Chapter One

Complex Numbers

[1] Definition:

A *complex number* ***z*** is an ordered pair (a, b) of real numbers such that

$$\mathbb{C} = \{ \mathbb{R} \times \mathbb{R} \} = \{(a, b) : a, b \in \mathbb{R} \}$$

where $\mathbb{R}$ denotes the Real Numbers set. The real numbers a, b are called the real and imaginary parts of the complex number $z = (a, b)$, that is $a = Re(z)$ and $b = Im(z)$. If $b = Im(z) = 0$ then $z = (a, 0) = a$ so that the set of complex numbers is a natural extension of real numbers, then we have:

$a = (a, 0)$ for any real number a. Thus

$$0 = (0,0), \qquad 1 = (1,0), \qquad 2 = (2,0), \dots$$

A pair $(0, b)$ is called a pure imaginary number and the pair $(0, 1)$ is called the imaginary $\boldsymbol{i}$, that is

$$(0,1) = i$$

Now any complex number z can be written as:

$$(a, 0) + (0, b) = (a, b) = z$$

The operation of addition $(z_1 + z_2)$ and multiplication $(z_1 . z_2)$ are defined as follows

$z_1 + z_2 = (a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$

$z_1 . z_2 = (a_1, b_1) . (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + b_1 a_2)$

Such that $z_1 = (a_1, b_1), z_2 = (a_2, b_2)$

Now,

$z = (a, 0) + (0, b) = (a, 0) + (0,1)(b, 0)$

Hence $(a, 0) + (0,1)(b, 0) = (a, b) = z$ where $(0,1) = i$

Then $z = a + ib$

Now, $z^2 = z.z,\ z^3 = z.z.z,\ z^n = \underbrace{z.z \ldots\ldots z}_{n-\text{times}}$

$i^2 = i.i = (0,1).(0,1) = -1$ or $i = \sqrt{-1}$

Then $i^2 = -1,\ i = \sqrt{-1}$

[2] Basic Algebraic Properties:

The following algebraic properties hold for all $z_1, z_2, z_3 \in \mathbb{C}$

1. $z_1 + z_2 = z_2 + z_1$
2. $z_1.z_2 = z_2.z_1$ (Commutative laws under addition and multiplication)
3. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Associative under addition)
4. $(z_1.z_2).z_3 = z_1.(z_2.z_3)$ (Associative under multiplication)
5. $z_1.(z_2 + z_3) = z_1.z_2 + z_1.z_3$ (Distribution laws)
6. $z_1 + z_3 = z_3 + z_2$ iff $z_1 = z_3$
7. $z_1.z_2 = z_3.z_2$ iff $z_1 = z_3$ (Cancelation law)

Note: the additive identity $0 = (0,0)$ and the multiplication identity $1 = (1,0)$, for any complex number. That is

$$z + 0 = 0 + z = z$$

$$1.z = z.1 = z$$

for any complex number.

Definition:

The additive inverse z^* of z is a complex number with the property that

$z + z^* = 0$ (1)

It is clear that (1) is satisfied if $z^* = (-x, -y)$, has an additive inverse.

Definition:

The multiplication inverse $z^{-1} (z \neq 0)$ of z is a complex number with the property that

$z.z^{-1} = z^{-1}.z = 1$ (2)

Such that:

$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \quad \text{(H.w)}$$

Note: the additive and multiplication identity are unique.

Note: if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \left(\frac{x_1x_2+y_1y_2}{x_2^2+y_2^2}, \frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}\right)$$

Exercise: show that $z = 0$ iff $Re(z) = 0$ and $Im(z) = 0$.

Example: verify that

1. $(\sqrt{2} - i) - i(1 - \sqrt{2}\, i)$

Solution:

$\sqrt{2} - i - i - \sqrt{2} = -2i$

2. $(2, -3)(-2,1)$

Solution:

$(2, -3)(-2, 1) = (-4 + 3, 2 + 6) = (-1, 8)$

3. $(3,1)(3, -1)\left(\frac{1}{5}, \frac{1}{10}\right)$

Solution:

$$(3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) = (9+1,-3+3)\left(\frac{1}{5},\frac{1}{10}\right)$$

$$= (10,0)\left(\frac{1}{5},\frac{1}{10}\right)$$

$$= \left(\frac{10}{5}-0,\ \frac{10}{10}+0\right)$$

$$= (2,1)$$

Example: show that each of the two numbers $z = 1 \mp i$ satisfies the equation

$$z^2 - 2z + 2 = 0$$

Proof: for $z = 1 + i$

$(1+i)^2 - 2(1+i) + 2 = 1 + 2i - 1 - 2 - 2i + 2 = 0$

for $z = 1 - i$ (H.w)

Example: show that $(1-i)^4 = -4$

Proof: $((1-i)^2)^2 = (1-2i-1)^2$

$$= 4i^2 = -4$$

Example: prove that $(1+z)^2 = 1 + 2z + z^2$

Proof: L.H.S → $(1+z)^2 = (1+z)(1+z)$

$$= \big((1,0)+(x,y)\big).\big((1,0)+(x,y)\big)$$

$$= (1+x,y)(1+x,y)$$

$$= (1+2x+x^2-y^2, 2y+2xy)$$

R.H.S → $1 + 2z + z^2 = (1,0) + 2(x,y) + (x,y).(x,y)$

$$= (1,0) + (2x,2y) + (x,y).(x,y)$$

$$= (1+2x+x^2-y^2, 2y+2xy)$$

$$= (1+z)^2$$

$$= \text{L.H.S}$$

Note: $(-z)$ is the only additive inverse of a given complex number.

[3] Properties of Complex Numbers:

1. $Im(iz) = Re(z)$

2. $Re(iz) = Im(z)$

3. $\frac{1}{1/z} = z,\ z \neq 0$

4. $(-1)z = -z$

5. $(z_1 z_2)(z_3 z_4) = (z_1 z_3)(z_2 z_4)$

6. $\frac{z_1+z_2}{z_3} = \frac{z_1}{z_3} + \frac{z_2}{z_3}, z_3 \neq 0$

Note:

$$(1+z)^n = 1 + nz + \frac{n(n+1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \cdots + z^n$$

[4] Vectors and Moduli

It is natural to associate any nonzero complex number $z = x + iy$ with the directed line segment or vector from the origin to the point (x, y) that represents z in the complex plane. In fact, we can often refer to z as the point z or the vector z, in Fig. 1 the number $z = x + iy$ and $-2 + i$ are displayed graphically as both two points and radius vector.

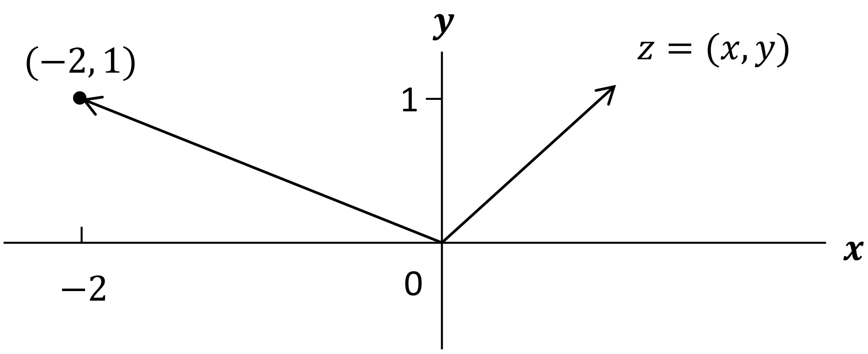


Figure 1

When $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

Corresponds to the point $(x_1 + x_2, y_1 + y_2)$, it is also corresponds to a vector with those coordinates as its components. Hence $z_1 + z_2$ may be obtained vectorially as shown in Fig. 2.

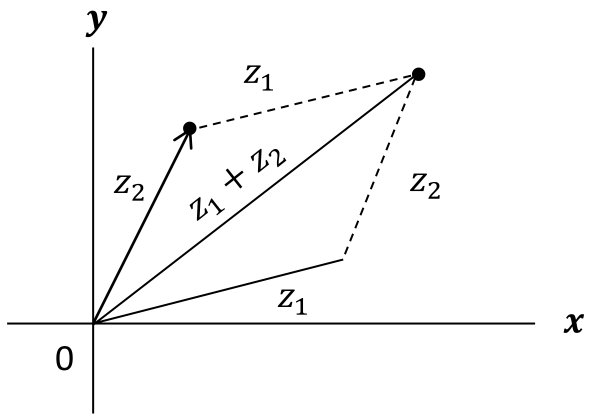


Figure 2

The distance between two points (x_1, y_1) and (x_2, y_2) is $|z_1 - z_2|$, this is clear from Fig. 3, since $|z_1 - z_2|$ is the length of the vector representing the number $z_1 - z_2 = z_1 + (-z_2)$,

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

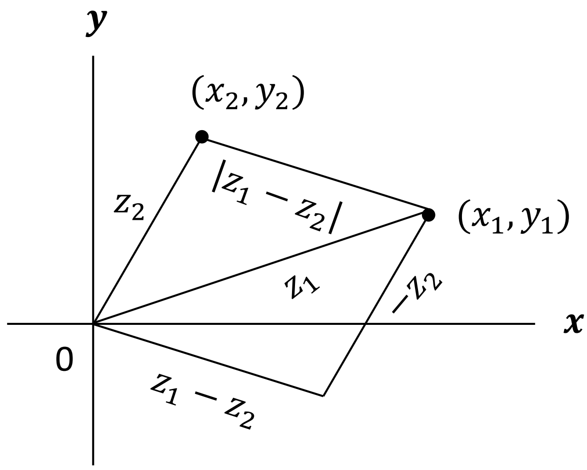


Figure 3

Example: the equation $|z-1+3i|=2$ represents the circle whose center is $z_0=(1,-3)$ and whose radius is $R=2$.

$|z-z_0|=R$, where z_0 represents the center of circle with radius R.

Definition: (The Absolute Value)

The modulus or absolute value of a complex number $z=x+iy$ is defined by $\sqrt{x^2+y^2}$ and also by $|z|$, such that

$$|z|=\sqrt{x^2+y^2}$$

we notice that the modulus $|z|$ is a distance from (0,0) to (x,y), the statement $|z_1|<|z_2|$ means that z_1 is closer to (0,0) than z_2. The distance between z_1 and z_2 is given by

$$|z_1-z_2|=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

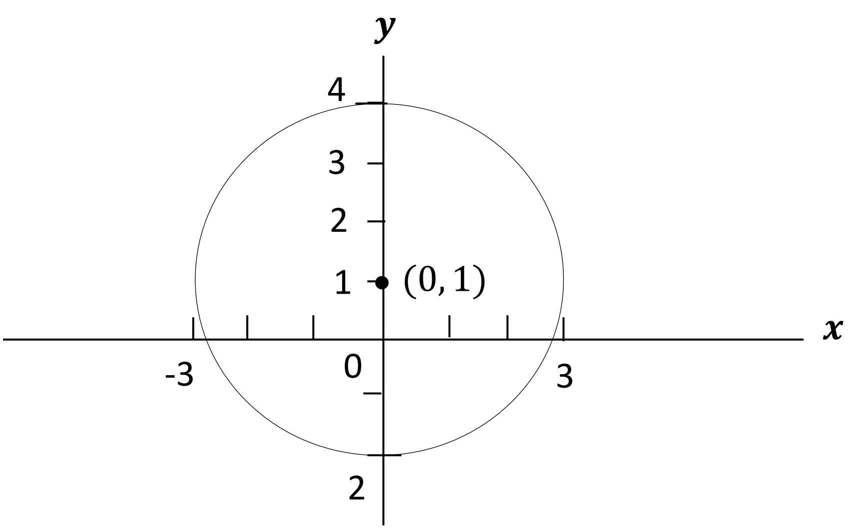
Example: $|z-i|=3$

Solution: we refer to $|z-i|=3$ as $|x+iy-i|=3$

$|x+i(y-1)|=3 \rightarrow \sqrt{x^2+(y-1)^2}=3$

$x^2+(y-1)^2=9 \Leftrightarrow (x-x_0)^2+(y-y_0)^2=r^2$

The complex number corresponding to the points lying on the circle with center (0,1) and radius 3



Note: the real numbers $|z|$, $Re(z)$ and $Im(z)$ are related by the equation:

$$|z|^2 = (Re(z))^2 + (Im(z))^2$$

As follows

$$|z| = \sqrt{x^2 + y^2} \rightarrow |z|^2 = x^2 + y^2 = (Re(z))^2 + (Im(z))^2$$

Since $y^2 \geq 0$, we have

$$|z|^2 \geq x^2 = \left(Re(z)\right)^2 = |Re(z)|^2$$

And since $|z| \geq 0$, we get

$$|z| \geq |Re(z)| \geq Re(z)$$

Similarly $|z| \geq |Im(z)| \geq Im(z)$.

[5] Complex Conjugates

The complex conjugate of z is defined by

$$\bar{z} = x - iy$$

The number is $\bar{z}$ represented by the point $(x, -y)$, which is the reflection in the real axis of the point (x, y) representing z (Fig. 4), note that

$$\bar{\bar{z}} = z \text{ and } |\bar{z}| = |z|, \qquad \text{for all } z$$

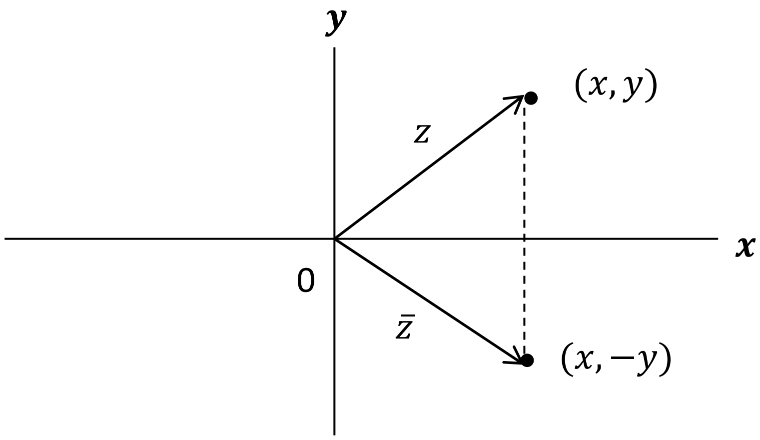


Figure 4

Some Properties of Complex Conjugates:

1. $\bar{\bar{z}} = z$

2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2, \quad \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

3. $\overline{z_1 . z_2} = \bar{z}_1 . \bar{z}_2$

4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad , z_2 \neq 0$

<u>Note:</u>

1. $z + \bar{z} = x + iy + x - iy = 2x = 2Re(z)$

$$Re(z) = \frac{z+\bar{z}}{2}$$

2. $z - \bar{z} = x + iy - x + iy = 2iy = 2Im(z)$

$$Im(z) = \frac{z-\bar{z}}{2}$$

Some Properties of Moduli

1. $|z_1 z_2| = |z_1||z_2|$

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad , z_2 \neq 0$

3. $|z_1 + z_2| \leq |z_1| + |z_2|$

4. $|z_1 + z_2 + \cdots z_n| \leq |z_1| + |z_2| \cdots |z_n|$

5. $\big||z_1| - |z_2|\big| \leq |z_1 + z_2|$

6. $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$

<u>Example:</u> If a point z lies on the unite circle $|z| = 1$ about the origin, show that $|z^2 - z + 1| \leq 3$ and $|z^3 - 2| \geq ||z|^3 - 2|$

<u>*Proof*</u>: $|z^2 - z + 1| = |(z^2 + 1) - z| \le |z^2 + 1| + |z|$

$$\le |z^2| + 1 + |z|$$

$$= |z|^2 + 1 + |z|$$

$$= 1^2 + 1 + 1$$

$$= 3$$

$$\to \ |z^2 - z + 1| \le 3$$

Prove that $\sqrt{2}\,|z| \ge |Re(z)| + |Im(z)|$

<u>*Solution*</u>*:*

$$\left(\sqrt{2}\,|z|\right)^2 = 2|z|^2 = 2(x^2 + y^2)$$

$$= (x^2 + y^2) + (x^2 + y^2)$$

$$\ge (x^2 + y^2) + 2|x||y| \ \cdots \ \text{(by } *\text{)}$$

$$= (|x| + |y|)^2$$

$$\therefore \left(\sqrt{2}\,|z|\right)^2 \ge (|x| + |y|)^2$$

$$\to \ \sqrt{2}\,|z| \ \ge |x| + |y| = |Re(z)| + |Im(z)|$$

$$\therefore \ \sqrt{2}\,|z| \ge |Re(z)| + |Im(z)|$$

<u>**Note:**</u> $(|x| - |y|)^2 \ge 0$

$\to \ |x|^2 + |y|^2 - 2|x||y| \ge 0$

$\to \ x^2 + y^2 \ge 2|x||y| \quad \dots (*)$

Prove that:

1. z is real iff $\bar{z} = z$ (H.w)

2. z is either real or pure imaginary iff $(\bar{z}\,)^2 = z^2$

Prove that: if $|z_2| \neq |z_3|$ then

$$\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{||z_2|-|z_3||}$$

Proof:

$$\left|\frac{z_1}{z_2+z_3}\right| = \frac{|z_1|}{|z_2+z_3|} \qquad \dots (1)$$

Since $|z_2 + z_3| \geq \big||z_2| - |z_3|\big|$

$$\rightarrow \ \frac{1}{|z_2+z_3|} \leq \frac{1}{||z_2|-|z_3||}$$

$$\rightarrow \ \frac{|z_1|}{|z_2+z_3|} \leq \frac{|z_1|}{||z_2|-|z_3||} \quad \dots (2)$$

From (1) and (2) we have

$$\left|\frac{z_1}{z_2+z_3}\right| \leq \frac{|z_1|}{||z_2|-|z_3||}$$

Example: If a point z lies on the unite circle $|z| = 2$ then show that

$$\frac{1}{|z^4-4z^3+3|} \leq \frac{1}{3}$$

Proof: $|z^4 - 4z^3 + 3| = |(z^2 - 1)(z^2 - 3)|$

$$= |z^2 - 1||z^2 - 3|$$

$$\geq ||z|^2 - 1|\,||z|^2 - 3|$$

$$= |4 - 1|\,|4 - 3|$$

$$= 3$$

$\therefore \ |z^4 - 4z^3 + 3| \geq 3$

$$\rightarrow \ \frac{1}{|z^4-4z^3+3|} \leq \frac{1}{3}$$

Exercises:

1. Show that the hyperbola $x^2 - y^2 = 1,$ can be written as

$$z^2 + \bar{z}^2 = 2$$

2. Show that $|z - 4i| + |z + 4i| = 10$ is an ellipse whose foci are $(0, \mp 4)$.

Proof: 1. $x^2 - y^2 = 1,\ x = \frac{z+\bar{z}}{2},\ y = \frac{z-\bar{z}}{2i}$

$$\left(\frac{z+\bar{z}}{2}\right)^2 - \left(\frac{z-\bar{z}}{2i}\right)^2 = 1$$

$$\frac{z^2+2z\bar{z}+\bar{z}^2}{4} - \frac{z^2-2z\bar{z}+\bar{z}^2}{4i^2} = 1$$

$$\frac{z^2+2z\bar{z}+\bar{z}^2}{4} + \frac{z^2-2z\bar{z}+\bar{z}^2}{4} = 1$$

$\rightarrow 2z^2 + 2\bar{z}^2 = 4$

$\rightarrow 2(z^2 + \bar{z}^2) = 4$

$\rightarrow z^2 + \bar{z}^2 = 2$

[6] Polar Form of Complex Numbers: (Exponential Form)

Let r and θ be polar coordinates of the point (x, y) that corresponds to a nonzero complex number $z = x + iy$,

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

The number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$tan\theta = \frac{y}{x}$, $x \neq 0,\ r^2 = x^2 + y^2,\ i\theta = \cos\theta + i\sin\theta$

This implies that for any complex number $z = x + iy,$ we have

$$|z| = \sqrt{x^2 + y^2} = \sqrt{r^2} = r$$

In fact r is the length of the vector represent z. In particular, since $z = x + iy$ we may express z in polar form by

$$z = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta)$$

The real number θ represents the angle, measured in radians, that z makes with the positive real axis (Fig. 5).

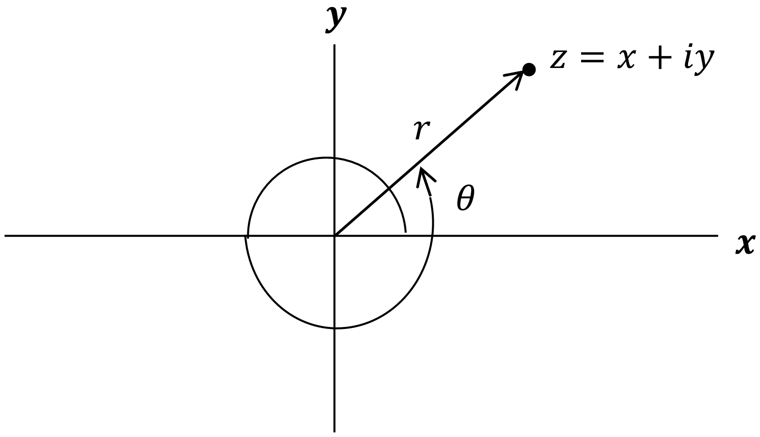


Figure 5

Each value of θ is called an argument of z and the set of all such values is denoted by $\arg z = \theta$.

Note: $arg\, z$ is not unique.

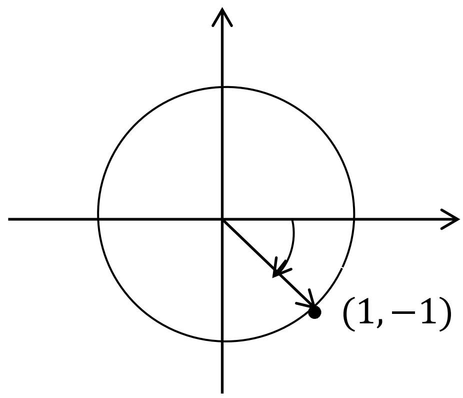
Definition: The principal value of $\arg z$ (Arg z)

If $-\pi < \theta < \pi$ and satisfy

$$\arg z = \operatorname{Arg} z + 2n\pi, \; n = 0, \mp 1, \mp 2, \ldots$$

Then this value of θ (which is unique) is called the principal value of $\arg z$ and denoted by $\operatorname{Arg} z$.

Example: Write $z = 1 - i$ in polar form



Solution: $r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$

$x = r\cos\theta \rightarrow 1 = \sqrt{2}\cos\theta \rightarrow \cos\theta = \frac{1}{\sqrt{2}}$

$y = r\sin\theta \rightarrow -1 = \sqrt{2}\sin\theta \rightarrow \sin\theta = \frac{-1}{\sqrt{2}}$

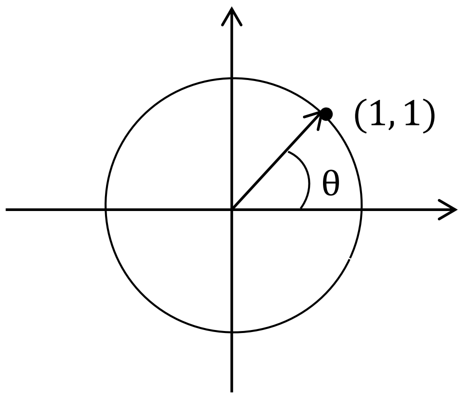
$\tan\theta = \frac{y}{x} = \frac{-1}{1} = -1$

$\theta = \tan^{-1}(-1) = \frac{-\pi}{4}$

$z = 1 - i = \sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right)$

$\quad = \sqrt{2}\left(\cos\left(\frac{-\pi}{4} + 2n\pi\right) + i\sin\left(\frac{-\pi}{4} + 2n\pi\right)\right)$

Example: Write $z = 1 + i$ in polar form



Solution: $r = \sqrt{2}$, $\tan\theta = \frac{y}{x} = 1$

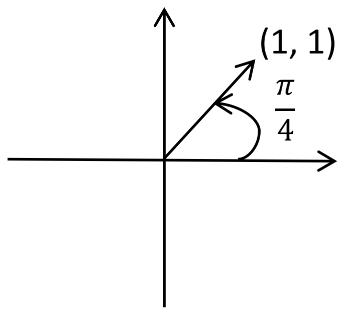
$\rightarrow \theta = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore \theta = arg\, z = \frac{\pi}{4} + 2n\pi$

$\therefore 1 + i = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2n\pi\right) + i\sin\left(\frac{\pi}{4} + 2n\pi\right)\right)$

Example: Find the principal argument Arg z when

1. $z = 1 + i$



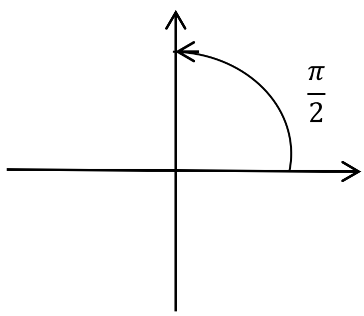
Solution: $\arg z = \text{Arg}\, z + 2n\pi$

$\quad = \frac{\pi}{4} + 2n\pi$

$\therefore \text{Arg}\, z = \frac{\pi}{4}$

2. $z = i$

Solution: $r = 1$, $\theta = \frac{\pi}{2} + 2n\pi = arg\, i$



$\arg z = \text{Arg}\, z + 2n\pi$

$\quad = \frac{\pi}{2} + 2n\pi$

$\therefore \text{Arg}\, z = \frac{\pi}{2}$

$\therefore i = z = 1.\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{2}\right)$

Exercises: Find the principal argument Arg z when $z = -i, 1, -1$.

Example: Let $z=-1-i$, write z in polar form and find $\operatorname{Arg} z$.

Solution: $r=\sqrt{1+1}=\sqrt{2}$

$x=r\cos\theta \rightarrow -1=\sqrt{2}\cos\theta \rightarrow \cos\theta=\frac{-1}{\sqrt{2}}$

$y=r\sin\theta \rightarrow -1=\sqrt{2}\sin\theta \rightarrow \sin\theta=\frac{-1}{\sqrt{2}}$

$\theta=\tan^{-1}(1)=\frac{\pi}{4}$

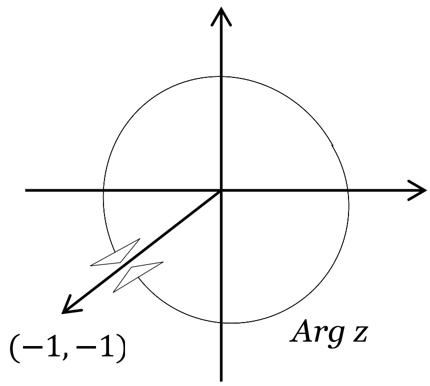
$\theta=\frac{\pi}{4}+\pi=\frac{5\pi}{4}+2n\pi$ (Since θ is located in the third quarter)

$=arg z$

$\therefore \operatorname{Arg} z=\arg z-2\pi$

$=\frac{5\pi}{4}-2\pi=\frac{-3\pi}{2}\in[-\pi,\pi]$

$z=-1-i=\sqrt{2}\left(\cos\frac{-3\pi}{2}+i\sin\frac{-3\pi}{2}\right)$



Example: Let $z_1=1+\sqrt{3}\,i$, $z_2=-1-\sqrt{3}\,i$, write z_1, z_2 in polar form and find $\operatorname{Arg} z_1$, $\operatorname{Arg} z_2$.

