want a Laurent series expansion of $z^3\cos\left(\frac{1}{z}\right)$ about this point (i.e. $z_0 = 0$), since:

$\cos z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$, we have

$$\cos\left(\frac{1}{z}\right) = 1 - \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^4}{4!} - \frac{\left(\frac{1}{z}\right)^6}{6!} + \cdots$$

$$\to z^3\cos\left(\frac{1}{z}\right) = z^3 - \frac{1}{2!}z + \frac{1}{4!}\frac{1}{z} - \frac{1}{6!}\frac{1}{z^3} + \cdots$$

$$\to b_1 = \frac{1}{4!}$$

$$\oint_C z^3\cos\left(\frac{1}{z}\right) dz = 2\pi i b_1 = \frac{2\pi i}{4!}$$

$$= \frac{\pi i}{12}$$

Example: Let C be a positively oriented unit circle $z_0 = 0$. Evaluate

$$\oint_C \frac{dz}{z^3+z^2}$$

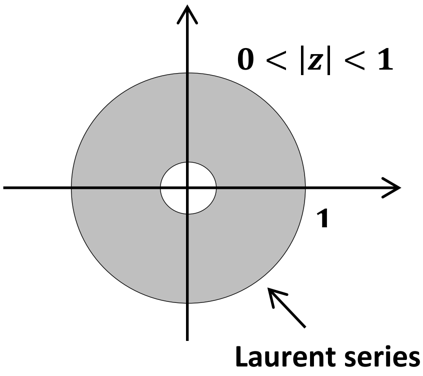
Solution: The isolated singular points are $z = 0$ and $z = -1, -1 \notin 0 < |z| < 1$

$f(z) = \frac{1}{z^3+z^2} = \frac{1}{z^2(z+1)}$

$= \frac{1}{z^2}\left(\frac{1}{1+z}\right)$

$= \frac{1}{z^2}\ (1 - z + z^2 - z^3 + \cdots)$

$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - \cdots$



$\mathbf{0 < |z| < 1}$

$\rightarrow b_1 = -1$, so

$\oint_C f(z)\,dz = \oint_C \frac{1}{z^3+z^2}\,dz$

$= 2\pi i b_1$

$= -2\pi i$

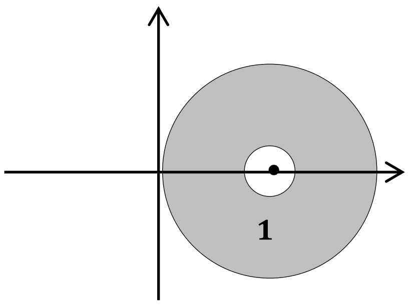
<u>Example</u>: Evaluate

$$\oint_C \frac{e^{-z}}{(z-1)^2}\,dz$$

where C is the circle $|z| = 2$,described in the positive sense.

<u>Solution</u>:

$f(z) = \frac{e^{-z}}{(z-1)^2}$ is analytic on C and its interior except at the isolated singular point $z = 1$, now



$e^{-z} = e^{-z+1-1}$

$= e^{-1}e^{1-z}$

$= e^{-1}\sum_{n=0}^{\infty}\frac{(1-z)^n}{n!}$

$= e^{-1}\sum_{n=0}^{\infty}(-1)^n\frac{(z-1)^n}{n!}$

$= e^{-1}\left[1 - (z-1) + \sum_{n=2}^{\infty}(-1)^n\frac{(z-1)^n}{n!}\right]$

$\therefore e^{-z} = e^{-1} - e^{-1}(z-1) + e^{-1}\sum_{n=2}^{\infty}(-1)^n\frac{(z-1)^n}{n!}$

Since $|z-1| > 0$, we can divide both sides by $(z-1)^2$

$\therefore \frac{e^{-z}}{(z-1)^2} = \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + e^{-1}\sum_{n=2}^{\infty}(-1)^n \frac{(z-1)^{n-2}}{n!}$

$\therefore b_1$ at $z = 1$ is equal to $-e^{-1}$, so

$$\oint_C \frac{e^{-z}}{(z-1)^2} dz = 2\pi i b_1$$

$$= -\frac{2\pi i}{e}$$

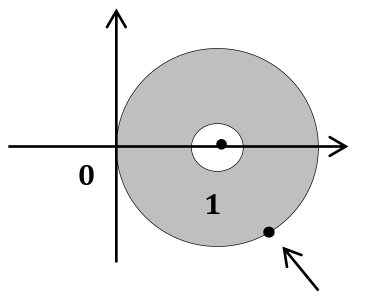
<u>Example</u>: Evaluate

$$\oint_C \frac{dz}{z(z-1)}$$

where C is the circle $|z-1| = 1$ (i.e.: or described in the positive sense as shown in the following figure).

