INTEGRATION

1. The Definite Integral.
2. The Fundamental Theorem of Calculus
3. Indefinite Integrals and the Substitution Rule.
4. Substitution and Area Between Curves
5. Natural Logarithms.
6. The Exponential Functions and logarithm functions.
7. Exponential Growth and Decay.
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10. Basic Integration Formulas.
11. Integration by Parts.
12. Integration of Rational Functions by Partial Fractions
13. Trigonometric Integrals.
14. Trigonometric Substitutions.
15. Integral Tables and Computer Algebra Systems.
16. Improper Integrals

References:

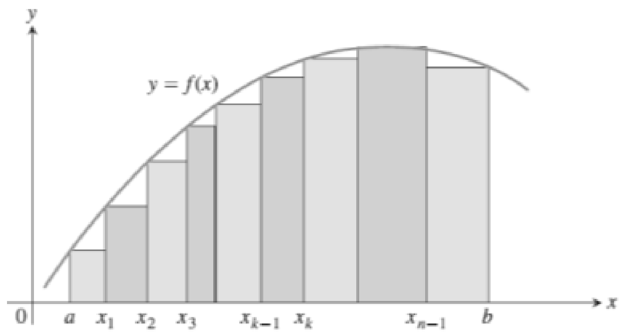
1. **Maurice Weir, Joel Hass, George B. Thomas,** *Thomas Calculus*, 12th ed. (2012).

2. **G Stephenson** *Mathematical Methods for Science Students* (1983).

3. Anton Bivens Davis *Calculus* (2002).

Integration:

1) The Definite Integral

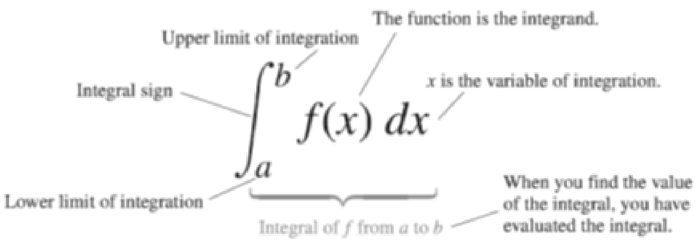


$$S_n = \sum_{k=1}^{n} f(c_k)\,\Delta x_k = \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right),$$

$\Delta x_k = \Delta x = (b-a)/n$ for all k

$$J = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\,\Delta x$$

$\Delta x = (b-a)/n$



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
6. $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

   $f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)

<u>EXAMPLE:</u>

Let $\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad$ and $\quad \int_{-1}^{1} h(x)\,dx = 7.$

Then: **1.** $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

$= 2(5) + 3(7) = 31$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

DEFINITION: If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the area under the curve $y = f(x)$ over $[a, b]$ is the integral of f from a to b.

$$A=\int_a^b f(x)dx$$

If f(x) is negative then $A=\int_a^b |f(x)|dx$

2) THEOREM (The Fundamental Theorem of Calculus 1):

If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$: $F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$

EXAMPLE:

Use the Fundamental Theorem to find dy/dx if:

(a) $y = \int_a^x (t^3 + 1)\,dt$ **(b)** $y = \int_x^5 3t \sin t\,dt$ **(c)** $y = \int_1^{x^2} \cos t\,dt$

Sol: **(a)** $\frac{dy}{dx} = \frac{d}{dx}\int_a^x (t^3 + 1)\,dt = x^3 + 1$

(b) $\frac{dy}{dx} = \frac{d}{dx}\int_x^5 3t \sin t\,dt = \frac{d}{dx}\left(-\int_5^x 3t \sin t\,dt\right)$

$$= -\frac{d}{dx}\int_5^x 3t \sin t\,dt$$

$$= -3x \sin x$$

(c) The upper limit of integration is not x. This makes y a composite of the two functions. We must therefore apply the Chain Rule when finding dy/dx.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \left(\frac{d}{du}\int_1^u \cos t\,dt\right) \cdot \frac{du}{dx}$$

$$= \cos u \cdot \frac{du}{dx}$$

$$= \cos(x^2) \cdot 2x$$

$$= 2x \cos x^2$$

THEOREM (**The Fundamental Theorem of Calculus 2**): If f is continuous at every point in $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