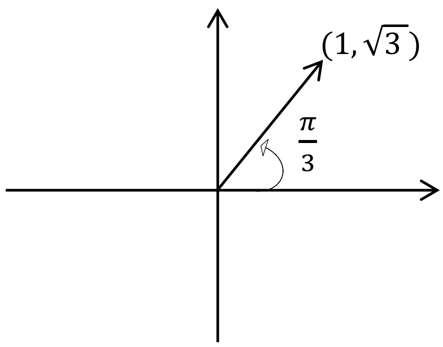
Solution: $z_1=r_1=\sqrt{x^2+y^2}=\sqrt{1^2+\left(\sqrt{3}\right)^2}=\sqrt{1+3}=2$

$x=r\cos\theta \rightarrow 1=2\cos\theta \rightarrow \cos\theta=\frac{1}{2}$

$y=r\sin\theta \rightarrow \sqrt{3}=2\sin\theta \rightarrow \sin\theta=\frac{\sqrt{3}}{2}$

$\therefore \theta=\tan^{-1}\frac{y}{x}=\frac{\pi}{3}+2n\pi$

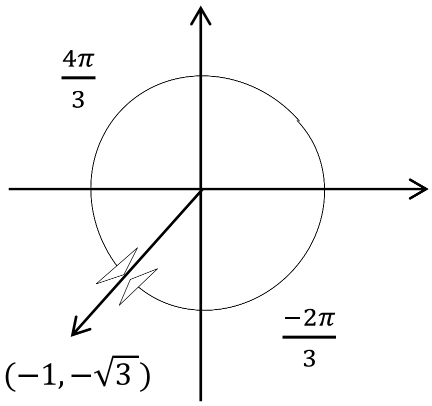
$z_1=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$



$\rightarrow z_2=r_2=\sqrt{(-1)^2+\left(-\sqrt{3}\right)^2}=2$

$x=r\cos\theta \rightarrow -1=2\cos\theta \rightarrow \cos\theta=\frac{-1}{2}$

$y = r\sin\theta \rightarrow -\sqrt{3} = 2\sin\theta \rightarrow \sin\theta = \frac{-\sqrt{3}}{2}$



$\therefore \theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3}$

$= \left(\pi + \frac{\pi}{3}\right) + 2n\pi$

$= \frac{4\pi}{3} + 2n\pi$

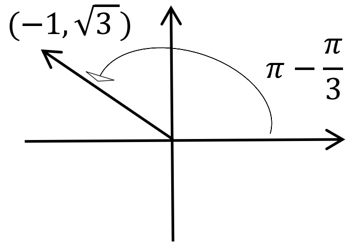
$\text{Arg } z_2 = \frac{4\pi}{3} - 2\pi$

$= \frac{-3\pi}{3}$

$z_2 = 2\left(\cos\left(\frac{-2\pi}{3}\right) + i\sin\left(\frac{-2\pi}{3}\right)\right)$

Example: $z_3 = -1 + \sqrt{3}\,i,\ z_4 = 1 - \sqrt{3}\,i$

Solution:

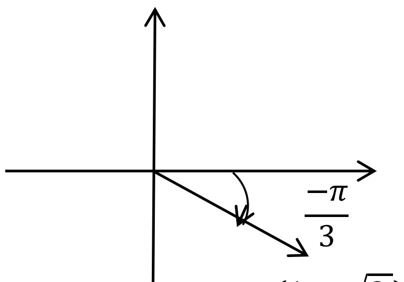


$\text{Arg } z_3 = \frac{2\pi}{3}$

$z_3 = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

$\rightarrow z_4 = 1 - \sqrt{3}\,i$

$= 2\left(\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right)$



Note:

$\left.\begin{matrix} 1 \mp i \\ -1 \mp i \end{matrix}\right\}$ Angle 45°

$\left.\begin{matrix} 1 \mp \sqrt{3}\,i \\ -1 \mp \sqrt{3}\,i \end{matrix}\right\}$ Angle 60°

$\left.\begin{matrix} \sqrt{3} \mp i \\ -\sqrt{3} \mp i \end{matrix}\right\}$ Angle 30°

- **Properties of $\arg z$:**

1. $\arg(z_1.z_2) = \arg z_1 + \arg z_2$

2. $\arg\left(\frac{1}{z}\right) = -\arg z$

3. $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

4. $\arg \bar{z} = -\arg z$

Proof:

1. Let $z_1 = r_1(\cos\theta_1 + i\ \sin\theta_1)$

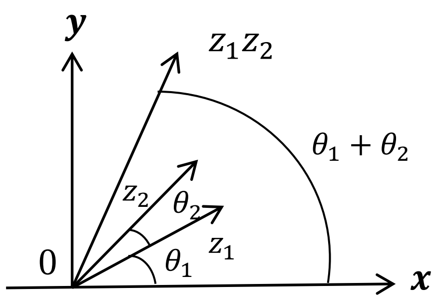
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$z_1.z_2 = r_1 r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i\ \cos\theta_1\ \sin\theta_2 + i\ \sin\theta_1\cos\theta_2)$

$= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$\therefore \arg z_1 z_2 = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$



Example: Find $\arg\left(i\left(1+\sqrt{3}\,i\right)\right)$

Solution:

$\arg\left(i\left(1+\sqrt{3}\,i\right)\right) = \arg i + \arg(1+\sqrt{3}\,i)$

$= \left(\frac{\pi}{2} + 2n\pi\right) + \left(\frac{\pi}{3} + 2n\pi\right)$

$= \frac{5}{6}\,\pi + 2k\pi\,, \quad k = n + m$

2. Let $z = r(\cos\theta + i\sin\theta)$

$\frac{1}{z} = \frac{1}{r(\cos\theta + i\sin\theta)} \cdot \frac{r(\cos\theta - i\sin\theta)}{r(\cos\theta - i\sin\theta)}$

$= \frac{r(\cos\theta - i\sin\theta)}{r^2(\cos^2\theta + \sin^2\theta)}$

$= \frac{r(\cos\theta - i\sin\theta)}{r^2}$

$\frac{1}{z} = \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right)$

$\therefore \arg\left(\frac{1}{z}\right) = -\arg z$

Note: $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

For example: Let $z_1 = i\,,\; z_2 = -1 + \sqrt{3}\,i$

$\arg z_1 = \left(\frac{\pi}{2} + 2n\pi\right),\; \arg z_2 = \left(\frac{\pi}{3} + 2n\pi\right)$

$\operatorname{Arg} z_1 = \frac{\pi}{2}$ $\quad ,\operatorname{Arg} z_2 = \frac{\pi}{3}$

$z_1 z_2 = i\left(-1 + \sqrt{3}\,i\right) = -\sqrt{3} - i$

$\arg z_1 z_2 = \pi + \frac{\pi}{6} = \frac{7}{6}\pi + 2n\pi$

$\operatorname{Arg} z_1 z_2 = \left(\pi + \frac{\pi}{6}\right) - 2\pi = \frac{-5}{6}\pi$

$\therefore \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \frac{7}{6}\pi \notin [-\pi, \pi]$

[7] Powers and Roots

Let $z = re^{i\theta}$ be a nonzero complex number, let n be an integer number then

$$z^n = r^n e^{in\theta}$$

Example: Find $(1+i)^{25}$

Solution: $r = \sqrt{x^2 + y^2} = \sqrt{2}\,,\; \theta = \frac{\pi}{4}$

$z^{25} = \left(re^{i\theta}\right)^{25}$

$= \left(\sqrt{2}\,e^{i\frac{\pi}{4}}\right)^{25}$

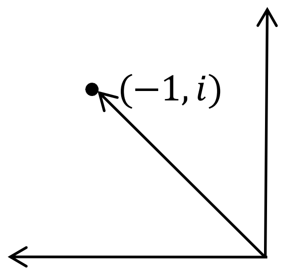
$= \left(\sqrt{2}\right)^{25} e^{i\,25.\frac{\pi}{4}}$

$= 12\sqrt{2}\left(cos\frac{\pi}{4} + i\,sin\frac{\pi}{4}\right)$

$= 12\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= 12(1 + i)$

Example: Find $(-1 + i)^4$



Solution: $r = \sqrt{2}$, $\theta = \pi - \frac{\pi}{4} = \frac{3}{4}\pi$

$z^n = r^n e^{in\theta} = \left(\sqrt{2}\right)^4 e^{i\,4.\frac{3\pi}{4}}$

$= 4e^{i3\pi}$

$= 4(\cos 3\pi + i\sin 3\pi)$

$= 4(-1 + 0) = -4$

[8] De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof: by mathematical induction

1. If $n = 1 \rightarrow (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$

2. Let it be true if $n = k$, we get

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta \ldots (*)$

3. We must proof it is true if $n = k + 1$

Multiplying $(*)$ by $(\cos\theta + i\sin\theta)$

$(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k = (\cos\theta + \mathrm{i}\sin\theta)(\cos k\theta + \mathrm{i}\sin k\theta)$

$= (\cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta - \sin\theta\sin k\theta)$

$(\cos\theta + i\sin\theta)^{k+1} = \cos(k + 1) + i\sin(k + 1)$

$\therefore$ It is true if $n = k + 1$

Note: If $z^n = z_0$ then $z = z_0^{\frac{1}{n}}$ and $z = re^{i\theta} = \sqrt[n]{r_0}\, e^{i\left(\frac{\theta_0 + 2k\pi}{n}\right)} = z^{1/n}$ is called nth – root of z.

Example: Calculate root of $z^3 = i$

Solution: $z^3 = i \rightarrow z = (i)^{1/3}$

$$\rightarrow re^{i\theta} = \left(1.e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right)^{1/3}$$

s.t $\theta_0 = \frac{\pi}{2} + 2k\pi$, $k = 0, \mp1, \mp2, \ldots$

$$\rightarrow re^{i\theta} = e^{i\frac{\pi}{6}+\frac{2}{3}k\pi}$$

$\therefore r = 1$, $\theta = \frac{\pi}{6} + 2k\pi$, $k = 0, \mp1, \mp2, \ldots$

To find the roots:

If $k = 0 \rightarrow \theta_1 = \frac{\pi}{6}$ (in the first quarter)

$\rightarrow z_1 = 1.e^{i\frac{\pi}{6}}$

If $k = 1 \rightarrow z_2 = 1.e^{i\frac{\pi}{6}+\frac{2\pi}{3}}$ (in the second quarter)

$$= \cos\frac{5}{6}\pi + i\sin\frac{5}{6}\pi$$

$$= \frac{-\sqrt{3}}{6} + \frac{i}{2}$$

If $k = 2 \rightarrow z_3 = 1.e^{i\frac{\pi}{6}+\frac{4\pi}{3}}$

$$= \cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}$$

$$= -i$$

Note:

1. If the complex number was raised to a fraction whether it was $\frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}$ then the number of roots is $3, 4, \ldots, n$. In the above example the number of roots is 3.

2. $z^n = z_0$ has n different roots only and they are located on the vertices of a regular polygon centered at the origin.

Example: $z^2 = 1 + i$ has two different roots

Solution:

$$z^2 = 1 + i \rightarrow z = (1+i)^{1/2}$$

$$r_0 = \sqrt{2}\,,\ \theta_0 = \frac{\pi}{4} + 2n\pi$$

Since $z = (1+i)^{1/2}$

$$\therefore re^{i\theta} = \left(\sqrt{2}\right)^{\frac{1}{2}} \left(e^{i\frac{\pi}{4}+2n\pi}\right)^{\frac{1}{2}}$$

$$= \sqrt[4]{2}\, e^{i\frac{\pi}{8}+n\pi}$$

$$r = \sqrt[4]{2}\,,\quad \theta = \frac{\pi}{8} + k\pi$$

If $k = 0 \rightarrow z_1 = \sqrt[4]{2}\, e^{i\frac{\pi}{8}}$

$$= \sqrt[4]{2}\left(\sqrt{\frac{1+\cos\frac{\pi}{8}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{8}}{2}}\right)$$

If $k = 1 \rightarrow z_2 = \sqrt[4]{2}\, e^{i\frac{\pi}{8}+\pi}$

$$= \sqrt[4]{2}\left(\cos\left(\frac{\pi}{8}+\pi\right) + i\sin\left(\frac{\pi}{8}+\pi\right)\right)$$

$$= \sqrt[4]{2}\left(-\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}\right)$$

$$= -\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$$

Note:

$$\cos\frac{\theta}{2} = \mp\sqrt{\frac{1+\cos\theta}{2}}$$

$$\sin\frac{\theta}{2} = \mp\sqrt{\frac{1-\cos\theta}{2}}$$

Note: Let $m, n \neq 0$ be any integer numbers, let z be any complex number then

$$(z)^{m/n} = \left(z^{\frac{1}{n}}\right)^{m} = \left(\sqrt[n]{r_0}\, e^{\left(\frac{i\theta_0+2k\pi}{n}\right)}\right)^{m}$$

$$= \left(\sqrt[n]{r_0}\right)^{m} e^{i\,\frac{m(\theta_0+2k\pi)}{n}},\ k = 0, \mp 1, \mp 2, \ldots$$

Example: Solve the following equation

$$z^{2/3} = i$$

Solution: $z^{3/2} = i \rightarrow z = (i)^{2/3} = \left(i^{1/3}\right)^{2}$

$$= (i)^{1/3}(i)^{1/3}$$

That is each one has three roots.

Let $w = (i)^{1/3} \rightarrow z = w^2$

Now, we find the roots of w

$r_0 = 1\,, \theta_0 = \frac{\pi}{2} + 2k\pi\,, k = 0, \mp 1, \mp 2, \ldots$

$$w = re^{i\theta} = 1.\left(e^{i\frac{\pi}{2}+2k\pi}\right)^{1/3}$$

$$= e^{i\frac{\pi}{6}+\frac{2k\pi}{3}}$$

$\therefore w_1 = e^{i\frac{\pi}{6}} = \cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right), k = 0$

$w_2 = e^{i\frac{\pi}{6}+\frac{2\pi}{3}} = e^{i\frac{5\pi}{6}}\,, k = 1$

$w_3 = e^{i\frac{\pi}{6}+\frac{4\pi}{3}} = e^{i\frac{3\pi}{2}}\,, k = 2$

$\therefore z = w^2$

$\therefore z_1 = (w_1)^2 = \left(e^{i\frac{\pi}{6}}\right)^{2} = e^{i\frac{\pi}{3}}$

$= cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}$

$= \frac{1}{2} + i\ \frac{\sqrt{3}}{2}$

$z_2 = (w_2)^2 = \left(e^{i\frac{5\pi}{6}}\right)^2 = e^{i\frac{5\pi}{3}}$

$$= \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$$

$z_3 = (w_3)^2 = \left(e^{i\frac{3\pi}{2}}\right)^2 = e^{i3\pi}$

$$= \cos 3\pi + i\sin 3\pi$$

H.w: Find the roots of $(-8i)^{1/3}$.

[9] Regions in the Complex Plane

Some definitions and concepts:

Definition: Let z be any point in the z-plane, let $\epsilon > 0$ then

1. $N_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| < \epsilon\}$

This set is called a neighborhood of z_0.

2. $S_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| = \epsilon\}$

This set is called sphere with center z_0.

3. $D_\epsilon(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq \epsilon\}$

This set is called the Disk with center z_0 and radius ϵ.

Definition: Let $U \subseteq \mathbb{C}$, we say that U is open set if

$$\forall\ w \in U, \exists\ N_\epsilon(w)\ \text{ s.t }\ N_\epsilon(w) \subseteq U.$$

For example: $\emptyset$, $\mathbb{C}$ are open sets.

Definition: Let $F \subseteq \mathbb{C}$, we say that F is closed set if $\mathbb{C} - F$ is open set.

Definition: An open set $S \subseteq \mathbb{C}$ is connected if each pair of points z_1, z_2 in it can be joined by a polygon line, consisting of a finite number of line segments joined end to end that lies entirely in S.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is Region if it is open and connected.

Example:

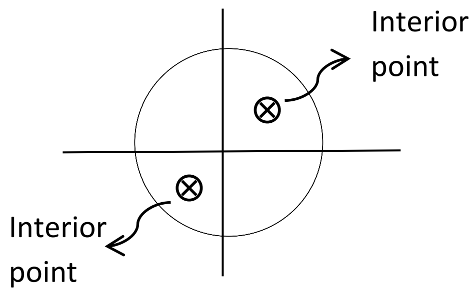
1. $|z| > 1, |z| < 1$ is Region.

2. Let $|z| = 0$ is not Region, since it is connected but not open set.

3. $\mathbb{R} \subset \mathbb{C}$ is connected but not open, since $\forall\, r \in \mathbb{R}, \exists\, N_\epsilon(r)$ contain some of complex points.

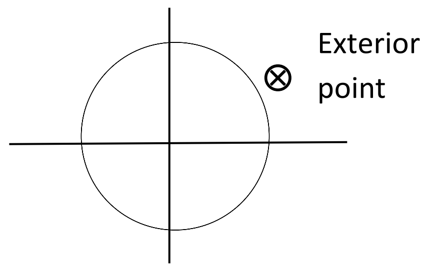
Definition: Let $z_0 \in S$, we say that z_0 is interior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \subseteq S$.

Example: $|z| < 1$

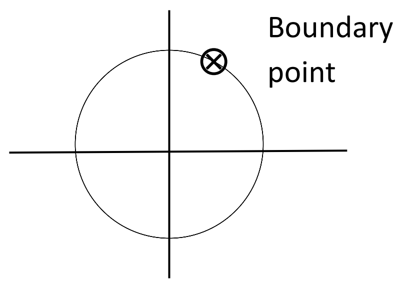


Definition: Let $z_0 \in S$, we say that z_0 is exterior point if there exist a neighborhood $N_\epsilon(z_0)$ s.t $N_\epsilon(z_0) \cap S = \emptyset$.

Example: $|z| > 1$



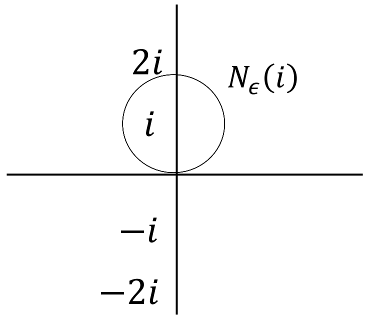
Definition: Let $z_0 \in S$, we say that z_0 is Boundary point if $\forall\, N_\epsilon(z_0)$ contain points from inside S and outside it.



Note: S is close set iff it contains all the boundary points.

Example: $S = \{ \mp i, \mp 2i\}$, is S open set ?

Note $N_\epsilon(i) \nsubseteq S$, therefore S is not open.



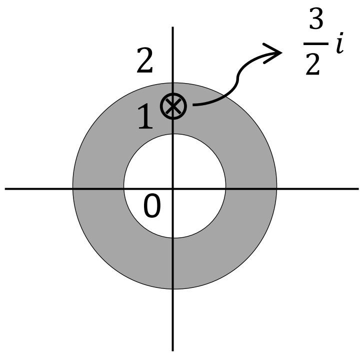
Example: $S = \{z \in \mathbb{C} : 1 < |z| < 2\}$

Note

0 is exterior point of S

1, 2 are boundary points of S

$\left(\frac{3}{2} i\right)$ is interior point of S

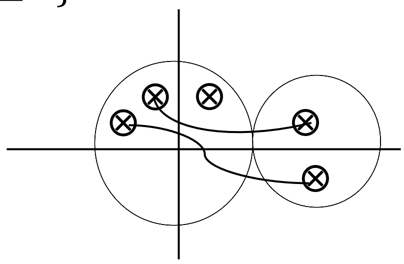


Example: $D = \{z \in \mathbb{C} : 2 < |z| \leq 3\}$

D is not open set since it contain all the boundary points.

Example: $S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| \leq 1\}$

Note S is connected set.



But if

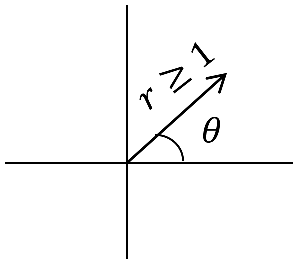
$S = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 2| < 1\}$,

then S is not a connected set.

Definition: Let $S \subseteq \mathbb{C}$, we say that S is bounded set if $\exists$ Disk D,

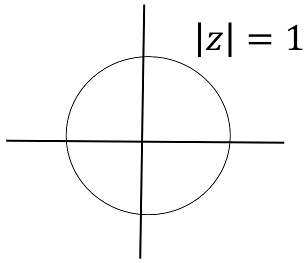
$D = \{z : |z| \leq \mathbb{R}\}$ such that $S \subseteq D$.

Example: $S = \left\{z \in \mathbb{C} : r \geq 1,\ 0 \leq \theta \leq \frac{\pi}{4}\right\}$



S is not bounded set since $\nexists$ Disk contain S.

Example: $|z| = 1$ is bounded set



Example: $S = \{ \mp i, \mp 2i\}$

1. S is not open set since every point of S is boundary point.

2. S is close set since every point of S is boundary point.

3. S is not connected set.

4. S is not bounded set.

Definition: Let $z_0 \in S$, we say that z_0 is limit point if

$$N_\epsilon(z_0) \cap (S - z_0) \neq \emptyset$$

Example: $S = \left\{z \in \mathbb{C} : z = \frac{1}{n},\ n = 1, 2, \dots\right\}$, 0 is the only limit point.

Chapter Two

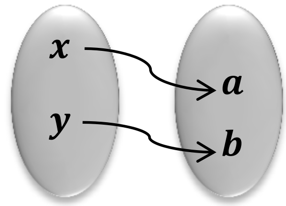
Analytic Functions

[1] Functions of a Complex Variable

Definition:

A function f defined on a set A to a set B is a rule assigns a unique element of B to each element of A; in this case we call f a single function. i.e.: $f: A \to B$, $A, B \subseteq \mathbb{C}$

$$\forall z \in A, \exists!\, w \in B \text{ s.t } w = f(z) \in B$$



Definition:

The domain of f in the above def. is A and the range is the set R of elements of B which f associate with elements of A.

Note: The elements in the domain of f are called independent variables and those in the range of f are called dependent variables.

Definition:

A f rule which assigns more than one number of B to any number of A is called a multiple valued function.

Example:

1. $f(z) = (z)^{1/2}$

Has two roots therefore $f(z)$ is a multiple function.

2. $f(z) = (z)^{3/5} = (z^3)^{1/5}$

Has five roots therefore $f(z)$ is a multiple function. In general, if $f(z) = \arg z$ then f is a multiple function.

3. If $f(z) = \operatorname{Arg} z$ then f is a single function.

Note:

1. Let $f: Z \to W$, if Z and W are complex, then f is called complex variables function (a complex function) or a complex valued function of a complex variable.

2. If A is a set of complex numbers and B is a set of real numbers then f is called real–valued function of a complex variable, conversely f is a complex–valued function of real variables.

Example: Find the domain of the following functions

1. $f(z) = \frac{1}{z}$

Ans.: $D_f = \mathbb{C} \setminus \{0\}$

2. $f(z) = \frac{1}{z^2+1}$

Ans.: $D_f = \mathbb{C} \setminus \{-i, i\}$

3. $f(z) = \frac{z+\bar{z}}{2}$

Ans.: $D_f = \mathbb{C}$, f is real–valued.

4. $f(z) = y \underbrace{\int_0^\infty e^{-xt}\, dt}_{\text{Improper integral}} + i \underbrace{\sum_{n=0}^{\infty} y^n}_{\text{Geometric series}}$

Ans.: $D_f = x \in (0, \infty)$ and $y \in (-1,1)$

(What are the conditions that must be satisfied for x so the integration will be converging?)

Definition: A complex function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

n is a positive integer and $a_0, a_1 \dots a_n \in \mathbb{C}$, is a polynomial of degree n ($a_n \neq 0$).

Definition: A function $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are two polynomials, is called a rational function.

Note: $D_f = \mathbb{C} \setminus \{z : Q(z) \neq 0\}$

♦ Suppose that:

$w = u + iv$ is the value of a function f at $z = x + iy$

$$\text{i.e.}: f(z) = f(x + iy) = u + i\,v$$

each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real–valued functions of real variables x and y.

$$f(z) = u(x, y) + i\,v(x, y)$$

If the polar coordinates r and θ are used instead of x and y, then

$$u + i\,v = f(re^{i\theta})$$

Where $w = u + iv$ and $z = re^{i\theta}$, in that case, we may write

$$f(z) = u(r, \theta) + i\,v(r, \theta)$$

Example: If $f(z) = z^2$, then

$$f(x + iy) = (x + iy)^2 = x^2 - y^2 + i\,2xy$$

Hence: $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$, when polar coordinates are used

$$f(re^{i\theta}) = (re^{i\theta})^2$$

$$= r^2 e^{i2\theta}$$

$$= r^2 \cos 2\theta + i\,r^2 \sin 2\theta$$

Therefore: $u(r, \theta) = r^2 \cos 2\theta$

$v(r, \theta) = r^2 \sin 2\theta$

Note: If $v(x, y) = 0$ then f is real, i.e. f is real–valued function.

[1] Limits

Let f be a function at all points z in some deleted neighborhood of z_0, the statement that the limit of $f(z)$ as z approaches z_0 is a number w_0, or that

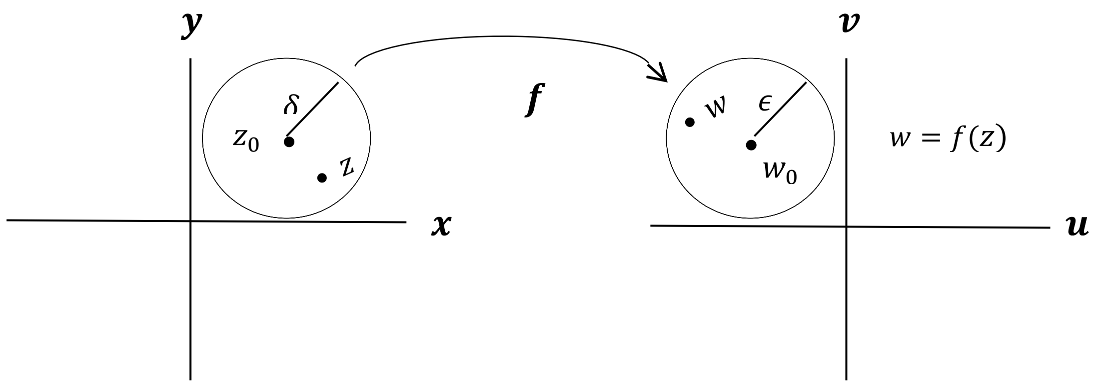
$$\lim_{z \to z_0} f(z) = w_0$$

Means that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

And this means: $z \to z_0$ in $z -$ plane

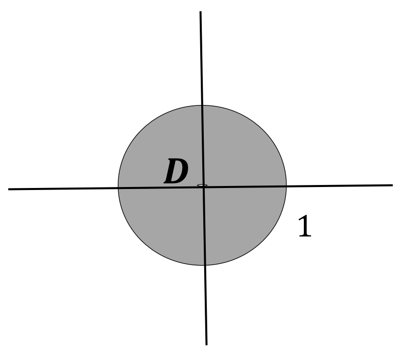
$w \to w_0$ in $w -$ plane



Example: Prove that

$$\lim_{z \to 1} \frac{iz}{2} = \frac{i}{2}$$

Such that f is defined on $|z| < 1$.



Proof: $f(z) = \dfrac{iz}{2}$

Let $\epsilon > 0$, T.p. $\exists\, \delta > 0$ such that

$$|z - 1| < \delta \to \left|f(z) - \frac{i}{2}\right| < \epsilon$$

To find δ

$\left|f(z)-\frac{i}{2}\right| = \left|\frac{iz}{2}-\frac{i}{2}\right| = \left|\frac{1}{2}i(z-1)\right|$

Let $\delta = 2\epsilon$ then:

$\left|f(z)-\frac{i}{2}\right| = |i|\left|\frac{z-1}{2}\right| < \frac{\delta}{2} < \epsilon$

Note: $|i| = 1$

Example: If $f(z) = z^2,\ |z| < 1,$ prove that

$$\lim_{z\to 1} z^2 = 1$$

Proof: Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$ s.t

$$|z^2-1| < \epsilon \ \text{ whenever } \ 0 < |z-1| < \delta$$

$|z^2-1| = |z+1||z-1| \le (|z|+1)|z-1|$

$$< 2|z-1| < \epsilon$$

$$= |z-1| < \frac{\epsilon}{2}$$

$\therefore$ chose $\delta = \frac{\epsilon}{2}$

$\therefore \lim_{z\to 1} z^2 = 1$

Example: Prove that

$$\lim_{z\to 1+2i}[(2x+y)+i(y-x)] = 4+i$$

Proof: $f(z) = (2x+y)+i(y-x)$

$z_0 = 1+2i\,, \qquad z = x+iy$

$L = 4+i$

Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$ s.t $0 < |z-z_0| < \delta \ \rightarrow |f(z)-L| < \epsilon$

$|z-z_0| = |x+iy-1-2i|$

$= |(x-1)+i(y-2)| < \delta$

$\rightarrow |x-1| \leq |(x-1)+i(y-2)|$

$$
\begin{aligned}
|f(z)-L| &= |2x+y+i(y-x)-4-i| \\
&\leq |2x+y-4+i(y-x-1)| \\
&\leq |2x-2+y-2|+|i(y-x-1)| \\
&= |2x-2+y-2|+|y-2-x+1| \\
&\leq 2|x-1|+|y-2|+|y-2|+|x-1| \\
&= 3|x-1|+2|y-2|
\end{aligned}
$$

Let $\delta = \min\left(\frac{\epsilon}{6}, \frac{\epsilon}{4}\right) = \frac{\epsilon}{6}$

Such that $|x-1| < \delta < \frac{\epsilon}{6}$

$|y-2| < \delta < \frac{\epsilon}{4}$

$\rightarrow |f(z)-L| \leq \frac{3\epsilon}{6} + \frac{2\epsilon}{4} < \epsilon$

Exercise: Prove that

$$\lim_{z\to z_0} z^2 = z_0^2$$

Properties of Limit:

1. If $f(z)=c$ then $\lim_{z\to z_0} f(z) = c$.
2. If $f(z)=z$ then $\lim_{z\to z_0} f(z) = z_0$.
3. $\lim_{z\to z_0}\big(f(z) \mp g(z)\big) = \lim_{z\to z_0} f(z) \mp \lim_{z\to z_0} g(z)$.
4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$
5. $\lim_{z\to z_0} f(z).g(z) = \lim_{z\to z_0} f(z).\ \lim_{z\to z_0} g(z)$

Proof:

1- Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$ s.t $|f(z) - c| < \epsilon$ whenever $|z - z_0| < \delta$

$\rightarrow |f(z) - c| = |c - c| = 0$

Let δ be any real number

$\therefore \lim_{z \to z_0} f(z) = c$

2- Let $\epsilon > 0$, T.p. $\exists\ \delta > 0$, $|f(z) - z_0| < \epsilon$ if $|z - z_0| < \delta$

$\rightarrow |f(z) - z_0| = |z - z_0| < \epsilon$

Chose $\epsilon = \delta$

$\therefore \lim_{z \to z_0} f(z) = z_0$

Example: Find limit $f(z)$ if its exist, such that

$$f(z) = \frac{2xy}{x^2+y^2} + \frac{x^2}{1+y}\ i$$

Proof: Assume that limit $f(z)$ exists.

Let $y = 0$, we get

$$\lim_{z \to z_0 = 0} f(z) = \lim_{(x,y) \to (0,0)} f(z) = \lim_{x \to 0} x^2 i = 0$$

Let $x = 0$, we get $\lim f(z) = 0$

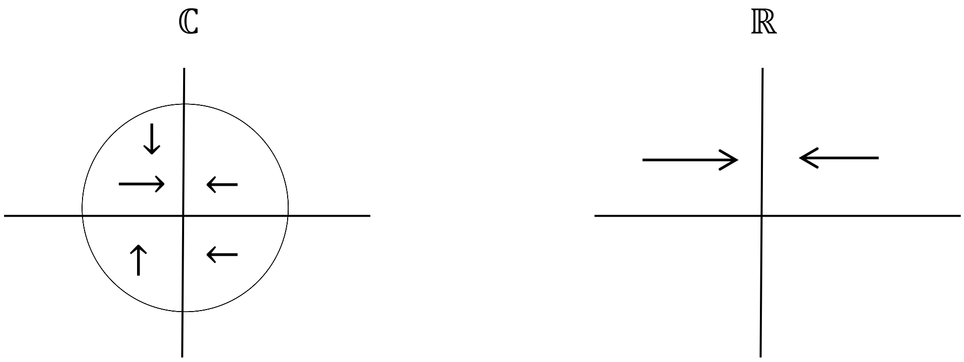
Let $y = x$, then

$$\lim_{z \to 0} f(z) = \lim_{(x,x) \to (0,0)} f(z) = \lim_{(x,x) \to (0,0)} \left(\frac{2x^2}{2x^2} + \frac{x^2}{1+x} i \right)$$

$$\lim_{(x,x) \to (0,0)} \left(1 + \frac{x^2}{1+x} i \right) = 1 + \lim_{(x,x) \to (0,0)} \frac{x^2}{1+x} i = 1 + 0 = 1$$

This is impossible; therefor this limit is not exist.

Note: The limit in the real numbers is studying the approaches from the right and left, but in the complex numbers is studying from every side of the circle.



Theorem: If $\lim_{z\to z_0} f(z) = w_1$, then $\lim_{z\to z_0} f(z) = w_2$

Then $w_1 = w_2$. (The limit is unique)

Proof: Let $\epsilon > 0$

Since

$\lim\limits_{z\to z_0} f(z) = w_1 \to \exists\ \delta_1 > 0, \text{if } |z - z_0| < \delta_1$

$\to |f(z) - w_1| < \frac{\epsilon}{2}$

Since

$\lim\limits_{z\to z_0} f(z) = w_2 \to \exists\ \delta_2 > 0, \text{if } |z - z_0| < \delta_2$

$\to |f(z) - w_2| < \frac{\epsilon}{2}$

$$|w_1 - w_2| = |w_1 - f(z) + f(z) - w_2|$$

$$\leq |w_1 - f(z)| + |f(z) - w_2|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Chose $\delta = \min(\delta_1, \delta_2)$

$\therefore |w_1 - w_2| < \epsilon$

$\to w_1 = w_2$

Theorem: Let $f(z) = u(x,y) + iv(x,y)$ such that $z = x + iy$,

$z_0 = x_0 + y_0$, $w_0 = u_0 + iv_0$, Then

$$\lim_{z\to z_0} f(z) = w_0 \text{ iff } \lim_{z\to z_0} u(x,y) = u_0, \lim_{z\to z_0} v(x,y) = v_0$$

Note: $\mathbb{C}$ is a complete space, since f is converge iff u, v are converge, but u, v are converge and u, v are real functions. Therefore it is Cauchy

$$\therefore f \text{ is converge} \to f \text{ is Cauchy}$$

$$\therefore \mathbb{C} \text{ is complete}$$

Note: $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ s.t $a_i \in \mathbb{C}, i = 0,1,\dots,n$

Then

$$\lim_{z\to z_0} p(z) = p(z_0)$$

Example: Find limit of $f(z)$ if it's exist

1. $\lim_{z\to 3-4i} \frac{4x^2y^2-1+i(x^2-y^2)-ix}{\sqrt{x^2+y^2}}$

Solution:

$\lim_{z\to 3-4i} \frac{(4x^2y^2-1)+i(x^2-y^2-x)}{\sqrt{x^2+y^2}} =$

$$= \lim_{z\to 3-4i} \frac{4x^2y^2-1}{\sqrt{x^2+y^2}} + i \lim_{z\to 3-4i} \frac{x^2-y^2-x}{\sqrt{x^2+y^2}}$$

$$= 115 - 2i$$

2. $\lim_{z\to i} \frac{z-i}{z^2+1}$

Solution:

$$\lim_{z\to i} \frac{z-i}{z^2+1} = \lim_{z\to i} \frac{z-i}{z^2-(-1)} = \lim_{z\to i} \frac{z-i}{z^2-i^2} = \lim_{z\to i} \frac{z-i}{(z-i)(z+i)}$$

$$= \lim_{z\to i} \frac{1}{(z+i)} = \frac{1}{2i}$$

3. $\lim_{z\to(-1,i)} \frac{z^2+(3-i)\,z+2-2i}{z+1-i}$

Solution:

Note: $z^2 + (3-i)\,z + 2 - 2i = (z+1-i)(z+2)$

$$\therefore \lim_{z\to(-1,i)} \frac{z^2+(3-i)\,z+2-2i}{z+1-i} = \lim_{z\to(-1,i)} \frac{(z+1-i)(z+2)}{(z+1-i)}$$

$$= \lim_{z\to(-1,i)} (z+2)$$

$$= -1 + i + 2$$

$$= 1 + i$$

[3] Continuity

Definition:

A function f is continuous at a point z_0 if all of the three following conditions are satisfied:

1. $\lim_{z\to z_0} f(z)$ exists,

2. $f(z_0)$ exists,

3. $\lim_{z\to z_0} f(z) = f(z_0)$

A function of a complex variable is said to be continuous in a region R if it is continuous at each point R.