<u>Solution</u>:

$f(z) = \frac{1}{z(z-1)}$, which is analytic on C and at all points inside C except at $z = 1$, which is an isolated singular point. The Laurent series expansion of $f(z)$ that converges in the annular region centered at $z = 1$, is



$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$$

$$= \frac{1}{z-1} - \frac{1}{(z-1)+1}$$

$$= (z-1)^{-1} - \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$= (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \cdots$$

$\therefore b_1 = 1$, so

$$\oint_C \frac{dz}{z(z-1)} = 2\pi i b_1 = 2\pi i$$

<u>Example</u>: Evaluate $\oint_C \frac{\sin z}{z \sinh z} dz$, around $|z| = 1$.

<u>Solution</u>:

$z = 0$ is the only isolated singular point inside $|z| = 1$, recall that:

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$, and

$z \sinh z = z\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]$

$= z^2 + \frac{z^4}{3!} + \frac{z^6}{5!} + \cdots$

$$f(z) = \frac{\sin z}{z \sinh z} = \frac{z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}{z\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)}$$

$$= \frac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

By using long division for $\frac{\sin z}{z \sinh z}$, we get:

$b_1 = 1$, so

$\oint_C \frac{\sin z}{z \sinh z}\, dz = 2\pi i b_1 = 2\pi i$

Note: $\frac{\sin z}{z \sinh z} = \frac{1}{z} - \frac{2}{3!} z + \cdots$

Residue Theorem:

Let C be a positively oriented simple closed contour. Let f be an analytic function within and on C except for a finite number of singular points $z_1, z_2, \ldots, z_n$. If $B_1, B_2, \ldots B_n$ are the residues of f at these points, then

$$\int_C f(z)\, dz = 2\pi i (B_1 + B_2 + \ldots + B_n)$$

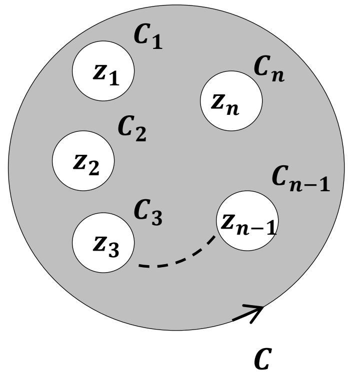
Proof:

Let the circles C_j , $0 < j < n$, be a positively oriented whose centers are $z_1, z_2, \ldots, z_n$, respectively and no intersection between any two of them. Now, these circles together with the simple closed contour C form the boundary of a closed region for which f is analytic. Therefore by Cauchy-Goursat theorem for multiply connected domain:

$$\int_C f(z)\, dz = \int_{C_1} f(z)\, dz + \int_{C_2} f(z)\, dz + \cdots + \int_{C_n} f(z)\, dz$$

$= 2\pi i B_1 + 2\pi i B_2 + \ldots + 2\pi i B_n$

$= 2\pi i(B_1 + B_2 + \ldots + B_n)$

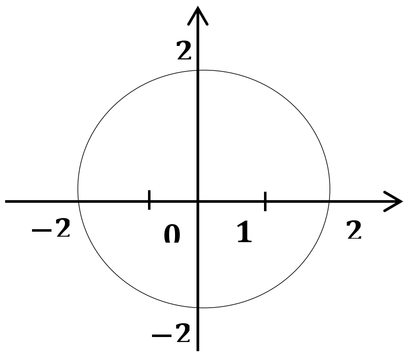


Example: Evaluate

$$\oint_C \frac{7z+11}{z(1-z)} dz$$

Where C is the circle $|z| = 2$ described in the positive sense.