EXAMPLE

(a) $\displaystyle\int_0^{\pi} \cos x\, dx = \sin x\Big]_0^{\pi}$

$= \sin \pi - \sin 0 = 0 - 0 = 0$

(b) $\displaystyle\int_{-\pi/4}^{0} \sec x \tan x\, dx = \sec x\Big]_{-\pi/4}^{0}$

$= \sec 0 - \sec\left(-\frac{\pi}{4}\right) = 1 - \sqrt{2}$

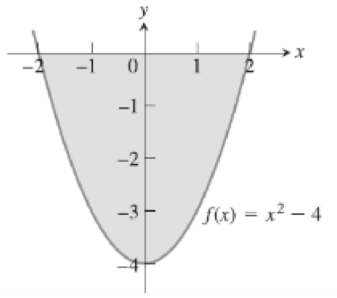
(c) $\displaystyle\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right) dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$

$= \left[(4)^{3/2} + \frac{4}{4}\right] - \left[(1)^{3/2} + \frac{4}{1}\right]$

$= [8 + 1] - [5] = 4.$

EXAMPLE

Let $f(x) = x^2 - 4$, compute (a) the definite integral over the interval [-2,2], and (b) the area between the graph and the x-axis over [-2,2].



Solution:

(a) $\displaystyle\int_{-2}^{2} f(x)\, dx = \left[\frac{x^3}{3} - 4x\right]_{-2}^{2} = \left(\frac{8}{3} - 8\right) - \left(-\frac{8}{3} + 8\right) = -\frac{32}{3},$

(b) The area between the graph and the x-axis is $\left|-\frac{32}{3}\right| = \frac{32}{3}$

EXAMPLE: Find the area between the graph $\boldsymbol{f(x) = x^3 - 2x^2 - x + 2}$ and the x-axis

SOL: f(x)=0 then $(\boldsymbol{x^2 - 1})(\boldsymbol{x - 2}) = \boldsymbol{0}$ that is x=1, -1 and x=2

A=$\boldsymbol{A_1 + A_2 = \int_{-1}^{1} |f(x)| dx + \int_1^2 |f(x)| dx}$

$=\left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right] + \left[\frac{x^4}{4} - 2\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]$

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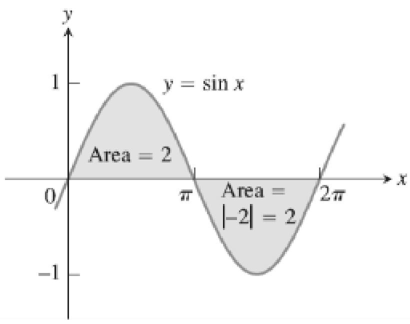
EXAMPLE: Let the *function f(x)* = sin *x* between *x* = 0 and *x* = 2π. Compute

(a) the definite integral of *f(x)* over [0, 2π].

(b) the area between the graph of *fix)* and the x-axis over [0, 2π].



Solution

(a) The definite integral for $f(x) = sinx$ is given by

$\displaystyle\int_0^{2\pi} \sin x\, dx = -\cos x\Big]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1 - 1] = 0.$

(b) To compute the area between the graph of *f(x)* and the x-axis over [0, 2π] we should find the points in which f is intersect x-axis i.e. f(x)=0 this implies to sin *x*=0 i.e. *x=0* , *x*=π *or* $x=2\pi$ Now subdivide [0, 2π] into two pieces: the interval [0, π] and the interval [π, 2π].

