Chapter One: Introductory Concepts

1.1 Definition

An integral equation is an equation in which the unknown function $u(x)$ to be determined appears under the integral sign. A typical form of an integral equation in $u(x)$ is of the form:

$$cu(x) = f(x) + \lambda \int_a^{b(x)} k\big(x,t,u(t)\big)dt \qquad \text{...(1.1)}$$

where the forcing function *f(x)* and the kernel function *k(x,t)* are prescribed, while *u(x)* is the unknown function to be determined, and *c* is constant. The parameter λ is often omitted; it is, however, of importance in certain theoretical investigations (e.g. stability) and the eigenvalue problem.

1.2 Classification of Linear Integral Equations

Definition (1.1):

The integral equation (1.1) is called ***linear integral equation*** if the kernel *k(x,t,u(t))=k(x,t)u(t)*, otherwise it is called ***nonlinear***.

i.e. $cu(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt$ (linear integral equation)

$cu(x) = f(x) + \int_a^{b(x)} k\big(x,t,u(t)\big)dt$ (nonlinear integral equation)

Definition (1.2):

The linear integral equation (1.1) is called ***homogeneous***, if $f(x) \equiv 0$, otherwise it is called ***nonhomogeneous***.

i.e. $cu(x) = \int_a^{b(x)} k(x,t)u(t)dt$ (homogeneous integral equation)

$cu(x) = f(x) + \int_a^{b(x)} k(x,t)u(t)dt$ (nonhomogeneous integral equation)

Definition (1.3):

The integral equation (1.1) is said to be an equation of the ***first kind*** if c=0

$$i.e.\ f(x)=\int_a^{b(x)} k(x,t)u(t)dt$$

Definition (1.4):

The integral equation (1.1) is said to be an equation of the ***second kind*** if c=1

$$i.e.\ u(x)=f(x)+\int_a^{b(x)} k(x,t)u(t)dt$$

Definition (1.5):

The integral equation (1.1) is called ***Volterra integral equation (VIE)*** when *b(x)=x*.

$$i.e.\ u(x)=f(x)+\int_a^{x} k(x,t)u(t)dt$$

Definition (1.6):

The integral equation (1.1) is called ***Fredholm integral equation (FIE)***, if *b(x)=b*, where *b* is constant such that $b \geq a$.

$$i.e.\ u(x)=f(x)+\int_a^{b} k(x,t)u(t)dt$$

Definition (1.7):

An ***integro-differential equation*** is an equation that involves one (or more) of an unknown function *u(x)*, together with differential and integral operations on *x*.

The following are examples of integro-differential equations:

1. $u''(x)=-x+\int_0^{x}(x-t)u(t)dt\ \ , u(0)=0, u'(0)=1,$ (2^{nd} order Volterra integro-differential equation)

2. $u'(x) = 1 - \frac{1}{3}x + \int_0^1 xtu(t)dt \ \ , u(0) = 1,$ (1st order Fredholm integro-differential equation)

Definition (1.8):

the integral equation is called ***singular*** if the lower limit, the upper limit or both limits of integration are *infinite*. In addition, the integral equation is also called a **singular integral equation** if the kernel $K(x, t)$ becomes *infinite* at one or more points in the domain of integration.
Examples of the second kind of *singular* integral equations are given by:

$$u(x) = 2x + 6\int_0^{\infty} \sin(x - t)u(t)dt$$

$$u(x) = x + \frac{1}{3}\int_{-\infty}^{0} \cos(x + t)u(t)dt$$

$$u(x) = 1 + x^2 + \frac{1}{6}\int_{-\infty}^{\infty} (x + t)u(t)dt$$

Examples of the first kind of *singular* integral equations are given by:

$$x^2 = \int_0^x \frac{1}{\sqrt{x - t}}u(t)dt$$

$$x = \int_0^x \frac{1}{(x - t)^{\alpha}}u(t)dt \quad 0 < \alpha < 1$$

1.3 Special Types of Kernels

The following special cases of the kernel of an integral equation are of main interest:

Definition (1.9):

The kernel $k(x,t)$ is called ***difference kernel***, if $k(x,t)=k(x-t)$. And the linear integral equation is called ***an integral equation of convolution type***.

i.e. $u(x) = f(x) + \int_a^b k(x - t)u(t)dt$

Definition (1.10):

The kernel *k(x,t)* is called ***the separable*** or ***degenerate kernel of rank n*** if it is of the form:

$$k(x,t)=\sum_{j=1}^{n}a_j(x)b_j(t)$$

where *n* is finite and the functions $\{a_j\}\ and\ \{b_j\}$ are sufficiently smooth functions.

Exercises 1.1.

Classify each of the following integral equations:

1. $u(x)=x+\int_0^1 xtu(t)dt$
2. $u(x)=1+x^2+\int_0^x (x-t)u(t)dt$
3. $u(x)=e^x+\int_0^x (tu^2(t)dt$
4. $u(x)=\int_0^1 (x-t)^2u(t)dt$
5. $u(x)=\frac{2}{3}x+\int_0^1 xtu(t)dt$
6. $u(x)=1+\frac{x}{4}\int_0^1 \frac{1}{x+t}\frac{1}{u(t)}dt$
7. $u'(x)=1+\int_0^x e^{-2t}u^3(t)dt \quad ,u(0)=0$
8. $u'''(x)=-\frac{1}{12}x^4+\int_0^x e^{x-t}u(t)dt \quad ,u(0)=u'(0)=0\,,u''(0)=2$

1.4 Solution of an Integral Equation

A solution of an integral equation or an integro-differential equation on the interval of integration is a function *u*(*x*) such that it satisfies the given equation. In other words, if the given solution is substituted on the right-hand side of the equation, the output of this direct substitution must yield on the left-hand side, i.e. we should verify that the given function *u*(*x*) satisfies the integral equation or the integro-differential equation under discussion. This important concept will be illustrated first by examining the following examples.

Example 1.1. Show that $u(x) = e^x$ is a solution of the Volterra integral equation:

$$u(x) = 1 + \int_0^x u(t)dt$$

Substituting $u(x) = e^x$ in the right-hand side (RHS) of the above integral equation yields**:**

RHS=$1 + \int_0^x u(t)dt$=$1 + \int_0^x e^t dt$=$1 + \int_0^x [e^t]_0^x$=$1+e^x - e^0$=$e^x = u(x) = LHS$

Example 1.2. Show that $u(x) = x$ is a solution of the following Fredholm integral equation:

$$u(x) = \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)u(t)dt$$

RHS= $\frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)u(t)dt$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\int_0^1 (x+t)tdt$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{xt^2}{2} + \frac{t^3}{3}\right]_0^1$

$= \frac{5}{6}x - \frac{1}{9} + \frac{1}{3}\left[\frac{x}{2} + \frac{1}{3}\right]$ = x = u(x)=LHS

Exercises 1.2.

verify that the given function is a solution of the corresponding integral equation:

1. $u(x) = \frac{2}{3}x + \int_0^1 xtu(t)dt \quad u(x) = x$
2. $u(x) = x - \int_0^x (x-t)u(t)dt \quad u(x) = \sin x$

1.5 Taylor Series

In this section, we will introduce a brief idea on the Taylor series. Recall that the Taylor series exists for analytic functions only. Let $f(x)$ be a function that is infinitely differentiable in an interval $[b, c]$ that contains an interior point a. The Taylor series of $f(x)$ generated at $x = a$ is given by the sigma notation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

which can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

The Taylor series of the function $f(x)$ at $a = 0$ is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

1.6 Infinite Geometric Series

A *geometric series* is a series with a constant ratio between successive terms. The standard form of an infinite geometric series is given by:

$$S_n = \sum_{k=0}^{n} a_1 r^k$$

An *infinite geometric series* converges if and only if $|r| < 1$, otherwise it diverges. The the sum of infinite geometric series, for $|r| < 1$, is given by:

$$S_n = \frac{a_1}{1-r}$$

Example 1.3. Find the sum of the infinite geometric series:

$$1 + \frac{3}{5} + \frac{9}{25} + \frac{27}{125} + \cdots$$

It is obvious that the first term is $a_1 = 1$ and the common ratio is $r = \frac{3}{5}$. The sum is therefore given by:

$$S = \frac{1}{1-\frac{3}{5}} = \frac{5}{2}$$

Example 1.4. Find the sum of the infinite geometric series:

$$1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$$

It is obvious that the first term is $a_1 = 1$ and the common ratio is $r = -\frac{1}{3}$, $|r| < 1$. The sum is therefore given by:

$$S = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$$

Chapter Two: Equivalence Between Integral Equations and Ordinary Differential Equations

2.1 Converting Volterra Equation to an ODE

In this section, we will present the technique that converts Volterra integral equations of the second kind to equivalent differential equations. This may be easily achieved by applying the important Leibniz Rule for differentiating an integral. It seems reasonable to review the basic outline of the rule.

2.1.1 Differentiating Any Integral: Leibniz Rule

To differentiate the integral $\int_{\alpha(x)}^{\beta(x)} G(x,t)dt$ with respect to x, we usually apply the useful Leibniz rule given by:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = G\big(x,\beta(x)\big)\frac{d\beta}{dx} - G\big(x,\alpha(x)\big)\frac{d\alpha}{dx} + \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}dt \qquad (2.1)$$

where $G(x, t)$ and $\frac{\partial G}{\partial x}$ are continuous functions in the domain D in the *xt*-plane that contains the rectangular region R, $a \le x \le b$, $t_0 \le t \le t_1$, and the limits of integration $\alpha(x)$ and $\beta(x)$ are defined functions having continuous derivatives for $a < x < b$. We note that the Leibniz rule is usually presented in most calculus books, and our concern will be on using the rule rather than its theoretical proof. The following examples are illustrative and will be mostly used in the coming approach that will be used to convert Volterra integral equations to differential equations.

Particular case: If α(x) and β(x) are absolute constants, then (2.1) reduces to:

$$\frac{d}{dx}\int_{\alpha(x)}^{\beta(x)} G(x,t)dt = \int_{\alpha(x)}^{\beta(x)} \frac{\partial G}{\partial x}dt$$

Example 2.1. Find $\boldsymbol{\frac{d}{dx}\int_0^x (x-t)^2 u(t)dt}$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = 2(x-t)u(t)$. Using Leibniz rule (2.1), we find:

$$\boldsymbol{\frac{d}{dx}\int_0^x (x-t)^2 u(t)dt = \int_0^x 2(x-t)u(t)dt}$$

Example 2.2. Find $\frac{d}{dx}\int_0^x (x-t)u(t)dt$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = u(t)$. Using Leibniz rule (2.1), we find:

$$\frac{d}{dx}\int_0^x (x-t)u(t)dt = \int_0^x u(t)dt$$

Example 2.3. Find $\frac{d}{dx}\int_0^x u(t)dt$

In this example α(x)=0 , β(x)=x, hence $\frac{d\alpha}{dx}$=0, $\frac{d\beta}{dx}$=1and $\frac{\partial G}{\partial x} = 0$. Using the Leibniz rule (2.1), we find:

$$\frac{d}{dx}\int_0^x u(t)dt = u(x)$$

We now turn to our main goal to convert a Volterra integral equation to an equivalent differential equation. This can be easily achieved by differentiating both sides of the integral equation, noting that the Leibniz rule should be used in differentiating the integral as stated above. The differentiating process should be continued as many times as needed until we obtain a pure differential equation with the integral sign removed. Moreover, the initial conditions needed can be obtained by substituting x =0 in the integral equation, and the resulting integro-differential equations will be shown. We are now ready to give the following illustrative examples.

Example 2.4. Find the initial value problem equivalent to the Volterra integral equation**:** $u(x) = 1 + \int_0^x u(t)dt$

Differentiating both sides of the integral equation and using the Leibniz rule we find:

$$u'(x) = u(x)$$

The initial condition can be obtained by substituting x = 0 into both sides of the integral equation; hence we find u(0) = 1. Consequently, the corresponding initial value problem of the first order is given by**:**

$$u'(x) - u(x) = 0, u(0) = 1$$

Example 2.5. Convert the following Volterra integral equation to an initial value problem**:** $u(x) = x + \int_0^x (t-x)u(t)dt$

Differentiating both sides of the integral equation, we obtain:

$$u'(x) = 1 - \int_0^x u(t)dt$$

We differentiate both sides of the resulting integro-differential equation to remove the integral sign, therefore, we obtain**:**

$$u''(x) = -u(x)$$

or equivalently

$$u''(x) + u(x) = 0$$

The related initial conditions are obtained by substituting $x = 0$ in $u(x)$ and in $u'(x)$ in the equations above, and as a result we find $u(0) = 0$ and $u'(0) = 1$. Combining the above results yields the equivalent initial value problem of the second order given by**:**

$$u''(x) + u(x) = 0\,, u(0) = 0, u'(0) = 1$$

Example 2. 6. Find the initial value problem equivalent to the Volterra integral equation: $u(x) = x^3 + \int_0^x (x-t)^2 u(t)dt$

Differentiating both sides of the above equation three times, we find**:**

$$u'(x) = 3x^2 + 2\int_0^x (x-t)u(t)dt$$

$$u''(x) = 6x + 2\int_0^x u(t)dt$$

$$u'''(x) = 6 + 2u(x)$$

The proper initial conditions can be easily obtained by substituting $x = 0$ in $u(x)$, $u'(x)$ and $u''(x)$ in the obtained equations above. Consequently, we obtain the nonhomogeneous initial value problem of third order given by:

$$u'''(x) - 2u(x) = 6\,, u(0) = 0\,, u'(0) = 0\,, u''(0) = 0$$

Exercises 2.1.