Theorem: If f, g are continuous functions at z_0 then

1. $f + g$ is continuous.

2. $f.g$ is continuous.

3. $\frac{f}{g}$, $g(z_0) \neq 0$ is continuous.

4. fog is continuous at z_0 if f is continuous at $g(z_0)$.

Example: $f(z) = z^2$ is continuous in complex plane since $\forall\, z_0 \in \mathbb{C}$

1. $f(z_0) = z_0^2$

2. $\lim_{z\to z_0} f(z) = z_0^2$

3. $\lim_{z\to z_0} f(z) = f(z_0)$

Example: Is $f(z) = \frac{z^2-1}{z-1}$ continuous at $z = 1$

Solution: f is not continuous since $f(1)$ not exist

$f(z_0) = \frac{z_0^2-1}{z_0-1} = \frac{(z_0-1)(z_0+1)}{z_0-1} = z_0 + 1$

$\therefore \lim_{z\to 1} f(z) = 2$

But $f(1) = \frac{0}{0}$

$\therefore \lim_{z\to 1} f(z) \neq f(1)$

Theorem: $f(z) = u(x,y) + iv(x,y)$ is continuous at z_0 iff $u(x,y)$ and $v(x,y)$ are continuous at (x_0, y_0).

Proof: Let f be continuous at z_0 , then

$$\lim_{z\to z_0} f(z) = f(z_0)$$

That means:

$\lim_{z\to z_0}(u(x,y) + iv(x,y)) = u(x_0,y_0) + i\,v(x_0,y_0)$

$\to \lim_{z\to z_0} u(x,y) + i\lim_{z\to z_0} v(x,y) = u(x_0,y_0) + i\,v(x_0,y_0)$

$\therefore \lim_{z\to z_0} u(x,y) = u(x_0,y_0)$

$\lim_{z\to z_0} v(x,y) = v(x_0,y_0)$

$\therefore u, v$ are continuous at z_0.

Example: Is $f(x+iy) = x^2 + y^2 + ixy$ continuous at $(1,1)$

Solution: $u(x,y) = x^2 + y^2,\ v(x,y) = xy$

By the above theorem

$$u(1,1) = 2\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} u(x,y) = 2 = u(1,1)$$

$$v(1,1) = 1\,, \qquad \lim_{\substack{x\to 1\\ y\to 1}} v(x,y) = 1 = v(1,1)$$

$\therefore u, v$ are continuous at (1,1)

$\therefore f(z)$ is continuous at (1,1).

Example: Find the limit if it's exists

$$\lim_{z\to 0} \frac{\bar{z}}{z}$$

Solution:

$$\lim_{z\to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} \frac{x-iy}{x+iy}$$

1. If $y = 0 \to \lim_{x\to 0} \frac{x}{x} = 1$

2. If $x = 0 \to \lim_{y\to 0} \frac{-iy}{iy} = -1$

$\therefore$ The limit is not exist.

Example: Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z-i}{z^2-1} & if \ \ z \neq i, -i \\ 2i & if \ \ z = \mp i \end{cases}$$

Solution: Note f is not continuous at $z = \mp i$.

(Since $f(\mp i)$ is undefined)

$$f(z) = 2i \ \text{ and } \lim_{z\to -i} f(z) = \lim_{z\to -i} \frac{z-i}{(z-i)(z+i)} = \lim_{z\to -i} \frac{1}{(z+i)} = \frac{1}{2i}$$

But f is not defined at $z = -i$, therefore f is not continuous at $z = i$, that is f is continuous at $\{z \in \mathbb{C}\backslash\{-i, i\}\}$

Example: Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z^2+4}{z+2i} & if \;\; z \neq -2i \\ -4i & if \;\; z = \mp i \end{cases}$$

Solution: f is continuous at $\forall\, z \neq -2i$.

When $z = -2i$

$$\lim_{z\to -2i} f(z) = f(-2i) = -4i$$

$$\lim_{z\to -2i} f(z) = \lim_{z\to -2i} \frac{(z-2i)(z+2i)}{(z+2i)} = -4i$$

But f is not defined at $z = -2i$

$\therefore$ f is not continuous at $z = -2i$.

Then is f is continuous at $\{z \in \mathbb{C} : z \neq -2i\}$

Exercise: Discuss the continuity of

$$f(z) = \begin{cases} \dfrac{z+2i}{z^2+4} & if \;\; z \neq \mp 2i \\ \dfrac{1}{4}i & if \;\; z = -2i \end{cases}$$

[4] Derivative

Let f be a function whose domain of definition contains a neighborhood $|z - z_0| < \epsilon$ of a point z_0. The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is said to be differentiable at z_0 when $f'(z_0)$ exists. If $\Delta z = z - z_0$, then $\Delta z \to 0$ when $z \to z_0$. Thus

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta\, z}$$

Theorem: If f is differentiable at z_0, then f is continuous at z_0.

Proof: To prove f is continuous, we must prove that

$$\lim_{z \to z_0} f(z) = f(z_0)$$

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \left[\frac{f(z) - f(z_0)}{z - z_0} (z - z_0) \right]$$

$$= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \,.\, \lim_{z \to z_0} (z - z_0)$$

$$= f'(z_0)\,.\,0$$

$$= 0$$

$$\therefore \lim_{z \to z_0} f(z) = f(z_0)$$

Differentiation Formulas:

In the following formulas, the derivative of a function f at a point z_0 is denoted by either $\frac{d}{dz}f(z)$ or $f'(z_0)$.

1. $\frac{d}{dz}\ c = 0,\ \ c$ is constant

2. $\frac{d}{dz}\ z = 1$

3. $\frac{d}{dz}\ \big(c\, f(z)\big) = c\, f^{'}(z)$

4. $\frac{d}{dz}\ [f + g] = \frac{d}{dz}\ f + \frac{d}{dz}\ g = f^{'} + g^{'}$

5. $\frac{d}{dz}\ [f.g] = f.g^{'} + g.f^{'}$

6. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g.f'-f.g'}{g^2}\ ,\ \ g \neq 0$

7. $\frac{d}{dz}\ (z^n) = n\, z^{n-1}$

8. $(gof)'(z_0) = g'\big(f(z_0)\big)\, .f'(z_0)$

Note: If $w = f(z)$ and $W = g(w)$, then

$$\frac{dW}{dz} = \frac{dW}{dw}\cdot\frac{dw}{dz} \qquad \text{(The Chain rule)}$$

Example: Find the derivative of $f(z) = (2z^2 + i)^5$

Solution: write $w = 2z^2 + i$ and $W = w^5$

Then:

$$\frac{d}{dz}\ (2z^2 + i)^5 = 5w^4.4z = 20\, z(2z^2 + i)^4$$

Examples: Find $f'(z)$ by using the definition of derivative:

1. $f(z) = z^2$

Solution:

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z^2 + 2z\,\Delta z + (\Delta z)^2 - z^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta z(2z + \Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} (2z + \Delta z)$$

$$= 2z$$

1. $f(z) = \bar{z}$

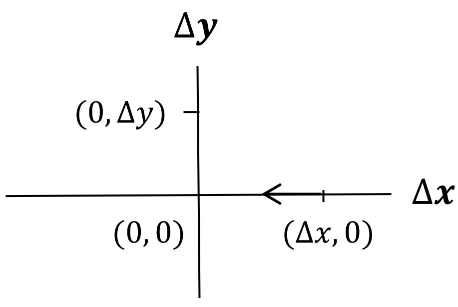
Solution:

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

Let $\Delta z = (\Delta x, \Delta y)$ approach the origin $(0,0)$ in the Δz −plane. In particular, as $\Delta z \to 0$ horizontally through the point $(\Delta x, 0)$ on the real axis, then



$$\overline{\Delta z} = \overline{\Delta x + \imath\, 0} = \Delta x - i\, 0$$

$$= \Delta x + i\, 0$$

$$= \Delta z$$

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta z}{\Delta z} = 1$$

When Δz approaches $(0, 0)$ vertically through the point $(0, \Delta y)$ on the imaginary axis, then

$$\overline{\Delta z} = \overline{0 + \imath\, \Delta y} = 0 - i\, \Delta y$$

$$= -(0 + i\, \Delta y)$$

$$= -\Delta z$$

$$\therefore \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta\, z} = \lim_{\Delta z \to 0} \frac{-\Delta z}{\Delta\, z} = -1$$

But the limit is unique, and then $\frac{dw}{dz}$ is not exist.

[5] Cauchy – Riemann Equations (C-R-E)

<u>Theorem:</u> Suppose that $f(z) = u(x, y) + iv(x, y)$ and $f\,'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y\,, \qquad u_y = -v_x$$

There is also

$$f'(z_0) = u_x + iv_x$$

Where these partial derivatives are to be evaluated at (x_0, y_0).

<u>Proof:</u>

Let f be differentiable at z_0 then

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta\, z}\,, \qquad \Delta z = \Delta x + i\Delta y$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y)+iv(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0)-iv(x_0, y_0)}{\Delta x+i\Delta y}$$

$$= \lim_{\Delta z \to 0} \frac{u(x_0+\Delta x, y_0+\Delta y)-u(x_0, y_0)}{\Delta x+i\Delta y} + i \lim_{\Delta z \to 0} \frac{v(x_0+\Delta x, y_0+\Delta y)-v(x_0, y_0)}{\Delta x+i\Delta y}$$

Let $y = 0 \Rightarrow \Delta y = 0 \Rightarrow \Delta z = \Delta x \to 0$

$$= \lim_{\Delta x \to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x}$$

$$= u_x(x_0, y_0) + i v_x(x_0, y_0) \quad \dots (1)$$

Let $x = 0 \Longrightarrow \Delta x = 0 \Longrightarrow \Delta z = i\Delta y \to 0$

$$= \lim_{i\Delta y \to 0} \frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{i\Delta y} + i \lim_{i\Delta y \to 0} \frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{i\Delta y}$$

$$= \frac{1}{i} u_y(x_0, y_0) + v_y(x_0, y_0)$$

$$= v_y(x_0, y_0) - i u_y(x_0, y_0) \quad \dots (2)$$

From (1) and (2) we get

$$u_x = v_y\,, \qquad u_y = -v_x$$

Note:

1. $f'(z) = u_x + i v_x$ or $f'(z) = u_y - i v_y$.
2. If $f'(z)$ exists then C-R-Eq. are satisfied, but the converse is not true.

The converse of the above theorem is not necessary true:

Example: Let

$$f(z) = \begin{cases} 0 & if \;\; z = 0 \\ \frac{(\bar{z})^2}{z} & if \;\; z \neq 0 \end{cases}$$

Solution: The C-R-Eq. are satisfied

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\frac{(\bar{z})^2}{z} - 0}{z - 0}$$

$$= \lim_{z \to 0} \left(\frac{\bar{z}}{z}\right)^2$$

$$= \lim_{z \to 0} \frac{(x-iy)^2}{(x+iy)^2}$$

Let $y = 0 \to f'(0) = 1$

Let $x = 0 \rightarrow f'(0) = 1$

Let $y = x \rightarrow f'(0) = \frac{y^2(1-i)^2}{y^2(1+i)^2} = \frac{1-2i-1}{1+2i-1}$

$$= \frac{-2i}{2i}$$

$$= -1$$

$\therefore f'(z)$ is not exist at $z = 0$.

Example: $f(z) = z^2 = x^2 - y^2 + 2\,ixy$

Solution:

$u(x,y) = x^2 - y^2 \rightarrow u_x = 2\text{x}$

$v(x,y) = 2xy \quad \rightarrow v_y = 2x$

$\rightarrow u_x = v_y$

$u_y = -2\text{y}\,,\; v_x = 2y$

$\rightarrow u_y = -v_x$

$\therefore f'(\text{z}) = u_x + iv_x = 2x + i2y = 2(x+iy) = 2z$

Example: $f(z) = \bar{z} = x - iy$

Solution: $u(x,y) = x \rightarrow u_x = 1$

$v(x,y) = -y \rightarrow v_y = -1$

$\therefore u_x \neq v_y \rightarrow f$ is not differentiable at z.

Note: The following theorem gives a necessary and sufficient condition to satisfy the converse of the previous theorem.

Theorem: Let $f(z) = u(x,y) + iv(x,y)$, and

1. $u,\; v,\; u_x,\; v_x,\; u_y,\; v_y$ are continuous at $N_\epsilon(z_0)$

2. $u_x = v_y$, $u_y = -v_x$

Then f is differentiable at z_0 and

$$f'(z_0) = u_x + iv_x$$

$$f'(z_0) = v_y - iu_y$$

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\, e^{-y}\sin x$$

Is differentiable z for all and find its derivative.

Solution:

Let $u(x,y) = e^{-y}\cos x$

$\rightarrow u_x = -e^{-y}\sin x$

$u_y = -e^{-y}\cos x$

$v(x,y) = e^{-y}\sin x$

$\rightarrow v_x = e^{-y}\cos x$

$v_y = -e^{-y}\sin x$

1. $u_x = v_y$ and $u_y = -v_x$

2. $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous

Then $f'(z)$ exist. To find $f'(z) = u_x + iv_x$

$$f'(z) = u_x + iv_x = -e^{-y}\sin x + ie^{-y}\cos x$$

$$= e^{-y}(i\cos x - \sin x)$$

$$= ie^{-y}(\cos x + i\sin x)$$

$$= ie^{-y}e^{ix}$$

$$= ie^{i\,x-y}$$

$$= ie^{i(x+iy)}$$

$$= ie^{iz}$$

[6] Polar Coordinates of Cauchy – Riemann Equations

Let $f(z) = u(r,\theta) + iv(r,\theta)$, then Cauchy-Riemann equations are:

$$u_r = \frac{1}{r} v_\theta \quad , \qquad u_\theta = -r\, v_r$$

And $f'(z_0) = e^{-i\theta}(u_r + i\, v_r)$.

Example: Use C-R equations to show that the functions

1. $f(z) = |z|^2$

2. $f(z) = z - \bar{z}$

are not differentiable at any nonzero point.

Solution:

1. $|z|^2 = x^2 + y^2$

$u(x,y) = x^2 + y^2 \;,\;\; v(x,y) = 0$

$u_x = 2\text{x} \qquad , \qquad v_x = 0$

$u_y = 2y \qquad , \qquad v_y = 2x$

C-R equations are not satisfied, therefore f' is not exist.

2. $z - \bar{z} = (x + iy) - (x - iy)$

$= x + iy - x + iy$

$= 2y\, i$

$u(x,y) = 0 \qquad , \quad v(x,y) = 2y$

$u_x = 0 \qquad , \qquad v_x = 0$

$u_y = 0 \qquad , \qquad v_y = 2$

C-R equations are not satisfied, hence f' is not exist.

Example: Use C-R equations to show that $f'(z)$ and $f''(z)$ are exist everywhere

1. $f(z) = z^3$

Solution:

$f(z) = z^3 = (x + iy)^3$

$= x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3i\,x^2y - 3xy^2 - iy^3$

$= x^3 - 3xy^2 + i\,(3x^2y - y^3)$

$u(x, y) = x^3 - 3xy^2 \to u_x = 3x^2 - 3y^2$

$u_y = -6xy$

$v(x, y) = 3x^2y - y^3 \to v_x = 6xy$

$v_y = 3x^2 - 3y^2$

$\therefore u_x = v_y \ , \qquad u_y = -v_x$

$\therefore$ C-R equations are satisfied

$f'(z) = u_x + iv_x$

$= 3x^2 - 3y^2 + i\,6xy$

$= 3(x^2 + i^2y^2 + 2i\,xy) = 3(x + iy)^2 = 3z^2$

$f''(z) = u'_x + iv'_x$

$= 6x + i\,6y$

$= 6(x + iy)$

$= 6z$

2. $f(z) = \cos x \cosh y - i \sin x \sinh y$

Solution:

$u(x,y) = \cos x \cosh y \ \ \rightarrow u_x = -\sin x \cosh y$

$u_y = \cos x \sinh y$

$v(x,y) = -\sin x \sinh y \rightarrow v_x = -\cos x \sinh y$

$v_y = -\sin x \cosh y$

$\therefore u_x = v_y \ , \qquad u_y = -v_x$

$\therefore$ C-R equations are satisfied

$f'(z) = u_x + iv_x$

$= -\sin x \cosh y - i \cos x \sinh y$

$f''(z) = u_x' + iv_x'$

$= -\cos x \cosh y + i \sin x \sinh y$

Example: Let $f(z) = z^3$, write f in polar form and then find $f'(\text{z})$

Solution: $f(z) = z^3 = \left(re^{i\theta}\right)^3 = r^3 e^{3i\theta}$

$= r^3 \cos 3\theta + i\, r^3 \sin 3\theta$

$u(r,\theta) = r^3 \cos 3\theta \rightarrow u_r = 3r^2 \cos 3\theta$

$u_\theta = -3r^3 \sin 3\theta$

$v(r,\theta) = r^3 \sin 3\theta \ \rightarrow v_r = 3r^2 \sin 3\theta$

$v_\theta = 3r^3 \cos 3\theta$

Now, $u_r = \frac{1}{r}\, v_\theta \ , \ \ u_\theta = -rv_r$

$f'(z) = e^{-i\theta}[u_r + i\, v_r]$

$= e^{-i\theta}[3r^2 \cos 3\theta + i3r^2 \sin 3\theta]$

$= 3r^2 e^{-i\theta}[\cos 3\theta + i \sin 3\theta]$

$= 3r^2 e^{-i\theta} e^{3\theta i}$

Example: Let $f(z) = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta\,,\ z \neq 0,\ f'(\text{z}).$

Solution:

$$u(r,\theta) = \left(r + \frac{1}{r}\right)\cos\theta$$

$$v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$$

$$\rightarrow u_r = \left(1 - \frac{1}{r^2}\right)\cos\theta\ ,\ u_\theta = -\left(r + \frac{1}{r}\right)\sin\theta$$

$$\rightarrow v_r = \left(1 + \frac{1}{r^2}\right)\sin\theta\ ,\ \ v_\theta = \left(r - \frac{1}{r}\right)\cos\theta$$

Since $u,\ v,\ u_x,\ v_x,\ u_y,\ v_y$ are continuous and C-R equations holds then

$$f'(z) = e^{-i\theta}[u_r + i\,v_r]$$

$$= e^{-i\theta}\left[\left(1 - \frac{1}{r^2}\right)\cos\theta + i\ \left(1 + \frac{1}{r^2}\right)\sin\theta\right]$$

[7] Analytic Functions

Definition:

A function f is said to be analytic at z_0 if $f'(z_0)$ exists and $f'(z)$ exists at each point z in the same neighborhood of z_0.

Note: f is analytic in a region R if it is analytic at every point in R.

Definition:

If f is analytic at each point in the entire plane, then we say that f is an entire function.

Example: $f(z) = z^2$, is an entire function since it is a polynomial.

Definition:

If f is analytic at every point in the same neighborhood of z_0 but f is not analytic at z_0, then z_0 is called singular point.

Example: Let $f(z) = \frac{1}{z}$, then $f'(z) = \frac{-1}{z^2}$ $(z \neq 0)$

Then f is not analytic at $z_0 = 0$, which is a singular point.

Note: If f is analytic in D, then f is continuous through D and C-R equations are satisfied.

Note: A sufficient conditions that f be analytic in $\mathbb{R}$ are that C-R equations are satisfied and $u_x,\ v_x,\ u_y,\ v_y$ are continuous in $\mathbb{R}$.

[8] Harmonic Functions

Definition:

A function h of two variables x and y is said to be harmonic in D if the first partial derivatives are continuous in D and

$$h_{xx} + h_{yy} = 0 \quad \text{(Laplace equation)}$$

Example: Show that $u(x,y) = 2x(1-y)$ is harmonic in some domain D.

Solution:

$u_x = 2(1-y) \rightarrow u_{xx} = 0$

$u_y = -2x \qquad \rightarrow u_{yy} = 0$

$\therefore u_{xx} + u_{yy} = 0$

Since u, u_x, u_y are continuous and satisfied Laplace equation then the function is harmonic.

Definition:

Let $w = u + iv$, we say that w is harmonic function if u, v are also harmonic functions and we say v is a harmonic conjugate of u and u is a harmonic conjugate of v.

Theorem: If a function $f(z) = u(x, y) + i\,v(x, y)$ is analytic in a domain D then its component functions u and v are harmonic in D.

Proof:

Since f is analytic then it satisfies C-R equations

i.e.: $u_x = v_y$, $u_y = -v_x$

$\rightarrow$ $u_{xx} = v_{yx}$, $u_{yy} = -v_{xy}$

$\therefore u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$

$\rightarrow u$ is harmonic function. By the same way we prove that v is harmonic function.

Note: The converse of the above theorem is not true, which means that if u and v are harmonic functions then f is not necessary analytic function.

Example: $u(x, y) = 2xy$, $v(x, y) = x^2 - y^2$

Solution: u , v are harmonic functions, but

$$f(z) = u + iv = 2xy + i(x^2 - y^2)$$

is not analytic function since it doesn't satisfy C-R equations

$u_x = 2y$, $v_x = 2x$

$u_y = 2x$, $v_y = -2y$

$\rightarrow u_x \neq v_y$

$\therefore f$ is not analytic function.

Definition:

Let u, v be two harmonic functions and $u_x = v_y$, $u_y = -v_x$, then we say that v is a harmonic conjugate of u .

Note:

1. If v is a harmonic conjugate of u and u is a harmonic conjugate of v then u , v are constant functions.

2. If v is a harmonic conjugate of u then u is a harmonic conjugate of $-v$.
3. $f = u + iv$ is analytic iff v a harmonic conjugate of u.

Example: Show that $u(x, y) = \sin x \cosh y$ is harmonic and find the harmonic conjugate.

Solution:

$u_x = \cos x \cosh y \;\rightarrow\; u_{xx} = -\sin x \cosh y$

$u_y = \sin x \sinh y \;\rightarrow\; v_{yy} = \sin x \cosh y$

$\rightarrow\; u_{xx} + v_{yy} = 0 \;\rightarrow u$ is harmonic

To find the harmonic conjugate v we must satisfy

$$u_x = v_y\,,\;\; u_y = -v_x$$

1. $u_x = \cos x \cosh y = v_y$

2. $v = \cos x \sinh y + \emptyset_x$

We obtain $\emptyset_x$ by integration and using the second equation of C-R:

$$v_x = -\sin x \sinh y + \emptyset'_x$$

But $-v_x = u_y$, then

$-\sin x \sinh y + \emptyset'_x = -\sin x \sinh y \;\rightarrow\; \emptyset'_x = 0 \overset{\int}{\rightarrow} \emptyset_x = c$

$\therefore\; v = \cos x \sinh y + c$

Example: Let $u(x,y) = xy$, find v such that $f(z) = u + iv$ is analytic.

Solution: Since f is an analytic, then C-R equation are satisfied

$u_x = v_y \rightarrow y = v_y \rightarrow v = \frac{y^2}{2} + \emptyset(x)$

But $u_y = -v_x \rightarrow x = -\emptyset'(x)$

$$\rightarrow \emptyset'(x) = -x$$

$$\overset{\int}{\rightarrow} \emptyset(x) = \frac{-x^2}{2} + c$$

$$\therefore v = \frac{y^2}{2} - \frac{x^2}{2} + c$$

If $c = 0$, then $f(z) = xy + i\left(\frac{y^2}{2} - \frac{x^2}{2}\right)$

Chapter Three

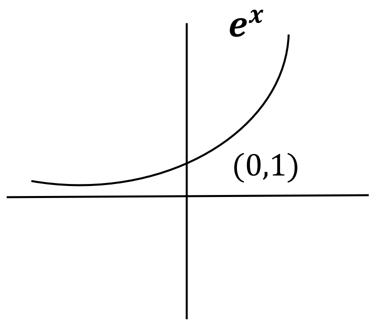
Elementary Functions

[1] The Exponential Functions

A real valued function $f(x) = e^x, f: \mathbb{R} \rightarrow \mathbb{R}^+$, is one-to-one and onto function, and

1. $e^{x_1}. e^{x_2} = e^{x_1 + x_2}$

2. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$



Definition:

Let $z = x + iy$, define

$$\text{Exp}(z) = e^z = e^{x+iy} = e^x . e^{iy} = e^x(\cos y + i \sin y)$$

If $f(z) = e^z = u + iv \rightarrow Re(z) = e^x \cos y \ , \ Im(z) = e^x \sin y$

If $y = 0 \rightarrow e^z = e^x$

If $x = 0 \rightarrow e^z = e^{iy} = \cos y + i \sin y$

Note: If $f(z) = e^z$, then

1. e^z is an analytic function, since

$$u = e^x \cos y \ , \qquad v = e^x \sin y$$

$$u_x = e^x \cos y = v_y \ , \qquad u_y = -e^x \sin y = -v_x$$

and u_x, u_y, v_y, v_x, u, v are continuous functions and satisfy C.R.E, therefore e^z is differentiable function $\forall z \in \mathbb{C}$.

2. $f'(z) = e^z$, since $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y$

$$= e^x(\cos y + i \sin y) = e^z$$

3. $|e^z| = e^x$, since
   $|e^z| = |e^x e^{iy}| = |e^x||e^{iy}|$

$$= |e^x|\sqrt{\cos^2 y + \sin^2 y}$$

$$= |e^x|.1$$

$$= |e^x|$$

   But $e^x > 0, \forall x \in \mathbb{R}$, so $|e^z| = e^x$

4. $|e^z| \neq 0$, since $|e^z| = e^x \neq 0, \forall x \in \mathbb{R}$
   Note: $e^z = 0$ iff $|e^z| = 0$

5. $e^z: \mathbb{R} \to \mathbb{C} - \{0\}$

Example: Let $w \neq 0$ and $w = re^{i\theta}$, find z if $z = re^{i\theta} = w$

Solution:

$e^z = e^x . e^{iy} = re^{i\theta}$

$\to r = e^x \quad , \ y = \theta + 2n\pi \ , n = 0, \mp 1, \ldots$

$\to x = \log r \ , \ y = \theta + 2n\pi$

$\therefore z = \ln r + i(\theta + 2n\pi)$

Therefore $\forall \ w \in \mathbb{Z}, \exists$ infinity number of values of z such that $w = e^z$, therefore e^z is not one-to-one.

Note: e^z is periodic function with period 2π

$$e^z = e^{z+2\pi i}$$

Proof: Let $z = x + iy$, hence

$e^{z+2\pi i} = e^{x+iy+2\pi i} = e^{x+i(y+2\pi)}$

$$= e^x(\cos(y + 2\pi) + i \ \sin(y + 2\pi)) = e^x(\cos y + i \sin y) = e^z$$

In general: e^z is not one-to-one only if $-\pi < Im(z) < \pi$.

Properties of Exponential Function:

1. $e^{z_1}.e^{z_2} = e^{z_1+z_2}$

2. $e^{1/z} = e^{-z}$

3. $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$

4. $(e^z)^n = e^{nz},\ n \in \mathbb{Z}$

Proof:

1. Let $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2$

$$e^{z_1}.e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1).e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1}.e^{x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$
$$= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$$
$$= e^{x_1+x_2}.e^{i(y_1+y_2)}$$
$$= e^{(x_1+iy_1)+(x_2+iy_2)}$$
$$= e^{z_1+z_2}$$

By the same way, we can prove 2 and 3.

4. $(e^z)^n = (e^x\cos y + ie^x\sin y)^n$

$$= \left(e^x(\cos y + i\sin y)\right)^n$$
$$= e^{nx}(\cos y + i\sin y)^n$$
$$= e^{nx}(\cos ny + i\sin ny)$$
$$= e^{nx}e^{iny}$$
$$= e^{nx+iny}$$
$$= e^{n(x+iy)}$$
$$= e^{nz}$$

5. $e^0 = 1$

6. $\arg e^z = y + 2n\pi$

7. $\overline{(e^z)} = e^{\bar{z}}$

Proof:

$\overline{(e^z)} = e^x(\cos y - i\sin y)$

$= e^x(\cos(-y) + i\sin(-y))$

$= e^{x-iy}$

$= e^{\bar{z}}$

Polar Coordinates of Exponential Function:

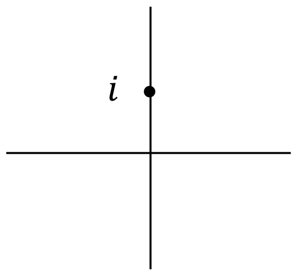
If $e^z = e^x(\cos y + i\sin y)$

$= r(\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi))$

Where $r = |e^z| = e^x, y = \theta + 2n\pi$

Example: Solve $e^z = i$

Solution: $z = \ln r + i(\theta + 2n\pi)$

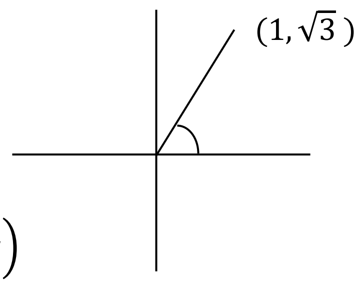


$r = |i| = 1$ and $\theta = \arg i = \frac{\pi}{2} + 2n\pi$

$\therefore z = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right), n = 0, \mp 1, \ldots$

$= i\left(\frac{\pi}{2} + 2n\pi\right)$

Example: Find the value of z such that



$$e^z = 1 + \sqrt{3}\, i$$

Solution: $z = \ln r + i(\theta + 2n\pi)$

$r = \sqrt{1+3} = 2$, $\theta = \frac{\pi}{3} + 2n\pi \rightarrow z = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$

Example: Prove that

$$e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{e}\ \left(\frac{1+i}{\sqrt{2}}\right)$$

Proof: $e^{\left(\frac{2+\pi i}{4}\right)} = e^{\left(\frac{1}{2}+\frac{\pi}{4}i\right)}$

$$= e^{1/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{e}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

$$= \sqrt{e}\left(\frac{1+i}{\sqrt{2}}\right)$$

Example: Prove that

$$e^{z+\pi i} = -e^{z}$$

Proof: $e^{z+\pi i} = e^{(x+iy)+\pi i}$

$$= e^{x+(y+\pi)i}$$

$$= e^{x}(\cos(y+\pi) + i\sin(y+\pi))$$

$$= e^{x}(-\cos y - i\sin y)$$

$$= -e^{x}(\cos y + i\sin y)$$

$$= -e^{z}$$

Example: Find all the complex solutions of

$$e^{z} = 1$$

Solution:

$e^{z} = 1 \rightarrow r = 1\ ,\ \theta = 0$

$\therefore z = \ln 1 + i(0 + 2n\pi) = i\ 2n\pi$

Example: Find all the complex solutions of

$$e^{4z} = i$$

Solution: $e^{4z} = i = e^{4x}(\cos 4y + i\ \sin 4y)$

$r = 1\ ,\ \theta = \frac{\pi}{2} + 2n\pi\ ,\ n = 0, \mp 1, \dots$

$e^{4z} = e^{4x}(\cos 4y + i \sin 4y)$

$\quad = 1.\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)$

$\therefore\ e^{4x} = 1 \rightarrow 4x = ln1 \rightarrow x = 0$

& $\cos 4y = \cos\frac{\pi}{2} \rightarrow 4y = \frac{\pi}{2} \rightarrow y = \frac{\pi}{8} + 2n\pi$

$\therefore\ z = x + iy = 0 + i\left(\frac{\pi}{8} + 2n\pi\right) = i\left(\frac{\pi}{8} + 2n\pi\right)$

Note:

1. $f(z) = e^{\bar{z}}$ is not analytic at any point (not analytic everywhere). (H.w)

2. $f(z) = e^{iz}$ is analytic function.