Solution:



$f(z) = \frac{7z+11}{z(1-z)}$

$= \frac{A}{z} + \frac{B}{1-z}$

$\rightarrow A = 11,\ B = 18$

$\oint_C \frac{7z+11}{z(1-z)} dz = \oint_C \frac{11}{z} dz + \oint_C \frac{18}{1-z} dz$

$= \oint_C \frac{11}{z} dz - \oint_C \frac{18}{z-1} dz$

$\therefore B_1 = 11\ ,\ B_2 = -18$

$\oint_C \frac{7z+11}{z(1-z)} dz = 2\pi i(B_1 + B_2)$

$= 2\pi i(11 - 18)$

$= -14\pi i$

Example: Find the following integral by means of residue theorem:

$$\oint_C \frac{1}{z(z-1)} dz$$

Where C is the circle $|z - 1| = 6$.

Solution:

The contour C encloses the singularities at $z=1$ and $z=0$ since:

$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$, we have

$$\oint_C \frac{1}{z(z-1)}dz = \oint_C \frac{-1}{z}dz + \oint_C \frac{1}{z-1}dz$$

$$= 2\pi i B_1 + 2\pi i B_2$$

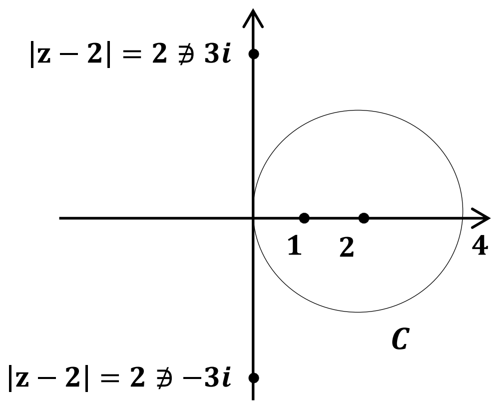
$$= 2\pi i(-1) + 2\pi i(1)$$

$$= 0$$

Example: Find $\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}dz\,;C : |z-2| = 2.$

Solution:

The function $f(z) = \frac{3z^2+2}{(z-1)(z^2+9)}$ is analytic on and inside C except $z=1$, which is an isolated singularity. Now, we shall find the residue B_1 at $z_0 = 1$. Next, we observe that



$$\frac{3z^2+2}{(z-1)(z^2+9)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+9}$$

$$\rightarrow A = \frac{1}{2}$$

$$\oint_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \oint_C \frac{1/2}{z-1}dz + \oint_C \frac{Bz+C}{z^2+9}dz$$

$$\oint_C \frac{1/2}{z-1}dz = \frac{1}{2}(2\pi i) = \pi i$$

$\oint_C \frac{Bz+C}{z^2+9}dz = 0$ (f is analytic on C)

$$\therefore \oint_C \frac{3z^2+2}{(z-1)(z^2+9)}dz = \pi i$$

Example: Evaluate $\oint_C \frac{e^z}{(z-i)^2}dz\,;C : |z-i| = 1.$

Solution: by Cauchy integral formula

$$\oint_C \frac{e^z}{(z-i)^2}dz = 2\pi i f'(i)$$

$f(z) = e^z \to f'(z) = e^z \to f'(i) = e^i$

$\therefore \oint_C \frac{e^z}{(z-i)^2} dz = 2\pi i e^i$

We can solve it by residue:

Note that the Taylor series of e^z at $z = i$, is:

$f(z) = e^z = e^i + \frac{e^i(z-i)}{1!} + \frac{e^i(z-i)^2}{2!} + \cdots, |z| < \infty$

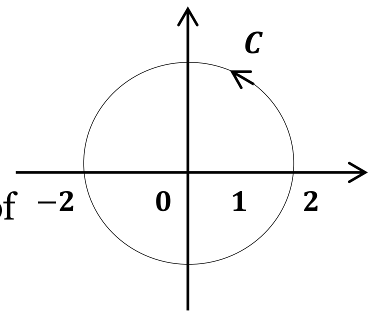
$\therefore \frac{e^z}{(z-i)^2} = \frac{e^i}{(z-i)^2} + \frac{e^i}{z-i} + \frac{e^i}{2!} + \cdots, 0 < \ |z-i| < \infty$

The coefficient of $(z-i)^{-1}$is $e^i = b_1$, so

$\oint_C \frac{e^z}{(z-i)^2} dz = 2\pi i b_1 = 2\pi i e^i$

Example: Evaluate $\oint_C \frac{5z-2}{z(z-1)} dz\,; C : |z| = 2.$

Solution:



The singular and isolated points are 0,1.