In exercises 1-4, find $\frac{d}{dx}$ for the given integrals by using the Leibniz rule:

1. $\int_0^x (x-t)^3 u(t)dt$
2. $\int_x^{x^2} e^{xt}dt$
3. $\int_0^x (x-t)^4 u(t)dt$
4. $\int_x^{4x} \sin(x+t)dt$

In exercises 5-8, convert each of the Volterra integral equations to an equivalent initial value problem:

5. $u(x) = e^x + \int_0^x (x-t)u(t)dt$
6. $u(x) = 2 + 3x + 5x^2 + \int_0^x [1 + 2(x-t)]u(t)dt$
7. $u(x) = x - cosx + \int_0^x (x-t)u(t)dt$
8. $u(x) = -5 + 6x + \int_0^x (5 - 6x + 6t)u(t)dt$

2.2 Converting IVP to Volterra Equation

In this section, we will study the method that converts an initial value problem to an equivalent Volterra integral equation. Before outlining the method needed, we wish to recall the useful transformation formula**:**

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_{n-1}} f(x_n)dx_n \dots dx_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt \qquad \textbf{(2.2)}$$

that converts any multiple integral to a single integral. This is an essential and useful formula that will be employed in the method that will be used in the conversion technique. We point out that this formula appears in most calculus texts. For practical considerations, the formulas:

$$\int_0^x \int_0^x f(t)dt\,dt = \int_0^x (x-t)f(t)dt \qquad \textbf{(2.3)}$$

$$\int_0^x \int_0^x \int_0^x f(t)dt\,dt\,dt = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt \qquad \textbf{(2.4)}$$

are two special cases of the formula given above, and the most used formulas that will transform double and triple integrals respectively to a single integral for each. For simplicity reasons, we prove the first formula (2.3) that converts a double integral to a single integral. Noting that the right-hand side of (2.3) is a function of *x* allows us to set the equation:

$$I(x) = \int_0^x (x-t)f(t)dt \qquad \textbf{(2.5)}$$

Differentiating both sides of (2.5), and using the Leibniz rule, we obtain:

$$I'(x) = \int_0^x f(t)dt \qquad \textbf{(2.6)}$$

Integrating both sides of (2.6) from 0 to *x*, noting that *I*(0) = 0 from (2.5), we find:

$$I(x) = \int_0^x \int_0^x f(t)dt\,dt$$

Exercises 2.2. Prove that $\int_0^x \int_0^x \int_0^x f(t)dt\,dt\,dt = \frac{1}{2}\int_0^x (x-t)^2 f(t)dt$

Example 2.7. Convert the following quadruple integral:

$$I(x) = \int_0^x \int_0^x \int_0^x \int_0^x u(t)dt\,dt\,dt\,dt$$

to a single integral.
Using the formula (2.2), noting that *n* = 4, we find:

$$\boldsymbol{I(x)=\frac{1}{3!}\int_0^x (x-t)^3 u(t)dt}$$

Returning to the main goal of this section, we discuss the technique that will be used to convert an initial value problem to an equivalent Volterra integral equation. Without loss of generality, and for simplicity reasons, we apply this technique to a third-order initial value problem given by:

$$y'''(x)+p(x)y''(x)+q(x)y'(x)+r(x)y(x)=g(x) \qquad (2.7)$$

subject to the initial conditions:

$$y(0)=\alpha\,, y'(0)=\beta\,, y''(0)=\gamma \quad ,\alpha,\beta \text{ and } \gamma \text{ are constant} \qquad (2.8)$$

The coefficient functions *p*(*x*), *q*(*x*), and *r*(*x*) are analytic functions by assuming that these functions have Taylor expansions about the origin. Besides, we assume that *g*(*x*) is continuous through the interval of discussion. To transform (2.7) into an equivalent Volterra integral equation, we first set:

$$y'''(x)=u(x) \qquad (2.9)$$

where *u*(*x*) is a continuous function on the interval of discussion. Based on (2.9), it remains to find other relations for *y* and its derivatives as single integrals involving *u*(*x*). This can be simply performed by integrating both sides of (2.9) from 0 to *x* where we find:

$$y''(x)-y''(0)=\int_0^x u(t)dt$$

or equivalently

$$y''(x)=\gamma+\int_0^x u(t)dt \qquad (2.10)$$

To obtain *y*′(*x*) we integrate both sides of (2.10) from 0 to *x*, to find that:

$$y'(x)=\beta+\gamma x+\int_0^x\int_0^x u(t)dt\,dt \qquad (2.11)$$

Similarly, we integrate both sides of (2.11) from 0 to *x* to obtain:

$$y(x)=\alpha+\beta x+\frac{1}{2}\gamma x^2+\int_0^x\int_0^x\int_0^x u(t)dt\,dt\,dt \qquad (2.12)$$

respectively. Substituting (2.9), (2.10), (2.11), and (2.12) into (2.7) leads to the following Volterra integral equation of the second kind:

$$y'''(x)+p(x)y''(x)+q(x)y'(x)+r(x)y(x)=g(x)$$

$$\Rightarrow u(x)+p(x)\left[\gamma+\int_0^x u(t)dt\right]+q(x)\left[\beta+\gamma x+\int_0^x\int_0^x u(t)dt\,dt\right]+$$

$$r(x)\left[\alpha+\beta x+\frac{1}{2}\gamma x^2+\int_0^x\int_0^x\int_0^x u(t)dt\,dt\,dt\right]=g(x)$$

$$\Rightarrow u(x)+p(x)\left[\gamma+\int_0^x u(t)dt\right]+q(x)\left[\beta+\gamma x+\int_0^x (x-t)u(t)dt\right]$$

$$+\ r(x)\left[\alpha+\beta x+\frac{1}{2}\gamma x^2+\frac{1}{2}\int_0^x (x-t)^2 dt\right]=g(x)$$

$$\Rightarrow u(x)=\left(g(x)-\left\{p(x)\gamma+q(x)(\beta+\gamma x)+r(x)\left(\alpha+\beta x+\frac{1}{2}\gamma x^2\right)\right\}\right)$$

$$+\int_0^x\left[-p(x)-(x-t)q(x)-\frac{1}{2}(x-t)^2 r(x)\right]u(t)dt$$

$$\Rightarrow u(x)=F(x)+\int_0^x K(x,t)u(t)dt$$

Where $F(x)=\left(g(x)-\left\{p(x)\gamma+q(x)(\beta+\gamma x)+r(x)\left(\alpha+\beta x+\frac{1}{2}\gamma x^2\right)\right\}\right)$
and $\left[K(x,t)=-p(x)-(x-t)q(x)-\frac{1}{2}(x-t)^2 r(x)\right]$
The following examples will be used to illustrate the above-discussed technique.
Example 2.8. Convert the following initial value problem

$$y'''-3y''-6y'+5y=0$$

Subject to the initial conditions: $y(0)=y'(0)=y''(0)=1$
to an equivalent Volterra integral equation.
As indicated before, we first set:

$$y'''(x)=u(x) \qquad (2.13)$$

Integrating both sides of (2.13) from 0 to x and using the initial condition $y''(0)=1$, we find:

$$y''(x)=1+\int_0^x u(t)dt \qquad (2.14)$$

And $y'(x)=1+x+\int_0^x\int_0^x u(t)dt\,dt$

$$y'(x)=1+x+\int_0^x (x-t)u(t)dt \qquad (2.15)$$

And $y(x)=1+x+\frac{1}{2}x^2+\int_0^x\int_0^x\int_0^x u(t)dt\,dtdt$

$$y(x)=1+x+\frac{1}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2 u(t)dt \qquad (2.16)$$

Substituting (2.13), (2.14), (2.15), and (2.16) into the IVP, we find:

$$y'''-3y''-6y'+5y=0$$

$$\Rightarrow u(x)-3\left[1+\int_0^x u(t)dt\right]-6\left[1+x+\int_0^x (x-t)u(t)dt\right]+5\left[1+x+\frac{1}{2}x^2+\frac{1}{2}\int_0^x (x-t)^2 u(t)dt\right]=0$$

$$\Rightarrow u(x) = 3\left[1 + \int_0^x u(t)dt\right] + 6\left[1 + x + \int_0^x (x-t)u(t)dt\right] - 5\left[1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt\right]$$

$$\Rightarrow u(x) = 4 + x - \frac{5}{2}x^2 + \int_0^x \left[3 + 6(x-t) - \frac{5}{2}(x-t)^2\right] u(t)dt$$

the equivalent Volterra integral equation.

Example 2.9. Find the equivalent Volterra integral equation to the following initial value problem:

$$y''(x) + y(x) = cosx \ , y(0) = 0, y'(0) = 1$$

As indicated before, we first set:

$$y''(x) = u(x) \qquad (2.17)$$

Integrating both sides of (2.17) from 0 to x and using the initial condition $y'(0) = 1$, we find:

$$y'(x) = 1 + \int_0^x u(t)dt$$

$$y(x) = x + \int_0^x (x-t)u(t)dt$$

And $y''(x) + y(x) = cosx \Rightarrow u(x) + x + \int_0^x (x-t)u(t)dt = cosx$

$$\Rightarrow u(x) = cosx - x - \int_0^x (x-t)u(t)dt$$

the equivalent Volterra integral equation.

Exercises 2.3.

Convert each of the following first-order initial value problems to a Volterra integral equation:

1. $y' + y = \sec 2x$, $y(0) = 0$
2. $y'' - \sin x\, y' + e^{xy} = x$, $y(0) = 1$, $y'(0) = -1$
3. $y''' - y'' - y' + y = 0$, $y(0) = 2$, $y'(0) = 0$, $y''(0) = 2$

2.3 Converting BVP to Fredholm Equation

So far we have discussed how an initial value problem can be transformed to an equivalent Volterra integral equation. In this section, we will present the technique that will be used to convert a boundary value problem to an equivalent Fredholm integral equation. The technique is similar to that discussed in the previous section with some exceptions that are related to the boundary conditions. It is important to point out here that the procedure of reducing the boundary value problem to the Fredholm integral equation is complicated and rarely used. The method is similar to the technique discussed above, which reduces the initial value problem to Volterra integral equation, with the exception that we are given boundary conditions.

Special attention should be taken to define $y'(0)$ since it is not always given, as will be seen later. This can be easily determined from the resulting equations. It seems useful and practical to illustrate this method by applying it to an example rather than proving it.

Example 2.10. We want to derive an equivalent Fredholm integral equation to the following boundary value problem**:** $y''(x) + y(x) = x \ , 0 < x < \pi$
subject to the boundary conditions**:** $y(0) = 1, y(\pi) = \pi - 1$
We first set**:** $y''(x) = u(x)$ (2.18)
Integrating both sides of the above equation from 0 to x gives:

$$\int_0^x y''(t)dt = \int_0^x u(t)dt \Rightarrow \mathbf{y'(x) = y'(0)} + \int_0^x \mathrm{u(t)dt} \qquad (2.19)$$

As indicated earlier, $y'(0)$ is not given in this boundary value problem. However, $y'(0)$ will be determined later by using the boundary condition at $x = \pi$. Integrating both sides of the last equation from 0 to x and using the given boundary condition at $x= 0$, we find**:**

$$\boldsymbol{y(x) = y(0) + y'(0)x} + \int_0^x \int_0^x u(t)dt\,dt \Rightarrow \boldsymbol{y(x) = 1 + y'(0)x} + \int_0^x (x-t)u(t)dt \qquad (2.20)$$

upon converting the resulting double integral to a single integral as discussed before. It remains to evaluate $y'(0)$, and this can be obtained by substituting $x = \pi$ on both sides of the last equation and using the boundary condition at $x = \pi$, hence, we find**:**

$$\boldsymbol{\pi y'(0) = y(\pi) - 1} - \int_0^\pi (\pi - t)u(t)dt$$

$$\Rightarrow \boldsymbol{y'(0)} = \frac{\mathbf{1}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi - 2} - \int_0^\pi (\pi - t)u(t)dt\right]$$

Substituting $y'(0)$ into (2.20) yields:

$$\boldsymbol{y(x) = 1 + x\left[\frac{1}{\pi}\left[\pi - 2 - \int_0^{\pi}(\pi - t)u(t)dt\right]\right] + \int_0^{x}(x - t)u(t)dt} \qquad (2.21)$$

Substituting (2.18) and (2.21) into BVP, we get**:**

$$y''(x) + y(x) = x$$

$$\Rightarrow u(x) + \mathbf{1} + \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{\pi}(\pi - t)u(t)dt\right] + \int_0^{x}(x - t)u(t)dt = x$$

$$\Rightarrow u(x) = x - \mathbf{1} - \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{\pi}(\pi - t)u(t)dt\right] - \int_0^{x}(x - t)u(t)dt$$

$$\Rightarrow u(x) = x - \mathbf{1} - \frac{\boldsymbol{x}}{\boldsymbol{\pi}}\left[\boldsymbol{\pi} - \mathbf{2} - \int_0^{x}(\pi - t)u(t)dt - \int_x^{\pi}(\pi - t)u(t)dt\right]$$

$$- \int_0^{x}(x - t)u(t)dt$$

or equivalently, after performing simple calculations and adding integrals with similar limits:

$$u(x) = \frac{\boldsymbol{2x - \pi}}{\boldsymbol{\pi}} - \int_{\mathbf{0}}^{\boldsymbol{x}} \frac{\boldsymbol{t(x - \pi)}}{\boldsymbol{\pi}}\boldsymbol{u(t)dt} - \int_{\boldsymbol{x}}^{\boldsymbol{\pi}} \frac{\boldsymbol{x(t - \pi)}}{\boldsymbol{\pi}}\boldsymbol{u(t)dt}$$

Consequently, the desired Fredholm integral equation of the second kind is given by

$$u(x) = \frac{\boldsymbol{2x - \pi}}{\boldsymbol{\pi}} - \int_{\mathbf{0}}^{\boldsymbol{\pi}} \boldsymbol{K(x,t)u(t)dt}$$

where the kernel $K(x, t)$ is defined by:

$$\boldsymbol{K(x,t)} = \begin{cases} \dfrac{\boldsymbol{t(x-\pi)}}{\boldsymbol{\pi}} & \boldsymbol{, for\ 0 \le t \le x} \\ \dfrac{\boldsymbol{x(t-\pi)}}{\boldsymbol{\pi}} & \boldsymbol{, for\ x \le t \le \pi} \end{cases}$$

Exercises 2.4.

Derive the equivalent Fredholm integral equation for the following boundary value problems:

$y'' + 4y = \sin x$, $0 < x < 1$, $y(0) = y(1) = 0$

Chapter Three: Fredholm Integral Equations

3.1 Introduction

In this chapter, we shall be concerned with the nonhomogeneous Fredholm integral equations of the second kind of the form**:**

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \quad , a \leq x \leq b \qquad (3.1)$$

where $K(x,t)$ is the kernel of the integral equation, and λ is a parameter. A considerable amount of discussion will be directed toward the various methods and techniques that are used for solving this type of equation starting with the most recent methods that proved to be highly reliable and accurate. To do this we will naturally focus our study on the *degenerate* or *separable* kernels all through this chapter. The standard form of the *degenerate* or *separable* kernel is given by**:**

$$K(x,t) = \sum_{j=1}^{n} g_j(x)h_j(t) \qquad (3.2)$$

The expressions $x - t$, $x + t$, xt, $x^2 - 3xt + t^2$, etc. are examples of separable kernels. For other well-behaved non-separable kernels, we can convert them to separable in the form (3.2) simply by expanding these kernels using Taylor's expansion.