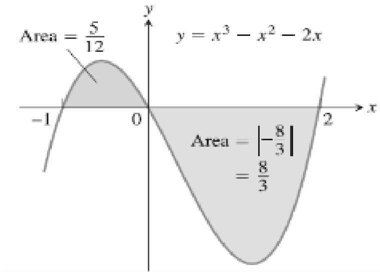
$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = -[\cos\pi - \cos 0] = -[-1-1] = 2$$
$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = -[\cos 2\pi - \cos\pi] = -[1-(-1)] = -2$$

$$\text{Area} = |2| + |-2| = 4.$$

EXAMPLE:

Find the area of the region between the x-axis and the graph of $\boldsymbol{f(x) = x^3 - x^2 - 2x,\ \ -1 \le x \le 2}$

Solution



First find the zeros of *f*. $f(x) = x^3 - x^2 - 2x = 0$

$$x(x^2 - x - 2) = 0$$
$$x(x+1)(x-2) = 0$$

$x = 0, -1$, and 2 . The zeros subdivide $[-1,2]$ into two subintervals: $[-1, 0]$, on which $f \ge 0$, and $[0, 2]$, on which $f \le 0$. We integrate *f* over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} = 0 - \left[\frac{1}{4} + \frac{1}{3} - 1\right] = \frac{5}{12}$$
$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{0}^{2} = \left[4 - \frac{8}{3} - 4\right] - 0 = -\frac{8}{3}$$

$$\text{Total enclosed area} = \frac{5}{12} + \left|-\frac{8}{3}\right| = \frac{37}{12}$$

EXAMPLE: Find $\boldsymbol{\int_{-1}^{2} |x-1|\,dx}$

Since $\boldsymbol{|x-1| = \begin{cases} x-1 & x \ge 1 \\ -x+1 & x < 1 \end{cases}}$ then $\boldsymbol{\int_{-1}^{2} |x-1|\,dx = \int_{-1}^{1} (-x+1)\,dx + \int_{1}^{2} (x-1)\,dx}$

3) Indefinite Integrals and the Substitution Method

Since any two antiderivatives of f differ by a constant, the indefinite integral notation means that for any antiderivative *F* of f,

$$\int f(x)\,dx = F(x) + C,$$

where C is any arbitrary constant.

THEOREM:

The Substitution Rule If u = g(x) is a differentiable function whose range is an interval I, and f is continuous on I, then $\int f(g(x))g'(x)\,dx = \int f(u)\,du.$

Substitution: Running the Chain Rule Backwards

If u is a differentiable function of x and n is any number different from -1, the Chain Rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right) = u^n \frac{du}{dx}.$$

Therefore $\int u^n \frac{du}{dx}\,dx = \frac{u^{n+1}}{n+1} + C.$

As well as $\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$ then $du = \frac{du}{dx}\,dx$

EXAMPLE:

Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx.$

Sol: let $u = x^3 + x$. then $du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$

so that by substitution we have :

$$\begin{aligned}\int (x^3 + x)^5(3x^2 + 1)\,dx &= \int u^5\,du && \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.\\ &= \frac{u^6}{6} + C && \text{Integrate with respect to } u.\\ &= \frac{(x^3 + x)^6}{6} + C && \text{Substitute } x^3 + x \text{ for } u.\end{aligned}$$

EXAMPLE:

Find the integral $\int \sqrt{2x + 1}\,dx.$

SOL: let u=2x+1 and n=1/2, $du = \frac{du}{dx}\,dx = 2\,dx$

because of the constant factor 2 is missing from the integral. So we write

$$\begin{aligned}\int \sqrt{2x + 1}\,dx &= \frac{1}{2}\int \underbrace{\sqrt{2x + 1}}_{u}\cdot \underbrace{2\,dx}_{du}\\ &= \frac{1}{2}\int u^{1/2}\,du && \text{Let } u = 2x + 1,\ du = 2\,dx.\\ &= \frac{1}{2}\frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u.\\ &= \frac{1}{3}(2x + 1)^{3/2} + C && \text{Substitute } 2x + 1 \text{ for } u.\end{aligned}$$

EXAMPLE: Find $\int \sec^2(5t + 1)\cdot 5\,dt.$

SOL: Let $u = 5t + 1$ and $du = 5\,dx$. Then,

$$\int \sec^2(5t+1)\cdot 5\,dt = \int \sec^2 u\,du \qquad \text{Let } u = 5t+1,\ du = 5\,dt.$$
$$= \tan u + C \qquad \frac{d}{du}\tan u = \sec^2 u$$
$$= \tan(5t+1) + C \qquad \text{Substitute } 5t+1 \text{ for } u.$$