Proof:

$e^{iz} = e^{-y}(\cos x + i \sin x)$

i. $u_x = -e^{-y} \sin x\ ,\ u_y = -e^{-y} \cos x$

$u_x = v_y \qquad ,\ u_y = -v_x \rightarrow$ C. R. E are satisfied.

ii. $u\ , v, u_x\ , u_y\ , v_y\ , v_x$ are continuous functions.

From (i) and (ii), we get e^{iz} is analytic function and

$(e^{iz})' = u_x + iv_x$

$\qquad = -e^{-y} \sin x + ie^{-y} \cos x$

$\qquad = ie^{-y}(\cos x + i \sin x)$

$\qquad = ie^{iz}$

[2] Trigonometric Functions

Definition: Let $z = x + iy$, define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}$$

Note: $\sin z$ and $\cos z$ are analytic functions in the complex plane, hence they're entire functions, but $\tan z, \sec z$ are analytic only when $\cos z \neq 0$.

Note:

1. $(\sin z)' = \frac{1}{2i}\left[ie^{iz} + ie^{-iz}\right]$

$$= \frac{e^{iz} + e^{-iz}}{2i} = \cos z$$

2. $(\cos z)' = \frac{1}{2}\left[ie^{iz} - ie^{-iz}\right] = \frac{i}{2}\left[e^{iz} - e^{-iz}\right]$

$$= -\left[\frac{e^{iz} - e^{-iz}}{2i}\right] = -\sin z$$

Note:

1. $\cos^2 z + \sin^2 z = 1$

Proof:

$$\cos^2 z + \sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2$$

$$= \frac{e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}}{4}$$

$$= \frac{4}{4}$$

$$= 1$$

2. $\cos z = \cos x \cosh y - i \sin x \sinh y$

   where $\cos iy = \cosh y$, $\sin iy = \sinh y$

3. $\sin z = \sin x \cosh y + i \cos x \sinh y$

4. $|\sin z|^2 = \sin^2 x + \sinh^2 y$

5. $|\cos z|^2 = \cos^2 x + \sinh^2 y$

Note: $\sin z$ and $\cos z$ are periodic, since

1. $\sin(z + 2\pi) = \sin z$

2. $\cos(z + 2\pi) = \cos z$

But

3. $\tan(z + \pi) = \tan z$

Proof: 1. (H.w)

2. $\cos(z + 2\pi) = \cos(x + iy + 2\pi) = \cos(x + 2\pi + iy)$

$$= \cos(x + 2\pi)\cosh y - i \sin(x + 2\pi)\sinh y$$

$$= \cos x \cosh y - i \sin x \sinh y$$

$$= \cos z$$

3. (H.w)

Note: The zeros of $\sin z$ and $\cos z$ are real.

Example: The zero of $\cos z$ is $z = \frac{\pi}{2} + n\pi$.

Solution:

$\cos z = 0$

$\rightarrow \cos x \cosh y - i \sin x \sinh y = 0 + 0i$

$\therefore \cos x \cosh y = 0 \quad \dots (1)$

$\& \ \sin x \sinh y = 0 \quad \dots (2)$

Since $\cos x \cosh y = 0 \ \rightarrow$ either $\cos x = 0$ or $\cosh y = 0$

If $\cos x = 0 \to x = \frac{\pi}{2} + n\pi$

Substituting in (2) we get

$\sinh y = 0 \to y = 0$

If $\cosh y = 0 \to$ this is not possible since ($\cosh y = \frac{e^{y} + e^{-y}}{2} \neq 0, \forall\, y$ and $\sinh y = \frac{e^{y} - e^{-y}}{2} = 0$ if $y = 0$).

$\therefore z = x + iy = \frac{\pi}{2} + n\pi + 0$

$\therefore z = \frac{\pi}{2} + n\pi$

Note: If we take equation (2) we get:

$\sin x \sinh y = 0 \to$ either $\sin x = 0$ or $\sinh y = 0$

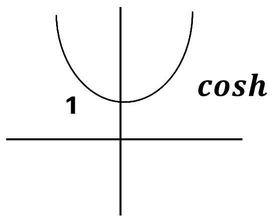
If $\sin x = 0 \to$ this is not possible since

$$\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$$

Then $\sinh y = 0 \to y = 0$

$\therefore z = \frac{\pi}{2} + n\pi + 0 = \frac{\pi}{2} + n\pi$

Note: Cosh$y$ (the range of cosh$y \geq 1$) is always positive.



Example: Find all the roots of

$$\sin z = 3$$

Solution:

$\sin z = \sin x \cosh y + i \cos x \sinh y$

$\sin z = 3 \to \sin x \cosh y + i \cos x \sinh y = 3 + 0i$

$\sin x \cosh y = 3 \quad \dots (1)$

$\cos x \sinh y = 0 \quad \dots (2)$

From (1) we get:

$\sin x \cosh y = 3$, then

Either $\sin x = 3 \rightarrow$ this is not possible since $(-1 \leq \sin x \leq 1)$

Or $\cosh y = 3 \rightarrow y \cong 1.8$

From (2) we get:

$\cos x \sinh y = 0$, then

Either $\cos x = 0 \rightarrow x = \frac{\pi}{2} + n\pi$

Or $\sinh y = 0 \rightarrow$ this is not possible

<u>Example</u>: Find all the roots of

$$\sin(\bar{z} + i) = 2i$$

<u>Solution</u>: $\sin(\bar{z} + i) = \sin(x - iy + i) = \sin(x + i(1 - y))$

$\rightarrow \sin\big(x + i(1 - y)\big) = 0 + 2i$

$\rightarrow \sin x \cosh(1 - y) + i \cos x \sinh(1 - y) = 0 + 2i$

$\sin x \cosh(1 - y) = 0 \quad \dots (1)$

$\cos x \sinh(1 - y) = 2 \quad \dots (2)$

From (1) we get:

$\sin x \cosh(1 - y) = 0$, then

Either $\cosh(1 - y) = 0 \rightarrow$ this is not possible

Or $\sin x = 0 \rightarrow x = n\pi$

From (2) we get:

$\cos x \sinh(1 - y) = 2$, then

Either $\cos x = 2 \rightarrow$ this is not possible since $(-1 \leq \cos x \leq 1)$

Or $\sinh(1-y) = 2 \rightarrow \sinh(1-y) = \mp 2$

$$\rightarrow \quad 1 - y = \sinh^{-1}(\mp 2)$$

$$\rightarrow \quad y = \mp \sinh^{-1} 2 + 1$$

$$\rightarrow \quad y = 1 \mp \sinh^{-1} 2$$

$\therefore z = n\pi + i\,(1 \mp \sinh^{-1} 2)$

Example: Prove that

$$\left|e^{2z+i} + e^{iz^2}\right| \leq e^{2x} + e^{-2xy}$$

Proof:

$$\left|e^{2z+i} + e^{iz^2}\right| = \left|e^{2x+i(2y+1)} + e^{i(x^2-y^2+2ixy)}\right|$$

$$\leq \left|e^{2x+i(2y+1)}\right| + \left|e^{i(x^2-y^2+2ixy)}\right|$$

$$= \left|e^{2x}\, e^{i(2y+1)}\right| + \left|e^{-2xy}\, e^{i(x^2-y^2)}\right|$$

$$= e^{2x} + e^{-2xy} \qquad \text{(Since } |e^{i\cdots}| = 1\text{)}$$

[3] Hyperbolic Functions

The hyperbolic Sine and Cosine of a complex variable defined as they are with a real variable; that is,

1. $\text{Sinh}\, z = \frac{e^z - e^{-z}}{2}$, $\text{Cosh}\, z = \frac{e^z + e^{-z}}{2}$

Since e^z and e^{-z} are entire functions, then it follows from definition (1) that $\sinh z$ and $\cosh z$ are entire functions, furthermore,

1. $\frac{d}{dz} \sinh z = \cosh z$

2. $\frac{d}{dz} \text{Cosh}\, z = \text{Sinh}\, z$

3. $\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$

$$= \frac{e^{2z} + 2 + e^{-2z} - e^{2z} + 2 - e^{-2z}}{4}$$

$$= 1$$

4. Sinh z and Cosh z are periodic functions with period $2\pi i$.

♦ Show that

$$\sinh(z + 2\pi i) = \sinh z$$

Proof:

$$\sinh(z + 2\pi i) = \frac{e^{z+2\pi i} - e^{-z-2\pi i}}{2}$$

$$= \frac{e^z . e^{2\pi i} - e^{-z} . e^{-2\pi i}}{2}$$

$$= \frac{e^z(\cos 2\pi i + i \sin 2\pi i) - e^{-z}(\cos(-2\pi i) + i \sin(-2\pi i))}{2}$$

$$= \frac{e^z - e^{-z}}{2} \qquad (\cos 2\pi i = 1 , \ \sin 2\pi i = 0)$$

$$= \sinh z$$

5. $|\sinh z|^2 = \sinh^2 x + \sin^2 y$

Proof:

$$|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$

$$= \sinh^2 x(1 - \sin^2 y) + (1 + \sinh^2 x) \sin^2 y$$

$$= \sinh^2 x - \sinh^2 x \ \sin^2 y + \sin^2 y + \sinh^2 x \sin^2 y$$

$$= \sinh^2 x + \sin^2 y$$

6. $|\cosh z|^2 = \cos^2 y + \sinh^2 x$ (H.w)

7. The zeros of $\text{Sinh}\, z$ are $z = n\pi i$

<u>*Proof*</u>:

$\sinh z = \sinh x \cos y + i \cosh x \sin y$

$\sinh z = 0 \rightarrow \sinh x \cos y + i \cosh x \sin y = 0 + 0i$

$\sinh x \cos y = 0 \quad \dots (1)$

$\cosh x \sin y = 0 \quad \dots (2)$

From (1), we get:

$\sinh x \cos y = 0$, then

Either $\sinh x = 0$ or $\cos y = 0$

If $\sinh x = 0 \rightarrow x = 0$

Substituting in (2), we get:

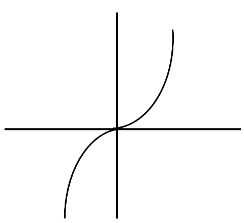
$\sin y = 0 \rightarrow y = n\pi$

If $\cos y = 0 \rightarrow$ this is not possible

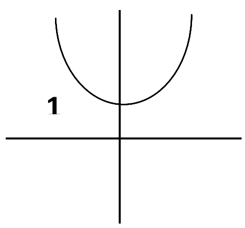
$\therefore z = x + iy = 0 + i(n\pi) = n\pi i$

<u>**Note:**</u> The *Cosh* cannot be negative in real numbers, but it can be in complex numbers.

sinhx



coshx



Example: Solve $e^{2z-1} = 1$

Solution:

$e^{2z-1} = e^{2(x+iy)-1} = e^{2x-1}.e^{2iy}$

$$= e^{2x-1}(\cos 2y + i \sin 2y)$$

$e^{2z-1} = 1 \rightarrow e^{2x-1}(\cos 2y + i \sin 2y) = \cos 0 + i \sin 0$

$e^{2x-1} \cos 2y = 1 \ \ldots (1)$

$e^{2x-1} \sin 2y = 0 \ \ldots (2)$

From (2), we get

Either $e^{2x-1} = 0$ or $\cos 2y = 0$

If $e^{2x-1} = 0 \rightarrow$ this is not possible

If $\sin 2y = 0 \rightarrow 2y = n\pi \rightarrow y = \frac{n\pi}{2}, n = 0, \mp 1, \ldots$

Substituting in (1), we get:

$e^{2x-1} = 1 \rightarrow 2x - 1 = 0 \rightarrow x = \frac{1}{2}$

$\therefore z = x + iy = \frac{1}{2} + i\frac{n\pi}{2} = \frac{1}{2}(1 + n\pi i)$

[4] Logarithmic Functions

The logarithmic function of a complex variable is defined by:

$\log z = \ln|z| + i\, arg z, z \neq 0$

$\log z = \ln r + i(\theta + 2n\pi), n = 0, \mp 1, \mp 2, \ldots$

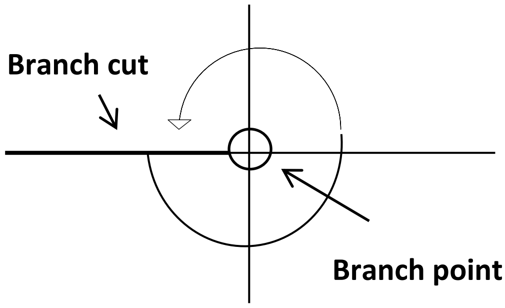
Definition: (Principal value)

The principal branch (Principal value) of the complex logarithmic function which is given by:

$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z = \ln r + i\theta$$

is continuous in the domain $\{r > 0, -\pi \leq \theta \leq \pi\}$.

Note: The nonpositive real axis is called a branch cut for $\operatorname{Log} z$ and the point 0 is called a branch point.



$$-\pi < \theta < \pi$$

Remarks:

1. The function $\log z = \ln r + i(\theta + 2n\pi)$ is a multiple-valued function.

2. The values of $\log z$ have the same real part, but their imaginary parts differ by interval multiple of 2π.

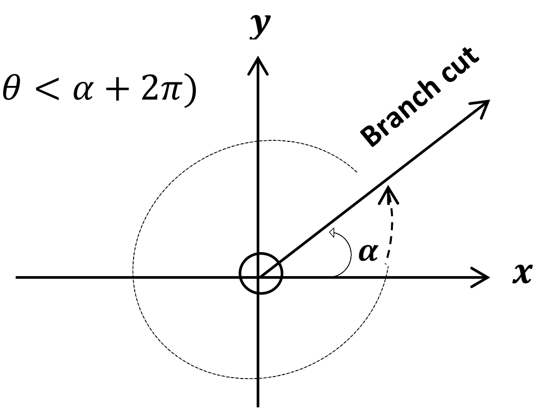
3. The function $\operatorname{Log} z = \ln r + i\theta$ is a single-valued function.

4. The principal branch of the complex logarithm ($\operatorname{Log} z$) is just one of many possible branches of the multiple-valued $\log z$, we can define other branches of $\log z$ as follows:

   Let $\alpha \in \mathbb{R}$ and $\alpha < \theta < \alpha + 2\pi$, then

   $$\operatorname{Log} z = \ln r + i\theta \quad (r > 0,\ \alpha < \theta < \alpha + 2\pi)$$

   is a single-valued function.



5. The principal branch of $\operatorname{Log} z$ is discontinuous at $z = 0$, since this function is not defined at $z = 0$. Also it is not continuous at every point in the negative real axis.

To verify that,

Let $z_0 \in$ branch cut, then

Arg $z \to \pi$ when $z \to z_0$ from the 2^{nd} quarter

And

Arg $z \to -\pi$ when $z \to z_0$ from the 3^{rd} quarter

Thus $\lim_{z \to z_0} \log z$ is not exist.

Examples:

1. Find $\log(1+\sqrt{3}\,i)$ and $\text{Log}(1+\sqrt{3}\,i)$

Solution:

$z = 1+\sqrt{3}\,i \to x = 1\,,\ y = \sqrt{3}$

$r = |z| = \sqrt{1+3} = 2$, and

$$\left.\begin{aligned} 1 &= 2\cos\theta \to \cos\theta = \tfrac{1}{2} \\ \sqrt{3} &= 2\sin\theta \to \sin\theta = \tfrac{\sqrt{3}}{2} \end{aligned}\right\} \to \theta = \frac{\pi}{3}$$

Thus:

$$\log(1+\sqrt{3}\,i) = \ln 2 + i\left(\frac{\pi}{3} + 2n\pi\right)$$

And:

$$\text{Log}(1+\sqrt{3}\,i) = \ln 2 + i\,\frac{\pi}{3}$$

2. $\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$

$\text{Log}(1+i) = \ln 2 + i\,\frac{\pi}{3}$

3. $\log(1) = \ln 1 + i(0 + 2n\pi) = 2n\pi\, i$

$\text{Log}(1) = \ln 1 + i0 = 0$

4. $\log(3i) = \ln 3 + i\left(\frac{\pi}{2} + 2n\pi\right)$

$\mathrm{Log}(3i) = \ln 3 + i\,\frac{\pi}{2}$

5. $\log(-3i) = \ln 3 + i\left(\frac{-\pi}{2} + 2n\pi\right)$

$\mathrm{Log}(-3i) = \ln 3 - i\,\frac{\pi}{2}$

Properties:

Let $z_1,\ z_2 \neq 0,\ n \in \mathbb{N}$, and $\alpha \in \mathbb{R}$, then

1. $\log(z_1 z_2) = \log z_1 + \log z_2$
2. $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$
3. $\log(z^n) = n \log z$ (Valid for certain values of Logarithms; i.e. it is not true in general).
4. $e^{\log z} = z, \forall\, z \neq 0$
5. a) $z^n = e^{n \log z},\ \ n = 1,2,3,\dots$

   b) $z^{1/n} = e^{1/n \log z}$
6. $\log e^z = z + 2n\pi i$
7. $\mathrm{Log}(e^z) = z$
8. $\frac{d}{dz}(\log z) = \frac{1}{z}\ \ ,\ \ \alpha < \theta < \alpha + 2\pi$
9. $\frac{d}{dz}(\mathrm{Log}\, z) = \frac{1}{z}\ ,\ -\pi < \theta < \pi\ ,\ r > 0$

Proof:

1. $\log(z_1 z_2) = \ln|z_1 z_2| + i \arg(z_1 z_2)$

$$= \ln|z_1| + \ln|z_2| + i(\arg z_1 + \arg z_2)$$

$$= \ln|z_1| + i \arg z_1 + \ln|z_2| + i \arg z_2$$

$$= \log z_1 + \log z_2$$

2. $\log\left(\frac{z_1}{z_2}\right) = \ln\left|\frac{z_1}{z_2}\right| + i\arg\left(\frac{z_1}{z_2}\right)$

$$= \ln|z_1| - \ln|z_2| + i(\arg z_1 - \arg z_2)$$

$$= \ln|z_1| + i\arg z_1 - \ln|z_2| - i\arg z_2$$

$$= \log z_1 - \log z_2$$

3. $\log(z^n) \neq \ln|z^n| + i\arg(z^n)$ in general

$$= n\ln|z| + i\,n\arg z$$

$$= n\,(\ln|z| + i\arg z)$$

$$= n\log z$$

4. $e^{\log z} = e^{\ln|z| + i\arg z} = e^{\ln|z|}e^{i\arg z}$

$$= |z|e^{i\arg z}$$

$$= |z|e^{i(\theta+2n\pi)}$$

$$= r\,e^{i\theta}e^{i2n\pi}$$

$$= re^{i\theta} = z$$

5. a) By induction

1. For $n = 1$, we have $z = e^{\log z}$which is true from (4).

2. For $2 \le k < n$, the result be true, that is

$$z^{n-1} = e^{(n-1)\log z}$$

3. $z^n = z.z^{n-1} = e^{\log z}.e^{(n-1)\log z} = e^{n\log z}$ as required.

b) $z^{1/n} = \left(re^{i\theta}\right)^{1/n}$

$$= r^{1/n}.e^{\frac{(i\theta)}{n}}$$

$$= e^{\ln r^{1/n}}.e^{\frac{[i\theta + i2n\pi]}{n}}$$

$$= e^{\ln r^{1/n}}.e^{i\frac{\theta}{n} + i2\pi}$$

$= e^{\frac{1}{n}\ln r} . e^{\frac{i}{n}(\theta+2n\pi)}$

$= e^{\frac{1}{n}[\ln r+i(\theta+2n\pi)]}$

$= e^{1/n \log z}$

6. $\log e^z = \ln|e^z| + i \arg(e^z)$

$= \ln\left|e^x e^{iy}\right| + i \arg\left(e^x . e^{i(y+2n\pi)}\right)$

$= \ln e^x + i(y + 2n\pi)$

$= x + iy + 2n\pi i$

$= z + 2n\pi i$

7. $\text{Log}(e^z) = \ln|e^z| + i\, \text{Arg}(e^z)$

$= \ln e^x + iy$

$= x + iy$

$= z$

8. $\log z = \ln r + i(\theta + 2n\pi)\ ,\ \ r > 0\ \ \&\ \ \alpha < \theta < \alpha + 2\pi$

Let $u = \ln r$, $v = \theta + 2n\pi$, then

$$\left.\begin{array}{ll} u_r = \frac{1}{r} \quad , & v_r = 0 \\ u_\theta = 0 \quad , & v_\theta = 1 \end{array}\right\} \Longrightarrow \begin{array}{l} u_r = \frac{1}{r} v_\theta \\ u_\theta = -r\, v_r \end{array}$$

$\therefore$ C.R. Eqs are satisfied and since $u_r , u_\theta , v_r , v_\theta, u , v$ are continuous functions, then $\log z$ is differentiable in its domain and

$\frac{d}{dz}(\log z) = e^{-i\theta}(u_r + iv_r)$

$= e^{-i\theta}\left(\frac{1}{r} + i0\right)$

$= \frac{1}{re^{i\theta}}$

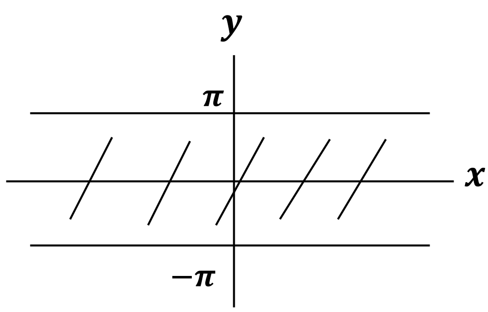
$= \frac{1}{z}$

9. Similar to 8.

Remark:

The function $\operatorname{Log} z$ is the inverse function of e^z, where $z = x + iy$, $x \in \mathbb{R}$ and $-\pi < y < \pi$, i.e. (e^z is one-to-one on the domain).

If $f(z) = e^z$ then $f^{-1}(z) = \operatorname{Log} z$



Exercise: Find $\frac{d}{dz}(\operatorname{Log} z) = \frac{1}{z}$.

Note: $(\operatorname{Log} f(z)) = \frac{f'(z)}{f(z)}$.

Example: Find $\frac{d}{dz}(\operatorname{Log} 3z^2)$

Solution: $f(z) = 3z^2 \to \frac{d}{dz}(\operatorname{Log} f(z)) = \frac{f'(z)}{f(z)} = \frac{6z}{3z^2} = \frac{2}{z}$.

Example: Show that $\operatorname{Log} z$ is analytic for all z except when $Re(z) \leq 0$, and $Im(z) = 0$.

Solution:

$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z$

$$= \ln\sqrt{x^2 + y^2} + i\left(\tan^{-1}\frac{y}{x}\right)$$

Let $u(x, y) = \frac{1}{2}\ln(x^2 + y^2)$, $v(x, y) = \tan^{-1}\frac{y}{x}$, then

$$\to u_x = \frac{x}{x^2+y^2} = v_y$$

$$\to u_y = \frac{y}{x^2+y^2} = -v_x$$

Since the C.R.Eqs hold for all $(x, y) \neq (0, 0)$ and u_x, u_y, v_x, v_y, u, v are continuous for all $(x, y) \neq (0, 0)$, then $\operatorname{Log} z$ is analytic everywhere except when $Re(z) \leq 0$, and $Im(z) = 0$.

Note: $\text{Log}\, z$ is not continuous function on the nonpositive real axis.

Example: Determine the domain of analyticity for the function

$$f(z) = \text{Log}(3z - i)$$

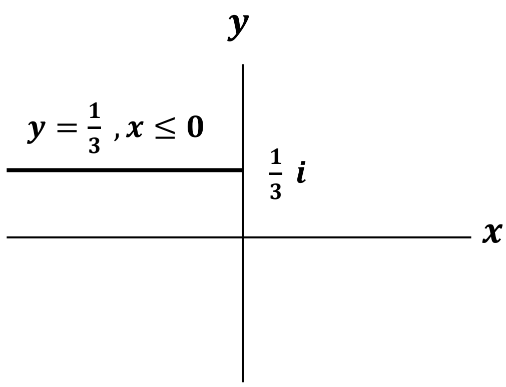
Solution:

The function $\text{Log}(3z - i)$ is analytic everywhere with $Re(3z - i) \leq 0$, and $Im(3z - i) = 0$, must be removed, i.e.

$$Re(3z - i) \leq 0 \rightarrow Re\big(3x + i(3y - 1)\big) = 3x \leq 0 \rightarrow x \leq 0$$

$$Im(3z - i) = 0 \rightarrow Im\big(3x + i(3y - 1)\big) = 3y - 1 = 0 \rightarrow y = \frac{1}{3}$$

Thus f is analytic everywhere except the horizontal line $x \leq 0,\ y = \frac{1}{3}$



Example: Find all the roots of the equation

$$\log z = \frac{\pi}{2} i$$

Solution:

1. Taking the e for both sides

$$e^{\log z} = e^{\frac{\pi}{2} i} \rightarrow z = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}$$

$$\rightarrow\ z = i$$

2. We can find the roots in other way as follows:

$$\log z = \frac{\pi}{2} i \rightarrow \ln r + i(\theta + 2n\pi) = 0 + \frac{\pi}{2} i$$

$$\rightarrow \ln r = 0 \rightarrow r = 1 \text{ and}$$

$\rightarrow \theta + 2n\pi = \frac{\pi}{2} \rightarrow \theta = \frac{\pi}{2} - 2n\pi$

$\therefore z = re^{i\theta} = e^{i\left(\frac{\pi}{2} - 2n\pi\right)}$

$= e^{i\frac{\pi}{2}}$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= i$

Example: Show that the function

$$f(z) = \frac{\text{Log}(z+4)}{z^2+i}$$

is analytic everywhere except for the point $\left(\frac{-(1-i)}{\sqrt{2}}, \frac{(1-i)}{\sqrt{2}}\right)$ and the portion $x \le -4$ of the real axis.

Solution: $\text{Log}(z+4)$ is analytic everywhere except for the points that satisfy the condition

$$Re(z+4) \le 0 \text{ and } Im(z+4) = 0$$

$\left.\begin{array}{l}\rightarrow x + 4 \le 0 \\ \quad x \le -4\end{array}\right\}, y = 0$ and $z^2 + i = 0 \rightarrow z^2 = -i \rightarrow z = \mp (-i)^{1/2}$

$z = re^{i\theta} = \mp \left(e^{-i\frac{\pi}{2}}\right)^{1/2}$

$= \mp e^{-i\frac{\pi}{4}}$

$= \mp \left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$= \mp \left[\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right]$

$= \mp \frac{(1-i)}{\sqrt{2}}$

Hence f is not analytic at the point $\mp\frac{(1-i)}{\sqrt{2}}$ and the half line $x \le -4$, $y = 0$.

Example: Show that if $Re(z_1) > 0$ and $Re(z_2) > 0$, then:

$$\text{Log}(z_1 z_2) = \log z_1 + \log z_2$$

Proof: Suppose that $Re(z_1) > 0,\ Re(z_2) > 0$, then

$$z_1 = r_1 e^{i\theta_1} \to \frac{-\pi}{2} < \theta_1 < \frac{\pi}{2}$$

$$z_2 = r_2 e^{i\theta_2} \to \frac{-\pi}{2} < \theta_2 < \frac{\pi}{2}$$

$\to -\pi < \theta_1 + \theta_2 < \pi$, which enables us to write

$$\begin{aligned}\text{Log}(z_1 z_2) &= \ln|z_1 z_2| + i\,\text{Arg}(z_1 z_2)\\ &= \ln(r_1 r_2) + i(\theta_1 + \theta_2)\\ &= \ln z_1 + \ln z_2 + i\theta_1 + i\theta_2\\ &= \ln z_1 + i\theta_1 + \ln z_2 + i\theta_2\\ &= \text{Log}\, z_1 + \text{Log}\, z_2\end{aligned}$$

Example: Show that:

a) If $\log z = \text{Log}\, r + i \arg z\ ,\left(r > 0\ ,\ \frac{\pi}{4} < \theta < \frac{9\pi}{4}\right)$, then

$$\log i^2 = 2 \log i$$

b) If $\log z = \text{Log}\, r + i \arg z\ ,\left(r > 0\ ,\ \frac{3\pi}{4} < \theta < \frac{11\pi}{4}\right)$, then

$$\log i^2 \neq 2 \log i$$

Solution:

a) $\log i^2 = \log(-1) \qquad (z = -1 + 0i)$

$= \ln(1) + i\pi$

$= i\pi \quad ,\quad \text{where} \quad \pi \in \left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$

And

$2 \log i = 2\left(\ln(1) + i\frac{\pi}{2}\right) = i\pi \qquad (z = 0 + i)$

$\therefore \log i^2 = 2 \log i$

b) $\log i^2 = i\pi$, where π is in the given interval $\left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$, and

$$2\log i = 2(\ln(1) + i\theta^*)$$

$$= 2i\theta^*$$

$$= 2i\left(\frac{\pi}{2}\right), \text{ which is not in } \frac{3\pi}{4} < \theta^* < \frac{11\pi}{4}$$

$$\rightarrow \theta^* = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2} \notin \left(\frac{\pi}{4}, \frac{9\pi}{4}\right)$$

$$\rightarrow 2\log i = 2i\left(\frac{5\pi}{2}\right) = 5\pi i$$

$$\therefore \log i^2 \neq 2\log i$$

Example: Show that

$$\text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$$

Solution:

$$\rightarrow \text{Log}(1+i)^2 = \text{Log}(1+2i+i^2)$$

$$= \text{Log}(1+2i-1)$$

$$= \text{Log}\,2i$$

$$= \ln 2 + i\frac{\pi}{2}$$

$$\rightarrow 2\,\text{Log}(1+i) = 2\left[\ln\sqrt{2} + i\frac{\pi}{4}\right]$$

$$= 2\ln(2)^{1/2} + i\frac{\pi}{2}$$

$$= \ln 2 + i\frac{\pi}{2}$$

$$\therefore \text{Log}(1+i)^2 = 2\,\text{Log}(1+i)$$

Example: Show that

$$\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$$

Solution:

$\rightarrow \operatorname{Log}(-1+i)^2 = \operatorname{Log}(-2i)$

$$= \ln 2 - i\frac{\pi}{2}$$

$\rightarrow 2\operatorname{Log}(-1+i) = 2\left[\ln\sqrt{2} + i\frac{3\pi}{4}\right]$

$$= \ln 2 + i\frac{3\pi}{2}$$

Hence

$$\operatorname{Log}(-1+i)^2 \neq 2\operatorname{Log}(-1+i)$$

In general:

1. $\operatorname{Log} z^n \neq n\log z$

Example: $\log i^2 \neq 2\log i$

Solution:

$\rightarrow \log i^2 = \log(-1)$

$$= \ln(1) + i(\pi + 2n\pi)$$

$$= (2n+1)\pi\, i\,, \quad n = 0, \mp1, \mp2, \dots$$

$\rightarrow 2\log i = 2\left[\ln(1) + i\left(\frac{\pi}{2} + 2n\pi\right)\right]$

$$= (4n+1)\pi i\,, \quad n = 0, \mp1, \mp2, \dots$$

It is clear that the set of values of $\log i^2$ is not the same as the set of values of $2\log i$.

$$\text{i.e.: } \log i^2 \neq 2\log i$$

2. $\text{Log}(z_1 z_2) \neq \text{Log}\, z_1 + \text{Log}\, z_2$

<u>*Example*</u>: Take $z_1 = z_2 = -1$

$\rightarrow \text{Log}(z_1 z_2) = \text{Log}(1) = \ln(1) + 0i = 0$

$\rightarrow \text{Log}\, z_1 + \text{Log}\, z_2 = \text{Log}(-1) + \text{Log}(-1) = 2\pi i$

$\rightarrow \text{Log}(1) \neq \text{Log}(-1) + \text{Log}(-1)$

Hence

$$\text{Log}(z_1 z_2) \neq \text{Log}\, z_1 + \text{Log}\, z_2$$

3. $\text{Log}\left(\frac{z_1}{z_2}\right) \neq \text{Log}\, z_1 - \text{Log}\, z_2$

<u>**Example**</u>: Show that when $n = 0, \mp 1, \mp 2, \ldots$

$$\log\left(i^{1/2}\right) = \left(n + \frac{1}{4}\right)\pi i$$

<u>*Solution*</u>: $\left(i^{1/2}\right) = e^{\frac{1}{2}\log i}$

$\rightarrow \log\left(i^{1/2}\right) = \log e^{\frac{1}{2}\log i} = \frac{1}{2}\text{Log}\, i \quad \ldots 1$

Since $\text{Log}\, i = i\left(\frac{\pi}{2} + 2n\pi\right)$, then

$\rightarrow \log\left(i^{1/2}\right) = \frac{1}{2}\, i\left(\frac{\pi}{2} + 2n\pi\right)$ (By 1)

$$= \left(\frac{1}{4} + n\right)\pi i$$

<u>**Exercise:**</u> Show that $\text{Log}(x^2 + y^2)$ is harmonic in $D/\{0\}$ two ways that is:

1) Show that $u_{xx} + u_{yy} = 0$, $u = \text{Log}(x^2 + y^2)$.