<u>Definition (3.1)</u>

The kernel $K(x, t)$ is defined to be ***square integrable*** in both x and t in the square $a \leq x \leq b$, $a \leq t \leq b$ if the following *regularity condition***:**

$$\int_a^b \int_a^b K(x,t)dx\,dt < \infty \qquad (3.3)$$

is satisfied.

This condition gives rise to the development of the solution of the Fredholm integral equation (3.1). It is also convenient to state, without proof, the so-called *Fredholm Alternative Theorem* that relates the solutions of homogeneous and nonhomogeneous Fredholm integral equations.

3.1.1 Fredholm Alternative Theorem

The nonhomogeneous Fredholm integral equation (3.1) has one and only one solution if the only solution to the homogeneous Fredholm integral equation**:**

$$u(x) = \lambda \int_a^b K(x,t)u(t)dt \qquad (3.4)$$

is the trivial solution $u(x) = 0$.

We end this section by introducing the necessary condition that will guarantee a unique solution to the integral equation (3.11) in the interval of discussion. Considering (3.2), if the kernel $K(x, t)$ is real, continuous, and bounded in the square $a \leq x \leq b$ and $a \leq t \leq b$, i.e. if**:**

$$|K(x,t)| \leq M \quad , a \leq x \leq b \text{ and } a \leq t \leq b \qquad (3.5)$$

and if $f(x) \neq 0$, and continuous in $a \leq x \leq b$, then the necessary condition that will guarantee that (3.1) has only a unique solution is given by**:**

$$|\lambda| M(b-a) < 1 \qquad (3.6)$$

It is important to note that a continuous solution to Fredholm integral equation may exist, even though the condition (3.6) is not satisfied. This may be seen by considering the equation:

$$u(x) = -4 + \int_0^1 (2x+3t)u(t)dt \qquad (3.7)$$

In this example, $\lambda = 1$, $|K(x,t)| \leq 5$ and $(b-a) = 1$; therefore :

$$|\lambda| M(b-a) = 5 \nless 1 \qquad (3.8)$$

Accordingly, the necessary condition (3.6) fails to hold, but the integral equation (3.7) has an exact solution given by:

$$u(x) = 4x \qquad (3.9)$$

and this can be justified through direct substitution.

In the following, we will discuss several methods that handle successfully the Fredholm integral equations of the second kind.

3.2 The Adomian Decomposition Method

Adomian developed the so-called Adomian decomposition method or simply the ***decomposition method*** **(ADM).** The method proved to be reliable and effective for a wide class of equations, differential and integral equations, and linear and nonlinear models. The method was applied mostly to ordinary and partial differential equations and was rarely used for integral equations.

In the decomposition method, we usually express the solution $u(x)$ of the integral equation (3.1) in a series form defined by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (3.10)$$

Substituting the decomposition (3.10) into both sides of (3.1) yields:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \qquad (3.11)$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = f(x) + \lambda \int_a^b K(x,t)u_0(t)dt + \lambda \int_a^b K(x,t)u_1(t)dt + \lambda \int_a^b K(x,t)u_2(t)dt + \cdots \qquad (3.12)$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the unknown function $u(x)$ are completely determined in a recurrent manner, if we set:

$$u_0(x) = f(x) \qquad (3.13)$$

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt \qquad (3.14)$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt \qquad (3.15)$$

$$u_3(x) = \lambda \int_a^b K(x,t)u_2(t)dt \qquad (3.16)$$

and so on. The above-discussed scheme for the determination of the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... of the solution $u(x)$ of Eq. (3.1) can be written recursively by:

$$u_0(x) = f(x) \qquad (3.17)$$

$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad ,n \geq 0 \qquad (3.18)$$

In view of (3.17) and (3.18), the components $u_0(x)$, $u_1(x)$, $u_2(x)$, $u_3(x)$, ... follow immediately. With these components determined, the solution $u(x)$ of (3.1) is readily determined in a series form using the decomposition (3.10). It is important to note that the obtained series for $u(x)$ converges to the exact solution in a closed form if such a solution exists as will be seen later. However, for concrete problems, where the exact solution cannot be evaluated, a truncated series $\sum_{n=0}^{k} u_n(x)$ is usually used to approximate the solution $u(x)$ and this can be used for numerical purposes. We point out here that a few terms of the truncated series usually provide a higher accuracy level of the approximate solution if compared with the existing numerical techniques.

In the following, we discuss some examples that illustrate the decomposition method outlined above.

Example 3.1. We first consider the Fredholm integral equation of the second kind

$$u(x) = \frac{9}{10}\boldsymbol{x^2} + \int_0^1 \frac{1}{2}\boldsymbol{x^2 t^2 u(t)dt} \qquad \textbf{(3.19)}$$

It is clear that $f(x) = \frac{9}{10}\boldsymbol{x^2}$, $\lambda = 1$, . To evaluate the components $u_0(x)$, $u_1(x)$, $u_2(x)$, ... of the series solution, we use the recursive scheme (3.17) and (3.18) to find**:**

$$u_0(x) = f(x) = \frac{9}{10}\boldsymbol{x^2} \qquad \textbf{(3.20)}$$

$$u_1(x) = \lambda \int_a^b K(x,t)u_0(t)dt = \int_0^1 \frac{1}{2}\boldsymbol{x^2 t^2}\left(\frac{9}{10}\boldsymbol{t^2}\right)\boldsymbol{dt} = \int_0^1 \frac{9}{20}\boldsymbol{x^2 t^4 dt} = \frac{9}{100}\boldsymbol{x^2} \qquad \textbf{(3.21)}$$

$$u_2(x) = \lambda \int_a^b K(x,t)u_1(t)dt = \int_0^1 \frac{1}{2}\boldsymbol{x^2 t^2}\left(\frac{9}{100}\boldsymbol{t^2}\right)\boldsymbol{dt} = \int_0^1 \frac{9}{200}\boldsymbol{x^2 t^4 dt} = \frac{9}{1000}\boldsymbol{x^2} \textbf{(3.22)}$$

and so on. Noting that:

$$u(x) = u_0(x) + u_1(x) + u_2(x) + \cdots \qquad \textbf{(3.23)}$$

We can easily obtain the solution in a series form given by:

$$u(x) = \frac{9}{10}\boldsymbol{x^2} + \frac{9}{100}\boldsymbol{x^2} + \frac{9}{1000}\boldsymbol{x^2} + \cdots \qquad \textbf{(3.24)}$$

so that the solution of (3.19) in a closed form:

$$u(x) = \boldsymbol{x^2} \qquad \textbf{(3.25)}$$

follows immediately upon using the formula for the sum of the infinite geometric series.

Example 3.2. Consider the Fredholm integral equation:

$$u(x) = cosx + 2x + \int_0^{\pi} xtu(t)dt \qquad \textbf{(3.26)}$$

Proceeding as in example 3.1, we set:

$$u_0(x) = cosx + 2x \qquad \textbf{(3.27)}$$

$$u_1(x) = \int_0^{\pi} xt(cost + 2t)dt = \left(-2 + \frac{2}{3}\pi^3\right)x \qquad \textbf{(3.28)}$$

$$u_2(x) = \int_0^{\pi} xt\left(-2 + \frac{2}{3}\pi^3\right)tdt = \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x \qquad \textbf{(3.29)}$$

Consequently, the solution of (3.26) in a series form is given by

$$u(x) = cosx + 2x + \left(-2 + \frac{2}{3}\pi^3\right)x + \left(-\frac{2}{3}\pi^3 + \frac{2}{9}\pi^6\right)x + \left(-\frac{2}{9}\pi^6 + \frac{2}{27}\pi^9\right)x + \cdots \qquad \textbf{(3.30)}$$

and in a closed form:

$$u(x) = cosx \qquad \textbf{(3.31)}$$

Example 3.3. We consider here the Fredholm integral equation**:**

$$u(x) = e^x - 1 + \int_0^1 tu(t)dt \qquad \textbf{(3.32)}$$

Applying the decomposition technique as discussed before, we find**:**

$$u_0(x) = e^x - 1 \qquad \textbf{(3.33)}$$

$$u_1(x) = \int_0^1 t(e^t - 1)dt = \frac{1}{2} \qquad \textbf{(3.34)}$$

$$u_2(x) = \int_0^1 \frac{1}{2}tdt = \frac{1}{4} \qquad \textbf{(3.35)}$$

The determination of the components (3.33)-(3.35) yields the solution of the equation (3.32) in a series form given by**:**

$$u(x) = e^x - 1 + \frac{1}{2}(1 + \frac{1}{2} + \frac{1}{4} + \cdots) \qquad \textbf{(3.36)}$$

where we can easily obtain the solution in a closed form given by**:**

$$u(x) = e^x \qquad \textbf{(3.37)}$$

Example 3.4. Solve the following Fredholm integral equation**:**

$$u(x) = 1 + \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)u(t)dt \qquad \textbf{(3.38)}$$

Applying the decomposition technique as discussed before, we find**:**

$$u_0(x) = 1 \qquad \textbf{(3.39)}$$

$$u_1(x) = \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)dt = \frac{\pi}{8}sec^2(x) \qquad \textbf{(3.40)}$$

$$u_2(x) = \frac{1}{2}\int_0^{\frac{\pi}{4}} sec^2(x)\left(\frac{\pi}{8}sec^2(t)\right)dt = \frac{\pi}{16}sec^2(x) \qquad \textbf{(3.41)}$$

The determination of the components (3.39)-(3.41) yields the solution of the equation (3.38) in a series form given by**:**

$$u(x) = 1 + \frac{\pi}{8}sec^2(x) + \frac{\pi}{16}sec^2(x) + \frac{\pi}{32}sec^2(x) + \cdots \qquad \textbf{(3.42)}$$

where we can easily obtain the solution in a closed form given by**:**

$$u(x) = 1 + \frac{\pi}{4}sec^2(x) \qquad \textbf{(3.43)}$$

Exercises 3.1. Solve the following Fredholm integral equations by using the *Adomian* *decomposition method*:

1. $u(x) = sinx - x + \int_0^{\frac{\pi}{2}} xtu(t)dt$
2. $u(x) = e^{x+2} - 2\int_0^1 e^{x+t}u(t)dt$
3. $u(x) = xsinx - \frac{1}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} u(t)dt$

3.3. The Modified Decomposition Method

As stated before, the Adomian decomposition method provides the solutions in an infinite series of components. The components u_j, $j \geq 0$ are easily computed if the inhomogeneous term $f(x)$ in the Fredholm integral equation:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt$$

consists of a polynomial of one or two terms. However, if the function $f(x)$ consists of a combination of two or more polynomials, trigonometric functions, hyperbolic functions, and others, the evaluation of the components u_j, $j \geq 0$ requires more work.

The modified decomposition method depends mainly on splitting the function $f(x)$ into two parts, therefore it cannot be used if the $f(x)$ consists of only one term. The modified decomposition method will be briefly outlined here,

The standard Adomian decomposition method employs the recurrence relation:

$$u_0(x) = f(x)$$
$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad ,n \geq 0 \quad \textbf{(3.44)}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad \textbf{(3.45)}$$

The modified decomposition method presents a slight variation to the recurrence relation (3.44) to determine the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set:

$$f(x) = f_1(x) + f_2(x) \qquad (3.46)$$

Because of (3.46), we introduce a qualitative change in the formation of the recurrence relation (3.44). The modified decomposition method identifies the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation. The modified decomposition method admits the use of the modified recurrence relation:

$$u_0(x) = f_1(x)$$
$$u_1(x) = f_2(x) + \lambda \int_a^b K(x,t)u_0(t)dt$$
$$u_{n+1}(x) = \lambda \int_a^b K(x,t)u_n(t)dt \quad , n \geq 1 \qquad \textbf{(3.47)}$$

Example 3.5. Solve the Fredholm integral equation by using the modified decomposition method.

$$u(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4) + \int_0^1 tu(t)dt$$

We first decompose *f*(*x*) given by

$$f(x) = 3x + e^{4x} - \frac{1}{16}(17 + 3e^4)$$

into two parts, namely

$$f_1(x) = 3x + e^{4x} \ , \quad f_2(x) = -\frac{1}{16}(17 + 3e^4)$$

We next use the modified recurrence formula (3.47) to obtain:

$$u_0(x) = 3x + e^{4x}$$
$$u_1(x) = -\frac{1}{16}(17 + 3e^4) + \int_0^1 t(3t + e^{4t})dt = 0$$
$$u_{n+1}(x) = \int_0^{b1} tu_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by: $u(x) = 3x + e^{4x}$

Example 3.6. Solve the Fredholm integral equation by using the modified decomposition method.

$$u(x) = \frac{1}{1 + x^2} - 2sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}u(t)dt$$

We first decompose *f*(*x*) given by

$$f(x) = \frac{1}{1 + x^2} - 2sinh\frac{\pi}{4}$$

into two parts, namely

$$f_1(x) = \frac{1}{1 + x^2} \ , \quad f_2(x) = -2sinh\frac{\pi}{4}$$

We next use the modified recurrence formula (3.47) to obtain:

$$u_0(x) = \frac{1}{1 + x^2}$$
$$u_1(x) = -2sinh\frac{\pi}{4} + \int_{-1}^1 e^{\tan^{-1} t}\left(\frac{1}{1+t^2}\right)dt = 0$$
$$u_{n+1}(x) = \int_0^{b1} e^{\tan^{-1} t}u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by: $u(x) = \frac{1}{1+x^2}$

Exercises 3.2. Use the *modified decomposition method* to solve the following Fredholm integral equations:

1. $u(x) = sinx - x + x\int_0^{\frac{\pi}{2}} tu(t)$
2. $u(x) = e^x + 12x^2 + (3 + e^1)x - 4 - \int_0^1 (x - t)u(t)dt$

3.4 The Successive Approximations Method

The *successive approximations method* or the *Picard iteration method* provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess as $u_0(x)$, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence in relation to determining the other approximations. The most commonly used values for the zeroth approximations are 0, 1, or x. Of course, other real values can be selected as well. Given Fredholm integral equation of the second kind: $u(x) = f(x) + \lambda\int_a^b K(x,t)u(t)dt$

where $u(x)$ is the unknown function to be determined, $K(x, t)$ is the kernel, and λ is a parameter. The successive approximations method introduces the recurrence relation:

$u_0(x)$ = any selective real-valued function,

$$u_{n+1}(x) = f(x) + \lambda\int_a^b K(x,t)u_n(t)dt \quad ,n \geq 0 \qquad \textbf{(3.48)}$$

the solution is determined by using the limit:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) \qquad \textbf{(3.49)}$$

3.4.1 The difference between The successive approximations method and the Adomian method can be summarized as follows:

1. The successive approximations method gives successive approximations of the solution $u(x)$, whereas the Adomian method gives successive components of the solution $u(x)$.
2. The successive approximations method admits the use of a selective real-valued function for the zeroth approximation u_0, whereas the Adomian decomposition method assigns all terms that are not inside the integral sign for the zeroth component $u_0(x)$. Recall that this assignment was modified when using the modified decomposition method.
3. The successive approximations method gives the exact solution, if it exists, by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x)$$

However, the Adomian decomposition method gives the solution as infinite series of components by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x)$$

This series solution converges rapidly to the exact solution if such a solution exists.