EXAMPLE: $\int \cos(7\theta + 3)\,d\theta.$

SOL: Let $u = 7\theta + 3$ so that $du = 7\,d\theta.$ The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7. Then,

$$\int \cos(7\theta+3)\,d\theta = \frac{1}{7}\int \cos(7\theta+3)\cdot 7\,d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$
$$= \frac{1}{7}\int \cos u\,du \qquad \text{Let } u = 7\theta+3,\ du = 7\,d\theta.$$
$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$
$$= \frac{1}{7}\sin(7\theta+3) + C \qquad \text{Substitute } 7\theta+3 \text{ for } u.$$

EXAMPLE: $\int x^2\sin(x^3)\,dx = \int \sin(x^3)\cdot x^2\,dx$

$$= \int \sin u\cdot\frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$
$$= \frac{1}{3}\int \sin u\,du$$
$$= \frac{1}{3}(-\cos u) + C \qquad \text{Integrate.}$$
$$= -\frac{1}{3}\cos(x^3) + C \qquad \text{Replace } u \text{ by } x^3.$$

EXAMPLE: Evaluate $\int x\sqrt{2x+1}\,dx$

SOL: $u = 2x + 1$ to obtain $x = (u - 1)/2$, and find that $x\sqrt{2x+1}\,dx = \frac{1}{2}(u-1)\cdot\frac{1}{2}\sqrt{u}\,du.$

The integration now becomes

$$\int x\sqrt{2x+1}\,dx = \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)u^{1/2}\,du \qquad \text{Substitute.}$$
$$= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du \qquad \text{Multiply terms.}$$
$$= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \qquad \text{Integrate.}$$
$$= \frac{1}{10}(2x + \int \frac{2z\,dz}{\sqrt[3]{z^2+1}} . + 1)^{3/2} + C \qquad \text{Replace } u \text{ by } 2x+1. \ \blacksquare$$

Let

$u = z^2 + 1$.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}} \qquad \text{Let } u = z^2+1,\ du = 2z\,dz.$$
$$= \int u^{-1/3}\,du \qquad \text{In the form } \int u^n\,du$$
$$= \frac{u^{2/3}}{2/3} + C \qquad \text{Integrate.}$$
$$= \frac{3}{2}u^{2/3} + C$$
$$= \frac{3}{2}(z^2+1)^{2/3} + C \qquad \text{Replace } u \text{ by } z^2+1.$$

The Integrals of $sin^2 x$ and $cos^2 x$

(a) $$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\,dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$ ■

4) SUBSTITUTION AND AREA BETWEEN CURVES:

THEOREM Substitution in Definite Integrals: If g' is continuous on the interval *[a,* b] and *f* is continuous on the range of *g(x)* = *u,* then $\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$

EXAMPLE: Evaluate $\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx.$

SOL:

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$
$$= \frac{2}{3}u^{3/2}\Big]_0^2 \qquad \text{Evaluate the new definite integral.}$$
$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

EXAMPLE: Find $\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta$

SOL: Let $u = \cot\theta$, $du = -\csc^2\theta\,d\theta$,
$-du = \csc^2\theta\,d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$\int_{\pi/4}^{\pi/2} \cot\theta\csc^2\theta\,d\theta = \int_1^0 u\cdot(-du)$$
$$= -\int_1^0 u\,du = -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$
$$= -\left[\frac{u^2}{2}\right]_1^0$$

THEOREM:

Let f be continuous on the symmetric interval [-a, a].

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

EXAMPLE: **Evaluate** $\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx$.