1. To calculate $Res[f,0]$ (to find the coefficient of z^{-1} and the series of power z):

$f(z) = \frac{5z-2}{z}\left(\frac{1}{z-1}\right)$

$= \left(5 - \frac{2}{z}\right)\left(\frac{-1}{1-z}\right)$

$= \left(\frac{2}{z} - 5\right)\sum_{n=0}^{\infty} z^n$

The coefficient of z^{-1} is $B_1 = 2$

$\therefore Res[f,0] = 2$

2. To calculate $Res[f,1]$ (find the coefficient of $(z-1)^{-1}$ and the series of power $z-1$):

$\frac{5z-2}{z(z-1)} = \frac{1}{z-1}\left(5 - \frac{2}{z}\right)$

$= \frac{1}{z-1}\left(5 - \frac{2}{1+(z-1)}\right)$

$$= \frac{1}{z-1}\left[5 - 2\sum_{n=0}^{\infty}(-1)^n(z-1)^n\right] \;, |z-1| < 1$$

$$= \frac{1}{z-1}[5 - 2(1 - (z-1) + (z-1)^2 - \cdots)]$$

$$= \frac{1}{z-1}[5 - 2 + 2(z-1) - 2(z-1)^2 + \cdots]$$

$$= \frac{1}{z-1}[3 + 2(z-1) - 2(z-1)^2 + \cdots]$$

$$= \frac{3}{z-1} + 2 - 2(z-1) + 2(z-1)^2 - \cdots$$

$$\therefore Res[f,1] = 3 = B_2$$

$$\oint_C \frac{5z-2}{z(z-1)}dz = 2\pi i(B_1 + B_2)$$

$$= 2\pi i(2+3)$$

$$= 10\pi i$$

Chapter Seven

Applications of Residues

[1] Evaluation of Improper Integrals

An important application of the theory of residues is the evaluation of certain types of real definite and improper integrals occurring in real analysis. It is known that if f is continuous real function for all x, then the improper integral

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{0} f(x)\,dx + \lim_{R\to\infty} \int_{0}^{R} f(x)\,dx \qquad \dots (1)$$

If the L.H.S of (1) exists, then we shall write

$$p.v. \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx \qquad \dots (2)$$

which is called Cauchy principal value provided the limit on the right of (2) exists.

Note: the convergence of $\int_{-\infty}^{\infty} f(x)\,dx$ imply the existence of $\lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = p.v. \int_{-\infty}^{\infty} f(x)\,dx$, but the converse is not necessary true. To see this,

Let $f(x) = x$, then

$$\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \frac{1}{2}x^2\Big|_{-R}^{R} = 0$$

But,

$$\lim_{R\to\infty} \int_{-R}^{0} x\,dx = \lim_{R\to\infty} \frac{1}{2}x^2\Big|_{-R}^{0} = \infty$$

So the integral is not exists and consequently the integral $\int_{-\infty}^{\infty} x\,dx$ is not convergent.

Note: if f is an even function, i.e. $f(-x) = f(x)\,, \forall x$, then

$$\int_{0}^{\infty} f(x)\,dx = \frac{1}{2} \int_{-\infty}^{\infty} f(x)\,dx$$

provided that the integral exists.

Example: Show that

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$$

Solution:

$$\int_0^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{x^2+1} = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx$$

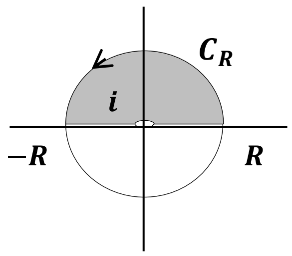
To find $\int_{-R}^{R} \frac{dx}{x^2+1}$, take $f(z) = \frac{1}{z^2+1}$ which has the simple poles $z = \mp i$.