The successive approximations method or iteration method will be illustrated by studying the following examples.

Example 3.7. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = x + e^x - \int_0^1 xtu(t)dt$$

For the zeroth approximation *u*0(*x*), we can select:

$$u_0(x) = 0$$

The method of successive approximations admits the use of the iteration formula:

$$u_{n+1}(x) = x + e^x - \int_0^1 xtu_n(t)dt \quad ,n \geq 0$$

Therefore, we obtain:

$$u_1(x) = x + e^x$$

$$u_2(x) = x + e^x - \int_0^1 xt(t + e^t)dt = e^x - \frac{1}{3}x$$

$$u_3(x) = x + e^x - \int_0^1 xt\left(e^t - \frac{1}{3}t\right)dt = e^x + \frac{1}{9}x$$

$$\vdots$$

$$u_{n+1}(x) = e^x + \frac{(-1)^n}{3^n}x$$

Consequently, the solution *u*(*x*) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty}\left(e^x + \frac{(-1)^n}{3^n}x\right) = e^x$$

Example 3.8. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = x + \lambda \int_{-1}^1 xtu(t)dt$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = x$$

The method of successive approximations admits the use of the iteration formula:

$$u_{n+1}(x) = x + \lambda \int_{-1}^{1} xtu_n(t)dt \quad , n \geq 0$$

Therefore, we obtain:

$$u_1(x) = x + \lambda \int_{-1}^{1} xt^2 dt \ = x + \frac{2}{3}\lambda x$$

$$u_2(x) = x + \lambda \int_{-1}^{1} xt\left(t + \frac{2}{3}\lambda t\right)dt \ = x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x$$

$$u_3(x) = x + \lambda \int_{-1}^{1} xt\left(t + \frac{2}{3}\lambda xt + \left(\frac{2}{3}\right)^2 \lambda^2 t\right)dt \ = x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x$$

$$\vdots$$

$$u_{n+1}(x) = x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x + \cdots + \left(\frac{2}{3}\right)^{n+1} \lambda^{n+1} x$$

The solution $u(x)$ is given by:

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_{n+1}(x) \\ &= \lim_{n\to\infty}\left(x + \frac{2}{3}\lambda x + \left(\frac{2}{3}\right)^2 \lambda^2 x + \left(\frac{2}{3}\right)^3 \lambda^3 x + \cdots + \left(\frac{2}{3}\right)^{n+1} \lambda^{n+1} x\right) \\ &= \frac{3x}{3 - 2\lambda} \quad , 0 < \lambda < \frac{3}{2} \end{aligned}$$

obtained upon using the infinite geometric series for the right side of the above equation.

Example 3.9. Solve the Fredholm integral equation by using the successive approximations method:

$$u(x) = sinx + sinx \int_{0}^{\frac{\pi}{2}} costu(t)dt$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0$$

We next use the iteration formula

$$u_{n+1}(x) = sinx + sinx \int_{0}^{\frac{\pi}{2}} costu_n(t)dt \quad , n \geq 0$$

Therefore, we obtain:

$$u_1(x) = sinx \quad , \quad u_2(x) = \frac{3}{2} sinx$$
$$u_3(x) = \frac{7}{4} sinx \quad . \quad u_4(x) = \frac{15}{8} sinx$$
$$\vdots$$
$$u_{n+1}(x) = \frac{2^{n+1} - 1}{2^n} sinx = \left(2 - \frac{1}{2^n}\right) sinx$$

The solution $u(x)$ is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \lim_{n\to\infty} \left(2 - \frac{1}{2^n}\right) sinx = 2sinx$$

Exercises 3.3. Use the *successive approximations method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + x^3 + \lambda \int_{-1}^{1} xtu(t)dt$
2. $u(x) = x + \sec^2 x - \int_0^{\frac{\pi}{4}} xu(t)dt$

3.5 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain:

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(b)}{n!}(x - b)^n \qquad (3.50)$$

converges to $u(x)$ in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (3.51)$$

The series solution method that stems mainly from the Taylor series for analytic functions, will be used for solving Fredholm integral equations. We will assume that the solution $u(x)$ of the Fredholm integral equations:

$$u(x) = f(x) + \lambda \int_a^b K(x,t)u(t)dt \qquad (3.52)$$

is analytic, and therefore possesses a Taylor series of the form given in (3.52), where the coefficients a_n will be determined recurrently. Substituting (3.51) into both sides of (3.52) gives:

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \lambda \int_a^b K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt \qquad (3.53)$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \lambda \int_a^b K(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt \qquad (3.54)$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (3.52) will be converted to a traditional integral in (3.53) or (3.54) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking a series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (3.53) or (3.54) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in a_j, $j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients a_j, $j \geq 0$. Having determined the coefficients a_j, $j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (3.51). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve. It is worth noting that using the series solution method for solving Fredholm integral equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x$, e^x, etc, the series method gives the exact solution after rounding a few of the coefficients a_j, $j \geq 0$. This will be illustrated by studying the following examples.

Example 3.10. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = (x+1)^2 + \int_{-1}^{1} (xt + x^2t^2)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side gives:

$$\begin{aligned} a_0 + a_1x + a_2x^2 + a_3x^2 + \cdots \\ = 1 + \left(2 + \frac{2}{3}a_1 + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7\right)x \\ + \left(1 + \frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8\right)x^2 \end{aligned}$$

Equating the coefficients of like powers of x on both sides gives:

$$a_0 = 1, a_1 = 6, a_2 = \frac{25}{9}\ , a_n = 0, n \geq 3$$

The exact solution is given by:

$$u(x) = 1 + 6x + \frac{25}{9}x^2$$

Example 3.11. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = x^2 - x^3 + \int_0^1 (1 + xt)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = x^2 - x^3 + \int_0^1 (1 + xt)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation we find

$a_0 = \frac{-29}{90}$, $a_1 = \frac{-1}{6}$, $a_2 = 1$, $a_3 = -1$, $a_n = 0$, $n \geq 4$.

Consequently, the exact solution is given by:

$$u(x) = \frac{-29}{90} - \frac{1}{6}x + x^2 - x^3$$

Example 3.12. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = -x^4 + \int_{-1}^1 (xt^2 - x^2t)u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n == -x^4 + \int_{-1}^1 (xt^2 - x^2t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation, we find:

$a_0 = 0$, $a_1 = \frac{-30}{133}$, $a_2 = \frac{20}{133}$, $a_3 = 0$, $a_4 = -1$, $a_n = 0$, $n \geq 5$.

Consequently, the exact solution is given by:

$$u(x) = \frac{-30}{133}x + \frac{20}{133}x^2 - x^4$$

Example 3.13. Solve the Fredholm integral equation by using the series solution method:

$$u(x) = -1 + \cos x + \int_0^{\frac{\pi}{2}} u(t)dt$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

leads to,

$$\sum_{n=0}^{\infty} a_n x^n = -1 + \cos x + \int_0^{\frac{\pi}{2}} \left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integral on the right side, and equating the coefficients of like powers of x on both sides of the resulting equation we find

$a_0 = 1, a_{2j+1} = 0, a_{2j+2} = \frac{(-1)^j}{(2j)!} \quad , j \geq 0$

Consequently, the exact solution is given by:

$$u(x) = cosx$$

Exercises 3.4. Use the *series solution method* to solve the following Fredholm integral equations:

1. $u(x) = 5x + \int_{-1}^{1}(1 - xt)u(t)dt$
2. $u(x) = \sec^2 x - 1 + \int_0^{\frac{\pi}{4}} u(t)dt$

3.6 The Direct Computation Method

In this section, the *direct computation method* will be applied to solve the Fredholm integral equations. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form:

$$K(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t) \qquad (3.55)$$

Examples of separable kernels are $x - t$, xt, $x^2 - t^2$, $xt^2 + x^2t$, etc.

The direct computation method can be applied as follows:

1. We first substitute (3.55) into the Fredholm integral equation of the form:

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt \qquad (3.56)$$

2. This substitution gives:

$$u(x) = f(x) + g_1(x)\int_a^b h_1(t)u(t)dt + g_2(x)\int_a^b h_2(t)u(t)dt + \cdots$$
$$+ g_n(x)\int_a^b h_n(t)u(t)dt \qquad (3.57)$$

3. Each integral at the right side depends only on the variable *t* with constant limits of integration for *t*. This means that each integral is equivalent to a constant. Based on this, Equation (3.57) becomes:
$$u(x) = f(x) + \alpha_1 g_1(x) + \alpha_2 g_2(x) + \cdots + \alpha_n g_n(x) \qquad (3.58)$$
Where
$$\alpha_i = \int_a^b h_i(t)u(t)dt \quad 1 \le i \le n \qquad (3.59)$$
4. Substituting (3.58) into (3.59) gives a system of *n* algebraic equations that can be solved to determine the constants α_i , $1 \le i \le n$. Using the obtained numerical values of α_i into (3.59), the solution $u(x)$ of the Fredholm integral equation (3.56) is readily obtained.

Example 3.14 Solve the Fredholm integral equation by using the direct computation method $u(x) = 3x + 3x^2 + \frac{1}{2}\int_0^1 \boldsymbol{x^2 t u(t) dt}$ (3.60)

The kernel $K(x, t) = x^2 t$ is separable. Consequently, we rewrite (3.60) as:
$$u(x) = 3x + 3x^2 + \frac{1}{2}\boldsymbol{x^2}\int_0^1 \boldsymbol{t u(t) dt} \qquad (3.61)$$
The integral at the right side is equivalent to a constant because it depends only on functions of the variable *t* with constant limits of integration. Consequently, Equation (3.61) can be rewritten as:
$$u(x) = 3x + 3x^2 + \frac{1}{2}\boldsymbol{\alpha x^2} \qquad \textbf{(3.62)}$$
Where $\boldsymbol{\alpha} = \int_0^1 \boldsymbol{t u(t) dt}$ **(3.63)**

To determine α, we substitute (3. 62) into (3.63) to obtain:
$$\boldsymbol{\alpha} = \int_0^1 \boldsymbol{t}\left(3t + 3t^2 + \frac{1}{2}\boldsymbol{\alpha t^2}\right)\boldsymbol{dt} \qquad \textbf{(3.64)}$$
Integrating the right side of (3.64) yields:
$$\boldsymbol{\alpha} = \frac{7}{4} + \frac{1}{8}\alpha$$
that gives $\boldsymbol{\alpha} = 2$

Substituting $\boldsymbol{\alpha} = 2$ into (3.62) leads to the exact solution: $u(x) = 3x + 4x^2$

Example 3.15 Solve the Fredholm integral equation by using the *direct computation method*
$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}x\int_0^{\pi/3} u(t)dt$$

The integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, we can rewrite the above equation as:

$$u(x) = \frac{1}{3}x + \sec x \tan x - \frac{1}{3}\alpha x$$

Where $\alpha = \int_0^{\pi/3} u(t)dt = \int_0^{\pi/3} \left(\frac{1}{3}t + \sec t \tan t - \frac{1}{3}\alpha t\right) dt = 1 + \frac{1}{54}\pi^2 - \frac{1}{54}\alpha\pi^2$
that gives $\alpha = 1$. Therefore, the exact solution is: $u(x) = \sec x \tan x$

Example 3.16 Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 11x + 10x^2 + x^3 - \int_0^1 (30xt^2 + 20x^2t)u(t)dt$$

The kernel $K(x, t) = 30xt^2 + 20x^2t$ is separable. Consequently, we rewrite the above equation as:

$$u(x) = 11x + 10x^2 + x^3 - 30x\int_0^1 t^2u(t)dt - 20x^2\int_0^1 tu(t)dt$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, the above the equation can be rewritten as:
$u(x) = 11x + 10x^2 + x^3 - 30\alpha x - 20\beta x^2 = (11 - 30\alpha)x + (10 - 20\beta)x^2 + x^3$,
Where $\alpha = \int_0^1 t^2u(t)dt$ and $\beta = \int_0^1 tu(t)dt$
And then, we have:

$$\alpha = \int_0^1 t^2[(11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3]dt = \frac{59}{12} - \frac{15}{2}\alpha - 4\beta$$

$$\beta = \int_0^1 t[(11 - 30\alpha)t + (10 - 20\beta)t^2 + t^3]dt = \frac{191}{30} - 10\alpha - 5\beta$$

Solving this system of algebraic equations gives:

$$\alpha = \frac{11}{30}\ , \qquad \beta = \frac{9}{20}$$

the exact solution is: $u(x) = x^2 + x^3$.