SOL: Since $f(x) = x^4 - 4x^2 + 6$ satisfies *f(-x) = f(x)* , it is even on the symmetric interval [- 2, 2] , so

$$\begin{aligned}\int_{-2}^{2}(x^4 - 4x^2 + 6)\,dx &= 2\int_{0}^{2}(x^4 - 4x^2 + 6)\,dx \\ &= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2 \\ &= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.\end{aligned}$$

AREAS BETWEEN CURVES:

DEFINITION: If f and g are continuous with f(x) ≥g(x) throughout [a, b], then the area of the region between the curves y = f(x) and y = g(x) from a to b is the integral of (f - g) from a to b:

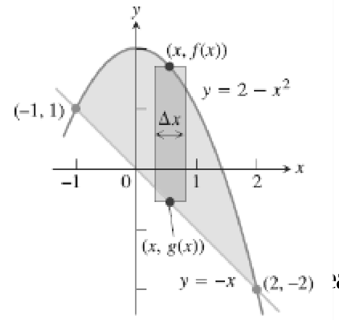
$$A = \int_a^b [f(x) - g(x)]\,dx.$$

EXAMPLE:

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution: First we sketch the two curves. The limits of integration are found from the intersection points $y = 2 - x^2$ and $y = -x$.

$2 - x^2 = -x$	Equate $f(x)$ and $g(x)$.
$x^2 - x - 2 = 0$	Rewrite.
$(x + 1)(x - 2) = 0$	Factor.
$x = -1, \quad x = 2.$	Solve.



The region runs from $x = -1$ *to* $x = 2$. The limits of integration a a between the curves is $A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2}[(2 - x^2) - (-x)]\,dx$

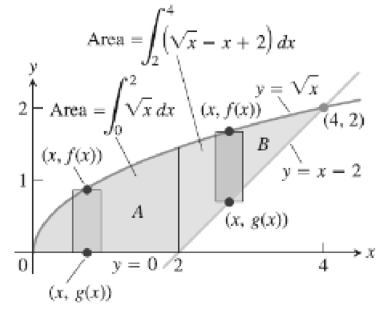
$$\begin{aligned}&= \int_{-1}^{2}(2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}\end{aligned}$$

EXAMPLE:

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution:

The sketch figure shows that the region's upper boundary is the graph of $\boldsymbol{f(x)} = \boldsymbol{\sqrt{x}}$. The lower boundary changes from g(x) = 0 for $0 \le x \le 2$ to g(x) = x – 2 for $2 \le x \le 4$. We subdivide the region at $x = 2$ into sub regions A and B, shown in the figure.



The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$
\begin{aligned}
\sqrt{x} &= x - 2 \\
x &= (x-2)^2 = x^2 - 4x + 4 \\
x^2 - 5x + 4 &= 0 \\
(x-1)(x-4) &= 0 \\
x = 1, &\quad x = 4.
\end{aligned}
$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. Therefore the right-hand limit is $b = 4$.

For $0 \le x \le 2$: $\quad f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x}$

For $2 \le x \le 4$: $\quad f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2$

We add the areas of subregions A and B to find the tota1 area:

$$
\begin{aligned}
\text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B} \\
&= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\
&= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\
&= \frac{2}{3}(8) - 2 = \frac{10}{3}.
\end{aligned}
$$

5) Natural Logarithms

DEFINITION: The natural logarithm is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0.$$

DEFINITION: The number e is that number in the domain of the natural logarithm satisfying In(e) = 1.

The Derivative of y = In x

By the first part of the Fundamental Theorem of Calculus,

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

$$\frac{d}{dx}\ln x = \frac{1}{x},$$

For every positive value of x, we have $\frac{d}{dx}lnx = \frac{1}{x}$ and the Chain Rule extends this formula for positive functions u(x): $\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$ → $\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.$

EXAMPLE:

(a) $\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$

(b) $\frac{d}{dx}\ln(x^2+3) = \frac{1}{x^2+3}\cdot\frac{d}{dx}(x^2+3) = \frac{1}{x^2+3}\cdot 2x = \frac{2x}{x^2+3}.$

Now if x<0 then –x>0 and hence

$\frac{d}{dx}\ln(-x) = \frac{1}{x} \qquad \text{for } x < 0.$

Since $|x| = \begin{cases} x & x > \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$

We have the following important result, which says that *ln |x|* is an antiderivative of *1/x, x ≠ 0*.