2) Show that $r^2 u_{rr} + r u_r + u_{\theta\theta} = 0$,

[5] Complex Exponents

We define z^c, where $z, c \in \mathbb{C}$ and $z \neq 0$, by

$$z^c = e^{c \log z} \qquad \dots (1)$$

And

$$c^z = e^{z \log c} \qquad (c \neq 0)$$

Example: Find i^{-2i}

Solution: $i^{-2i} = e^{-2i \log i}$

$$= e^{-2i\left(\frac{\pi}{2}+2n\pi\right)i}$$

$$= e^{(4n+1)\pi}\ ,\ n = 0, \mp 1, \mp 2, \dots$$

Which is multiple valued.

Note: In a view of the property $e^{-z} = \frac{1}{e^z}$, we have $z^{-c} = \frac{1}{z^c}$ $(z \neq 0)$ and so

$$(i)^{-2i} = \frac{1}{i^{2i}} = e^{(4n+1)\pi}\ ,\ n = 0, \mp 1, \mp 2, \dots$$

We notice that the function $\log z = \ln r + i(\theta + 2n\pi)$, $r > 0$, $\alpha < \theta < \alpha + 2\pi$, is a single-valued and analytic function in the domain, thus when the branch of $\log z$ is used, it follows that

$$z^c = e^{c \log z}$$

is also single-valued and analytic in the same domain, and

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\left(e^{c \log z}\right) = \frac{c}{z} e^{c \log z}$$

Since $z = e^{\log z}$, then

$$\frac{d}{dz}(z^c) = c\ \frac{e^{c \log z}}{e^{\log z}} = c e^{c \log z} e^{-\log z}$$

$$= c e^{c \log z - \log z}$$

$$= c e^{(c-1)\log z}$$

$$= c z^{c-1}$$

$$\therefore \frac{d}{dz}(z^c) = cz^{c-1} \quad (r > 0\,,\ \alpha < \arg z < \alpha + 2\pi)$$

When $\alpha = -\pi$ then $-\pi < \arg z < \pi$, the function

$$z^c = e^{c \log z}\,, \qquad z \neq 0$$

Is called principal value of z^c.

Example: Find the principal value of the following:

a) $(i)^i$

Solution: p.v. $(i)^i = e^{i \operatorname{Log} i} = e^{i\left(\ln 1 + i\frac{\pi}{2}\right)} = e^{-\frac{\pi}{2}}$

b) $\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i}$

Solution:

p.v. $\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]^{3\pi i} = e^{3\pi i \operatorname{Log}\left[\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right]}$

$$= e^{3\pi i\left[\ln\left|\frac{e}{2}\left(-1-\sqrt{3}\,i\right)\right| - i\frac{2\pi}{3}\right]}$$

$$= e^{3\pi i\left(\ln e - i\frac{2\pi}{3}\right)}$$

$$= e^{3\pi i\left(1 - i\frac{2\pi}{3}\right)}$$

$$= e^{3\pi i + 2\pi^2}$$

$$= e^{2\pi^2} . e^{3\pi i}$$

$$= -e^{2\pi^2} \qquad (e^{3\pi i} = \cos 3\pi + i \sin 3\pi = -1)$$

c) $z^{2/3}$

Solution: p.v $z^{2/3} = e^{\frac{2}{3}\operatorname{Log} z} = e^{\frac{2}{3}(\ln|z| + i\theta)}$

$$= e^{\frac{2}{3}\ln r + \frac{2}{3}\theta i}$$

$$= e^{\ln r^{2/3}} . e^{\frac{2}{3}\theta i}$$

$$= \sqrt[3]{r^2}\, e^{\frac{2}{3}\theta i}$$

Note: One can show that the above p.v. is analytic in the domain $r > 0\,, -\pi < \theta < \pi$.

Finally,

$$\frac{d}{dz}(c^z) = \frac{d}{dz}\left(e^{z\log c}\right) = e^{z\log c}.\log c = c^z \log c$$

Which is analytic when the value of $\log c$ is specified, i.e.: it is analytic everywhere.

[6] Inverse of Trigonometric and Hyperbolic Functions

In this section, we shall show the following identities:

1. $\sin^{-1} z = -i\log\left(iz + \sqrt{1 - z^2}\right)$

2. $\cos^{-1} z = -i\log\left(z + i\sqrt{1 - z^2}\right)$

3. $\tan^{-1} z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$

4. $\sinh^{-1} z = \log\left(z + \sqrt{z^2 + 1}\right)$

5. $\cosh^{-1} z = \log\left(z + \sqrt{z^2 - 1}\right)$

6. $\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$

7. $\frac{d}{dz}\sin^{-1} z = \frac{1}{\sqrt{1-z^2}}$

8. $\frac{d}{dz}\cos^{-1} z = \frac{-1}{\sqrt{1-z^2}}$

9. $\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2}$

10. $\frac{d}{dz}\sinh^{-1} z = \frac{1}{\sqrt{1+z^2}}$

11. $\frac{d}{dz}\cosh^{-1} z = \frac{1}{\sqrt{z^2-1}}$

12. $\frac{d}{dz}\tanh^{-1} z = \frac{1}{1-z^2}$

Example: Find the values of the following:

1) $\sin^{-1}(-i)$ 2) $\tan^{-1} 2i$ 3) $\cosh^{-1}(-1)$ 4) $\tanh^{-1}(0)$

Solution:

1) $\sin^{-1}(-i) = -i \log\left[i\,(-i) + \sqrt{1-(-i)^2}\right]$

$$= -i \log\left[1+\sqrt{2}\right] \qquad \dots(1)$$

Now: $\log(1+\sqrt{2}) = \ln(1+\sqrt{2}) + i2n\pi$

And:

$\log(1-\sqrt{2}) = \ln\left|1-\sqrt{2}\right| + i(\pi + 2n\pi)$

$$= -\ln\left|1-\sqrt{2}\right| + i(2n+1)\pi \quad \dots(2)$$

Since $(-1)^n \ln(1+\sqrt{2}) + n\pi i$, constitute the set of values of $\ln(1 \mp \sqrt{2})$ and $n\pi i$ is the same as $2k\pi i$ when n is even and $(2k+1)\pi i$ when n is odd, so

$\sin^{-1}(-i) = -i\left[(-1)^n \ln(1+\sqrt{2}) + n\pi i\right]$

$$= n\pi + i(-1)^{n+1} \ln(1+\sqrt{2})$$

2) $\tan^{-1} 2i = \frac{i}{2} \log\left(\frac{i+2i}{i-2i}\right)$

$$= \frac{i}{2}\log(-3)$$

$$= \frac{i}{2}[\ln 3 + i(\pi + 2n\pi)]$$

$$= \frac{-1}{2}(2n+1)\pi + \frac{i}{2}\ln 3$$

3) $\cosh^{-1}(-1) = \log\left[-1 \mp \sqrt{(-1)^2 - 1}\right] = \log(-1)$

$$= \ln 1 + i(\pi + 2n\pi)$$

$$= (2n+1)\pi i\,,\ \ n = 0, \mp 1, \mp 2, \dots$$

4) $\tanh^{-1}(0) = \frac{1}{2}\log\left(\frac{i+0}{i-0}\right)$

$$= \ln 1 + 2n\pi i$$

$$= 2n\pi i\ ,\ n = 0, \mp 1, \mp 2, \ldots$$

Example: Solve

$$\sin z = 2$$

Solution: $\sin z = 2 \rightarrow z = \sin^{-1} 2$

$$= -i\log\left(2i + \sqrt{1-4}\right)$$

$$= -i\log\left(2i + \sqrt{3}\, i\right)$$

$$= -i\log\left(\left(2 + \sqrt{3}\,\right)i\right)$$

$$\rightarrow -i\log\left(\left(2 + \sqrt{3}\,\right)i\right) = -i\left[\log i + \log\left(2 + \sqrt{3}\,\right)\right]$$

$$= -i\left[\left(\ln 1 + \left(\frac{\pi}{2} + 2n\pi\right)i\right) + \log\left(2 + \sqrt{3}\,\right)\right]$$

$$= \frac{\pi}{2} + 2n\pi - i\log\left(2 + \sqrt{3}\,\right)$$

$$= \pi(1 + 2n) - i\log\left(2 + \sqrt{3}\,\right)$$

Example: Solve

$$\cos z = \sqrt{2}$$

Solution: $\cos z = \sqrt{2} \rightarrow z = \cos^{-1}\sqrt{2}$

$$\cos^{-1} z = -i\log\left(z + i\sqrt{1-z^2}\right)$$

$$\cos^{-1}\sqrt{2} = -i\log\left(\sqrt{2} + i\sqrt{1-\left(\sqrt{2}\right)^2}\right)$$

$$= -i\log\left(\sqrt{2} + i\sqrt{1-2}\right)$$

$$= -i\log\left(\sqrt{2} - 1\right)$$

$$= -i\log\left(\sqrt{2} - 1\right) + 2n\pi$$

Chapter Four

Complex Integration

[1] Definite Integration of $f(t)$

Definition:

Let $f(t)$ be a complex-valued function of real variable t and it can be written as

$$f(t) = u(t) + iv(t)$$

where u and v are real-valued functions. The definite integral of $f(t)$ over an interval $a \le t \le b$, is defined as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Thus:

1. $Re\int_a^b f(t)\,dt = \int_a^b \big(Re(f(t))\big)\,dt = \int_a^b u(t)\,dt$

2. $Im\int_a^b f(t)\,dt = \int_a^b \big(Im(f(t))\big)\,dt = \int_a^b v(t)\,dt$

3. $\int_a^b z_0 f(t)\,dt = z_0\int_a^b f(t)\,dt\,,\ z_0 = x_0 + iy_0$

Proof:

$$\begin{aligned}\int_a^b z_0 f(t)\,dt &= \int_a^b (x_0 + iy_0)(u + iv)\,dt \\ &= \int_a^b [(x_0 u - y_0 v) + i(x_0 v + y_0 u)]\,dt \\ &= \int_a^b (x_0 u - y_0 v)\,dt + i\int_a^b (x_0 v + y_0 u)\,dt \\ &= \int_a^b x_0 u\,dt - \int_a^b y_0 v\,dt + i\int_a^b x_0 v\,dt + i\int_a^b y_0 u\,dt \\ &= x_0\left(\int_a^b u\,dt + i\int_a^b v\,dt\right) + iy_0\left(\int_a^b u\,dt + i\int_a^b v\,dt\right) \\ &= (x_0 + iy_0)\int_a^b f(t)\,dt\end{aligned}$$

4. $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt\ ,\ a < c < b$

5. $\int_a^b (f(t) \mp g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

6. $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$

<u>*Proof*</u>: Suppose that $\int f(t)\,dt \neq 0$

$\because\ \int_a^b f(t)\,dt \neq 0$, then it can be written in polar form:

$\int_a^b f(t)\,dt = r_0 e^{i\theta_0}$ s.t $r_0 = |\int f(t)|$

$\therefore\ r_0 = e^{-i\theta_0} \int_a^b f(t)\,dt = \int_a^b e^{-i\theta_0} f(t)\,dt \qquad (1)$

$\therefore Re \int_a^b e^{-i\theta_0} f(t)\,dt = r_0$

Since both sides of (1) is real number

$\therefore r_0 = \int_a^b Re\left(e^{-i\theta_0} f(t)\right) dt \leq \int_a^b \left|e^{-i\theta_0} f(t)\right| dt$ (by $Rez \leq |Rez| \leq |z|$)

$= \int_a^b \left|e^{-i\theta_0}\right| |f(t)|\,dt$

$= \int_a^b |f(t)|\,dt \qquad$ (Since $\left|e^{-i\theta_0}\right| = 1$)

7. Let $f(t)$ be a continuous function or piecewise continuous function such that $f' = F(t)\ ,\ t \in [a, b]$, then

$$\int_a^b F(t)\,dt = f(b) - f(a)$$

<u>*Proof*</u>:

Let $F(t) = u(t) + iv(t)\ ,\ f(t) = f_1(t) + if_2(t)$

$f'(t) = F(t) \rightarrow f_1'(t) = u(t)\ ,\ f_2'(t) = v(t)$

Integrating both sides with respect to t, we get:

$\int u(t)\,dt = f_1(t)\ ,\ \int v(t)\,dt = f_2(t)$

$\therefore \int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

$= f_1(t)|_a^b + if_2(t)|_a^b$

$= f_1(b) - f_1(a) + if_2(b) - if_2(a)$

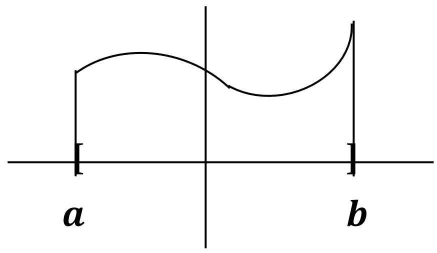
$= \big(f_1(b) + if_2(b)\big) - \big(f_1(a) + if_2(a)\big)$

$= f(b) - f(a)$

Note: Every continuous function from $[a,b]$ to $\mathbb{C}$ represents a curve and it's denoted by

$$z(t) = x(t) + iy(t) \ , t \in [a,b]$$

where $x(t)$ and $y(t)$ are continuous. And $z(a)$, $z(b)$ represent the starting point and end point of the arc.



$[\boldsymbol{a},\boldsymbol{b}] \curvearrowright \mathbb{C}$

For example:

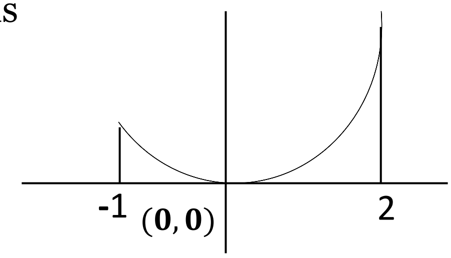
$z(t) = t + i\,t^2 \ , \ -1 \le t \le 2$

$x(t) = t \,,\ y(t) = t^2$, are continuous functions

$z(-1) = -1 + i(-1)^2 = -1 + i = (-1,1)$

$z(2) = 2 + i(2)^2 = 2 + 4i = (2,4)$

$z(0) = (0,0)$



$z(t)$ is a curve which represents all the points in the form (x, x^2).

Example: Calculate the following integrals

1. $\int_0^{\frac{\pi}{6}} e^{2it}\,dt$

Solution:

$$\int_0^{\frac{\pi}{6}} e^{2it}\,dt = \int_0^{\frac{\pi}{6}} (\cos 2t + i\sin 2t)\,dt$$

$$= \int_0^{\frac{\pi}{6}} \cos 2t\,dt + i\int_0^{\frac{\pi}{6}} \sin 2t\,dt$$

$$= \frac{1}{2}\sin 2t\Big|_0^{\frac{\pi}{6}} - \frac{1}{2}i\cos 2t\Big|_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4}i$$

2. $\int_0^1 (1+it)^2\,dt$

Solution:

$$(1+it)^2 = 1 + 2ti - t^2 = (1-t^2) + i2t$$

$$\rightarrow \int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)\,dt + i\int_0^1 2t\,dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^1 + i[t^2]_0^1$$

$$= 1 - \frac{1}{3} + i$$

$$= \frac{2}{3} + i$$

3. $\int_0^{\frac{\pi}{4}} e^{it}\,dt$

Solution: $\int_0^{\frac{\pi}{4}} e^{it}\,dt = \int_0^{\frac{\pi}{4}} (\cos t + i\sin t)\,dt$

$$= \int_0^{\frac{\pi}{4}} \cos t\,dt + i\int_0^{\frac{\pi}{4}} \sin t\,dt$$

$$= \sin t\Big|_0^{\frac{\pi}{4}} - i\cos t\Big|_0^{\frac{\pi}{4}}$$

$$= \left[\sin\frac{\pi}{4} - \sin 0\right] - i\left[\cos\frac{\pi}{4} - \cos 0\right]$$

$= \frac{1}{\sqrt{2}} - i\left[\frac{1}{\sqrt{2}} - 1\right]$

$= \frac{1}{\sqrt{2}} - i\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$

[2] Contours

Definition:

A set of points $z = (x, y)$ in the complex plane is said to be an **arc** if

$$x = x(t), \qquad y = y(t)\ , \qquad a \le t \le b$$

where $x(t)$ and $y(t)$ are continuous functions of the real variable.

Definition:

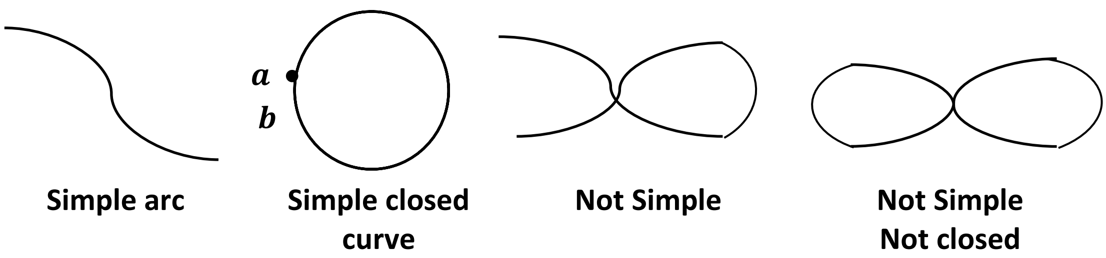
An arc is called simple arc or Jordan arc if it doesn't cross itself, that is simple if

$$z(t_1) \neq z(t_2)\ \text{, when } \ t_1 \neq t_2$$

When the arc C is simple except for the fact that

$$z(b) = z(a)$$

Then we say that C is simple closed curve or Jordan closed curve.



Simple arc **Simple closed curve** **Not Simple** **Not Simple Not closed**

Example: Graph and classify the following

1. $z = \begin{cases} t + it\ , & 0 \le t \le 1 \\ t + i\ , & 1 \le t \le 2 \end{cases}$

Solution:

$z = t + it \rightarrow x = t\ , y = t\ ,\ 0 \le t \le 1$

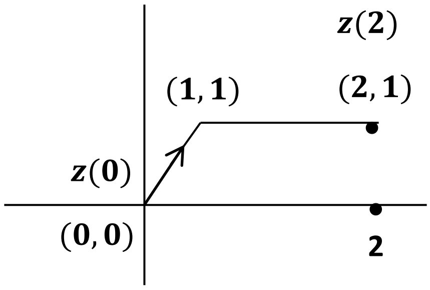
If $t=1 \rightarrow z(1)=1+i=(1,1)$

If $t=0 \rightarrow z(0)=0+0i=(0,0)$

$z=t+i \ \rightarrow x=t\,, y=1\,,\ \ 1\leq t\leq 2$

If $t=1 \rightarrow z(1)=1+i=(1,1)$

If $t=2 \rightarrow z(2)=2+i=(2,1)$



Note: $z(0)\neq z(2)$, i. e: $z(a)\neq z(b)$

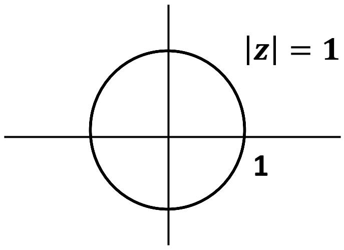
$z(0)\neq z(1)\,,\ 0\neq 1$

$\therefore C$ is simple but not closed curve (the starting point $\neq$ the end point)

2. $z=e^{i\theta}$, $0\leq\theta\leq 2\pi$

Solution:

$|z|=\left|e^{i\theta}\right|=|\cos\theta+i\sin\theta|=1$



It is a unit circle about the origin, since $z(0)=1$ and $z(2\pi)=1$ then the unite circle is a simple closed curve (Jordan curve).

Definition:

Let $z(t)=x(t)+iy(t)$, such that $a\leq t\leq b$ is a curve equation. Then

$$z'(t)=x'(t)+iy'(t)$$

provided that $x'(t), y'(t)$ are exist.

Definition:

We say that $z(t)=x(t)+iy(t), a\leq t\leq b$ is differentiable if $x'(t), y'(t)$ are exist and continuous on $[a,b]$.

Definition:

A differentiable curve $z(t)=x(t)+iy(t), a\leq t\leq b$ is called smooth if $z'(t)\neq 0\ \ \forall\, t\in[a,b]$.

Definition:

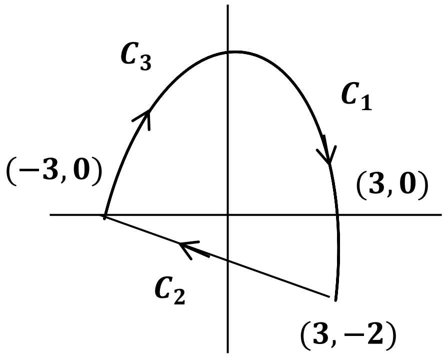
A curve $z(t)$ is called piecewise smooth (contour) if it consists of a finite number of smooth arcs joined end to end.

Example: $C = C_1 + C_2 + C_3$ is a smooth arc

$C_1: z_1(t) = 3 - it\ ,\ 0 \le t \le 2$

$C_2: z_2(t) = -6t + 3 + i(2t - 2)\ ,\ 0 \le t \le 1$

$C_3: z_3(t) = -3\cos t + i3\sin t\ ,\ 0 \le t \le \pi$



$z_1(0) = 3\ ,\ z_1(2) = 3 - 2i$

$z_2(0) = 3 - 2i\ ,\ z_2(1) = -3$

$z_3(0) = -3\ ,\ z_3(\pi) = 3$

Note: $\arg z' = \tan^{-1}\frac{y'(t)}{x'(t)} = \tan^{-1}\frac{dy}{dx}$

Notes:

1. If the derivative exists then it means that there is a tangent to the curve.
2. $z'(t)$ represents a smooth tangent to the arc.
3. The smooth arc is the arc that has a tangent at each point.

Example: $C : z(t) = \begin{cases} t + it^3 , -1 \le t \le 1 \\ t + i \quad , \quad 1 \le t \le 2 \end{cases}$

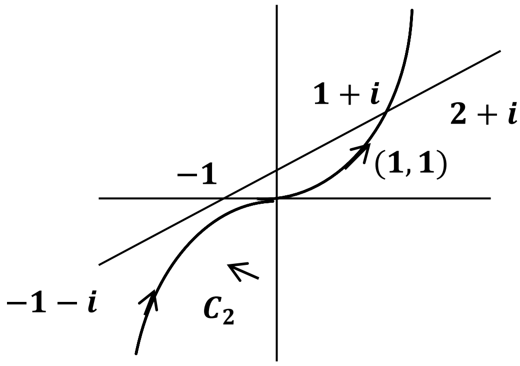
Check that $z(t)$ is simple, smooth?

Solution:

Note that $z(t)$ is simple arc (check?), but not smooth arc since $z'(t)$ is not exist

$z'(t) = 1\ \ ,\ \ 1 \le t \le 2 \rightarrow z'(1) = 0$

(Sharp ends don't make a smooth arc).



Note:

$|z'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$\rightarrow \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = L$ (Length of C)

[3] Contour Integral

Suppose that the equation $z = z(t),\ a \le t \le b,$ represents the contour C connecting $z_1 = z(a)$ to $z_2 = z(b)$.

Let the function $f(z(t))$ be a piecewise on $[a, b]$, we define the line integral or contour integral of f along C as follows:

$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$ (2)

Note that, since C is a contour, $z'(t)$ is piecewise continuous on $[a, b]$, so the existence of integral (2) is ensured from 2, we have

$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$ (3)

$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$

Note:

1. $(-C)$ is the contour connecting $z_2 = z(b)$to $z_1 = z(a)$ and it has a parametric representation (i.e.:$z = z(-t), -b \le t \le -a$)

   Thus:

   $\int_C f(z)dz = \int_C f(z(-t))dz$

   $= \int_{-a}^{-b} f\big(z(-t)\big)\, z'(-t)\, dz$

   $= -\int_C f(z)dz$

   Note: if it is counterclockwise, then multiply by (-1).

2. Suppose that C consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_0 to z_2. Then there is a real number $a \le c \le b$, where $z(c) = z_0$.

   C_1: is represented by $z = z(t)$, $(a \le t \le c)$

   C_2: is represented by $z = z(t)$, $(c \le t \le b)$

   Since:

$$\int_C f(z)dz = \int_a^c f(z(t))\, z'(t)\, dt + \int_c^b f(z(t))\, z'(t)\, dt$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Theorem: If $|f(z)| \le M$, then:

$$\left|\int_C f(z)dz\right| \le ML$$

such that M is constant (bounded) and L is length of contour.

Proof:

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\, z'(t)\, dt\right|$$

$$\le \int_a^b |f(z(t))|\, |z'(t)|\, dt$$

$$\le M \int_a^b |z'(t)|\, dt$$

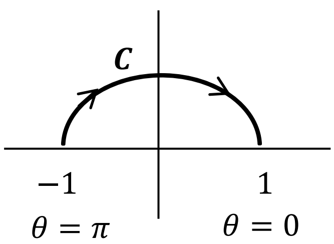
$$= M \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= ML$$

Example: Evaluate the following integrals:

1. $\int_C \bar{z}\, dz$, where C is the upper half of the circle $|z| = 1$ from

   $z = -1$ to $z = 1$



Solution:

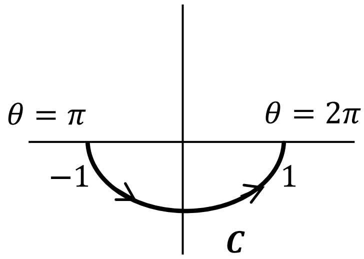
$z = re^{i\theta} = e^{i\theta} \to \bar{z} = e^{-i\theta}$

$\to dz = ie^{i\theta} d\theta$

$\therefore \int_C \bar{z}\, dz = \int_{\pi}^{0} e^{-i\theta}\ (ie^{i\theta} d\theta)$

2. $I = \int_C \bar{z}\, dz$, where C is the lower half of the circle $|z| = 1$ from

$z = -1$ to $z = 1$



Solution:

$r = 1\ ,\ z = e^{i\theta} \rightarrow \bar{z} = e^{-i\theta}$

$\therefore \int_C \bar{z}\, dz = \int_{\pi}^{2\pi} e^{-i\theta}\ (ie^{i\theta} d\theta)$

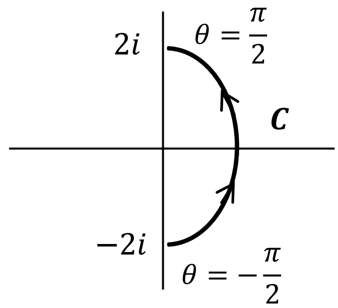
$= i\theta|_{\pi}^{2\pi}$

$= i[2\pi - \pi]$

$= i\pi$

2. $I = \int_C \bar{z}\, dz$, where C is the right half of the circle $|z| = 2$ from

$z = -2i$ to $z = 2i$



Solution:

$r = 2\ ,\ z = 2e^{i\theta} \rightarrow \bar{z} = 2e^{-i\theta}$

$\therefore \int_C \bar{z}\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\, e^{-i\theta}\ (2ie^{i\theta} d\theta)$

$= 4i\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 4i\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$

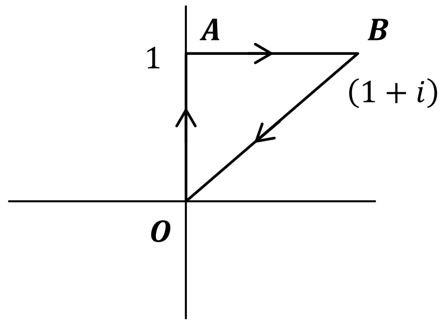
$= 4i\pi$

Example: Evaluate $\int_C \bar{z}\, dz$, where C is the contour OAB:

1. Shown in the accompanied figure and $f(z) = y - x - 3ix^2$

Solution: Take the integration of all paths (arc).