Example 3.17 Solve the Fredholm integral equation by using the *direct computation method*

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 (1 + 30xt^2 + 12x^2t)u(t)dt$$

The kernel $K(x, t) = 1+30xt^2+12x^2t$ is separable. Consequently, we rewrite the above equation as:

$$u(x) = 4 + 45x + 26x^2 - \int_0^1 u(t)dt - 30x \int_0^1 t^2u(t)dt - 12x^2 \int_0^1 tu(t)dt$$

Each integral at the right side is equivalent to a constant because it depends only on functions of the variable t with constant limits of integration. Consequently, the above an equation can be rewritten as:

$$u(x) = (4 - \alpha) + (45 - 30\beta)x + (26 - 12\gamma)x^2$$

where $\alpha = \int_0^1 u(t)dt$, $\beta = \int_0^1 t^2u(t)dt$ and $\gamma = \int_0^1 tu(t)dt$.

And then, we have:

$$\alpha = \int_0^1 ((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt = \frac{211}{6} - \alpha - 15\beta - 4\gamma$$

$$\beta = \int_0^1 t^2((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt$$

$$= \frac{1067}{60} - \frac{1}{3}\alpha - \frac{15}{2}\beta - \frac{12}{5}\gamma$$

$$\gamma = \int_0^1 t((4 - \alpha) + (45 - 30\beta)t + (26 - 12\gamma)t^2)dt = \frac{47}{2} - \frac{1}{2}\alpha - 10\beta - 3\gamma$$

Solving this system of algebraic equations gives:

$\alpha = 3, \beta = \frac{43}{30}$ *and* $\gamma = \frac{23}{12}$, and the exact solution is : $u(x) = 1+2x+ 3x^2$

Exercises 3.5. Use the *direct computation method* to solve the following Fredholm integral equations:

1. $u(x) = 1 + 9x + 2x^2 + x^3 - \int_0^1 (20xt + 10x^2t^2)u(t)dt$
2. $u(x) = \left(\frac{2}{\sqrt{3}} - 1\right)x + \sec x \tan x - \int_0^{\pi/6} xu(t)dt$

Chapter Four: Volterra Integral Equations

Volterra integral equations arise in many scientific applications such as population dynamics, the spread of epidemics, and semiconductor devices. It was also shown in chapter two that Volterra integral equations can be derived from initial value problems. We will study Volterra integral equations of the second kind given:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \qquad (4.1)$$

The unknown function $u(x)$, which will be determined, occurs inside and outside the integral sign. The kernel $K(x, t)$ and the function $f(x)$ are given real-valued functions, and λ is a parameter. In what follows we will present the methods that will be used.

4.1 The Adomian Decomposition Method

The Adomian decomposition method consists of decomposing the unknown function $u(x)$ of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.2)$$

where the components $u_n(x)$, $n \geq 0$ are to be determined recursively. The decomposition method concerns itself with finding the components u_0, u_1, u_2, . . . individually. The determination of these components can be achieved easily through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (4.2) into the Volterra integral equation (4.1) to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x K(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt \qquad (4.3)$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, the components $u_j(x)$, $j \geq 1$ of the unknown function $u(x)$ are completely determined by setting the recurrence relation:

$$u_0(x) = f(x) \qquad (4.4)$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad , n \geq 0 \qquad (4.5)$$

Example 4.1. Solve the following Volterra integral equation:

$$u(x) = 1 - \int_0^x u(t)dt \qquad (4.6)$$

We notice that $f(x) = 1$, $\lambda = -1$, $K(x, t) = 1$. Recall that the solution $u(x)$ is assumed to have a series form given in (4.2). Substituting the decomposition series (4.2) into both sides of (4.6) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 - \int_0^x \left(\sum_{n=0}^{\infty} u_n(t)\right) dt$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation:

$$u_0(x) = 1$$

$$u_1(x) = -\int_0^x u_0(t)dt == -\int_0^x 1dt = -x$$

$$u_2(x) = -\int_0^x u_1(t)dt = \int_0^x tdt = \frac{1}{2!}x^2$$

$$u_3(x) = -\int_0^x u_2(t)dt = -\int_0^x t^2dt = -\frac{1}{3!}x^3$$

$$u_4(x) = -\int_0^x u_3(t)dt = \int_0^x t^3dt = \frac{1}{4!}x^4$$

and so on. Using (4.2) gives the series solution:

$$u(x) = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots$$

that converges to the closed form solution:

$$u(x) = e^{-x}$$

Example 4.2. Solve the following Volterra integral equation:

$$u(x) = 1 + \int_0^x (t - x)u(t)dt \qquad (4.7)$$

We notice that $f(x) = 1$, $\lambda = 1$, $K(x, t) = t - x$. Substituting the decomposition series (4.2) into both sides of (4.7) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 + \int_0^x (t - x)\left(\sum_{n=0}^{\infty} u_n(t)\right) dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 1 + \int_0^x (t - x)(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

Proceeding as before we set the following recurrence relation:

$$u_0(x) = 1$$
$$u_k(x) = \int_0^x (t-x)u_{k-1}(t)dt \quad ,k \geq 1$$

that gives

$$u_0(x) = 1$$
$$u_1(x) = \int_0^x (t-x)u_0(t)dt = \int_0^x (t-x)dt = -\frac{1}{2!}x^2$$
$$u_2(x) = \int_0^x (t-x)u_1(t)dt = \int_0^x (t-x)\left(-\frac{1}{2!}t^2\right)dt = \frac{1}{4!}x^4$$
$$u_3(x) = \int_0^x (t-x)u_2(t)dt = \int_0^x (t-x)\left(\frac{1}{4!}t^4\right)dt = -\frac{1}{6!}x^6$$
$$u_4(x) = \int_0^x (t-x)u_3(t)dt = \int_0^x (t-x)\left(-\frac{1}{6!}t^6\right)dt = \frac{1}{8!}x^8$$

and so on. The solution in a series form is given by:

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 + \cdots$$

and in a closed form by:

$$u(x) = \cos x$$

obtained upon using the Taylor expansion for $\cos x$.

Example 4.3. Solve the following Volterra integral equation:

$$u(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)u(t)dt \qquad (4.8)$$

We notice that $f(x) = 1-x-\frac{1}{2}x^2$, $\lambda = -1$, $K(x,t) = t-x$. Substituting the decomposition series (4.2) into both sides of (4.8) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots$$
$$= 1 - x - \frac{1}{2}x^2 - \int_0^x (t-x)(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

This allows us to set the following recurrence relation:

$$u_0(x) = 1 - x - \frac{1}{2}x^2$$

$$u_k(x) = \int_0^x (t-x)u_{k-1}(t)dt \quad , k \geq 1$$

that gives:

$$u_0(x) = 1 - x - \frac{1}{2}x^2$$

$$u_1(x) = \int_0^x (t-x)u_0(t)dt = \int_0^x (t-x)\left(1 - t - \frac{1}{2}t^2\right)dt = \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4$$

$$u_2(x) = \int_0^x (t-x)u_1(t)dt = \int_0^x (t-x)\left(\frac{1}{2!}t^2 - \frac{1}{3!}t^3 - \frac{1}{4!}t^4\right)dt$$
$$= \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6$$

$$u_3(x) = \int_0^x (t-x)u_2(t)dt = \int_0^x (t-x)\left(\frac{1}{4!}t^4 - \frac{1}{5!}t^5 - \frac{1}{6!}t^6\right)dt$$
$$= \frac{1}{6!}x^6 - \frac{1}{7!}x^7 - \frac{1}{8!}x^8$$

and so on. The solution in a series form is given by:

$$u(x) = 1 - x - \frac{1}{2}x^2 + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 + \cdots$$
$$= 1 - (x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots)$$

and in a closed form by:

$$u(x) = 1 - \sinh x$$

obtained upon using the Taylor expansion for sinh *x*.

Example 4.4. Solve the following Volterra integral equation:

$$u(x) = 5x^3 - x^5 + \int_0^x tu(t)dt \qquad (4.9)$$

We notice that $f(x) = 5x^3 - x^5$, $\lambda = 1$, $K(x, t) = t$. Substituting the decomposition series (4.2) into both sides of (4.9) gives:

$$\sum_{n=0}^{\infty} u_n(x) = 5x^3 - x^5 + \int_0^x t\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = 5x^3 - x^5 - \int_0^x t(u_0(t) + u_1(t) + u_2(t) + \cdots)dt$$

This allows us to set the following recurrence relation:

$$u_0(x) = 5x^3 - x^5$$

$$u_k(x) = \int_0^x t u_{k-1}(t)dt \quad , k \geq 1$$

that gives:

$$u_0(x) = 5x^3 - x^5$$

$$u_1(x) = \int_0^x t u_0(t)dt = \int_0^x t(5t^3 - t^5)dt = x^5 - \frac{1}{7}x^7$$

$$u_2(x) = \int_0^x t u_1(t)dt = \int_0^x t\left(t^5 - \frac{1}{7}t^7\right)dt = \frac{1}{7}x^7 - \frac{1}{63}x^9$$

$$u_3(x) = \int_0^x t u_2(t)dt = \int_0^x t\left(\frac{1}{7}t^7 - \frac{1}{63}t^9\right)dt = \frac{1}{63}x^9 - \frac{1}{693}x^{11}$$

The solution in a series form is given by:

$$u(x) = (5x^3 - x^5) + \left(x^5 - \frac{1}{7}x^7\right) + \left(\frac{1}{7}x^7 - \frac{1}{63}x^9\right) + \left(\frac{1}{63}x^9 - \frac{1}{693}x^{11}\right) + \cdots$$

We can easily notice the appearance of identical terms with opposite signs. Such terms are called **noise terms** which will be discussed later. Canceling the identical terms with opposite signs gives the exact solution:

$$u(x) = 5x^3$$

Example 4.5. We finally solve the Volterra integral equation:

$$u(x) = 2 + \frac{1}{3}\int_0^x x t^3 u(t)dt \qquad (4.10)$$

Proceeding as before, we set the recurrence relation:

$$u_0(x) = 2$$

$$u_k(x) = \frac{1}{3}\int_0^x x t^3 u_{k-1}(t)dt \quad , k \geq 1$$

This in turn gives:

$$u_0(x) = 2$$

$$u_1(x) = \frac{1}{3}\int_0^x x t^3 u_0(t)dt = \frac{2}{3}\int_0^x x t^3 dt = \frac{1}{6}x^5$$

$$u_2(x) = \frac{1}{3}\int_0^x x t^3 u_1(t)dt = \frac{1}{3}\int_0^x x t^3\left(\frac{1}{6}t^5\right)dt = \frac{1}{162}x^{10}$$

$$u_3(x) = \frac{1}{3}\int_0^x x t^3 u_2(t)dt = \frac{1}{3}\int_0^x x t^3\left(\frac{1}{162}t^{10}\right)dt = \frac{1}{6804}x^{15}$$

and so on. The solution in a series form is given by:

$$u(x) = 2 + \frac{1}{6}x^5 + \frac{1}{162}x^{10} + \frac{1}{6804}x^{15} + \cdots$$

It seems that an exact solution is not obtainable. The obtained series solution can be used for numerical purposes. The more components that we determine the higher the accuracy level that we can achieve.

Exercises 4.1. solve the following Volterra integral equations by using the *Adomian decomposition method*:

1. $u(x) = 6x - 3x^2 + \int_0^x u(t)dt$
2. $u(x) = 1 + x + \int_0^x (x-t)u(t)dt$
3. $u(x) = 1 + x^2 + \int_0^x (x-t+1)^2 u(t)dt$

4.2 The Modified Decomposition Method

To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation:

$$u_0(x) = f(x)$$
$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad ,n \ge 0 \qquad (4.11)$$

where the solution $u(x)$ is expressed by an infinite sum of components defined before by:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (4.12)$$

In view of (4.11), the components $u_n(x)$, $n \ge 0$ can be easily evaluated. The modified decomposition method introduces a slight variation to the recurrence relation (4.11) that will lead to the determination of the components of $u(x)$ in an easier and faster manner. For many cases, the function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x) \qquad (4.13)$$

In view of (4.13), we introduce a qualitative change in the formation of the recurrence relation (4.11). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$. The other part of $f(x)$ can be added to the component $u_1(x)$ among other terms. In other words, the modified decomposition method introduces the modified recurrence relation:

$$u_0(x) = f_1(x)$$
$$u_1(x) = f_2(x) + \lambda \int_a^x K(x,t)u_0(t)dt$$
$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt \quad ,n \ge 1 \qquad (4.14)$$

Example 4.6. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sin x + (e^1 - e^{\cos x}) - \int_0^x e^{\cos t} u(t)dt \qquad (4.15)$$

We first split $f(x)$ given by:

$$f(x) = \sin x + (e^1 - e^{\cos x})$$

into two parts, namely

$$f_1(x) = \sin x \quad and \quad f_2(x) = (e^1 - e^{\cos x})$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = \sin x$$

$$u_1(x) = (e^1 - e^{\cos x}) - \int_0^x e^{\cos t} u_0(t)dt = (e^1 - e^{\cos x}) - \int_0^x e^{\cos t}(\sin t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = \sin x$$

Example 4.7. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = \sec x \tan x + (e^{\sec x} - e^1) - \int_0^x e^{\sec t} u(t)dt \quad , x < \frac{\pi}{2} \qquad (4.16)$$

Proceeding as before we split $f(x)$ into two parts:

$$f_1(x) = \sec x \tan x \quad and \quad f_2(x) = (e^{\sec x} - e^1)$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = \sec x \tan x$$

$$u_1(x) = (e^{\sec x} - e^1) - \int_0^x e^{\sec t} u_0(t)dt = (e^{\sec x} - e^1) - \int_0^x e^{\sec t}(\sec t \tan t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of $u_j, j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = \sec x \tan x$$

Example 4.8. Solve the Volterra integral equation by using the modified decomposition method:

$$u(x) = 1 + x^2 + \cos x - x - \frac{1}{3}x^3 - \sin x + \int_0^x u(t)dt \qquad (4.17)$$

Proceeding as before we split $f(x)$ into two parts:

$$f_1(x) = 1 + x^2 + \cos x \quad and \quad f_2(x) = -\left(x + \frac{1}{3}x^3 + \sin x\right)$$

We next use the modified recurrence formula (4.14) to obtain:

$$u_0(x) = f_1(x) = 1 + x^2 + \cos x$$

$$u_1(x) = -\left(x + \frac{1}{3}x^3 + \sin x\right) + \int_0^x u_0(t)dt$$

$$= -\left(x + \frac{1}{3}x^3 + \sin x\right) + \int_0^x (1 + t^2 + \cos t)dt = 0$$

$$u_{n+1}(x) = \lambda \int_a^x K(x,t)u_n(t)dt = 0, n \geq 1$$

It is obvious that each component of u_j, $j \geq 1$ is zero. This in turn gives the exact solution by:

$$u(x) = 1 + x^2 + \cos x$$

Exercises 4.2. Use the *modified decomposition method* to solve the following Volterra integral equations:

1. $u(x) = \sinh x + \cosh x - 1 - \int_0^x u(t)dt$
2. $u(x) = 2x + \left(1 - e^{-x^2}\right) - \int_0^x e^{-x^2+t^2} u(t)dt$

4.3 The Successive Approximations Method

The *successive approximations method* also called the *Picard iteration method* provides a scheme that can be used for solving initial value problems or integral equations. This method solves any problem by finding successive approximations to the solution by starting with an initial guess, called the zeroth approximation. As will be seen, the zeroth approximation is any selective real-valued function that will be used in a recurrence in relation to determining the other approximations. The successive approximations method introduces the recurrence relation:

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)u_{n-1}(t)dt \quad , n \geq 1 \qquad (4.18)$$

We always start with an initial guess for $u_0(x)$, mostly we select 0, 1, x for $u_0(x)$, and by using (4.18), several successive approximations u_k , $k \geq 1$ will be determined as:

$$u_1(x) = f(x) + \lambda \int_a^x K(x,t)u_0(t)dt$$

$$u_2(x) = f(x) + \lambda \int_a^x K(x,t)u_1(t)dt$$

$$\vdots$$

$$u_n(x) = f(x) + \lambda \int_a^x K(x,t)u_{n-1}(t)dt$$

The successive approximations method or the Picard iteration method will be illustrated by the following examples.