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0$$

THEOREM -Algebraic Properties of the Natural Logarithm: For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln bx = \ln b + \ln x$

2. *Quotient Rule*: $\ln \frac{b}{x} = \ln b - \ln x$

3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$

4. *Power Rule*: $\ln x^r = r \ln x$

EXAMPLE:

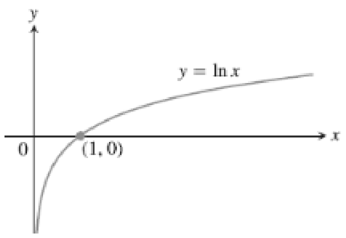
(a) $\ln 4 + \ln \sin x = \ln (4 \sin x)$

(b) $\ln \frac{x+1}{2x-3} = \ln (x+1) - \ln (2x-3)$

(c) $\ln \frac{1}{8} = -\ln 8$

$= -\ln 2^3 = -3 \ln 2$

Graph lnx



DEFINITION: If u is a differentiable function that is never zero, $\int \frac{1}{u}\,du = \ln |u| + C.$

In general $\int \frac{f'(x)}{f(x)}\,dx = \ln |f(x)| + C$

EXAMPLE $\int_0^2 \frac{2x}{x^2-5}\,dx = \int_{-5}^{-1} \frac{du}{u} = \ln |u| \Big]_{-5}^{-1}$ $\quad u = x^2 - 5, \quad du = 2x\,dx, \quad u(0) = -5, \quad u(2) = -1$

$= \ln |-1| - \ln |-5| = \ln 1 - \ln 5 = -\ln 5$

<u>The Integrals of tan x, cot x, sec x, and esc x</u>

1- $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$ $\quad u = \cos x > 0 \text{ on } (-\pi/2, \pi/2), \quad du = -\sin x\,dx$

$= -\ln |u| + C = -\ln |\cos x| + C$

$= \ln \frac{1}{|\cos x|} + C = \ln |\sec x| + C.$ $\quad$ Reciprocal Rule

2- $\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u}$ $\quad u = \sin x, \quad du = \cos x\,dx$

$= \ln |u| + C = \ln |\sin x| + C = -\ln |\csc x| + C.$

3- $\int \sec x\,dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$

$= \int \frac{du}{u} = \ln |u| + C = \ln |\sec x + \tan x| + C$ $\quad u = \sec x + \tan x, \quad du = (\sec x \tan x + \sec^2 x)\,dx$

4- $\int \csc x\, dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)} dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x} dx$

$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$ $\quad$ $u = \csc x + \cot x$, $du = (-\csc x \cot x - \csc^2 x)\, dx$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\, du = \ln|\sec u| + C \qquad \int \sec u\, du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\, du = \ln|\sin u| + C \qquad \int \csc u\, du = -\ln|\csc u + \cot u| + C$$

EXAMEL: $\int_0^{\pi/6} \tan 2x\, dx = \int_0^{\pi/3} \tan u \cdot \frac{du}{2} = \frac{1}{2}\int_0^{\pi/3} \tan u\, du$ $\quad$ Substitute $u = 2x$, $dx = du/2$, $u(0) = 0$, $u(\pi/6) = \pi/3$

$$= \frac{1}{2}\ln|\sec u|\Big]_0^{\pi/3} = \frac{1}{2}(\ln 2 - \ln 1) = \frac{1}{2}\ln 2$$

Logarithmic Differentiation:

EXAMPLE 1: Find dy/dx if $\quad y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \quad x > 1.$

Solution: We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned} \ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\ &= \ln((x^2 + 1)(x + 3)^{1/2}) - \ln(x - 1) \\ &= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) \\ &= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). \end{aligned}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

6) The Exponential Functions

DEFINITION: For every real number *x*, we define the **natural** exponential function to be

$$e^x = \exp x$$

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$
$$\ln(e^x) = x \qquad (\text{all } x)$$

EXAMPLE 1: Solve the equation $e^{2x-6} = 4$ for x.