$z = x + iy$, on OA, we have



$z = iy\,,\ x = 0$

$-dz = -idy\,,\ f(z) = y$

$\int_{OA} f(z)dz = \int_0^1 y\, idy$

$= i\,\frac{y^2}{2}\Big|_0^1$

$= \frac{i}{2}$

On AB, we have $y = 1$ and $z = x + i$

$\rightarrow dz = dx\,,\ f(z) = 1 - x - 3ix^2$

$\int_{AB} f(z)dz = \int_0^1 (1 - x - 3ix^2)\, dx$

$= \left[x - \frac{x^2}{2} - ix^3\right]\Big|_0^1$

$= 1 - \frac{1}{2} - i$

$= \frac{1}{2} - i$

$\therefore \int_{OAB} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$

$= \frac{1}{2}i + \frac{1}{2} - i$

$= \frac{1}{2} - \frac{1}{2}i$

2. If C is the contour $OABO$

<u>Solution</u>:

On BO, we have $x = y \rightarrow z = x + ix = (1 + i)x$

$\rightarrow dz = dx + idx = (1 + i)dx$

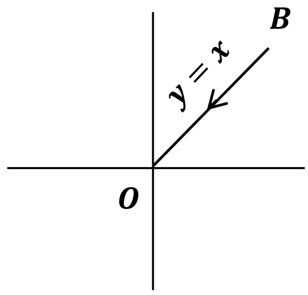
$f(z) = x - x - 3ix^2 = -3ix^2$

$\int_{BO} f(z)dz = \int_1^0 (-3ix^2)\,(1 + i)dx$

$= (1 + i)(-ix^3)|_1^0$

$= 0 + (1+i)i$

$= i - 1$



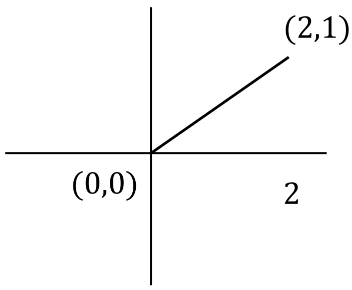
$\therefore \int_{OABO} f(z)dz = \int_{OAB} f(z)dz - \int_{BO} f(z)dz$

$= \left(\frac{1}{2} - \frac{1}{2}i\right) - (i-1)$

$= \frac{3}{2} - \frac{3}{2}i$

Example: Evaluate $\int_C z^2\, dz$, where:

1. C is the line segment from $z = 0$ to $z = 2 + i$.



Solution:

$\frac{x-x_1}{y-y_1} = \frac{x-x_2}{y-y_2}$

$\rightarrow \frac{y}{x} = \frac{2}{1} \rightarrow x = 2y\,,\ 0 \le y \le 1$

$\rightarrow z = x + iy = 2y + iy$

$\rightarrow dz = 2dy + idy = (2+i)dy$

$f(z) = z^2 = (2y + iy\,)^2$

$= ((2+i)y)^2$

$= (4 - 1 + 4i)y^2$

$= (3 + 4i)y^2$

$\therefore \int_C f(z)\, dz = \int_0^1 (3+4i)(2+i)y^2\, dy$

$= (3+4i)(2+i)\left.\frac{y^3}{3}\right|_0^1$

$= \frac{1}{3}(6 - 4 + 3i + 8i)$

$= \frac{1}{3}(2 + 11i)$

2. Find $I_2 = \int_{C_2} z^2\, dz + \int_{C_3} z^2\, dz$

<u>*Solution*</u>:

On C_2, we have

$y = 0\ ,\ z = x\ \rightarrow dz = dx\ , f(x) = x^2$

$\int_{C_2} f(z)\, dz = \int_0^2 x^2\, dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$

On C_3, we have

$x = 2\ ,\ z = 2 + iy\ \rightarrow dz = idy\ , f(x) = (2 + iy)^2$

$\int_{C_3} f(z)\, dz = \int_0^1 (2 + iy)^2\, i\, dy$

$$= i\int_0^1 [4 + 4iy - y^2]\ dy$$

$$= i\left[4y + 2iy^2 - \frac{y^3}{3}\right]\Big|_0^1$$

$$= i\left[4 + 2i - \frac{1}{3}\right]$$

$$= \frac{11}{3}i - 2$$

$\therefore I_2 = \frac{8}{3} + \frac{11}{3}i - 2 = \frac{2}{3} + \frac{11}{3}i$

<u>Example</u>: Show that if C is the circle

$$z - z_0 = re^{i\theta}\ ,\ \ 0 \le \theta \le 2\pi$$

Then

a) $\int_C f(z)\, dz = ir\int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{i\theta} d\theta$

<u>*Solution*</u>: $z - z_0 = re^{i\theta} \rightarrow z = z_0 + re^{i\theta}$

$$\rightarrow dz = ire^{i\theta} d\theta$$

$\int_C f(z)\, dz = \int_0^{2\pi} f(z_0 + re^{i\theta})\, ire^{i\theta} d\theta$

$$= ir\int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{i\theta} d\theta$$

b) $\int_C \frac{dz}{z-z_0}$

Solution:

$$\int_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{z_0+re^{i\theta}-z_0}$$

$$= \int_0^{2\pi} i\, d\theta$$

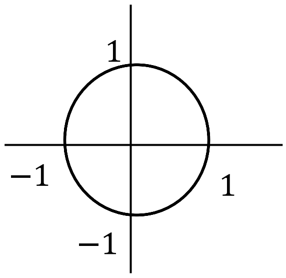
$$= i\theta|_0^{2\pi}$$

$$= 2\pi i$$

Example: Evaluate $\int_C z^n\, dz$, such that C is the circle $|z| = 1$,

i.e.: $z(t) = e^{it}$, $0 \leq t \leq 2\pi$, $n = 0, \mp 1, \ldots$

Solution:



$$\int_C z^n\, dz = \int_0^{2\pi} f(e^{it})\, ie^{it} dt$$

$$\Leftrightarrow \int f(z(t))z' = \int e^{int} ie^{it}$$

$$= i\int_0^{2\pi} e^{it(n+1)}\, dt$$

If $n+1 = 0 \longrightarrow \int z^n dz = i\int_0^{2\pi} dt = 2\pi i$

If $n+1 \neq 0$, let $t(n+1) = k \rightarrow dt = \frac{dk}{n+1}$, then

$\int_0^{2\pi} e^{it(n+1)}\, dt = 0$, since

$$\frac{1}{n+1}\int_0^{2\pi} e^{ik}\, dk = \frac{1}{n+1}\int_0^{2\pi}(\cos k + i\sin k)\, dk$$

$$= \frac{1}{n+1}[\sin k - \cos k]|_0^{2\pi}$$

$$= 0$$

In general,

$$\int_C z^n\, dz = \begin{cases} 0 & if\ n \neq -1 \\ 2\pi i & if\ n = -1 \end{cases}$$

Example: Find $\int_C \frac{dz}{z}$, $C : |z| = 1$

Solution: This example can be solved by two ways:

1. $\int_C \frac{dz}{z} = \int_C z^{-1} dz$

i.e.: $n = -1$, then:

$\int_C \frac{dz}{z} = 2\pi i$

2. $z(t) = re^{i\theta} = 1.e^{i\theta} = e^{i\theta}$

$z'(t) = ie^{i\theta} d\theta$, $0 \le \theta \le 2\pi$

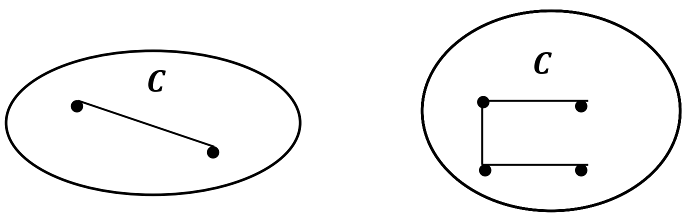
$\int_C \frac{dz}{z} = \int_0^{2\pi} i \frac{e^{i\theta}}{e^{i\theta}} d\theta$

$= i\theta|_0^{2\pi}$

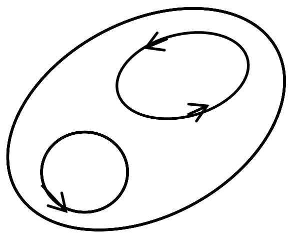
$= 2\pi i$

Definition:

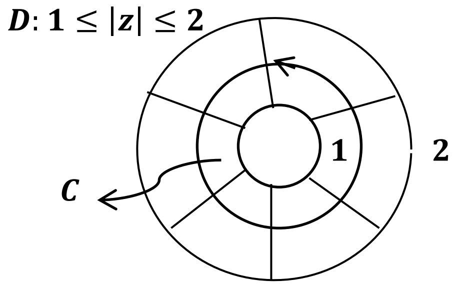
A region D is said to be simply connected if C is a piecewise smooth (closed) curve contained completely in D and then $Int\ C \subset D$.



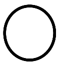
* D is called simply connected if we can connect any two points by a path which is contained completely in D.
* The region D is called simply connected if every closed path in the region contains points from the region, otherwise D is non-simply connected or complex connected.



Simply connected region



Non-simply connected region

The region $D: 1 \leq |z| \leq 2$ is multiply connected since $int\ C \not\subset D$, and the internal circle  $\notin D$. Note that is complex connected since it contained a closed path C which contains points from outside D.

<u>Theorem</u>:

Let D be a simply connected region and let $f(z)$ be an analytic function on D, then

$$\oint_C f(z)\ dz = 0$$

For each simple piecewise smooth curve C contained inside D.

<u>Note</u>:

If the region D is complex connected then it is not necessary that $\oint_C f(z)\ dz = 0$.

The converse of the above theorem is not true as in the following example:

<u>Example</u>:

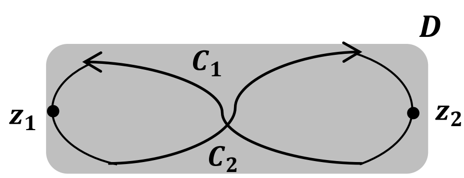
$$\oint_C \frac{dz}{z^2} = 0\ ,\ C:\ |z| = r$$

But $\frac{1}{z^2}$ is not analytic function at $z = 0$.

<u>Note</u>:

Let D be a simply connected region and let $f(z)$ be an analytic function on D. Let $z_1, z_2 \in D$, then

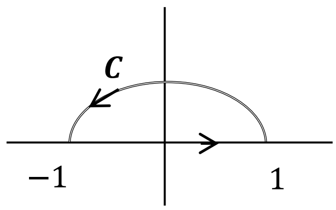
$$\oint_{C_1} f(z)\ dz = \oint_{C_2} f(z)\ dz$$



Such that C_1, C_2 are simple smooth curve which connect z_1 and z_2 , and $C_1, C_2 \subset D$.

Example: Calculate

$$\oint_{C_1+C_2}(3z^2+2z-5)\ dz$$



Such that C_1, C_2 are clear from the graph:

$C_1 : z(t) = t \quad -1 \leq t \leq 1,$

C_2 is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$

Solution:

$f(z) = 3z^2 + 2z - 5$, is analytic $\forall\ \mathbb{C}$, and $z_1 = -1,\ z_2 = 1 \in D$, then

$\oint_{C_1}(3z^2+2z-5)\ dz = \oint_{C_2}(3z^2+2z-5)\ dz$

$\therefore \oint_C f(z)\ dz = \oint_{C_1+C_2} f(z)\ dz = 0$

Note:

The equation of circle with center z_0 and radius r is:

$$C : |z - z_0| = r$$

And the polar form becomes:

$$C : z_0 + re^{i\theta}\ , \qquad 0 \leq \theta \leq 2\pi$$

In general, we can prove:

$$\oint_C (z-z_0)^n\ dz = \begin{cases} 0 & if\ n \neq -1 \\ 2\pi i & if\ n = -1 \end{cases}$$

Proof:

$C : z(t) = z_0 + re^{it}\ , \qquad 0 \leq t \leq 2\pi$

$z'(t) = ire^{it}$

$\oint_C (z-z_0)^n\ dz = \oint_0^{2\pi} r^n e^{int}\ ire^{it} dt = \oint_0^{2\pi} (ir^{n+1}) e^{it(n+1)}\ dt$

If $n + 1 = 0 \to \oint_C (z-z_0)^n\ dz = 2\pi i$

If $n+1 \neq 0 \rightarrow \oint_C (z-z_0)^n \, dz = \frac{r^{n+1}}{n+1} \; e^{it(n+1)}\Big|_0^{2\pi}$

$$= \frac{r^{n+1}}{n+1}\left[\cos(n+1)t + i\sin(n+1)t\right]\Big|_0^{2\pi}$$

$$= 0$$

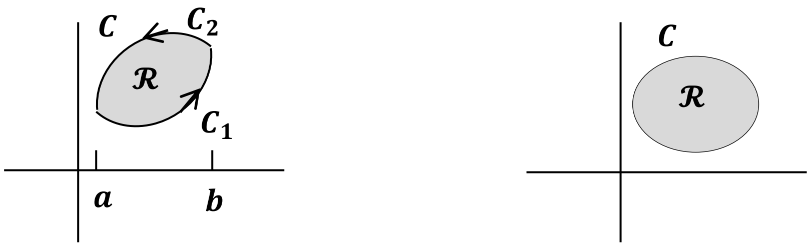
[4] Cauchy Goursat Theorem

The following theorem will be needed through this section:

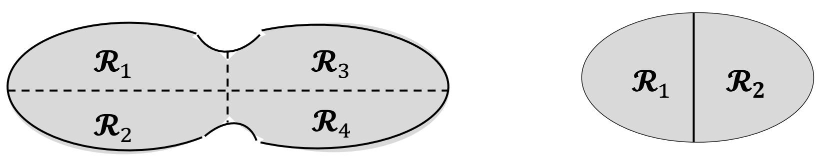
Green's theorem:

Suppose that $p(x,y)$ and $\emptyset(x,y)$ are two real-valued functions and $p, \emptyset$ are continuous with their first partial derivatives, throughout a closed region $\mathcal{R}$ consisting of points interior within and on a simple closed contour C in the xy-plane, then

$$\oint_C (p dx + \emptyset dy) = \iint_{\mathcal{R}} \left(\emptyset_x - p_y\right) dxdy$$



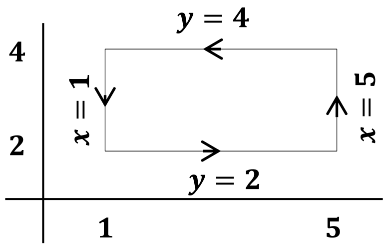
Note: Green's theorem might be extended to a finite union of closed regions.



Example: Evaluate

$$\oint_C \left(\left(e^{x^2} + y\right)dx + \left(x^2 + \tan^{-1}\sqrt{y}\right)dy\right)$$

Where C is the boundary of the rectangle having the vertices $(1,2)$, $(5,2), (5,4)$, and $(1,4)$.



Solution: By using Green's theorem

$p(x, y) = e^{x^2} + y \;,\; \emptyset(x, y) = x^2 + \tan^{-1}\sqrt{y}$

$p_y(x, y) = 1 \qquad , \; \emptyset_x(x, y) = 2x$

$$\therefore \oint_C \left(\left(e^{x^2} + y\right)dx + \left(x^2 + \tan^{-1}\sqrt{y}\right)dy\right) = \int_2^4 \int_1^5 (2x - 1)\, dxdy$$

$$= \int_2^4 (x^2 - x)\Big|_1^5 dy$$

$$= \int_2^4 20\, dy = 20y|_2^4 = 40$$

Note: If $f(z) = u(x, y) + iv(x, y)$ is analytic on $\mathcal{R}$, where u, v and their first partial derivatives are continuous in $\mathcal{R}$, then

$$\int_C f(z)\, dz = 0$$

Proof: $z = x + iy \to dz = dx + idy$

$\int_C f(z)\, dz = \int_C (u + iv)\,(dx + idy)$

$\qquad = \int_C (udx - vdy) + i\int_C (vdx + udy)$

By using Green's theorem, we get:

$\int_C f(z)\, dz = \iint_{\mathcal{R}} \left(-v_x - u_y\right) dxdy + i\iint_{\mathcal{R}} \left(u_x - v_y\right) dxdy$

But f is analytic, then f satisfies C-R equations

i.e.: $u_x = v_y \;,\; u_y = -v_x$

$\therefore \int_C f(z)\, dz = 0$

Cauchy-Goursat theorem: **(C.G.T)**

If f is analytic function at each point within and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Note:

The C.G.T can be stated in the following alternative form:

If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

For every simple closed contour C lying in D.

Example: Determine the domain of analyticity of the function f and apply the C.G.T to show that

$$\int_C f(z)\,dz = 0$$

where C is the circle $|z| = 1$, when

a. $f(z) = \frac{z^2}{z-3}$

Solution:

D_f is $\mathbb{C} \setminus \{3\}$

$\therefore$ So f is analytic everywhere except at $z = 3$ which is not in the circle $|z| = 1$.

$\therefore$ By C.G.T, we have:

$\int_C \frac{z^2}{z-3}\,dz = 0$

Since C is simple closed contour.

b. $f(z) = ze^{-z}$

Solution:

$f(z) = ze^{-z} = \frac{z}{e^z}$

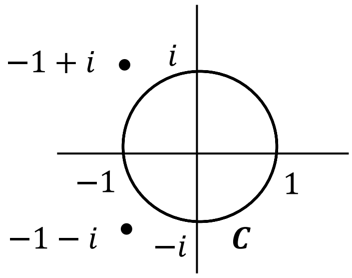
D_f is $\mathbb{C}$, f is analytic everywhere (entire function), so by C.G.T:

$\int_C f(z)\, dz = 0$

Since C is simple closed contour.

c. $f(z) = \frac{1}{z^2+2z+2}$

Solution:



$f(z) = \frac{1}{z^2+2z+2}$

$= \frac{1}{z^2+2z+1+1}$

$= \frac{1}{(z+1)^2+1}$

D_f is $\mathbb{C} \setminus \{-1+i, -1-i\}$

f is analytic function everywhere except at the point $-1+i, -1-i$ which both aren't belonging to the circle $|z| = 1$, so by C.G.T we have:

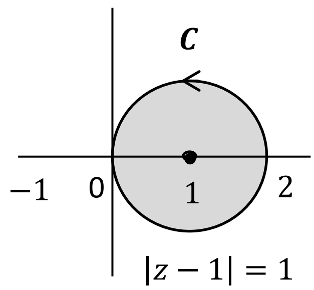
$\int_C f(z)\, dz = 0$

Since C is simple closed contour.

Example: Evaluate the following integral

$$\oint \frac{1}{z^2-1}\, dz \ , \ \ C : |z-1| = 1$$

Solution:



$f(z) = \frac{1}{z^2-1}$

$= \frac{1}{(z-1)(z+1)}$

$= \frac{1/2}{z-1} - \frac{1/2}{z+1}$

Inside path   Outside path

$\therefore \int \frac{1}{z^2-1}\, dz = \frac{1}{2}\int \frac{1}{z-1}\, dz - \frac{1}{2}\int \frac{1}{z+1}\, dz$

Note: $\frac{1}{z+1}$ is analytic function in $|z-1| = 1$

$\therefore \int \frac{1}{z+1}\, dz = 0$

But $\frac{1}{z-1}$ is not analytic in $|z-1| = 1$

Let: $z - 1 = re^{i\theta} \rightarrow dz = ire^{i\theta} d\theta$

$$\therefore \frac{1}{2}\int \frac{1}{z-1}\, dz = \frac{1}{2}\int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}}$$

$$= \frac{i}{2}\int_0^{2\pi} d\theta$$

$$= \frac{i}{2}\theta\Big|_0^{2\pi}$$

$$= i\pi$$

$$\therefore \int_C \frac{1}{z^2-1}\, dz = \frac{1}{2}\int \frac{1}{z-1}\, dz - \frac{1}{2}\int \frac{1}{z+1}\, dz$$

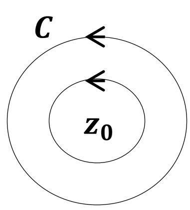
$$= i\pi - 0$$

$$= i\pi$$

[5] The Cauchy Integral Formula

<u>Theorem 1</u>: The Cauchy integral formula states that:

If a function f is analytic everywhere in and within a simple closed contour C and if z_0 is any interior point of C, then



$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz$$

or $\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$

And the integral is taken in the positive direction around C.

Remark: The general formula of Cauchy integral C.I.F is called general Cauchy integral formula and it says that:

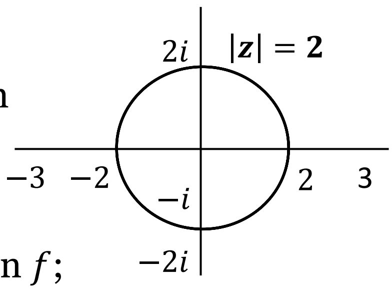
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\text{i. e.: } \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Example: Evaluate the following integrals

1. $\oint_C \frac{z}{(9-z^2)(z+i)}\, dz$, where $C: |z| = 2$, taken in the positive sense.

Solution:



It is clear that only $z = -i$ lies within the given circle, so the function $f(z) = \frac{z}{9-z^2}$ is analytic within and on C, thus we can apply the C.I.F on f;

$$\text{i. e.: } \oint_C \frac{z}{(9-z^2)(z+i)}\, dz = 2\pi i\, f(-i) = \frac{\pi}{5}$$

2. $\oint_C \frac{z^3+2z+1}{(z-1)^3}\, dz$, where $C : |z| = 3$, taken in the positive sense.

Solution:

It is clear that $z = 1$ is inside the circle $|z| = 3$, we will use the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

If $z_0 = 1$ and $n = 2$, then we have:

$$f^{(2)}(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-1)^3}\, dz$$

where $f(z) = z^3 + 2z + 1$, thus

$$\oint_C \frac{f(z)}{(z-1)^3}\,dz = \frac{2\pi i}{2} f^{(2)}(1) = \pi i f^{(2)}(1)$$

$$\rightarrow \frac{d^2}{dz^2}[z^3+2z+1]\Big|_{z=1} = 6z|_{z=1} = 6$$

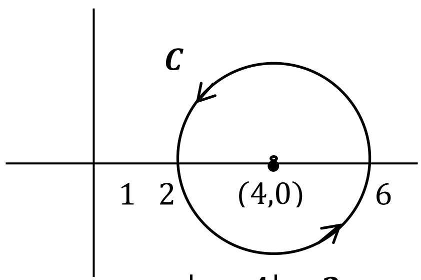
$$\therefore \oint_C \frac{z^3+2z+1}{(z-1)^3}\,dz = 6\pi i$$

3. $\oint_C \frac{\cos z}{(z-1)^3(z-5)^2}\,dz$, where $C: |z-4| = 2$ taken in the positive sense.

Solution:

It is clear that the term $(z-1)^3$ is nonzero on and inside the given contour of integration, but the term $(z-5)^2$ equals zero at $z = 5$ inside C. Then we rewrite the integral as:

$$\oint_C \frac{\frac{\cos z}{(z-1)^3}}{(z-5)^2}\,dz$$



Applying the formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

with $z_0 = 5,\ n = 1$, and $f(z) = \frac{\cos z}{(z-1)^3}$, thus:

$$\oint_C \frac{\cos z/(z-1)^3}{(z-5)^2}\,dz = 2\pi i\ \frac{d}{dz}\left[\frac{\cos z}{(z-1)^3}\right]\Big|_{z=5}$$

$$= 2\pi i\left[\frac{-(z-1)^3\sin z - 3\cos z(z-1)^2}{(z-1)^6}\right]\Big|_{z=5}$$

$$= 2\pi i\left[\frac{-4\sin 5 - 3\cos 5}{256}\right]$$

4. $\oint_C \frac{dz}{z(z+\pi i)}$, where $C: z(t) = z_0 + re^{it}\ ,\ 0 \le t \le 2\pi$

Solution:

Note that the singular points are $0, -\pi i$, thus we take first

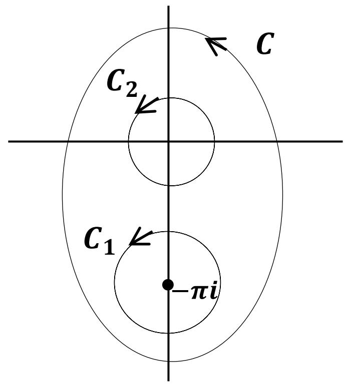
$$f(z) = \frac{1}{z}\ ,\ z_0 = -\pi i$$

Then: $\oint_C \frac{f(z)}{z-z_0}\, dz = \oint \frac{1/z}{z-(-\pi i)}\, dz$

$= 2\pi i\, f(-\pi i)$

$= 2\pi i\, \frac{1}{-\pi i}$

$= -2$



Now, let $f(z) = \frac{1}{z+\pi i}\ ,\ z_0 = 0$

$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint \frac{1/(z+\pi i)}{z}\, dz$

$= 2\pi i\, f(0)$

$= 2\pi i\, \frac{1}{\pi i}$

$= 2$

By Cauchy Goursat theorem, we find

$\int_C \frac{f(z)}{z-z_0}\, dz = \int_{C_1} \frac{f(z)}{z-z_0}\, dz + \int_{C_2} \frac{f(z)}{z-z_0}\, dz$

$= -2 + 2$

$= 0$

5. $\oint_C \frac{e^z}{z-i}\, dz$, where $C : |z| = 2$

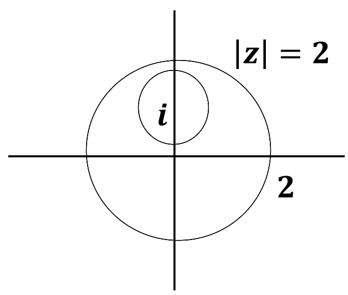
<u>*Solution*</u>:

Note $f(z) = e^z$ is analytic function and $z_0 = i$ is the only singular point $\in Int\ C$

$\oint_C \frac{e^z}{z-i}\, dz = 2\pi i\, f(z_0)$

$= 2\pi i\, f(i)$

$= 2\pi i\, e^i$



Note:

1. If z_0 is outside the path then we use Cauchy Goursat Theorem ($\int_C f(z)\,dz = 0$).
2. If z_0 is inside the path then we use Cauchy integral formula.
3. If z_0 is on the path then we divide the path and apply the integration.

Example: find $\oint_C \frac{\sin z}{z}\,dz\,,\ C : |z| = 1$

Solution:

$$f(z) = \frac{\sin z}{z}\ ,\ z_0 = 0 \in C$$

$$\oint_C \frac{\sin z}{z}\,dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(0)$$

$$= 2\pi i \sin 0$$

$$= 0$$

Cauchy's Inequality:

If $f(z)$ is analytic function on and within C, such that C: $|z - z_0| = r$ then:

$$\left|f^{(n)}(z_0)\right| = \frac{n!M}{r^n}$$

where $|f(z)| \le M \quad \forall z \in \mathbb{C}$.

Proof:

By the general Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}\right|$$

$$\le \frac{n!}{2\pi} \oint_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$\leq \frac{n!\,M}{2\pi} \oint_C \frac{|dz|}{r^{n+1}}$

$= \frac{n!\,M}{2\pi} \frac{2\pi r}{r^{n+1}}$

$= \frac{n!M}{r^n}$

Where $\oint_C |dz| = 2\pi r$, circumference of the circle (length of the path)

If $n = 1$, then:

$|f'(z_0)| = \frac{M}{r}$

[6] Derivatives of Analytic Functions

Now, we are ready to prove the following theorem:

Theorem:

If f is analytic function at a point then its derivatives of all orders are analytic functions at that point.

Proof: Let f be an analytic function within and on a positively oriented simple closed contour C. Let z be any point inside C. Letting s denotes the points on C, and then by C.I.F, we have:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds \qquad \ldots (1)$$

We will show that $f'(z)$ exists and

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds \qquad \ldots (2)$$

To do this, using formula (1), we have:

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \left(\frac{1}{s-\Delta z-z} - \frac{1}{s-z}\right) f(s)ds$$

$$\frac{f(s)ds}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{(s-z-s+z+\Delta z)}{(s-\Delta z-z)(s-z)\Delta z} f(s)ds$$

$$= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-\Delta z-z)(s-z)}\, ds \quad \ldots (3)$$

If d is the smallest distance from z to s on C, then

$$|s - z| \geq d$$

And if $|\Delta z| < d$, then

$$|s - z - \Delta z| \geq |s - z| - |\Delta z| \geq d - |\Delta z|$$

Since f is analytic within and on C, it is also continuous and so it is bounded on C. i.e.: $|f(s)| \leq K$, and if the length of C is L, then

$$\left|\int_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right] f(s)ds\right| = \left|\Delta z \int_C \frac{f(s)ds}{(s-\Delta z-z)(s-z)^2}\right|$$

$$\leq |\Delta z| \int_C \frac{|f(s)||ds|}{(d-|\Delta z|)d^2}$$

$$\leq \frac{|\Delta z|K}{(d-|\Delta z|)d^2} \int_C |dz|$$

$$= \frac{|\Delta z|\ K\ L}{(d-|\Delta z|)d^2}$$

Hence, when $\Delta z \to 0$, then

$$\frac{|\Delta z|\ K\ L}{(d-|\Delta z|)d^2} \to 0$$

Or:

$$\int_C \frac{f(s)ds}{(s-\Delta z-z)(s-z)} - \int_C \frac{f(s)ds}{(s-z)^2} \to 0$$

That means, the integral (3) approaches the integral (2) as $\Delta z \to 0$, so

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^2}$$

Or:

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\ ds$$

If we apply the same technique to formula (2), we find that:

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)}{(s-z)^3}\ ds \quad ...(4)$$

In general, one can show that:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}}\, ds$$

This is called the extension of C.I.F.

Theorem:

Suppose that f is a continuous function on a simply connected domain D, then the following statements are equivalent:

a) There exists a function F such that $F' = f$.
b) $\int_C f(z)\, dz = 0$, for any simple closed contour C.
c) $\int_C f(z)\, dz$ depends only on the end points of C for any contour C.

Remark:

Part (c) in the above theorem means that the integral $\int_C f(z)\, dz$ is independent of path connecting the end points of contour C.

[7] Morera's Theorem

If f is continuous function through a simply connected domain D and if

$$\int_C f(z)\, dz = 0$$

for every simple closed contour C lying in D, then f is analytic through out D.

Proof:

Since $\int_C f(z)\, dz = 0$, for every simple closed contour C in D, and the values of the contour integrals are independent of the contour in D, then:

By part (a) of the previous theorem, the function f has an antiderivative everywhere in D, that is there exists an analytic function F such that $F' = f$, then it follows that f is analytic in D since it's the derivative of an analytic function.

Maximum Moduli of Function

Theorem 1:

Let f be analytic and not constant in some domain D such that $|f(z)|$ is constant, and then $f(z)$ is also constant in D

Theorem 2:

Let f be analytic and not constant in a $\epsilon -$ ngh of z_0, then there is at least one point z in that ngh. Such that

$$|f(z)| \geq |f(z_0)|$$

Maximum Principle

Theorem:

Let f be analytic and not constant in a domain D, then $|f(z)|$ has no maximum value in D.

Proof:

Since f is analytic and not constant in a domain D, then f is not constant over any ngh of any point in D.

Suppose that $|f(z)|$ has a maximum value at z_0 in D, it follows that:

$$|f(z_0)| \geq |f(z)|$$

For each point z in a ngh of z_0, but this contradicts the fact that

$$|f(z)| \geq |f(z_0)| \qquad \text{(Th. 2)}$$

Thus $|f(z)|$ has no maximum value for any ngh of D, so that $|f(z)|$ has no maximum value in D.

Corollary:

If f is a continuous function in a closed bounded region $\mathcal{R}$ and analytic, and not constant in the interior of $\mathcal{R}$, then $|f|$ has a maximum value on the boundary of $\mathcal{R}$ and never in the interior.

Proof:

Since f is continuous in a closed bounded region $\mathcal{R}$, then $|f|$ has a maximum value in $\mathcal{R}$, and by the maximum principle theorem $|f|$ has no maximum value in the interior of $\mathcal{R}$, then $|f|$ has no maximum value on the boundary of $\mathcal{R}$.

Minimum Principle

Theorem:

Let f be a continuous function in a closed bounded region $\mathcal{R}$, and let f be analytic and not constant throughout the interior of $\mathcal{R}$. If $|f(z)| \neq 0$ anywhere in $\mathcal{R}$, then $|f(z)|$ has a minimum value in $\mathcal{R}$ which occurs on the boundary of $\mathcal{R}$, and never in the interior of $\mathcal{R}$.