Example 4.9. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (4.19)$$

The method of successive approximations admits the use of the iteration formula:

$$u_n(x) = 1 - \int_0^x (x-t)u_{n-1}(t)dt \quad , n \geq 1 \qquad (4.20)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 1 \qquad (4.21)$$

Substituting (4.21) into (4.20), we obtain:

$$u_1(x) = 1 - \int_0^x (x-t)u_0(t)dt = 1 - \int_0^x (x-t)dt = 1 - \frac{1}{2!}x^2$$

$$u_2(x) = 1 - \int_0^x (x-t)u_1(t)dt = 1 - \int_0^x (x-t)\left(1 - \frac{1}{2!}t^2\right)dt = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$u_3(x) = 1 - \int_0^x (x-t)u_2(t)dt = 1 - \int_0^x (x-t)\left(1 - \frac{1}{2!}t^2 + \frac{1}{4!}t^4\right)dt$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

Consequently, we obtain:

$$u_{n+1}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!}$$

The solution $u(x)$ of (4.19):

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \cos x$$

Example 4.10. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt \qquad (4.22)$$

The method of successive approximations admits the use of the iteration formula:

$$u_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_{n-1}(t)dt \quad , n \geq 1 \qquad (4.23)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0 \qquad (4.24)$$

Substituting (4.24) into (4.23), we obtain:

$$u_1(x) = 1 + x + \frac{1}{2!}x^2$$

$$u_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u_1(t)dt = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$\vdots$$

and so on. The solution $u(x)$ of (4.22) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = e^x$$

Example 4.11. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = -1 + e^x + \tfrac{1}{2}x^2 e^x - \tfrac{1}{2}\int_0^x tu(t)dt \qquad (4.25)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = 0 \qquad (4.26)$$

We next use the iteration formula:

$$u_{n+1}(x) = -1 + e^x + \tfrac{1}{2}x^2 e^x - \tfrac{1}{2}\int_0^x tu_n(t)dt \ , \ n \ge 0 \qquad (4.27)$$

Substituting (4.26) into (4.27), we obtain:

$$u_1(x) = -1 + e^x + \frac{1}{2}x^2 e^x$$

$$u_2(x) = -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x tu_1(t)dt$$

$$= -1 + e^x + \frac{1}{2}x^2 e^x - \frac{1}{2}\int_0^x t\left(-1 + e^t + \frac{1}{2}t^2 e^t\right)dt$$

$$= -3 + \frac{1}{4}x^2 + e^x\left(3 - 2x + \frac{5}{4}x^2 - \frac{1}{4}x^3\right)$$

$$u_3(x) = x\left(1 + x + \frac{1}{2!}x^2\right)$$

Example 4.12. Solve the Volterra integral equation by using the successive approximations method:

$$u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt \qquad (4.28)$$

For the zeroth approximation $u_0(x)$, we can select:

$$u_0(x) = x \qquad (4.29)$$

We next use the iteration formula:

$$u_{n+1}(x) = 1 - x\sin x + x\cos x + \int_0^x tu_n(t)dt\, , \ n \ge 0 \qquad (4.30)$$

Substituting (4.29) into (4.30), we obtain:

$$u_1(x) = 1 + \frac{1}{3}x^3 - x\sin x + x\cos x$$

$$u_2(x) = 3 + \frac{1}{2}x^2 + \frac{1}{15}x^3 - (2 + 3x - x^2)\sin x - (2 - 3x - x^2)\cos x$$

$$u_3(x) = \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7\right) + \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6\right)$$

$$\vdots$$

$$u_{n+1}(x) = \sum_{k=0}^{n}(-1)^k \frac{x^{2k+1}}{(2k+1)!} + \sum_{k=0}^{n}(-1)^k \frac{x^{2k}}{(2k)!}$$

Notice that we used the Taylor expansion for sin x and cos x to determine the approximations $u_3(x)$, $u_4(x)$, ... The solution $u(x)$ of (4.28) is given by:

$$u(x) = \lim_{n\to\infty} u_{n+1}(x) = \sin x + \cos x$$

Exercises 4.3.

Use the *successive approximations method* to solve the following Volterra integral equations:

1. $u(x) = x + \int_0^x (x-t)u(t)dt$
2. $u(x) = x\cosh x - \int_0^x tu(t)dt$
3. $u(x) = 1 - x\sin x + \int_0^x tu(t)dt$
4. $u(x) = 1 + \sinh x - \sin x + \cos x - \cosh x + \int_0^x u(t)dt$

4.4 The Laplace Transform Method

The *Laplace transform method* is a powerful technique that can be used for solving initial value problems and integral equations as well. The details and properties of the Laplace method can be found in ordinary differential equations texts.

Before we start applying this method, we summarize some of the concepts presented in Section 1.3. In the convolution theorem for the Laplace transform, it was stated that if the kernel $K(x, t)$ of the integral equation:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt$$

depends on the difference $x-t$, then it is called a ***difference kernel***. Examples of the difference kernel are e^{x-t}, $\cos(x-t)$, and $x-t$. The integral equation can thus be expressed as:

$$u(x) = f(x) + \lambda \int_a^x K(x-t)u(t)dt \qquad (4.31)$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions needed for the existence of Laplace transform for each. Let the Laplace transforms for the functions $f_1(x)$ and $f_2(x)$ be given by:

$$\mathcal{L}\{f_1(x)\} = F_1(s)$$
$$\mathcal{L}\{f_2(x)\} = F_2(s)$$

The *Laplace convolution product* of these two functions is defined by:

$$(f_1 * f_2)(x) = \int_0^x f_1(x-t)f_2(t)dt$$

or

$$(f_2 * f_1)(x) = \int_0^x f_2(x-t)f_1(t)dt$$

Recall that

$$(f_1 * f_2)(x) = (f_2 * f_1)(x)$$

We can easily show that the Laplace transform of the convolution product $(f_1 * f_2)(x)$ is given by:

$$\mathcal{L}\{(f_1 * f_2)(x)\} = \mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)dt\right\} = F_1(s)F_2(s)$$

Based on this summary, we will examine specific Volterra integral equations where the kernel is a difference kernel. Recall that we will apply the Laplace transform method and the inverse of the Laplace transform using the following Table :

f(x)	F(s)=$\mathcal{L}\{f(x)\}$
C	$\frac{c}{s}, s > 0$
X	$\frac{1}{s^2}, s > 0$
x^n	$\frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}, S > 0, Re\,(n) > -1$
e^{ax}	$\frac{1}{s-a}, s > a$
$\sin ax$	$\frac{a}{s^2+a^2}$
$\cos ax$	$\frac{s}{s^2+a^2}$
$sin^2 ax$	$\frac{2a^2}{s(s^2+4a^2)}, Re\,(s) > \lvert Im(a)\rvert$
$cos^2 ax$	$\frac{s^2+2a^2}{s(s^2+4a^2)}, Re\,(s) > \lvert Im(a)\rvert$
x $\sin ax$	$\frac{2as}{(s^2+a^2)^2}$
x $\cos ax$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sinh ax$	$\frac{a}{s^2-a^2}, s > \lvert a\rvert$

$\cosh ax$	$\frac{s}{s^2-a^2}$, $s>\lvert a\rvert$
$sinh^2 ax$	$\frac{2a^2}{s(s^2-4a^2)}$, $Re\ (s)>\lvert Im(a)\rvert$
$cosh^2 ax$	$\frac{s^2-2a^2}{s(s^2-4a^2)}$, $Re\ (s)>\lvert Im(a)\rvert$
x $\sinh ax$	$\frac{2as}{(s^2-a^2)^2}$, $s>\lvert a\rvert$
x $\cosh ax$	$\frac{s^2+a^2}{(s^2-a^2)^2}$, $s>\lvert a\rvert$
$x^n e^{ax}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$, n *is a positive integer*
$e^{ax}\sin bx$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{ax}\cos bx$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$e^{ax}\sinh bx$	$\frac{b}{(s-a)^2-b^2}$, $s>a$
$e^{ax}\cosh bx$	$\frac{s-a}{(s-a)^2-b^2}$, $s>a$

By taking Laplace transform of both sides of (4.31), we find:

$$U(s)=F(s)+\lambda K(s)U(s) \qquad (4.32)$$

Where

$$U(s)=\mathcal{L}\{u(x)\}\,,F(s)=\mathcal{L}\{f(x)\}\ \,,K(s)=\mathcal{L}\{K(x)\}$$

Solving (4.32) for $U(s)$ gives:

$$U(s)=\frac{F(s)}{1-\lambda K(s)}\ ,\ \lambda K(s)\neq 1 \qquad (4.33)$$

The solution $u(x)$ is obtained by taking the inverse Laplace transform of both sides of (4.33), where we find:

$$u(x)=\mathcal{L}^{-1}\left\{\frac{F(s)}{1-\lambda K(s)}\right\} \qquad (4.34)$$

Recall that the right side of (4.34) can be evaluated by using the above Table. The Laplace transform method for solving Volterra integral equations will be illustrated by studying the following examples.

Example 4.13. Solve the Volterra integral equation by using the Laplace transform method

$$u(x)=1+\int_0^x u(t)dt \qquad (4.35)$$

Notice that the kernel $K(x-t)=1$, $\lambda=1$. Taking Laplace transform of both sides (4.35) gives:

$$\mathcal{L}\{u(x)\}=\mathcal{L}\{1\}+\mathcal{L}\{1*u(x)\}$$

So that

$$U(s) = \frac{1}{s} + \frac{1}{s}U(s)$$

$$U(s) = \frac{1}{s-1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = e^x$$

Example 4.14. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = 1 - \int_0^x (x-t)u(t)dt \qquad (4.36)$$

Notice that the kernel $K(x-t)$ = x-t, λ = -1. Taking Laplace transform of both sides (4.36) gives:

$$\mathcal{L}\{u(x)\} = \mathcal{L}\{1\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{s} - \frac{1}{s^2}U(s)$$

$$U(s) = \frac{s}{s^2+1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = \cos x$$

Example 4.15. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = \frac{1}{3!}x^3 - \int_0^x (x-t)u(t)dt \qquad (4.37)$$

Taking Laplace transform of both sides (4.37) gives:

$$\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{3!}\frac{3!}{s^4} - \frac{1}{s^2}U(s)$$

$$U(s) = \frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = x - \sin x$$

Example 4.16. Solve the Volterra integral equation by using the Laplace transform method

$$u(x) = \sin x + \cos x + 2\int_0^x \sin(x-t)\,u(t)dt \qquad (4.38)$$

Taking Laplace transform of both sides (4.38) gives:

$$\mathcal{L}\{u(x)\} = \frac{1}{3!}\mathcal{L}\{x^3\} - \mathcal{L}\{(x-t) * u(x)\}$$

So that

$$U(s) = \frac{1}{s^2+1} + \frac{s}{s^2+1} + \frac{2}{s^2+1}U(s)$$
$$U(s) = \frac{1}{s-1}$$

By taking the inverse Laplace transform of both sides of the above equation, the exact solution is therefore given by:

$$u(x) = e^x$$

Exercises 4.4.

Use the *Laplace transform method* to solve the Volterra integral equations:

1. $u(x) = 1 - x - \int_0^x (x-t)u(t)dt$
2. $u(x) = \cos x - \sin x + 2\int_0^x \cos(x-t)u(t)dt$
3. $u(x) = e^x - \cos x - 2\int_0^x e^{x-t}u(t)dt$
4. $u(x) = 1 - \int_0^x ((x-t)^2 - 1)u(t)dt$
5. $u(x) = \sin x - \cos x + \cosh x - 2\int_0^x \cosh(x-t)u(t)dt$

4.5 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{\infty} \frac{u^n(b)}{n!}(x-b)^n$$

converges to $u(x)$ in a neighborhood of b. For simplicity, the generic form of the Taylor series at $x = 0$ can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \qquad (4.39)$$

In this section, we will present a useful method, that stems mainly from the Taylor series for analytic functions, for solving Volterra integral equations. We will assume that the solution $u(x)$ of the Volterra integral equation:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt \qquad (4.40)$$

is analytic, and therefore possesses a Taylor series of the form given in (4.40), where the coefficients a_n will be determined recurrently. Substituting (4.39) into both sides of (4.40) gives:

$$\sum_{n=0}^{\infty} a_n x^n = T\big(f(x)\big) + \lambda \int_a^x K(x,t)\left(\sum_{n=0}^{\infty} a_n t^n\right)dt$$

or for simplicity we use

$$a_0 + a_1 x + a_2 x^2 + \cdots = T\big(f(x)\big) + \lambda \int_a^x K(x,t)(a_0 + a_1 t + a_2 t^2 + \cdots)dt \qquad (4.41)$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (4.40) will be converted to a traditional integral in(4.41) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking a series solution, then if $f(x)$ includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (4.41) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in a_j, $j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients a_j, $j \geq 0$. Having determined the coefficients a_j, $j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (4.39). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve.