Solution: We take the natural logarithm of both sides of the equation and use the second inverse equation:

$$\ln(e^{2x-6}) = \ln 4$$
$$2x - 6 = \ln 4$$
$$2x = 6 + \ln 4$$
$$x = 3 + \frac{1}{2}\ln 4 = 3 + \ln 4^{1/2}$$
$$x = 3 + \ln 2$$

<u>The Derivative and Integral of e^x</u>

$$\ln(e^x) = x$$
$$\frac{d}{dx}\ln(e^x) = 1$$
$$\frac{1}{e^x}\cdot\frac{d}{dx}(e^x) = 1$$
$$\frac{d}{dx}e^x = e^x.$$

If u is ... unction of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

EXAMPLE 2: We find derivatives of the exponential

(a) $\frac{d}{dx}(5e^x) = 5\frac{d}{dx}e^x = 5e^x$

(b) $\frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ Eq. (2) with $u = -x$

(c) $\frac{d}{dx}e^{\sin x} = e^{\sin x}\frac{d}{dx}(\sin x) = e^{\sin x}\cdot\cos x$ Eq. (2) with $u = \sin x$

(d) $\frac{d}{dx}\left(e^{\sqrt{3x+1}}\right) = e^{\sqrt{3x+1}}\cdot\frac{d}{dx}\left(\sqrt{3x+1}\right)$ Eq. (2) with $u = \sqrt{3x+1}$

$$= e^{\sqrt{3x+1}}\cdot\frac{1}{2}(3x+1)^{-1/2}\cdot 3 = \frac{3}{2\sqrt{3x+1}}e^{\sqrt{3x+1}}$$

The general antiderivative of the exponential function

$$\int e^u\,du = e^u + C$$

EXAMPLE 3:

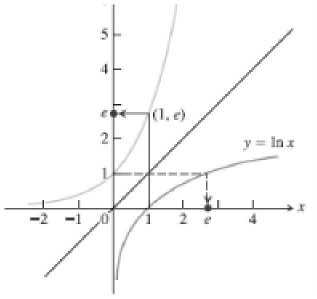
(a) $\int_0^{\ln 2} e^{3x}\,dx = \int_0^{\ln 8} e^u\cdot\frac{1}{3}\,du$ $\quad u = 3x,\ \frac{1}{3}du = dx,\ u(0) = 0,$ $u(\ln 2) = 3\ln 2 = \ln 2^3 = \ln 8$

$$= \frac{1}{3}\int_0^{\ln 8} e^u\,du$$
$$= \frac{1}{3}e^u\Big]_0^{\ln 8}$$
$$= \frac{1}{3}(8 - 1) = \frac{7}{3}$$

(b) $\int_0^{\pi/2} e^{\sin x}\cos x\,dx = e^{\sin x}\Big]_0^{\pi/2}$ Antiderivative from Example 2c

$$= e^1 - e^0 = e - 1$$

Graph e^x



Laws of Exponents:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2}$
2. $e^{-x} = \frac{1}{e^x}$
3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1 - x_2}$
4. $(e^{x_1})^r = e^{r x_1}$, if r is rational

Proof of Law 1 Let $y_1 = e^{x_1}$ and $y_2 = e^{x_2}$. Then

$$x_1 = \ln y_1 \quad \text{and} \quad x_2 = \ln y_2 \qquad \text{Inverse equations}$$

$$x_1 + x_2 = \ln y_1 + \ln y_2$$

$$= \ln y_1 y_2 \qquad \text{Product Rule for logarithms}$$

$$e^{x_1 + x_2} = e^{\ln y_1 y_2} \qquad \text{Exponentiate.}$$

$$= y_1 y_2 \qquad e^{\ln u} = u$$

$$= e^{x_1} e^{x_2}. \qquad \blacksquare$$

Proof of Law 4 Let $y = (e^{x_1})^r$. Then

$$\ln y = \ln (e^{x_1})^r$$

$$= r \ln (e^{x_1}) \qquad \text{Power Rule for logarithms, rational } r$$

$$= r x_1 \qquad \ln e^u = u \text{ with } u = x_1$$

The General Exponential Function a^x

Since $a = e^{\ln a}$ **then** $a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$

DEFINITION: For any numbers a > 0 and x, the exponential function with base a is $a^x = e^{x \ln a}$

Power Rule (General Version)

DEFINITION: For any $x > 0$ and for any real number n, $x^n = e^{n \ln x}$.

General Power Rule for Derivatives

For *all* x and any real number n, $\frac{d}{dx} x^n = n x^{n-1}$.