Proof: Define a function F by:

$$F(z) = \frac{1}{f(z)} \ , \ f(z) \neq 0 \text{ in } \mathcal{R}$$

F is analytic and not constant throughout the interior of $\mathcal{R}$, so by corollary, $|F|$ has a maximum value on the boundary of $\mathcal{R}$. This implies that there is z_0 on the boundary of in $\mathcal{R}$, such that

$$|F(z)| \leq |F(z_0)|$$

$$\left|\frac{1}{f(z)}\right| \leq \left|\frac{1}{f(z_0)}\right|$$

Or

$$|f(z)| \geq |f(z_0)|$$

Thus, $|f(z)|$ has a minimum value in $\mathcal{R}$ which occurs on the boundary of $\mathcal{R}$, and never in the interior of $\mathcal{R}$.

[8] Liouville's Theorem

Theorem:

If f is entire function and bounded for all values of z in the complex plane $\mathbb{C}$, then $f(z)$ is constant throughout the plane.

Proof: Since f is entire function in $\mathbb{C}$, then f is analytic in $\mathbb{C}$, so Cauchy's inequality holds,

$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$, $n = 1,2,3,\ldots$

$\rightarrow |f'(z_0)| = \frac{M}{r}$

Since $|f(z)| \leq M,\ \forall z \in \mathbb{C}$. If we chose r large enough, we should have $f'(z_0) = 0$ for any z, since z_0 is any arbitrary point, then

$$f'(z_0) = 0\,, \qquad \forall z \in \mathbb{C}$$

So f is constant.

[9] The Fundamental Theorem of Algebra

Theorem:

Any polynomial $p(z)$, such that

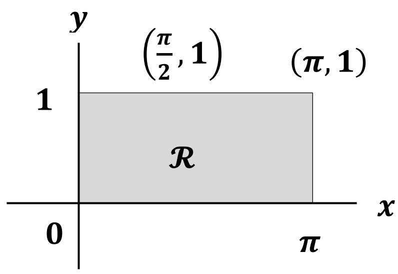
$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \ \ , a_n \neq 0$$

for all $n \geq 0$, has at least one zero that is there exists at least one point z_0 such that $p(z_0) = 0$.

Example:

1. Let $\mathcal{R}$ denotes the rectangular region $0 \leq x \leq \pi\,, 0 \leq y \leq 1$, find the maximum and minimum values of f, when

$$f(z) = \sin z$$



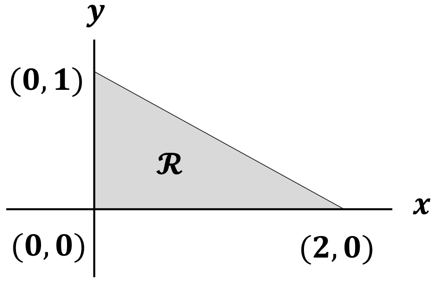
Solution:

$|f(z)| = |\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$

It is clear that the term $\sin^2 x$ is greatest when $x = \frac{\pi}{2}$, and the increasing function $\sinh^2 y$ is greatest when $y = 1$, then the maximum value of $|f(z)|$ in $\mathcal{R}$ occurs at the boundary point $z = \left(\frac{\pi}{2}, 1\right)$ and the minimum value of $|f(z)|$ in $\mathcal{R}$ occurs at the boundary point $z = (0,0)$.

2. Let $f(z) = (z+1)^2$, and the region $\mathcal{R}$ is the triangle with vertices at the points $z = 0,\ z = 2$ and $z = i$. Find points in $\mathcal{R}$ where $|f(z)|$ have its maximum and minimum values.

<ins>*Solution*</ins>:



$$|f(z)| = |(z+1)^2| = |(x+iy+1)^2|$$
$$= \left|\left((x+1)+iy\right)^2\right|$$
$$= |(x+1)+iy|^2$$
$$= (x+1)^2 + y^2\ ,\ \ 0 \le x \le 2\,, 0 \le y \le 1$$

Since the maximum and minimum values occur on the boundary of $\mathcal{R}$, so it is clear that $|f(z)|$ takes maximum value when $x = 2$ and $y = 0$, i.e. at $z = 2$, and takes its minimum value when $x = 0$ and $y = 0$, i.e. at $z = 0$.

3. Let $f(z) = e^z$ in the region $|z| \le 1$. Find the points in this region, where $|f(z)|$ achieves its maximum and minimum values.

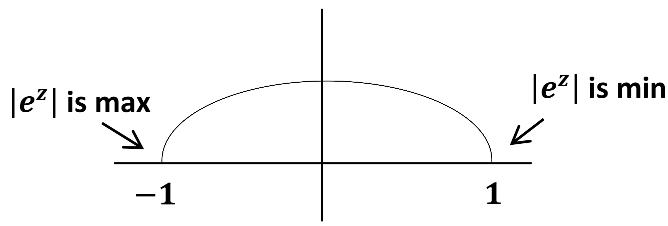
<ins>*Solution*</ins>:

Since e^z is entire function, $e^z \ne 0\ , \forall z$ in the region, both maximum and minimum points are guaranteed by our results.

Now, we have

$$|f(z)| = |e^z| = \left|e^x . e^{iy}\right| = |e^x|$$

Then, its maximum value will occur at the boundary points $(x, y) = (1,0)$ and $|f(z)|$ takes minimum value at the boundary points $(x, y) = (-1,0)$, as in the Fig.



Chapter Four

Complex Integration

[1] Definite Integration of $f(t)$

Definition:

Let $f(t)$ be a complex-valued function of real variable t and it can be written as

$$f(t) = u(t) + iv(t)$$

where u and v are real-valued functions. The definite integral of $f(t)$ over an interval $a \le t \le b$, is defined as

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Thus:

1. $Re\int_a^b f(t)\,dt = \int_a^b \big(Re(f(t))\big)\,dt = \int_a^b u(t)\,dt$

2. $Im\int_a^b f(t)\,dt = \int_a^b \big(Im(f(t))\big)\,dt = \int_a^b v(t)\,dt$

3. $\int_a^b z_0 f(t)\,dt = z_0\int_a^b f(t)\,dt\,,\ z_0 = x_0 + iy_0$

Proof:

$$\begin{aligned}\int_a^b z_0 f(t)\,dt &= \int_a^b (x_0 + iy_0)(u + iv)\,dt\\ &= \int_a^b [(x_0u - y_0v) + i(x_0v + y_0u)]\,dt\\ &= \int_a^b (x_0u - y_0v)\,dt + i\int_a^b (x_0v + y_0u)\,dt\\ &= \int_a^b x_0u\,dt - \int_a^b y_0v\,dt + i\int_a^b x_0v\,dt + i\int_a^b y_0u\,dt\\ &= x_0\left(\int_a^b u\,dt + i\int_a^b v\,dt\right) + iy_0\left(\int_a^b u\,dt + i\int_a^b v\,dt\right)\\ &= (x_0 + iy_0)\int_a^b f(t)\,dt\end{aligned}$$

4. $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt\,,\ a < c < b$

5. $\int_a^b (f(t) \mp g(t))\,dt = \int_a^b f(t)\,dt + \int_a^b g(t)\,dt$

6. $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$

Proof: Suppose that $\int f(t)\,dt \neq 0$

$\because\ \int_a^b f(t)\,dt \neq 0$, then it can be written in polar form:

$\int_a^b f(t)\,dt = r_0 e^{i\theta_0}$ s.t $r_0 = |\int f(t)|$

$\therefore\ r_0 = e^{-i\theta_0} \int_a^b f(t)\,dt = \int_a^b e^{-i\theta_0} f(t)\,dt \qquad (1)$

$\therefore\ Re \int_a^b e^{-i\theta_0} f(t)\,dt = r_0$

Since both sides of (1) is real number

$\therefore\ r_0 = \int_a^b Re\left(e^{-i\theta_0} f(t)\right) dt \le \int_a^b \left|e^{-i\theta_0} f(t)\right| dt$ (by $Rez \le |Rez| \le |z|$)

$= \int_a^b \left|e^{-i\theta_0}\right| |f(t)|\,dt$

$= \int_a^b |f(t)|\,dt \qquad$ (Since $\left|e^{-i\theta_0}\right| = 1$)

7. Let $f(t)$ be a continuous function or piecewise continuous function such that $f' = F(t)$, $t \in [a, b]$, then

$$\int_a^b F(t)\,dt = f(b) - f(a)$$

Proof:

Let $F(t) = u(t) + iv(t)$, $f(t) = f_1(t) + if_2(t)$

$f'(t) = F(t) \rightarrow f_1'(t) = u(t)$, $f_2'(t) = v(t)$

Integrating both sides with respect to t, we get:

$\int u(t)\,dt = f_1(t)$, $\int v(t)\,dt = f_2(t)$

$\therefore\ \int_a^b F(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$

$= f_1(t)|_a^b + if_2(t)|_a^b$

$= f_1(b) - f_1(a) + if_2(b) - if_2(a)$

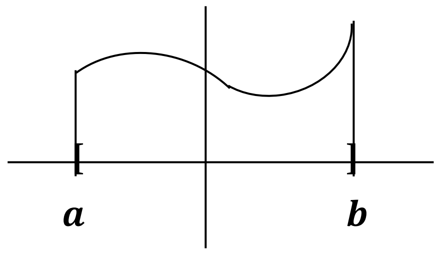
$= \big(f_1(b) + if_2(b)\big) - \big(f_1(a) + if_2(a)\big)$

$= f(b) - f(a)$

<u>Note</u>: Every continuous function from $[a,b]$ to $\mathbb{C}$ represents a curve and it's denoted by

$$z(t) = x(t) + iy(t) \ , t \in [a,b]$$

where $x(t)$ and $y(t)$ are continuous. And $z(a)$, $z(b)$ represent the starting point and end point of the arc.



$[\boldsymbol{a},\boldsymbol{b}] \curvearrowright \mathbb{C}$

<u>For example</u>:

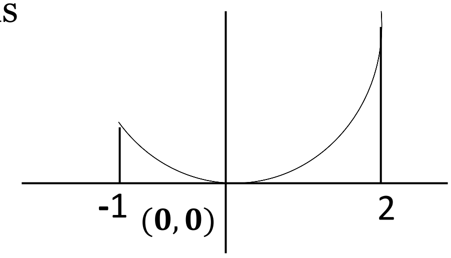
$z(t) = t + i\,t^2 \ , \ -1 \leq t \leq 2$

$x(t) = t \ , \ y(t) = t^2$, are continuous functions

$z(-1) = -1 + i(-1)^2 = -1 + i = (-1,1)$

$z(2) = 2 + i(2)^2 = 2 + 4i = (2,4)$

$z(0) = (0,0)$



$z(t)$ is a curve which represents all the points in the form (x, x^2).

Example: Calculate the following integrals

1. $\int_0^{\frac{\pi}{6}} e^{2it}\, dt$

Solution:

$$\int_0^{\frac{\pi}{6}} e^{2it}\, dt = \int_0^{\frac{\pi}{6}} (\cos 2t + i \sin 2t)\, dt$$

$$= \int_0^{\frac{\pi}{6}} \cos 2t\, dt + i \int_0^{\frac{\pi}{6}} \sin 2t\, dt$$

$$= \frac{1}{2} \sin 2t \Big|_0^{\frac{\pi}{6}} - \frac{1}{2} i \cos 2t \Big|_0^{\frac{\pi}{6}}$$

$$= \frac{\sqrt{3}}{4} - \frac{1}{4} i$$

2. $\int_0^1 (1 + it)^2\, dt$

Solution:

$$(1 + it)^2 = 1 + 2ti - t^2 = (1 - t^2) + i2t$$

$$\rightarrow \int_0^1 (1 + it)^2\, dt = \int_0^1 (1 - t^2)\, dt + i \int_0^1 2t\, dt$$

$$= \left[t - \frac{t^3}{3}\right]_0^1 + i[t^2]_0^1$$

$$= 1 - \frac{1}{3} + i$$

$$= \frac{2}{3} + i$$

3. $\int_0^{\frac{\pi}{4}} e^{it}\, dt$

Solution: $\int_0^{\frac{\pi}{4}} e^{it}\, dt = \int_0^{\frac{\pi}{4}} (\cos t + i \sin t)\, dt$

$$= \int_0^{\frac{\pi}{4}} \cos t\, dt + i \int_0^{\frac{\pi}{4}} \sin t\, dt$$

$$= \sin t \Big|_0^{\frac{\pi}{4}} - i \cos t \Big|_0^{\frac{\pi}{4}}$$

$$= \left[\sin \frac{\pi}{4} - \sin 0\right] - i \left[\cos \frac{\pi}{4} - \cos 0\right]$$

$= \frac{1}{\sqrt{2}} - i\left[\frac{1}{\sqrt{2}} - 1\right]$

$= \frac{1}{\sqrt{2}} - i\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$

[2] Contours

<u>Definition</u>:

A set of points $z = (x, y)$ in the complex plane is said to be an **arc** if

$$x = x(t), \qquad y = y(t)\ , \qquad a \le t \le b$$

where $x(t)$ and $y(t)$ are continuous functions of the real variable.

<u>Definition</u>:

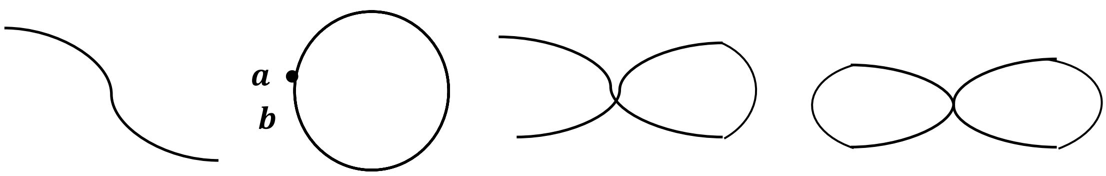
An arc is called simple arc or Jordan arc if it doesn't cross itself, that is simple if

$$z(t_1) \neq z(t_2)\ , \text{when } t_1 \neq t_2$$

When the arc C is simple except for the fact that

$$z(b) = z(a)$$

Then we say that C is simple closed curve or Jordan closed curve.



Simple arc | Simple closed curve | Not Simple | Not Simple Not closed

<u>Example</u>: Graph and classify the following

1. $z = \begin{cases} t + it\ ,\ 0 \le t \le 1 \\ t + i\ , 1 \le t \le 2 \end{cases}$

<u>Solution</u>:

$z = t + it \to x = t\ , y = t\ ,\ 0 \le t \le 1$

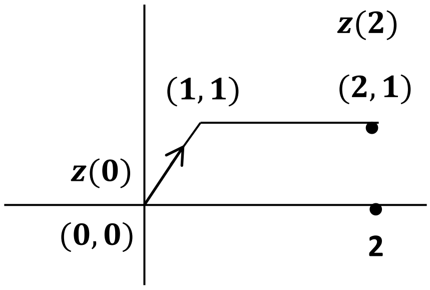
If $t=1 \rightarrow z(1)=1+i=(1,1)$

If $t=0 \rightarrow z(0)=0+0i=(0,0)$

$z=t+i \ \rightarrow x=t\,, y=1\,,\ 1\leq t\leq 2$

If $t=1 \rightarrow z(1)=1+i=(1,1)$

If $t=2 \rightarrow z(2)=2+i=(2,1)$

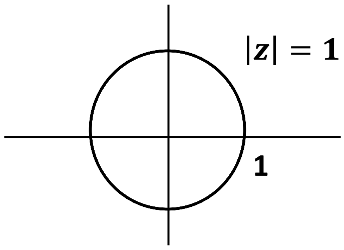


Note: $z(0)\neq z(2)$, i.e: $z(a)\neq z(b)$

$z(0)\neq z(1)\,,\ 0\neq 1$

$\therefore C$ is simple but not closed curve (the starting point $\neq$ the end point)

2. $z=e^{i\theta}\ ,\ 0\leq\theta\leq 2\pi$



Solution:

$|z|=\left|e^{i\theta}\right|=|\cos\theta+i\sin\theta|=1$

It is a unit circle about the origin, since $z(0)=1$ and $z(2\pi)=1$ then the unite circle is a simple closed curve (Jordan curve).

Definition:

Let $z(t)=x(t)+iy(t)$, such that $a\leq t\leq b$ is a curve equation. Then

$$z'(t)=x'(t)+iy'(t)$$

provided that $x'(t), y'(t)$ are exist.

Definition:

We say that $z(t)=x(t)+iy(t), a\leq t\leq b$ is differentiable if $x'(t), y'(t)$ are exist and continuous on $[a,b]$.

Definition:

A differentiable curve $z(t)=x(t)+iy(t), a\leq t\leq b$ is called smooth if $z'(t)\neq 0\ \ \forall\, t\in[a,b]$.

<u>Definition</u>:

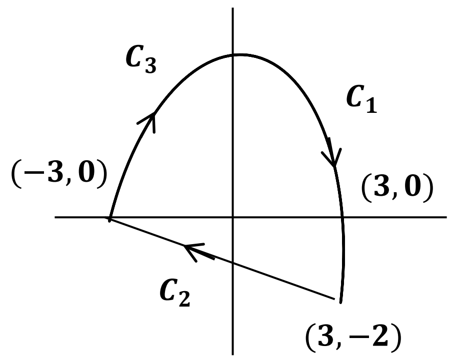
A curve $z(t)$ is called piecewise smooth (contour) if it consists of a finite number of smooth arcs joined end to end.

<u>Example</u>: $C = C_1 + C_2 + C_3$ is a smooth arc

$C_1: z_1(t) = 3 - it\ ,\ 0 \le t \le 2$

$C_2: z_2(t) = -6t + 3 + i(2t - 2)\ ,\ 0 \le t \le 1$

$C_3: z_3(t) = -3\cos t + i3\sin t\ ,\ 0 \le t \le \pi$



$z_1(0) = 3\ ,\ z_1(2) = 3 - 2i$

$z_2(0) = 3 - 2i\ ,\ z_2(1) = -3$

$z_3(0) = -3\ ,\ z_3(\pi) = 3$

Note: $\arg z' = \tan^{-1}\frac{y'(t)}{x'(t)} = \tan^{-1}\frac{dy}{dx}$

<u>Notes</u>:

1. If the derivative exists then it means that there is a tangent to the curve.
2. $z'(t)$ represents a smooth tangent to the arc.
3. The smooth arc is the arc that has a tangent at each point.

<u>Example</u>: $C : z(t) = \begin{cases} t + it^3\ , -1 \le t \le 1 \\ t + i\quad ,\quad 1 \le t \le 2 \end{cases}$

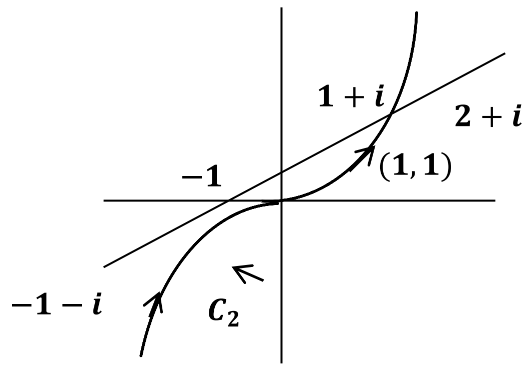
Check that $z(t)$ is simple, smooth?

<u>Solution</u>:

Note that $z(t)$ is simple arc (check?), but not smooth arc since $z'(t)$ is not exist

$z'(t) = 1\ \ ,\ \ 1 \le t \le 2 \to z'(1) = 0$

(Sharp ends don't make a smooth arc).



Note:

$$|z'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\rightarrow \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = L \qquad \text{(Length of } C \text{)}$$

[3] Contour Integral

Suppose that the equation $z = z(t),\ a \le t \le b,$ represents the contour C connecting $z_1 = z(a)$ to $z_2 = z(b)$.

Let the function $f(z(t))$ be a piecewise on $[a, b]$, we define the line integral or contour integral of f along C as follows:

$$\int_C f(z)dz = \int_a^b f\big(z(t)\big)\, z'(t)\, dt \qquad (2)$$

Note that, since C is a contour, $z'(t)$ is piecewise continuous on $[a, b]$, so the existence of integral (2) is ensured from 2, we have

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz \qquad (3)$$

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

Note:

1. $(-C)$ is the contour connecting $z_2 = z(b)$to $z_1 = z(a)$ and it has a parametric representation (i.e.:$z = z(-t), -b \le t \le -a$)

   Thus:

   $$\int_C f(z)dz = \int_C f(z(-t))dz$$

   $$= \int_{-a}^{-b} f\big(z(-t)\big)\, z'(-t)\, dz$$

   $$= -\int_C f(z)dz$$

   Note: if it is counterclockwise, then multiply by (-1).

2. Suppose that C consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_0 to z_2. Then there is a real number $a \leq c \leq b$, where $z(c) = z_0$.

   C_1: is represented by $z = z(t)$, $(a \leq t \leq c)$

   C_2: is represented by $z = z(t)$, $(c \leq t \leq b)$

   Since:

$$\int_C f(z)dz = \int_a^c f(z(t))\, z'(t)\, dt + \int_c^b f(z(t))\, z'(t)\, dt$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

Theorem: If $|f(z)| \leq M$, then:

$$\left|\int_C f(z)dz\right| \leq ML$$

such that M is constant (bounded) and L is length of contour.

Proof:

$$\left|\int_C f(z)dz\right| = \left|\int_a^b f(z(t))\, z'(t)\, dt\right|$$

$$\leq \int_a^b |f(z(t))|\, |z'(t)|\, dt$$

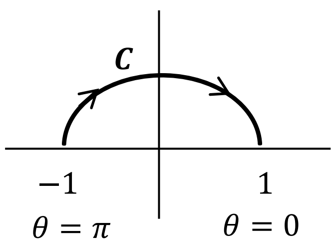
$$\leq M \int_a^b |z'(t)|\, dt$$

$$= M \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

$$= ML$$

Example: Evaluate the following integrals:

1. $\int_C \bar{z}\, dz$, where C is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$



Solution:

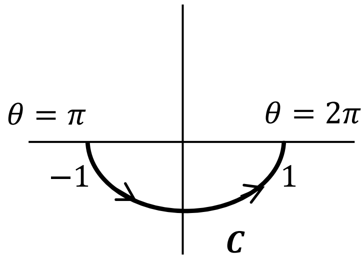
$z = re^{i\theta} = e^{i\theta} \rightarrow \bar{z} = e^{-i\theta}$

$\rightarrow dz = ie^{i\theta} d\theta$

$\therefore \int_C \bar{z}\,dz = \int_\pi^0 e^{-i\theta}\,(ie^{i\theta}d\theta)$

2. $I = \int_C \bar{z}\,dz$, where C is the lower half of the circle $|z| = 1$ from

$z = -1$ to $z = 1$



Solution:

$r = 1\,,\; z = e^{i\theta} \rightarrow \bar{z} = e^{-i\theta}$

$\therefore \int_C \bar{z}\,dz = \int_\pi^{2\pi} e^{-i\theta}\,(ie^{i\theta}d\theta)$

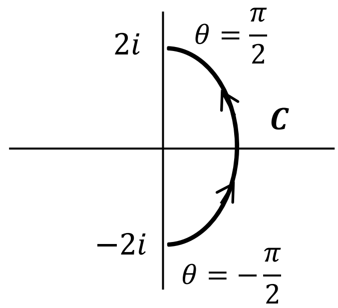
$= i\theta|_\pi^{2\pi}$

$= i[2\pi - \pi]$

$= i\pi$

2. $I = \int_C \bar{z}\,dz$, where C is the right half of the circle $|z| = 2$ from

$z = -2i$ to $z = 2i$



Solution:

$r = 2\,,\; z = 2e^{i\theta} \rightarrow \bar{z} = 2e^{-i\theta}$

$\therefore \int_C \bar{z}\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\,e^{-i\theta}\,(2ie^{i\theta}d\theta)$

$= 4i\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 4i\left[\frac{\pi}{2} + \frac{\pi}{2}\right]$

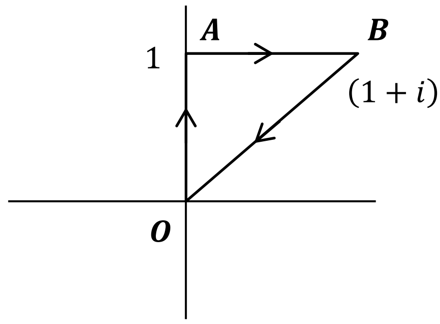
$= 4i\pi$

Example: Evaluate $\int_C \bar{z}\,dz$, where C is the contour OAB:

1. Shown in the accompanied figure and $f(z) = y - x - 3ix^2$

Solution: Take the integration of all paths (arc).

$z = x + iy$, on OA, we have



$z = iy\ ,\ x = 0$

$-dz = -idy\ ,\ f(z) = y$

$\int_{OA} f(z)dz = \int_0^1 y\, idy$

$= i\,\frac{y^2}{2}\Big|_0^1$

$= \frac{i}{2}$

On AB, we have $y = 1$ and $z = x + i$

$\rightarrow dz = dx\ ,\ f(z) = 1 - x - 3ix^2$

$\int_{AB} f(z)dz = \int_0^1 (1 - x - 3ix^2)\, dx$

$= \left[x - \frac{x^2}{2} - ix^3\right]\Big|_0^1$

$= 1 - \frac{1}{2} - i$

$= \frac{1}{2} - i$

$\therefore \int_{OAB} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz$

$= \frac{1}{2}i + \frac{1}{2} - i$

$= \frac{1}{2} - \frac{1}{2}i$

2. If C is the contour $OABO$

<u>Solution</u>:

On BO, we have $x = y \rightarrow z = x + ix = (1 + i)x$

$\rightarrow dz = dx + idx = (1 + i)dx$

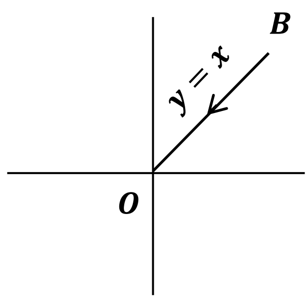
$f(z) = x - x - 3ix^2 = -3ix^2$

$\int_{BO} f(z)dz = \int_1^0 (-3ix^2)\,(1 + i)dx$

$= (1 + i)(-ix^3)|_1^0$

$= 0 + (1 + i)i$

$= i - 1$



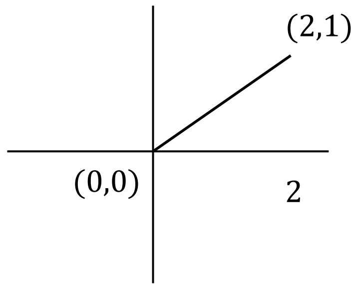
$\therefore \int_{OABO} f(z)dz = \int_{OAB} f(z)dz - \int_{BO} f(z)dz$

$= \left(\frac{1}{2} - \frac{1}{2}i\right) - (i - 1)$

$= \frac{3}{2} - \frac{3}{2}i$

<u>Example</u>: Evaluate $\int_C z^2\, dz$, where:

1. C is the line segment from $z = 0$ to $z = 2 + i$.



<u>Solution</u>:

$\frac{x - x_1}{y - y_1} = \frac{x - x_2}{y - y_2}$

$\rightarrow \frac{y}{x} = \frac{2}{1} \rightarrow x = 2y\,,\ \ 0 \le y \le 1$

$\rightarrow z = x + iy = 2y + iy$

$\rightarrow dz = 2dy + idy = (2 + i)dy$

$f(z) = z^2 = (2y + iy\,)^2$

$= ((2 + i)y)^2$

$= (4 - 1 + 4i)y^2$

$= (3 + 4i)y^2$

$\therefore \int_C f(z)\, dz = \int_0^1 (3 + 4i)(2 + i)y^2\, dy$

$= (3 + 4i)(2 + i)\left.\frac{y^3}{3}\right|_0^1$

$= \frac{1}{3}(6 - 4 + 3i + 8i)$

$= \frac{1}{3}(2 + 11i)$

2. Find $I_2 = \int_{C_2} z^2\, dz + \int_{C_3} z^2\, dz$

Solution:

On C_2, we have

$y = 0\ ,\ z = x\ \rightarrow dz = dx\ , f(x) = x^2$

$\int_{C_2} f(z)\, dz = \int_0^2 x^2\ dx = \frac{x^3}{3}\Big|_0^2 = \frac{8}{3}$

On C_3, we have

$x = 2\ ,\ z = 2 + iy\ \rightarrow dz = idy\ , f(x) = (2 + iy)^2$

$\int_{C_3} f(z)\, dz = \int_0^1 (2 + iy)^2\, i\, dy$

$$= i \int_0^1 [4 + 4iy - y^2]\ dy$$

$$= i\left[4y + 2iy^2 - \frac{y^3}{3}\right]\Big|_0^1$$

$$= i\left[4 + 2i - \frac{1}{3}\right]$$

$$= \frac{11}{3} i - 2$$

$\therefore I_2 = \frac{8}{3} + \frac{11}{3} i - 2 = \frac{2}{3} + \frac{11}{3} i$

Example: Show that if C is the circle

$$z - z_0 = re^{i\theta}\ ,\ \ 0 \le \theta \le 2\pi$$

Then

a) $\int_C f(z)\, dz = ir \int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{i\theta} d\theta$

Solution: $z - z_0 = re^{i\theta} \rightarrow z = z_0 + re^{i\theta}$

$$\rightarrow dz = ire^{i\theta} d\theta$$

$\int_C f(z)\, dz = \int_0^{2\pi} f(z_0 + re^{i\theta})\, ire^{i\theta} d\theta$

$$= ir \int_0^{2\pi} f(z_0 + re^{i\theta})\, e^{i\theta} d\theta$$

b) $\int_C \frac{dz}{z-z_0}$

Solution:

$$\int_C \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{z_0+re^{i\theta}-z_0}$$

$$= \int_0^{2\pi} i\, d\theta$$

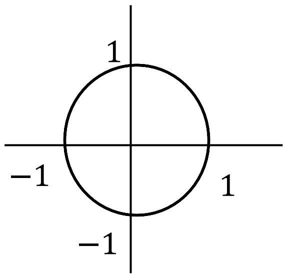
$$= i\theta|_0^{2\pi}$$

$$= 2\pi i$$

Example: Evaluate $\int_C z^n\, dz$, such that C is the circle $|z| = 1$,

i.e.: $z(t) = e^{it}$, $0 \le t \le 2\pi$, $n = 0, \mp 1, \ldots$

Solution:



$$\int_C z^n\, dz = \int_0^{2\pi} f\left(e^{it}\right) ie^{it} dt$$

$$\Leftrightarrow \int f\big(z(t)\big)z' = \int e^{int} ie^{it}$$

$$= i\int_0^{2\pi} e^{it(n+1)}\, dt$$

If $n+1 = 0 \longrightarrow \int z^n dz = i\int_0^{2\pi} dt = 2\pi i$

If $n+1 \neq 0$, let $t(n+1) = k \rightarrow dt = \frac{dk}{n+1}$, then

$\int_0^{2\pi} e^{it(n+1)}\, dt = 0$, since

$$\frac{1}{n+1}\int_0^{2\pi} e^{ik}\, dk = \frac{1}{n+1}\int_0^{2\pi}(\cos k + i\sin k)\, dk$$

$$= \frac{1}{n+1}\left[\sin k - \cos k\right]\Big|_0^{2\pi}$$

$$= 0$$

In general,

$$\int_C z^n\, dz = \begin{cases} 0 & if\ n \neq -1 \\ 2\pi i & if\ n = -1 \end{cases}$$

Example: Find $\int_C \frac{dz}{z}$ $, C : |z| = 1$

Solution: This example can be solved by two ways:

1. $\int_C \frac{dz}{z} = \int_C z^{-1} dz$

$\text{i.e.}: n = -1$, then:

$$\int_C \frac{dz}{z} = 2\pi i$$

2. $z(t) = re^{i\theta} = 1. e^{i\theta} = e^{i\theta}$

$z'(t) = ie^{i\theta} d\theta \quad , \quad 0 \leq \theta \leq 2\pi$

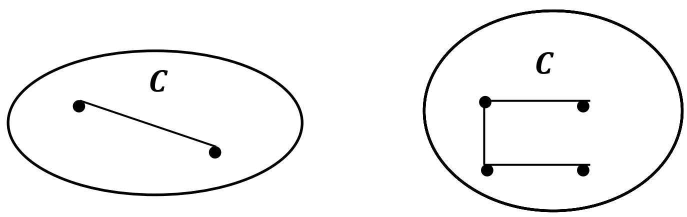
$$\int_C \frac{dz}{z} = \int_0^{2\pi} i \frac{e^{i\theta}}{e^{i\theta}} d\theta$$

$$= i\theta|_0^{2\pi}$$

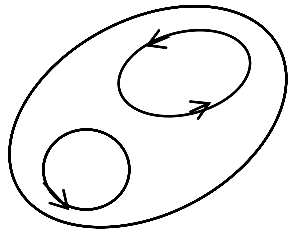
$$= 2\pi i$$

Definition:

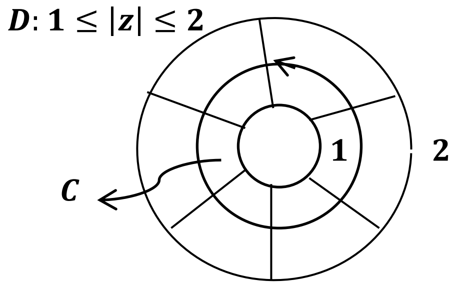
A region D is said to be simply connected if C is a piecewise smooth (closed) curve contained completely in D and then $Int\ C \subset D$.