Example 4.17 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 1 + \int_0^x u(t)dt \qquad (4.42)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 42) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n \right) dt$$

Evaluating the integral on the right side gives:

$$\sum_{n=0}^{\infty} a_n x^n = 1 + \sum_{n=0}^{\infty} \frac{1}{n+1} a_n x^{n+1}$$

that can be rewritten as:

$$a_0 + \sum_{n=1}^{\infty} a_n x^n = 1 + \sum_{n=1}^{\infty} \frac{1}{n} a_{n-1} x^n$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = 1 + a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 1, a_n = \frac{1}{n} a_{n-1}, n \geq 1$$

where this result gives:

$$a_n = \frac{1}{n!} \quad ,n \geq 0$$

Substituting this result into (4.39) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

that converges to the exact solution $u(x) = e^x$.

Example 4.18 Solve the Volterra integral equation by using the series solution method:

$$u(x) = x + \int_0^x (x - t)u(t)dt \qquad (4.43)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 43) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = x + \int_0^x x\left(\sum_{n=0}^{\infty} a_n t^n\right) dt - \int_0^x \left(\sum_{n=0}^{\infty} a_n t^{n+1}\right) dt$$

Evaluating the integral on the right side gives:

$$\sum_{n=0}^{\infty} a_n x^n = x + \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+1)} a_n x^{n+2}$$

that can be rewritten as:

$$a_0 + a_1 x + \sum_{n=2}^{\infty} a_n x^n = x + \sum_{n=2}^{\infty} \frac{1}{n(n-1)} a_{n-2} x^n$$

or equivalently

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots = x + \frac{1}{2} a_0 x^2 + \frac{1}{6} a_1 x^3 + \frac{1}{12} a_2 x^4 + \cdots$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 0, a_1 = 1, a_n = \frac{1}{n(n-1)} a_{n-2} \,, n \geq 2$$

where this result gives:

$$a_n = \frac{1}{(2n+1)!} \quad ,n \geq 0$$

Substituting this result into (4.39) gives the series solution:

$$u(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$$

that converges to the exact solution $u(x) = \sinh x$.

Example 4.19 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 1 - x\sin x + \int_0^x tu(t)dt \qquad (4.44)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 44) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 1 - x\sin x + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integral on the right side gives:

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = 1 - x\left(x - \frac{x^3}{3!} + \cdots\right) + \int_0^x t(a_0 + a_1 t + a_2 t^2 + \cdots)dt$$

Integrating the right side and collecting the like terms of x we find

$$(a_0 + a_1 x + a_2 x^2 + \cdots) = 1 + \left(\frac{1}{2}a_0 - 1\right)x^2 + \frac{1}{3}a_1 x^3 + \left(\frac{1}{6} + \frac{1}{4}a_2\right)x^4 + ..$$

Equating the coefficients of like powers of x on both sides of the above equation gives the recurrence relation:

$$a_0 = 1, a_1 = 0, a_2 = \left(\frac{1}{2}a_0 - 1\right) = -\frac{1}{2!}, a_3 = \frac{1}{3}a_1 = 0, a_4 = \left(\frac{1}{6} + \frac{1}{4}a_2\right) = \frac{1}{4!}, \ldots$$

and generally

$$a_{2n+1} = 0, a_{2n} = \frac{(-1)^n}{(2n)!}, n \geq 0$$

The solution in a series form is given by:

$$u(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \cdots$$

that converges to the exact solution $u(x) = \cos x$.

Example 4.20 Solve the Volterra integral equation by using the series solution method:

$$u(x) = 2e^x - 2 - x + \int_0^x (x - t)u(t)dt \qquad (4.45)$$

Substituting $u(x)$ by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (4. 45) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 2e^x - 2 - x + \int_0^x (x - t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integral on the right side gives:

$$(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$= x + \left(1 + \frac{1}{2}a_0\right)x^2 + \left(\frac{1}{3} + \frac{1}{6}a_1\right)x^3 + \left(\frac{1}{12} + \frac{1}{12}a_2\right)x^4 + \cdots$$

Equating the coefficients of like powers of *x* on both sides of the above equation gives the recurrence relation:

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = \frac{1}{2!}, a_4 = \frac{1}{3!}, \ldots$$

The solution in a series form is given by:

$$u(x) = x\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \cdots\right)$$

that converges to the exact solution $u(x) = xe^x$

Exercises 4.5 Use the *series solution method* to solve the Volterra integral equations:

1. $u(x) = 1 + xe^x - \int_0^x tu(t)dt$
2. $u(x) = 2\cosh x - 2 + \int_0^x (x-t)u(t)dt$
3. $u(x) = \sec x + \tan x - \int_0^x \sec t\, u(t)dt$
4. $u(x) = 3 + x^2 - \int_0^x (x-t)u(t)dt$

4.6 The Variational Iteration Method

In this section, we will study the newly developed *variational iteration method* that proved to be effective and reliable for analytic and numerical purposes. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in the Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need for specific restrictions such as the so-called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed-form solution if an exact solution exists. The obtained series can be employed for numerical purposes if an exact solution is not obtainable. In what follows, we present the main steps of the method.

Consider the differential equation:

$$\mathcal{L}u + \aleph u = g(t) \qquad (4.46)$$

where $\mathcal{L}$ and $\aleph$ are linear and nonlinear operators respectively, and $g(t)$ is the source inhomogeneous term. The variational iteration method presents a correction functional for equation (4.46) in the form:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)\big(\mathcal{L}u_n(\psi) + \aleph\tilde{u}_n(\psi) - g(\psi)\big)d\psi \qquad (4.47)$$

where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n$ = 0, where δ is the variational derivative. The Lagrange multiplier λ can be identified optimally via the variational theory.

The determination of the Lagrange multiplier plays a major role in the determination of the solution to the problem. In what follows, we summarize some iteration formulae that show ODE, its corresponding Lagrange multipliers, and its correction functional respectively:

$$(i)\begin{cases} u' + f\big(u(\psi), u'(\psi)\big) = 0, \lambda = -1 \\ u_{n+1} = u_n - \int_0^x [u_n' + f(u_n, u_n')]d\psi \end{cases}$$

$$(ii)\begin{cases} u'' + f\big(u(\psi), u'(\psi), u''(\psi)\big) = 0, \lambda = (\psi - x) \\ u_{n+1} = u_n + \int_0^x (\psi - x)[u_n'' + f(u_n, u_n', u_n'')]d\psi \end{cases}$$

$$(iii)\begin{cases} u''' + f\big(u(\psi), u'(\psi), u''(\psi), u'''(\psi)\big) = 0, \lambda = \frac{1}{2!}(\psi - x)^2 \\ u_{n+1} = u_n - \int_0^x \frac{1}{2!}(\psi - x)^2[u_n''' + f(u_n, u_n', u_n'', u_n''')]d\psi \end{cases}$$

and generally

$$\begin{cases} \boldsymbol{u^{(n)} + f\left(u(\psi), u'(\psi), u''(\psi), \dots, u^{(n)}(\psi)\right) = 0, \lambda = (-1)^n \frac{1}{(n-1)!}(\psi - x)^{(n-1)}} \\ \boldsymbol{u_{n+1} = u_n + (-1)^n \int_0^x \frac{1}{(n-1)!}(\psi - x)^{(n-1)} \left[u_n^{(n)} + f\left(u_n, u_n', u_n'', \dots, u_n^{(n)}\right)\right] d\psi} \end{cases} \boldsymbol{, for\ n \geq 1}$$

To use the variational iteration method for solving Volterra integral equations, it is necessary to convert the integral equation to an equivalent initial value problem or an equivalent integro-differential equation. As defined before, an integro-differential equation is an equation that contains differential and integral operators in the same equation.

Example 4.21 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + \int_0^x u(t)dt \qquad (4.48)$$

Using the Leibnitz rule to differentiate both sides of (4.48)gives:

$$u'(x) - u(x) = 0 \qquad (4.49)$$

Substituting x = 0 into (4.48) gives the initial condition $u(0) = 1$.

<u>Using the variational iteration method</u>

The correction functional for equation (4.49)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u'(\psi) - u(\psi)]d\psi \qquad (4.50)$$

Using the formula (i) given above leads to:

$$\lambda = -1$$

Substituting this value of the Lagrange multiplier $\lambda = -1$ into the functional (4.50) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x [u'(\psi) - u(\psi)]d\psi$$

As stated before, we can use the initial condition to select $u_0(x) = u(0) = 1$.
Using this selection into (4.50) gives the following successive approximations:

$$u_0 = 1$$

$$u_1 = 1 - \int_0^x [u_0'(\psi) - u_0(\psi)]d\psi = 1 + x$$

$$u_2 = 1 + x - \int_0^x [u_1'(\psi) - u_1(\psi)]d\psi = 1 + x + \frac{1}{2!}x^2$$

$$u_3 = 1 + x + \frac{1}{2!}x^2 - \int_0^x [u_2'(\psi) - u_2(\psi)]d\psi = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

and so on. The VIM admits the use of

$$u(x) = \lim_{n\to\infty} u_n(x)$$

$$= \lim_{n\to\infty} 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$$

that gives the exact solution by: $u(x) = e^x$.

Example 4.22 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = x + \int_0^x (x-t)u(t)dt \qquad (4.51)$$

Using the Leibnitz rule to differentiate both sides of (4.51) once with respect to x gives the integro-differential equation:

$$u'(x) = 1 + \int_0^x u(t)dt \qquad (4.52)$$

However, by differentiating (4.52) with respect to x we obtain the differential equation:

$$u''(x) - u(x) = 0 \qquad (4.53)$$

Substituting $x = 0$ into (4.51) and (4.52) gives the initial conditions $u(0) = 0$and u'(0)=1.
The resulting initial value problem, which consists of a second order ODE and initial conditions is given by:

$$u''(x) - u(x) = 0 \ , u(0) = 0 \text{and } u'(0) = 1 \qquad (4.54)$$

<u>Using the variational iteration method</u>

The correction functional for equation (4.54)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.55)$$

Using the formula (ii) given above leads to:

$$\lambda = \psi - x$$

Substituting this value of the Lagrange multiplier $\lambda = \psi - x$ into the functional (4.55) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x (\psi - x)[u'(\psi) - u(\psi)]d\psi \qquad (4.56)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = x$. Using this selection in (4.56) gives the following successive approximations:

$$u_0 = x$$

$$u_1 = x + \int_0^x (\psi - x)[u_0''(\psi) - u_0(\psi)]d\psi = x + \frac{1}{3!}x^3$$

$$u_2 = x + \frac{1}{3!}x^3 + \int_0^x (\psi - x)[u_1''(\psi) - u_1(\psi)]d\psi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$$

$$u_3 = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \int_0^x (\psi - x)[u_2''(\psi) - u_2(\psi)]d\psi = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7$$

$$\vdots$$

$$u_n = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{1}{7!}x^7 + \cdots + \frac{1}{(2n+1)!}x^{2n+1}$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = \sinh x$

Example 4.23 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{3!}x^3 - \int_0^x (x - t)u(t)dt \qquad (4.57)$$

Using the Leibnitz rule to differentiate both sides of (4.57) once with respect to x gives the integro-differential equation:

$$u'(x) = 1 + \frac{1}{2!}x^2 - \int_0^x u(t)dt \qquad (4.58)$$

However, by differentiating (4.58) with respect to x we obtain the differential equation:

$$u''(x) + u(x) = x \qquad (4.59)$$

Substituting $x = 0$ into (4.57) and (4.58) gives the initial conditions $u(0) = 1$and $u'(0)=1$. The resulting initial value problem, which consists of a second order ODE and initial conditions is given by:

$$u''(x) + u(x) = x \ , u(0) = 1 \text{and } u'(0) = 1 \qquad (4.60)$$

<u>Using the variational iteration method</u>

The correction functional for equation (4.60)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u''(\psi) + \tilde{u}(\psi) - \psi]d\psi \qquad (4.61)$$

Using the formula (ii) given above leads to:

$$\lambda = \psi - x$$

Substituting this value of the Lagrange multiplier $\lambda = \psi - x$ into the functional (4.60) gives the iteration formula:

$$u_{n+1} = u_n - \int_0^x (\psi - x)[u'(\psi) + u(\psi) - \psi]d\psi \qquad (4.62)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) = 1+x$. Using this selection in (4.62) gives the following successive approximations:

$$u_0 = 1 + x$$

$$u_1 = 1 + x + \int_0^x (\psi - x)[u_0''(\psi) + u_0(\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2$$

$$u_2 = 1 + x - \frac{1}{2!}x^2 + \int_0^x (\psi - x)[u_1''(\psi) + u_1(\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4$$

$$u_3 = 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \int_0^x (\psi - x)[u_2''(\psi) + (\psi) - \psi]d\psi$$

$$= 1 + x - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6$$

$$\vdots$$

$$u_n = x + (1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = x + \cos x$

Example 4.24 Solve the Volterra integral equation by using the variational iteration method

$$u(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2}\int_0^x (x-t)^2 u(t)dt \qquad (4.63)$$

Using the Leibnitz rule to differentiate both sides of (4.63) three times with respect to x gives the two integro-differential equations:

$$u'(x) = 1 + x + \int_0^x (x-t)u(t)dt \qquad (4.64)$$

$$u''(x) = 1 + \int_0^x u(t)dt \qquad (4.65)$$

However, by differentiating (4.65) with respect to x we obtain the differential equation:

$$u'''(x) - u(x) = 0 \qquad (4.66)$$

Substituting $x = 0$ into (4.63) , (3.64) and (4.65) gives the initial conditions:

$u(0) = u'(0) = u''(0) = 1.$

The resulting initial value problem, which consists of a third order ODE and initial conditions is given by:

$$u''(x) - u(x) = 0, u(0) = u'(0) = u''(0) = 1 \qquad (4.67)$$

Using the variational iteration method

The correction functional for equation (4.67)is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\psi)[u'''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.68)$$