We take a semicircle C_R such that it contains the pole, specifically a semicircle contains the positive poles, $|R| > 0$ and then we find $Res[f, i]$.

$$Res[f, i] = \lim_{z\to i}(z-i)\frac{1}{(z-i)(z+i)}$$

$$= \frac{1}{2i}$$

$$= -\frac{i}{2}$$



$$\therefore \int_{-R}^{R} \frac{dx}{x^2+1} = 2\pi i\left(-\frac{i}{2}\right) - \int_{C_R} f(z)\,dz$$

$$= \pi - \int_{C_R} \frac{dz}{z^2+1}$$

$$|z^2+1| \geq |z^2| - 1 \geq R^2 - 1$$

$$\to \left|\int_{C_R} \frac{dz}{z^2+1}\right| \leq \int_{C_R} \frac{|dz|}{R^2-1} \leq \frac{\pi R}{R^2-1}$$

$$p.v. \int_{-\infty}^\infty \frac{dx}{x^2+1} = \lim_{R\to\infty} \int_{-R}^{R} \frac{dx}{x^2+1} = \pi - 0$$

$$\therefore \int_{-\infty}^\infty \frac{dx}{x^2+1} = \pi$$

$$\to \int_0^\infty \frac{dx}{x^2+1} = \frac{\pi}{2}$$

Note: let $f(x) = \frac{p(x)}{q(x)}$, where p and q are real polynomials with no factors in common and $q(x)$ has no real zeros, i.e. $q(x) \neq 0$. If the degree $q(x)$ is at least two greater than the degree of $p(x)$ the then integral converges. The value of the integral can be found by using the theory of residue.

Example: Evaluate

$$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx$$

Solution: the above integral can be written as:

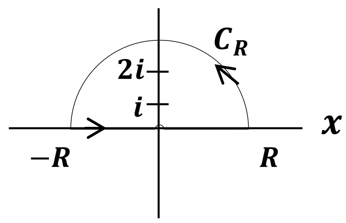
$$\int_0^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{2x^2-1}{x^4+5x^2+4}\,dx$$

Note that the integral on the right represents an integration of the function

$$f(z) = \frac{2z^2-1}{z^4+5z^2+4} = \frac{2z^2-1}{(z^2+1)(z^2+4)}$$

along the entire real axis. The function f is analytic everywhere in the upper semicircle with radius $R > 2$ except at $z = i$ and $z = 2i$ which are inside the semicircle bounded by $-R \le x \le R,\ y = 0$ and the upper half of $|z| = R$.

Hence,



$$\int_C f(z)\,dz = \int_{-R}^R f(x)\,dx + \int_{C_R} f(z)\,dz$$

And thus we have

$$2\pi i(B_1 + B_2) = \int_{-R}^R f(x)\,dx + \int_{C_R} f(z)\,dz$$

where B_1 is the residue of f at $z = i$

and B_2 is the residue of f at $z = 2i$, such that:

$$B_1 = \frac{i}{2} \text{ and } B_2 = -\frac{3i}{4}$$

* To find B_1, we write

$$f(z) = \frac{(2z^2-1)/(z^2+4)}{(z^2+1)} = \frac{(2z^2-1)/(z^2+4)(z+i)}{(z-i)}$$

$$= \frac{\varphi(z)}{z-i}$$

$$\therefore \varphi(i) = B_1 = \frac{2(i)^2-1}{((i)^2+4)(2i)} = -\frac{3}{6i} = -\frac{1}{2i} = \frac{i}{2}$$