* D is called simply connected if we can connect any two points by a path which is contained completely in D.
* The region D is called simply connected if every closed path in the region contains points from the region, otherwise D is non-simply connected or complex connected.



Simply connected region



Non-simply connected region

The region $D: 1 \le |z| \le 2$ is multiply connected since $int\ C \not\subset D$, and the internal circle ◯ $\notin D$. Note that is complex connected since it contained a closed path C which contains points from outside D.

Theorem:

Let D be a simply connected region and let $f(z)$ be an analytic function on D, then

$$\oint_C f(z)\ dz = 0$$

For each simple piecewise smooth curve C contained inside D.

Note:

If the region D is complex connected then it is not necessary that $\oint_C f(z)\ dz = 0$.

The converse of the above theorem is not true as in the following example:

Example:

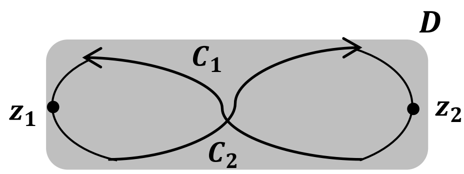
$$\oint_C \frac{dz}{z^2} = 0 \ , \ C : \ |z| = r$$

But $\frac{1}{z^2}$ is not analytic function at $z = 0$.

Note:

Let D be a simply connected region and let $f(z)$ be an analytic function on D. Let $z_1, z_2 \in D$, then

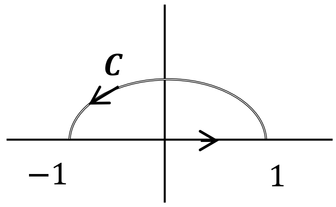
$$\oint_{C_1} f(z)\ dz = \oint_{C_2} f(z)\ dz$$



Such that C_1, C_2 are simple smooth curve which connect z_1 and z_2 , and $C_1, C_2 \subset D$.

Example: Calculate

$$\oint_{C_1 + C_2} (3z^2 + 2z - 5)\ dz$$



Such that C_1, C_2 are clear from the graph:

$C_1 : z(t) = t \quad -1 \leq t \leq 1,$

C_2 is the upper half of the circle $|z| = 1$ from $z = -1$ to $z = 1$

Solution:

$f(z) = 3z^2 + 2z - 5$, is analytic $\forall\ \mathbb{C}$, and $z_1 = -1,\ z_2 = 1 \in D$, then

$\oint_{C_1} (3z^2 + 2z - 5)\ dz = \oint_{C_2} (3z^2 + 2z - 5)\ dz$

$\therefore \oint_C f(z)\ dz = \oint_{C_1 + C_2} f(z)\ dz = 0$

Note:

The equation of circle with center z_0 and radius r is:

$$C : |z - z_0| = r$$

And the polar form becomes:

$$C : z_0 + re^{i\theta} \ , \qquad 0 \leq \theta \leq 2\pi$$

In general, we can prove:

$$\oint_C (z - z_0)^n\, dz = \begin{cases} 0 & if\ n \neq -1 \\ 2\pi i & if\ n = -1 \end{cases}$$

Proof:

$C : z(t) = z_0 + re^{it} \ , \qquad 0 \leq t \leq 2\pi$

$z'(t) = ire^{it}$

$\oint_C (z - z_0)^n\, dz = \oint_0^{2\pi} r^n e^{int}\, ire^{it} dt = \oint_0^{2\pi} (ir^{n+1}) e^{it(n+1)}\, dt$

If $n + 1 = 0 \rightarrow \oint_C (z - z_0)^n\, dz = 2\pi i$

If $n+1 \neq 0 \rightarrow \oint_C (z-z_0)^n\, dz = \frac{r^{n+1}}{n+1}\, e^{it(n+1)}\Big|_0^{2\pi}$

$$= \frac{r^{n+1}}{n+1}[\cos(n+1)t + i\sin(n+1)t]\Big|_0^{2\pi}$$

$$= 0$$

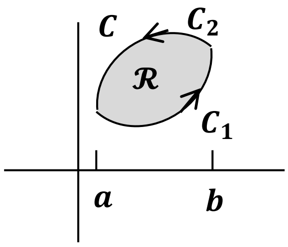
[4] Cauchy Goursat Theorem

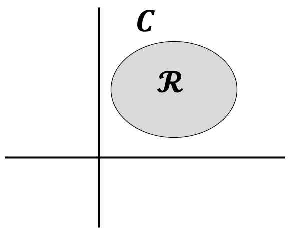
The following theorem will be needed through this section:

Green's theorem:

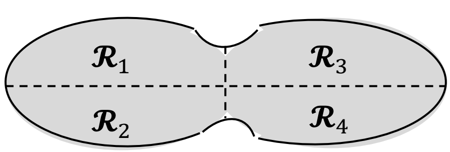
Suppose that $p(x,y)$ and $\emptyset(x,y)$ are two real-valued functions and $p, \emptyset$ are continuous with their first partial derivatives, throughout a closed region $\mathcal{R}$ consisting of points interior within and on a simple closed contour C in the xy-plane, then

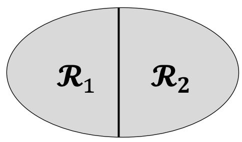
$$\oint_C (p dx + \emptyset dy) = \iint_{\mathcal{R}} \left(\emptyset_x - p_y\right) dxdy$$





Note: Green's theorem might be extended to a finite union of closed regions.

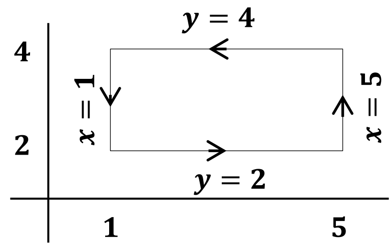




Example: Evaluate

$$\oint_C \left(\left(e^{x^2} + y\right)dx + \left(x^2 + \tan^{-1}\sqrt{y}\right)dy\right)$$

Where C is the boundary of the rectangle having the vertices $(1,2)$, $(5,2)$, $(5,4)$, and $(1,4)$.



Solution: By using Green's theorem

$p(x,y) = e^{x^2} + y$, $\emptyset(x,y) = x^2 + \tan^{-1}\sqrt{y}$

$p_y(x,y) = 1$, $\emptyset_x(x,y) = 2x$

$$\therefore \oint_C \left(\left(e^{x^2} + y\right)dx + \left(x^2 + \tan^{-1}\sqrt{y}\right)dy \right) = \int_2^4 \int_1^5 (2x-1)\,dxdy$$

$$= \int_2^4 (x^2 - x)\Big|_1^5\, dy$$

$$= \int_2^4 20\,dy = 20y|_2^4 = 40$$

Note: If $f(z) = u(x,y) + iv(x,y)$ is analytic on $\mathcal{R}$, where u, v and their first partial derivatives are continuous in $\mathcal{R}$, then

$$\int_C f(z)\,dz = 0$$

Proof: $z = x + iy \rightarrow dz = dx + idy$

$\int_C f(z)\,dz = \int_C (u + iv)\,(dx + idy)$

$= \int_C (udx - vdy) + i\int_C (vdx + udy)$

By using Green's theorem, we get:

$\int_C f(z)\,dz = \iint_{\mathcal{R}} (-v_x - u_y)\,dxdy + i\iint_{\mathcal{R}} (u_x - v_y)\,dxdy$

But f is analytic, then f satisfies C-R equations

i.e.: $u_x = v_y$, $u_y = -v_x$

$\therefore \int_C f(z)\,dz = 0$

Cauchy-Goursat theorem: **(C.G.T)**

If f is analytic function at each point within and on a simple closed contour C, then

$$\int_C f(z)\,dz = 0$$

Note:

The C.G.T can be stated in the following alternative form:

If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)\,dz = 0$$

For every simple closed contour C lying in D.

Example: Determine the domain of analyticity of the function f and apply the C.G.T to show that

$$\int_C f(z)\,dz = 0$$

where C is the circle $|z| = 1$, when

a. $f(z) = \frac{z^2}{z-3}$

Solution:

D_f is $\mathbb{C} \setminus \{3\}$

$\therefore$ So f is analytic everywhere except at $z = 3$ which is not in the circle $|z| = 1$.

$\therefore$ By C.G.T, we have:

$\int_C \frac{z^2}{z-3}\,dz = 0$

Since C is simple closed contour.

b. $f(z) = ze^{-z}$

Solution:

$f(z) = ze^{-z} = \frac{z}{e^z}$

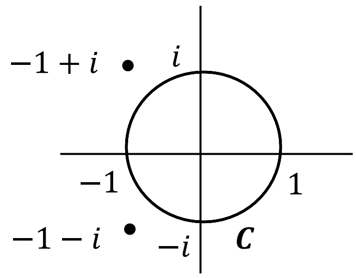
D_f is $\mathbb{C}$, f is analytic everywhere (entire function), so by C.G.T:

$\int_C f(z)\,dz = 0$

Since C is simple closed contour.

c. $f(z) = \frac{1}{z^2+2z+2}$

Solution:



$f(z) = \frac{1}{z^2+2z+2}$

$= \frac{1}{z^2+2z+1+1}$

$= \frac{1}{(z+1)^2+1}$

D_f is $\mathbb{C} \setminus \{-1+i, -1-i\}$

f is analytic function everywhere except at the point $-1+i, -1-i$ which both aren't belonging to the circle $|z| = 1$, so by C.G.T we have:

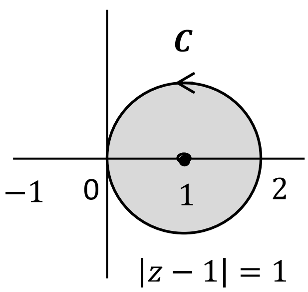
$\int_C f(z)\,dz = 0$

Since C is simple closed contour.

Example: Evaluate the following integral

$$\oint \frac{1}{z^2-1}\,dz \ , \ C : |z-1| = 1$$

Solution:



$f(z) = \frac{1}{z^2-1}$

$= \frac{1}{(z-1)(z+1)}$

$= \frac{1/2}{z-1} - \frac{1/2}{z+1}$

Inside path   Outside path

$\therefore \int \frac{1}{z^2-1}\, dz = \frac{1}{2}\int \frac{1}{z-1}\, dz - \frac{1}{2}\int \frac{1}{z+1}\, dz$

Note: $\frac{1}{z+1}$ is analytic function in $|z-1| = 1$

$\therefore \int \frac{1}{z+1}\, dz = 0$

But $\frac{1}{z-1}$ is not analytic in $|z-1| = 1$

Let: $z - 1 = re^{i\theta} \rightarrow dz = ire^{i\theta} d\theta$

$$\therefore \frac{1}{2}\int \frac{1}{z-1}\, dz = \frac{1}{2}\int_0^{2\pi} \frac{ire^{i\theta} d\theta}{re^{i\theta}}$$

$$= \frac{i}{2}\int_0^{2\pi} d\theta$$

$$= \frac{i}{2}\theta\Big|_0^{2\pi}$$

$$= i\pi$$

$$\therefore \int_C \frac{1}{z^2-1}\, dz = \frac{1}{2}\int \frac{1}{z-1}\, dz - \frac{1}{2}\int \frac{1}{z+1}\, dz$$

$$= i\pi - 0$$

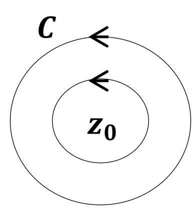
$$= i\pi$$

[5] The Cauchy Integral Formula

Theorem 1: The Cauchy integral formula states that:

If a function f is analytic everywhere in and within a simple closed contour C and if z_0 is any interior point of C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\, dz$$



or $\oint_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)$

And the integral is taken in the positive direction around C.

Remark: The general formula of Cauchy integral C.I.F is called general Cauchy integral formula and it says that:

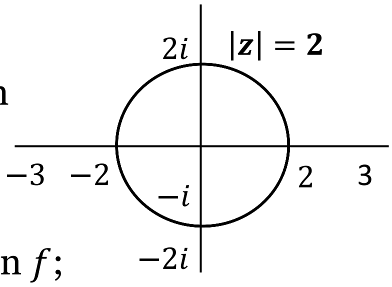
$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

i. e.: $\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$

Example: Evaluate the following integrals

1. $\oint_C \frac{z}{(9-z^2)(z+i)}\, dz$, where $C: |z| = 2$, taken in the positive sense.

Solution:



It is clear that only $z = -i$ lies within the given circle, so the function $f(z) = \frac{z}{9-z^2}$ is analytic within and on C, thus we can apply the C.I.F on f;

i. e.: $\oint_C \frac{z}{(9-z^2)(z+i)}\, dz = 2\pi i\, f(-i) = \frac{\pi}{5}$

2. $\oint_C \frac{z^3+2z+1}{(z-1)^3}\, dz$, where $C : |z| = 3$, taken in the positive sense.

Solution:

It is clear that $z = 1$ is inside the circle $|z| = 3$, we will use the formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

If $z_0 = 1$ and $n = 2$, then we have:

$$f^{(2)}(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-1)^3}\, dz$$

where $f(z) = z^3 + 2z + 1$, thus

$\oint_C \frac{f(z)}{(z-1)^3}\, dz = \frac{2\pi i}{2} f^{(2)}(1) = \pi i f^{(2)}(1)$

$\rightarrow \frac{d^2}{dz^2}[z^3 + 2z + 1]\big|_{z=1} = 6z|_{z=1} = 6$

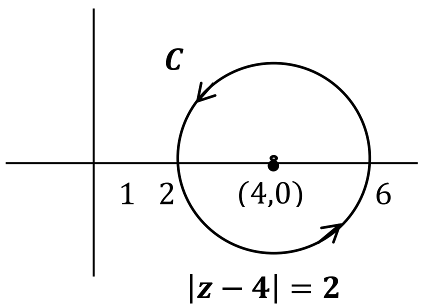
$\therefore \oint_C \frac{z^3+2z+1}{(z-1)^3}\, dz = 6\pi i$

3. $\oint_C \frac{\cos z}{(z-1)^3(z-5)^2}\, dz$, where $C: |z-4| = 2$ taken in the positive sense.

Solution:

It is clear that the term $(z-1)^3$ is nonzero on and inside the given contour of integration, but the term $(z-5)^2$ equals zero at $z = 5$ inside C. Then we rewrite the integral as:

$\oint_C \frac{\frac{\cos z}{(z-1)^3}}{(z-5)^2}\, dz$



Applying the formula:

$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$

with $z_0 = 5,\ n = 1$, and $f(z) = \frac{\cos z}{(z-1)^3}$, thus:

$\oint_C \frac{\cos z/(z-1)^3}{(z-5)^2}\, dz = 2\pi i\ \frac{d}{dz}\left[\frac{\cos z}{(z-1)^3}\right]\Big|_{z=5}$

$= 2\pi i \left[\frac{-(z-1)^3 \sin z - 3\cos z (z-1)^2}{(z-1)^6}\right]\Big|_{z=5}$

$= 2\pi i \left[\frac{-4\sin 5 - 3\cos 5}{256}\right]$

4. $\oint_C \frac{dz}{z(z+\pi i)}$, where $C : z(t) = z_0 + re^{it}\ ,\ 0 \le t \le 2\pi$

Solution:

Note that the singular points are $0, -\pi i$, thus we take first

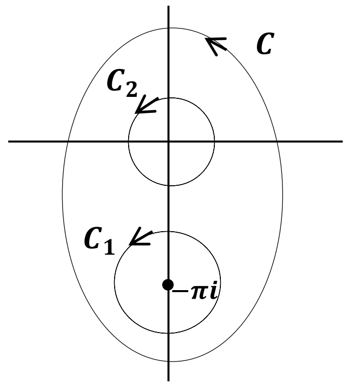
$f(z) = \frac{1}{z}\ ,\ z_0 = -\pi i$

Then: $\oint_C \frac{f(z)}{z-z_0}\, dz = \oint \frac{1/z}{z-(-\pi i)}\, dz$

$= 2\pi i\, f(-\pi i)$

$= 2\pi i\, \frac{1}{-\pi i}$

$= -2$



Now, let $f(z) = \frac{1}{z+\pi i}$, $z_0 = 0$

$\oint_C \frac{f(z)}{z-z_0}\, dz = \oint \frac{1/(z+\pi i)}{z}\, dz$

$= 2\pi i\, f(0)$

$= 2\pi i\, \frac{1}{\pi i}$

$= 2$

By Cauchy Goursat theorem, we find

$\int_C \frac{f(z)}{z-z_0}\, dz = \int_{C_1} \frac{f(z)}{z-z_0}\, dz + \int_{C_2} \frac{f(z)}{z-z_0}\, dz$

$= -2 + 2$

$= 0$

5. $\oint_C \frac{e^z}{z-i}\, dz$, where $C : |z| = 2$

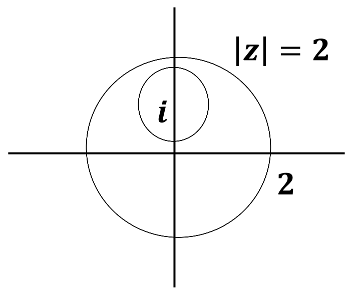
<u>*Solution*</u>:

Note $f(z) = e^z$ is analytic function and $z_0 = i$ is the only singular point $\in Int\ C$

$\oint_C \frac{e^z}{z-i}\, dz = 2\pi i\, f(z_0)$

$= 2\pi i\, f(i)$

$= 2\pi i\, e^i$



Note:

1. If z_0 is outside the path then we use Cauchy Goursat Theorem ($\int_C f(z)\, dz = 0$).
2. If z_0 is inside the path then we use Cauchy integral formula.
3. If z_0 is on the path then we divide the path and apply the integration.

Example: find $\oint_C \frac{\sin z}{z}\, dz\,,\ C : |z| = 1$

Solution:

$$f(z) = \frac{\sin z}{z}\ ,\ z_0 = 0 \in C$$

$$\oint_C \frac{\sin z}{z}\, dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(0)$$

$$= 2\pi i \sin 0$$

$$= 0$$

Cauchy's Inequality:

If $f(z)$ is analytic function on and within C, such that $C: |z - z_0| = r$ then:

$$\left| f^{(n)}(z_0) \right| = \frac{n!M}{r^n}$$

where $|f(z)| \leq M \quad \forall z \in \mathbb{C}$.

Proof:

By the general Cauchy integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz$$

$$\left| f^{(n)}(z_0) \right| = \left| \frac{n!}{2\pi i} \oint_C \frac{f(z)\, dz}{(z-z_0)^{n+1}} \right|$$

$$\leq \frac{n!}{2\pi} \oint_C \frac{|f(z)||dz|}{|z-z_0|^{n+1}}$$

$\leq \frac{n!\,M}{2\pi} \oint_C \frac{|dz|}{r^{n+1}}$

$= \frac{n!\,M}{2\pi} \frac{2\pi r}{r^{n+1}}$

$= \frac{n!M}{r^n}$

Where $\oint_C |dz| = 2\pi r$, circumference of the circle (length of the path)

If $n = 1$, then:

$|f'(z_0)\,| = \frac{M}{r}$

[6] Derivatives of Analytic Functions

Now, we are ready to prove the following theorem:

Theorem:

If f is analytic function at a point then its derivatives of all orders are analytic functions at that point.

Proof: Let f be an analytic function within and on a positively oriented simple closed contour C. Let z be any point inside C. Letting s denotes the points on C, and then by C.I.F, we have:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{s-z}\, ds \qquad \ldots (1)$$

We will show that $f'(z)$ exists and

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\, ds \qquad \ldots (2)$$

To do this, using formula (1), we have:

$$\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \left(\frac{1}{s-\Delta z-z} - \frac{1}{s-z}\right) f(s)ds$$

$$\frac{f(s)ds}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{(s-z-s+z+\Delta z)}{(s-\Delta z-z)(s-z)\Delta z}\, f(s)ds$$

$$= \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-\Delta z-z)(s-z)}\, ds \quad \ldots (3)$$

If d is the smallest distance from z to s on C, then

$$|s - z| \geq d$$

And if $|\Delta z| < d$, then

$$|s - z - \Delta z| \geq |s - z| - |\Delta z| \geq d - |\Delta z|$$

Since f is analytic within and on C, it is also continuous and so it is bounded on C. i.e.: $|f(s)| \leq K$, and if the length of C is L, then

$$\left|\int_C \left[\frac{1}{(s-z-\Delta z)(s-z)} - \frac{1}{(s-z)^2}\right] f(s)ds\right| = \left|\Delta z \int_C \frac{f(s)ds}{(s-\Delta z-z)(s-z)^2}\right|$$

$$\leq |\Delta z| \int_C \frac{|f(s)||ds|}{(d-|\Delta z|)d^2}$$

$$\leq \frac{|\Delta z|K}{(d-|\Delta z|)d^2} \int_C |dz|$$

$$= \frac{|\Delta z|\; K\; L}{(d-|\Delta z|)d^2}$$

Hence, when $\Delta z \to 0$, then

$$\frac{|\Delta z|\; K\; L}{(d-|\Delta z|)d^2} \to 0$$

Or:

$$\int_C \frac{f(s)ds}{(s-\Delta z-z)(s-z)} - \int_C \frac{f(s)ds}{(s-z)^2} \to 0$$

That means, the integral (3) approaches the integral (2) as $\Delta z \to 0$, so

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{1}{2\pi i} \int_C \frac{f(s)ds}{(s-z)^2}$$

Or:

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-z)^2}\; ds$$

If we apply the same technique to formula (2), we find that:

$$f''(z) = \frac{1}{\pi i} \int_C \frac{f(s)}{(s-z)^3}\; ds \quad \ldots (4)$$

In general, one can show that:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(s)}{(s-z)^{n+1}}\, ds$$

This is called the extension of C.I.F.

Theorem:

Suppose that f is a continuous function on a simply connected domain D, then the following statements are equivalent:

a) There exists a function F such that $F' = f$.
b) $\int_C f(z)\, dz = 0$, for any simple closed contour C.
c) $\int_C f(z)\, dz$ depends only on the end points of C for any contour C.

Remark:

Part (c) in the above theorem means that the integral $\int_C f(z)\, dz$ is independent of path connecting the end points of contour C.

[7] Morera's Theorem

If f is continuous function through a simply connected domain D and if

$$\int_C f(z)\, dz = 0$$

for every simple closed contour C lying in D, then f is analytic through out D.

Proof:

Since $\int_C f(z)\, dz = 0$, for every simple closed contour C in D, and the values of the contour integrals are independent of the contour in D, then:

By part (a) of the previous theorem, the function f has an antiderivative everywhere in D, that is there exists an analytic function F such that $F' = f$, then it follows that f is analytic in D since it's the derivative of an analytic function.

Maximum Moduli of Function

<u>Theorem 1</u>:

Let f be analytic and not constant in some domain D such that $|f(z)|$ is constant, and then $f(z)$ is also constant in D

<u>Theorem 2</u>:

Let f be analytic and not constant in a $\epsilon -$ ngh of z_0, then there is at least one point z in that ngh. Such that

$$|f(z)| \geq |f(z_0)|$$

Maximum Principle

<u>Theorem</u>:

Let f be analytic and not constant in a domain D, then $|f(z)|$ has no maximum value in D.

<u>Proof</u>:

Since f is analytic and not constant in a domain D, then f is not constant over any ngh of any point in D.

Suppose that $|f(z)|$ has a maximum value at z_0 in D, it follows that:

$$|f(z_0)| \geq |f(z)|$$

For each point z in a ngh of z_0, but this contradicts the fact that

$$|f(z)| \geq |f(z_0)| \qquad \text{(Th. 2)}$$

Thus $|f(z)|$ has no maximum value for any ngh of D, so that $|f(z)|$ has no maximum value in D.

<u>Corollary</u>:

If f is a continuous function in a closed bounded region $\mathcal{R}$ and analytic, and not constant in the interior of $\mathcal{R}$, then $|f|$ has a maximum value on the boundary of $\mathcal{R}$ and never in the interior.

<u>Proof</u>:

Since f is continuous in a closed bounded region $\mathcal{R}$, then $|f|$ has a maximum value in $\mathcal{R}$, and by the maximum principle theorem $|f|$ has no maximum value in the interior of $\mathcal{R}$, then $|f|$ has no maximum value on the boundary of $\mathcal{R}$.

Minimum Principle

Theorem:

Let f be a continuous function in a closed bounded region $\mathcal{R}$, and let f be analytic and not constant throughout the interior of $\mathcal{R}$. If $|f(z)| \neq 0$ anywhere in $\mathcal{R}$, then $|f(z)|$ has a minimum value in $\mathcal{R}$ which occurs on the boundary of $\mathcal{R}$, and never in the interior of $\mathcal{R}$.

Proof: Define a function F by:

$$F(z) = \frac{1}{f(z)} \;,\; f(z) \neq 0 \text{ in } \mathcal{R}$$

F is analytic and not constant throughout the interior of $\mathcal{R}$, so by corollary, $|F|$ has a maximum value on the boundary of $\mathcal{R}$. This implies that there is z_0 on the boundary of in $\mathcal{R}$, such that

$$|F(z)| \leq |F(z_0)|$$

$$\left|\frac{1}{f(z)}\right| \leq \left|\frac{1}{f(z_0)}\right|$$

Or

$$|f(z)| \geq |f(z_0)|$$

Thus, $|f(z)|$ has a minimum value in $\mathcal{R}$ which occurs on the boundary of $\mathcal{R}$, and never in the interior of $\mathcal{R}$.

[8] Liouville's Theorem

Theorem:

If f is entire function and bounded for all values of z in the complex plane $\mathbb{C}$, then $f(z)$ is constant throughout the plane.

Proof: Since f is entire function in $\mathbb{C}$, then f is analytic in $\mathbb{C}$, so Cauchy's inequality holds,

$\left|f^{(n)}(z_0)\right| \leq \frac{n!M}{r^n}$, $n = 1,2,3, \dots$

$\rightarrow |f'(z_0)| = \frac{M}{r}$

Since $|f(z)| \leq M, \ \forall z \in \mathbb{C}$. If we chose r large enough, we should have $f'(z_0) = 0$ for any z, since z_0 is any arbitrary point, then

$$f'(z_0) = 0 \,, \qquad \forall z \in \mathbb{C}$$

So f is constant.

[9] The Fundamental Theorem of Algebra

Theorem:

Any polynomial $p(z)$, such that

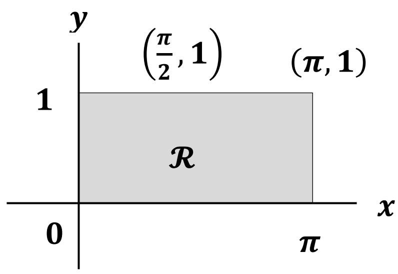
$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \ \ , a_n \neq 0$$

for all $n \geq 0$, has at least one zero that is there exists at least one point z_0 such that $p(z_0) = 0$.

Example:

1. Let $\mathcal{R}$ denotes the rectangular region $0 \leq x \leq \pi \,, 0 \leq y \leq 1$, find the maximum and minimum values of f, when

$$f(z) = \sin z$$



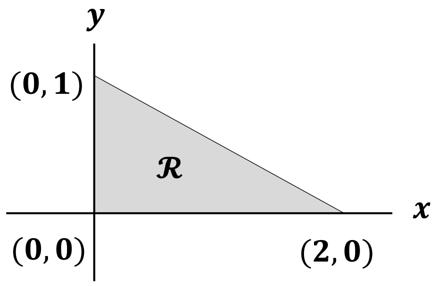
Solution:

$|f(z)| = |\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$

It is clear that the term $\sin^2 x$ is greatest when $x = \frac{\pi}{2}$, and the increasing function $\sinh^2 y$ is greatest when $y = 1$, then the maximum value of $|f(z)|$ in $\mathcal{R}$ occurs at the boundary point $z = \left(\frac{\pi}{2}, 1\right)$ and the minimum value of $|f(z)|$ in $\mathcal{R}$ occurs at the boundary point $z = (0,0)$.

2. Let $f(z) = (z+1)^2$, and the region $\mathcal{R}$ is the triangle with vertices at the points $z = 0,\ z = 2$ and $z = i$. Find points in $\mathcal{R}$ where $|f(z)|$ have its maximum and minimum values.

<u>Solution</u>:



$$|f(z)| = |(z+1)^2| = |(x+iy+1)^2|$$

$$= \left|\left((x+1)+iy\right)^2\right|$$

$$= |(x+1)+iy|^2$$

$$= (x+1)^2 + y^2\ ,\ \ 0 \le x \le 2\ , 0 \le y \le 1$$

Since the maximum and minimum values occur on the boundary of $\mathcal{R}$, so it is clear that $|f(z)|$ takes maximum value when $x = 2$ and $y = 0$, i.e. at $z = 2$, and takes its minimum value when $x = 0$ and $y = 0$, i.e. at $z = 0$.

3. Let $f(z) = e^z$ in the region $|z| \le 1$. Find the points in this region, where $|f(z)|$ achieves its maximum and minimum values.

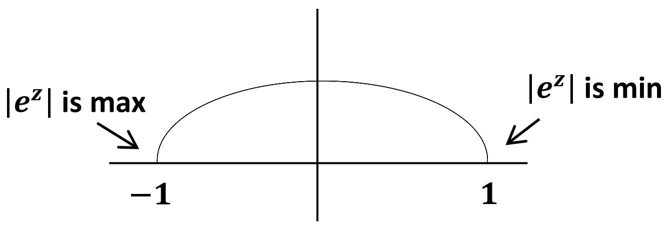
<u>Solution</u>:

Since e^z is entire function, $e^z \neq 0\ , \forall z$ in the region, both maximum and minimum points are guaranteed by our results.

Now, we have

$$|f(z)| = |e^z| = \left|e^x . e^{iy}\right| = |e^x|$$

Then, its maximum value will occur at the boundary points $(x, y) = (1,0)$ and $|f(z)|$ takes minimum value at the boundary points $(x, y) = (-1,0)$, as in the Fig.



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