Using the formula (iii) given above leads to:

$$\lambda = -\frac{1}{2!}(\psi - x)^2$$

Substituting this value of the Lagrange multiplier $\lambda = -\frac{1}{2!}(\psi - x)^2$ into the functional (4.68) gives the iteration formula:

$$u_{n+1} = u_n - \frac{1}{2!}\int_0^x (\psi - x)^2[u'''(\psi) - \tilde{u}(\psi)]d\psi \qquad (4.69)$$

We can use the initial conditions to select $u_0(x) = u(0) + xu'(0) + \frac{x^2}{2}u''(0) = 1 + x + \frac{x^2}{2}$. Using this selection in (4.69) gives the following successive approximations:

$$u_0 = 1 + x + \frac{x^2}{2}$$

$$u_1 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

$$u_2 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + + \frac{x^6}{6!} + + \frac{x^7}{7!} + \frac{x^8}{8!}$$

$$\vdots$$

$$u_n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + + \frac{x^6}{6!} + + \frac{x^7}{7!} + \frac{x^8}{8!} + \cdots$$

The VIM admits the use of $u(x) = \lim_{n\to\infty} u_n(x)$

that gives the exact solution by: $u(x) = e^x$

Exercises 4.6 Use the *variational iteration method* to solve the following Volterra integral equations:

1. $u(x) = x + x^4 + \frac{1}{2}x^2 + \frac{1}{5}x^5 - \int_0^x u(t)dt$
2. $u(x) = 2 + x - 2\cos x - \int_0^x (x - t + 2)u(t)dt$
3. $u(x) = 1 - x\sin x + x\cos x + \int_0^x tu(t)dt$
4. $u(x) = 1 - 2\sinh x + \int_0^x (x - t + 2)u(t)dt$

Chapter Five Volterra-Fredholm Integral Equations

The Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the Spatio-temporal development of an epidemic, and from various physical and biological models. The Volterra-Fredholm integral equations appear in the literature in two forms, namely:

$$u(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt \quad (5.1)$$

or,

$$u(x) = f(x) + u(x) = f(x) + \lambda \int_a^x \int_a^b k(x,t)u(t)dt \quad (5.2)$$

where *f*(*x*) and *K*(*x*, *t*) are analytic functions. It is interesting to note that (5.1) contains disjoint Volterra and Fredholm integrals, whereas (5.2) contains mixed Volterra and Fredholm integrals. Moreover, the unknown functions $u(x)$ appear inside and outside the integral signs. This is a characteristic feature of the second kind of integral equation. If the unknown functions appear only inside the integral signs, the resulting equations are of the first kind.

In this chapter, we will study some of the reliable methods that will be used for the analytic treatment of the Volterra-Fredholm integral equations of the form (5.1).

This type of equation will be handled by using the Taylor series method and the Adomian decomposition method combined with the noise terms phenomenon or the modified decomposition method.

5.1 The Series Solution Method:

The series solution method was examined before. A real function u(x) is called analytic if it has derivatives of all orders such that the generic form of the Taylor series at x = 0 can be written as:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad (5.3)$$

In this section, we will apply the series solution method, which stems mainly from the Taylor series for analytic functions, for solving Volterra-Fredholm integral equations. We will assume that the solution u(x) of the Volterra-Fredholm integral equation (5.1) is analytic, and therefore possesses a Taylor series of the form given in (5.3), where the coefficients an will be determined recurrently. In this method, we usually substitute the Taylor series (5.3) into both sides of (5.1) to obtain:

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda_1 \int_a^x k_1(x,t)\left(\sum_{n=0}^{\infty} a_n x^n\right)dt + \lambda_2 \int_a^b k_2(x,t)\left(\sum_{n=0}^{\infty} a_n x^n\right)dt \quad (5.4)$$

where T (f(x)) is the Taylor series for f(x). The Volterra-Fredholm integral equation (5.1) will be converted to a regular integral in (5.4) where instead of integrating the unknown function u(x), terms of the form $t^n, n \geq 0$, will be integrated. Notice that because we are seeking a series solution, then if f(x) includes elementary functions such as trigonometric functions, exponential functions, etc., Taylor expansions for functions involved in f(x) should be used.

We first integrate the right side of the integrals in (5.4) and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x into both sides of the resulting equation to determine a recurrence relation in $a_j, j \geq 0$. Solving the recurrence relation will lead to a complete determination of the coefficients $a_j, j \geq 0$. Having determined the coefficients $a_j, j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (5.3). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher the accuracy level we achieve.

Example 5.1

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt \quad (5.5)$$

Substituting u(x) by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (5.5) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = -5 - x + 12x^2 - x^3 - x^4 + \int_0^x (x-t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt + \int_0^1 (x+t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integrals at the right side, using a few terms from both sides, and collecting the coefficients of like powers of x, we find:

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots)\\ &= -5 + \frac{1}{2}a_0 + \frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{5}a_3 + \frac{1}{6}a_4\\ &+ \left(-1 + a_0 + \frac{1}{2}a_1 + \frac{1}{3}a_2 + \frac{1}{4}a_3 + \frac{1}{5}a_4\right)x + \left(12 + \frac{1}{2}a_0\right)x^2 + \left(-1 + \frac{1}{6}a_1\right)x^3\\ &+ \left(-1 + \frac{1}{12}a_2\right)x^4 + \cdots\end{aligned}$$

Equating the coefficients of like powers of *x* on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0 = 0, a_1 = 6, a_2 = 12, a_3 = a_4 = a_5 = \cdots = 0$$

the exact solution is therefore given by:

$$u(x) = 6x + 12x^3$$

Example 5.2

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = 2 - x - x^2 - 6x^3 + x^5 + \int_0^x tu(t)dt + \int_{-1}^1 (x+t)u(t)dt \quad (5.6)$$

Substituting u(x) by the series:

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (5.6) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = 2 - x - x^2 - 6x^3 + x^5 + \int_0^x t\left(\sum_{n=0}^{\infty} a_n t^n\right) dt + \int_{-1}^{1} (x+t)\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

Evaluating the integrals at the right side, using a few terms from both sides, and collecting the coefficients of like powers of x, we find:

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= 2 + \frac{2}{3}a_1 + \frac{2}{5}a_3 + \left(-1 + 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4\right)x + \left(-1 + \frac{1}{2}a_0\right)x^2 \\ &+ \left(-6 + \frac{1}{3}a_1\right)x^3 + \frac{1}{4}a_2 x^4 + \left(1 + \frac{1}{5}a_3\right)x^5 + \cdots\end{aligned}$$

Equating the coefficients of like powers of x on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0 = 2, a_1 = 3, a_2 = 0, a_3 = -5, a_4 = a_5 = \cdots = 0$$

the exact solution is therefore given by:

$$u(x) = 2 + 3x - 5x^3$$

Example 5.3

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = e^x - 1 - x + \int_0^x u(t)dt + \int_0^1 xu(t)dt \qquad (5.7)$$

Using the Taylor polynomial for e^x, substituting u(x) by the Taylor polynomial

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

into both sides of Eq. (5.7) leads to:

$$\sum_{n=0}^{\infty} a_n x^n = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1 - x + \int_0^x \left(\sum_{n=0}^{\infty} a_n t^n\right) dt + \int_0^1 x\left(\sum_{n=0}^{\infty} a_n t^n\right) dt$$

and proceeding as before leads to:

$$\begin{aligned}(a_0 + a_1 x + a_2 x^2 + \cdots) \\ &= \left(2a_0 + \sum_{n=1}^{\infty} \frac{1}{n+1} a_n\right)x + \frac{1 + a_1}{2!}x^2 + \frac{(1 + 2!\,a_2)}{3!}x^3 + \frac{1 + 3!\,a_3}{4!}x^4 \\ &+ \frac{(1 + 4!\,a_4)}{5!}x^5 + \frac{(1 + 5!\,a_5)}{6!}x^6 + \cdots\end{aligned}$$

Equating the coefficients of like powers of x on both sides of the above equation and solving the resulting system of equations, we obtain:

$$a_0 = 0, a_1 = 1, a_2 = 1, a_3 = \frac{1}{2!}, a_4 = \frac{1}{3!}, a_5 = \frac{1}{4!}, \ldots$$

the exact solution is therefore given by:

$$u(x) = xe^x$$

Example 5.4

Solve the Volterra-Fredholm integral equation by using the series solution method:

$$u(x) = 1 - \int_0^x (x-t)u(t)dt + \int_0^1 u(t)dt \qquad (5.8)$$

Using the Taylor polynomial for e^x, substituting u(x) by the Taylor polynomial

$$u(x) = \sum_{n=0}^{\infty} a_n x^n$$

and proceeding as before we obtain that:

$$a_0 = 1, a_1 = a_3 = a_5 = a_7 = 0, \ldots$$

$$a_2 = -\frac{1}{2!}, a_4 = \frac{1}{4!}, a_6 = -\frac{1}{6!}, \ldots$$

the exact solution is therefore given by:

$$u(x) = \cos x$$

Exercises 5.1

Use the series solution method to solve the following Volterra-Fredholm integral equations:

1. $u(x) = 4 - x - 4x^2 - x^3 + \int_0^x (x-t+1)u(t)dt + \int_0^1 (x+t-1)u(t)dt$
2. $u(x) = 2 + x - 2\cos x - \int_0^x (x-t)u(t)dt - \int_0^{\frac{\pi}{2}} xu(t)dt$

5.2 The Adomian Decomposition Method

The Adomian decomposition method (ADM) was introduced thoroughly in this text for handling independently Volterra and Fredholm integral equations. The method consists of decomposing the unknown function u(x) of any equation into a sum of an infinite number of components defined by the decomposition series:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (5.9)$$

where the components $u_n(x), n \geq 0$ are to be determined recursively. To establish the recurrence relation, we substitute the decomposition series into the Volterra-Fredholm integral equation (5.1) to obtain:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda_1 \int_a^x k_1(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt + \lambda_2 \int_a^b k_2(x,t)\left(\sum_{n=0}^{\infty} u_n(t)\right)dt$$

The zeroth component $u_0(x)$ is identified by all terms that are not included under the integral sign. Consequently, we set the recurrence relation:

$$u_0(x) = f(x) \qquad (5.10)$$

$$u_{n+1}(x) = \lambda_1 \int_a^x k_1(x,t)u_n(t)dt + \lambda_2 \int_a^b k_2(x,t)u_n(t)dt \quad , n \geq 0 \qquad (5.11)$$

Having determined the components $u_0(x), u_1(x), u_2(x), \dots$, the solution in a series form is readily obtained upon using (5.9). The series solution converges to the exact solution if such a solution exists. We point out here that the noise terms phenomenon and the modified decomposition method will be employed in this section. This will be illustrated by using the following examples.

Example 5.5

Use the Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = e^x + 1 + x + \int_0^x (x-t)u(t)dt - \int_0^x e^{x-t}u(t)dt \qquad (5.12)$$

Using the decomposition series (5.9), and using the recurrence relation (5.10) and (5.11), we obtain:

$$u_0(x) = e^x + 1 + x$$

$$u_1(x) = \int_0^x (x-t)u_0(t)dt - \int_0^x e^{x-t}u_0(t)dt = -x - 1 + \frac{1}{2}x^2 + \cdots,$$

and so on. We notice the appearance of the noise terms ± 1 and $\pm x$ between the components $u_0(x)$ and $u_1(x)$. By canceling these noise terms from $u_0(x)$, the non-canceled term of $u_0(x)$ gives the exact solution $u(x) = e^x$, that satisfies the given equation.

It is to be noted that the modified decomposition method can be applied here. Using the modified recurrence relation:

$$u_0(x) = e^x$$

$$u_1(x) = 1 + x + \int_0^x (x-t)u_0(t)dt - \int_0^x e^{x-t}u_0(t)dt = 0$$

The exact solution $u(x) = e^x$ follows immediately.

Example 5.6

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = x^2 - \frac{1}{12}x^4 - \frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u(t)dt + \int_0^1 (x+t)u(t)dt \qquad (5.13)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = x^2 - \frac{1}{12}x^4$$

$$u_1(x) = -\frac{1}{4} - \frac{1}{3}x + \int_0^x (x-t)u_0(t)dt + \int_0^1 (x+t)u_0(t)dt$$

$$= \frac{1}{12}x^4 - \frac{1}{360}x^6 - \frac{1}{60}x - \frac{1}{72}$$

and so on. We notice the appearance of the noise terms $\pm\frac{1}{12}x^4$ between the components $u_0(x)$ and $u_1(x)$. By canceling the noise term from the $u_0(x)$, the non-canceled term gives the exact solution $u(x) = x^2$, that satisfies the given equation.

Example 5.7

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = \cos x - \sin x - 2 + \int_0^x u(t)dt + \int_0^\pi (x-t)u(t)dt \qquad (5.14)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = \cos x$$

$$u_1(x) = -\sin x - 2 + \int_0^x u_0(t)dt + \int_0^\pi (x-t)u_0(t)dt = 0$$

Consequently, the exact solution is given by: $u(x) = \cos x$.

Example 5.8

Use the modified Adomian decomposition method to solve the following Volterra-Fredholm integral equation:

$$u(x) = 3x + 4x^2 - x^3 - x^4 - 2 + \int_0^x tu(t)dt + \int_{-1}^1 tu(t)dt \qquad (5.15)$$

Using the modified decomposition method gives the recurrence relation:

$$u_0(x) = 3x + 4x^2 - x^3$$

$$u_1(x) = -x^4 - 2 + \int_0^x tu_0(t)dt + \int_{-1}^1 tu_0(t)dt = -\frac{2}{5} - \frac{1}{5}x^5 + x^3$$

Canceling the noise term $-x^3$ from $u_0(x)$ gives the exact solution $u(x) = 3x + 4x^2$

Exercises 5.2

Use the modified decomposition method to solve the following Volterra-Fredholm integral equations:

1. $u(x) = x - \frac{1}{3}x^3 + \int_0^x tu(t)dt + \int_{-1}^1 t^2u(t)dt$
2. $u(t) = \sec^2 x - \tan x - 1 + \int_0^x u(t)dt + \int_0^{\frac{\pi}{4}} u(t)dt$
3. $u(x) = x^3 - \frac{9}{20}x^5 - \frac{1}{4}x + \frac{1}{5} + \int_0^x (x+t)u(t)dt + \int_0^1 (x-t)u(t)dt$