* To find B_2, we write

$$f(z) = \frac{(2z^2-1)/(z^2+1)}{(z^2+4)} = \frac{(2z^2-1)/(z^2+1)(z+2i)}{(z-2i)}$$

$$= \frac{\varphi(z)}{z-2i}$$

$$\therefore \varphi(2i) = B_2 = \frac{2(2i)^2-1}{((2i)^2+1)(2i+2i)} = \frac{9}{12i} = \frac{3}{4i} = \frac{-3i}{4}$$

$$\to B_1 + B_2 = \frac{i}{2} - \frac{3i}{4} = \frac{-i}{4}$$

Hence:

$$2\pi i\left(\frac{-i}{4}\right) = \int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz$$

$$\to \int_{-R}^{R} f(x)\,dx = \frac{\pi}{2} - \int_{C_R} f(z)\,dz$$

Next, we will show that $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$. Since

$$|z^4 + 5z^2 + 4| = |z^2 + 1||z^2 + 4| \geq (|z|^2 - 1)(|z|^2 - 4)$$

when z is on C_R, we have

$$|z^4 + 5z^2 + 4| \geq (R^2 - 1)(R^2 - 4)$$

Also, on C_R, we have:

$$|2z^2 - 1| \leq 2|z|^2 + 1 = 2R^2 + 1$$

Thus

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4}\,dz\right| \leq \int_{C_R} \frac{2R^2+1}{(R^2-1)(R^2-4)}\,|dz|$$

$$= \frac{2R^2+1}{(R^2-1)(R^2-4)}\,\pi R \quad \left(\int_{C_R} |dz| = \pi R\right)$$

And therefore

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$$

And then we have:

$\lim_{R\to\infty} \int_{-R}^{R} \frac{2x^2-1}{x^4+5x^2+4} dx = \frac{\pi}{2}$

Or:

$p.v. \int_{-\infty}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx = \frac{\pi}{2}$

and the desired result is:

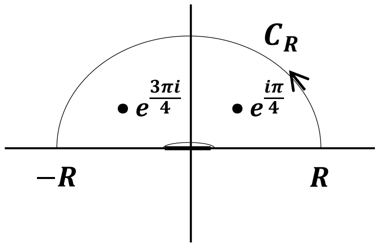
$\int_{0}^{\infty} \frac{2x^2-1}{x^4+5x^2+4} dx = \frac{\pi}{4}$

Example: Find $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$ by using residues.

Solution:

Consider $\oint_C \frac{z^2}{z^4+1} dz$, taken around the closed contour C consisting of the line segment $y = 0,\ -R \le x \le R$, and the semicircle $|z| = R$, $0 < \theta < \pi$.

Let $R > 1$, which means that C enclose all the poles of $f(z) = \frac{z^2}{z^4+1}$ in the upper half plane which are



$z_1 = e^{\frac{i\pi}{4}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$z_2 = e^{i\left(\frac{3\pi}{4}\right)} = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

$(z^4 = -1 = e^{i(\pi+2k\pi)} \to z = e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)}, k = 0,1,2, \ldots)$

Hence,

$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$

$2\pi i(B_1 + B_2) = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$

where B_1 is the residue of f at $z_1 = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

and B_2 is the residue of f at $z_2 = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

Such that: $B_1 = \frac{1}{4} e^{-\frac{i\pi}{4}}$ and $B_2 = \frac{1}{4} e^{-i\frac{3\pi}{4}}$

$$\left(f(z) = \frac{p(z)}{q(z)} = \frac{z^2}{z^4+1} \to B_1 = \frac{p(z_1)}{q'(z_1)} \, , B_2 = \frac{p(z_2)}{q'(z_2)} \right)$$

Hence:

$$2\pi i \left(\frac{1}{4} e^{-\frac{i\pi}{4}} + \frac{1}{4} e^{-i\frac{3\pi}{4}} \right) = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz \text{ , then}$$

$$\frac{2\pi i}{4} \left(e^{-\frac{i\pi}{4}} + e^{-i\frac{3\pi}{4}} \right) = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$$

$$\frac{\pi i}{2} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$$

$$\to \frac{\pi}{\sqrt{2}} = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$$

Or:

$$\int_{-R}^{R} f(x)\, dx = \frac{\pi}{\sqrt{2}} - \int_{C_R} f(z)\, dz$$

Next, we will show that $\int_{C_R} f(z)\, dz \to 0$ as $R \to \infty$. Since, when z is on C_R, we have

$$|z^4 + 1| \geq |z|^4 - 1 = R^4 - 1 \text{ and } z^2 = R^2$$

$$\left| \int_{C_R} \frac{z^2}{z^4+1} dz \right| \leq \int_{C_R} \left| \frac{z^2}{z^4+1} \right| |dz|$$

$$\leq \frac{R^2}{R^4-1} \int_{C_R} |dz|$$

$$= \frac{R^2 \pi R}{R^4-1} \qquad \left(\int_{C_R} |dz| = \pi R \right)$$

$$= \frac{\pi R^3}{R^4-1}$$

And therefore

$$\lim_{R\to\infty} \int_{C_R} f(z)\, dz = 0$$

Thus, we have:

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$$

Or:

$p.v.\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$

Note:

Since $f(x) = \frac{x^2}{x^4+1}$ is an even function, then we get:

$2\int_{0}^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{\sqrt{2}}$

$\therefore \int_{0}^{\infty} \frac{x^2}{x^4+1} dx = \frac{\pi}{2\sqrt{2}}$

[2] Improper Integrals Involving Sine and Cosine

This section is devoted to illustrate how the theorem of residues can be used to evaluate convergent integrals of the forms

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)} \sin x \, dx \text{ and } \int_{-\infty}^{\infty} \frac{p(z)}{q(z)} \cos x \, dx$$

where $q(z) \neq 0$ and the functions p and q are real polynomials and have no factors in common.

Since $e^{ix} = \cos x + i \sin x$

$\therefore \int_{-R}^{R} \frac{p(z)}{q(z)} \cos x \, dx + i \int_{-R}^{R} \frac{p(z)}{q(z)} \sin x \, dx = \int_{-R}^{R} \frac{p(z)}{q(z)} e^{ix} \, dx$ ... (1)

Equation (1) will be used for evaluating any of

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)} \sin x \, dx \text{ or } \int_{-\infty}^{\infty} \frac{p(z)}{q(z)} \cos x \, dx$$

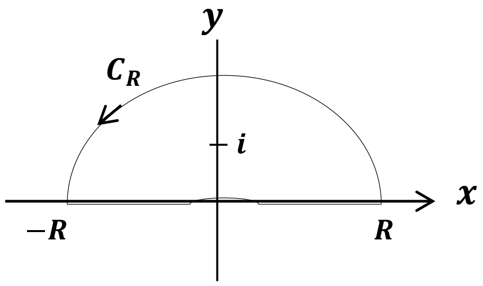
Example: Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)^2} dx$$

Solution: the above integral is the real part of the integral

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+1)^2} dx$$

which represents an integration of the function $f(z) = \frac{e^{iz}}{(z^2+1)^2}$ along the real axis. The singularities of f are $\pm i$ and so we may integrate around the simple contour as shown,



where $R > 1$, note that the I.S.P. $z = i$ is a pole od order two, hence

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx + \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz$$

Therefore,

$$2\pi i B_1 = \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx + \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz$$

Where B_1 is the residue of f at $z = i$. And to find B_1, we write

$$f(z) = \frac{e^{iz}}{(z+i)^2(z-i)^2} = \frac{e^{iz}/(z+i)^2}{(z-i)^2}$$

$$= \frac{\varphi(z)}{(z-i)^2}$$

$\rightarrow B_1 = \frac{\varphi'(z)}{1!}$, where

$$\varphi'(z) = \frac{(z+i)^2.ie^{iz} - 2e^{iz}(z+i)}{(z+i)^4}$$

$$\rightarrow \varphi'(i) = \frac{(2i)^2(i)e^{-1} - 2e^{-1}(2i)}{(2i)^4}$$

$$= \frac{(-4i-4i)e^{-1}}{2^4}$$

$$= -\frac{i}{2e}$$

$\rightarrow B_1 = -\frac{i}{2e}$, and so

$$\int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx = 2\pi i\left(\frac{-i}{2e}\right) - \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz$$

$$= \frac{\pi}{e} - \int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz \qquad \ldots (2)$$

Next, we will show that $\int_{C_R} f(z)\,dz \rightarrow 0$ as $R \rightarrow \infty$. Since

$|z^2 + 1| \geq |z|^2 - 1 = R^2 - 1$, so

$\frac{1}{|z^2+1|} \leq \frac{1}{R^2-1}$

Hence:

$$\left|\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz\right| \leq \int_{C_R} \frac{|e^{iz}|}{(R^2-1)^2}\,|dz|$$

Since, $\left|e^{iz}\right| \leq 1$ and $\int_{C_R} |dz| = \pi R$, then

$$\left|\int_{C_R} \frac{e^{iz}}{(z^2+1)^2}\,dz\right| \leq \frac{\pi R}{(R^2-1)^2} \rightarrow 0 \text{ as } R \rightarrow \infty$$

Using (2), we get

$$Re \int_{-R}^{R} \frac{e^{ix}}{(x^2+1)^2}\,dx = \frac{\pi}{e}$$

Or

$\int_{-R}^{R} \frac{\cos x}{(x^2+1)^2}\,dx = \frac{\pi}{e}$, and thus

$$\lim_{R\rightarrow\infty} \int_{-R}^{R} \frac{\cos x}{x^4+1}\,dx = \frac{\pi}{e}$$

Or:

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx = \frac{\pi}{e}$$

And hence the Cauchy p.v. exists and equals to $\frac{\pi}{e}$.

Example: Find $\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+4} dx$ by using residues.

Solution: the above integral is the imaginary part of the integral

$$\int_{-\infty}^{\infty} \frac{xe^{ix}}{x^4+4} dx$$

which represents an integration of the function $f(z) = \frac{ze^{iz}}{z^4+4}$, along the real axis. The function f is analytic everywhere except at $z^4 + 4 = 0$, so

$z^4 = -4 = re^{i\theta} = 4e^{i(\pi+2k\pi)}$

$\rightarrow z = (4)^{\frac{1}{4}} e^{i\left(\frac{\pi}{4}+\frac{k\pi}{2}\right)}, k = 0,1,2, \dots$

The simple poles are: $z_1 = \sqrt{2}\, e^{i\frac{\pi}{4}}$ and $z_2 = \sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)}$, lie inside the semicircle region whose boundary are the segment $-R \le x \le R$, $y = 0$ of the real axis, and the upper half C_R of the circle $|z| = R$, where $R > \sqrt{2}$. Hence,

$\int_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{C_R} f(z)\, dz$

$2\pi i(B_1 + B_2) = \int_{-R}^{R} \frac{xe^{ix}}{x^4+4} dx + \int_{C_R} \frac{ze^{iz}}{z^4+4} dz$

Where B_1 is the residue of f at $z_1 = \sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}$ and B_2 is the residue of f at $z_2 = \sqrt{2}\, e^{i\left(\frac{3\pi}{4}\right)}$, such that

$B_1 = \left.\frac{p(z)}{q'(z)}\right|_{z=z_1}, p(z) = \frac{ze^{iz}}{z^2+2i}$ and $q(z) = z^2 - 2i$

$B_1 = \left.\frac{ze^{iz}/z^2+2i}{2z}\right|_{z=\sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}}$

$= \frac{\sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}.e^{i\sqrt{2}e^{i\frac{\pi}{4}}}/(2i+2i)}{2\sqrt{2}\, e^{i\left(\frac{\pi}{4}\right)}}$

$= \frac{e^{i-1}}{8i}$

$= \frac{-e^{-1}ie^{i}}{8}$

On the other hand if we write $f(z) = \frac{p(z)}{q(z)}$, where $p(z) = \frac{ze^{iz}}{z^2-2i}$ and $q(z) = z^2 + 2i$, then one can show that

$$B_2 = \left.\frac{p(z)}{q'(z)}\right|_{z=z_1} = \frac{e^{-1}ie^{-i}}{8}$$

And

$$B_1 + B_2 = \frac{e^{-1}}{8}\left(-ie^{i} + ie^{-i}\right) = \frac{e^{-1}\sin 1}{4}$$

Similarly, then

$$Im\int_{-R}^{R}\frac{xe^{ix}}{x^4+4}dx + Im\int_{C_R} f(z)\,dz = Im\big(2\pi i(B_1 + B_2)\big)$$

$$= \frac{\pi}{2}e^{-1}\sin 1$$

Since one can show that

$Im\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^2\pi}{R^4-4} \to 0$ as $R \to \infty$, it follows that

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{x\sin x}{x^4+4}dx = \frac{\pi}{2}e^{-1}\sin 1.$$