Proof: for $x > 0$

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n \ln x} \qquad \text{Definition of } x^n,\ x > 0$$

$$= e^{n \ln x} \cdot \frac{d}{dx}(n \ln x) \qquad \text{Chain Rule for } e^u \text{, Eq. (2)}$$

$$= x^n \cdot \frac{n}{x} \qquad \text{Definition and derivative of } \ln x$$

$$= nx^{n-1}. \qquad x^n \cdot x^{-1} = x^{n-1}$$

for x<0

if $y = x^n$, y', and x^{n-1} all exist, then

$$\ln|y| = \ln|x|^n = n \ln|x|.$$

$$\frac{y'}{y} = \frac{n}{x}.$$

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}.$$

It can be shown directly from the definition of the derivative that the derivative equals 0 when $x = 0$.

EXAMPLE 4: Differentiate $f(x) = x^x, x > 0$.

Solution: $f(x) = x^x = e^{x \ln x}$, $\quad f'(x) = \frac{d}{dx}(e^{x \ln x})$

$$= e^{x \ln x}\frac{d}{dx}(x \ln x)$$

$$= e^{x \ln x}\left(\ln x + x \cdot \frac{1}{x}\right)$$

$$= x^x(\ln x + 1).$$

__**The Number e Expressed as a Limit**__

Theorem: The number e can be calculated as the limit $e = \lim_{x\to 0}(1 + x)^{1/x}$.

Proof If $f(x) = \ln x$, then $f'(x) = 1/x$, so $f'(1) = 1$. But, by the definition of derivative,

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{1/x} = \ln\left[\lim_{x\to 0}(1+x)^{1/x}\right]$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1+x)^{1/x}\right] = 1$$

Therefore, exponentiating both sides we get

$$\lim_{x\to 0}(1+x)^{1/x} = e.$$

The Derivative of a^x

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x\ln a} = e^{x\ln a}\cdot\frac{d}{dx}(x\ln a)$$
$$= a^x \ln a.$$

If a = e then $\frac{d}{dx}e^x = e^x \ln e = e^x.$

In general $\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$ .where u= f(x)

The integral of a^u

$$\int a^u\, du = \frac{a^u}{\ln a} + C.$$

EXAMPLE 5: **(a)** $\frac{d}{dx}3^x = 3^x \ln 3$

(b) $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$

(c) $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

(d) $\int 2^x\, dx = \frac{2^x}{\ln 2} + C$

(e) $\int 2^{\sin x}\cos x\, dx = \int 2^u\, du = \frac{2^u}{\ln 2} + C$
$$= \frac{2^{\sin x}}{\ln 2} + C$$

Logarithms with Base a

For any positive number a ≠1, $\log_a x$ is the inverse function of a^x.

$$a^{\log_a x} = x \qquad (x > 0)$$
$$\log_a(a^x) = x \qquad (\text{all } x)$$

Property: $\log_a x = \frac{\ln x}{\ln a}.$

Proof : $y = \log_a x$ then $a^y = x$ hence $y\ln a = \ln x.$ therefore $\log_a x = \frac{\ln x}{\ln a}.$

Rules:

1. *Product Rule*:
$\log_a xy = \log_a x + \log_a y$

2. *Quotient Rule*:
$\log_a \frac{x}{y} = \log_a x - \log_a y$

3. *Reciprocal Rule*:
$\log_a \frac{1}{y} = -\log_a y$

4. *Power Rule*:
$\log_a x^y = y\log_a x$

Derivative and Integral

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

Example:

(a) $\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10} \cdot \frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$

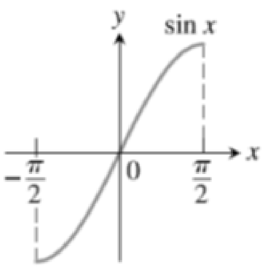
(b) $\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$

$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}dx$

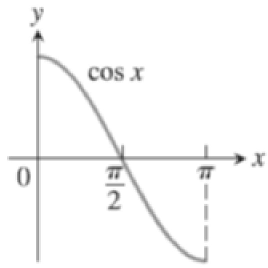
$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$

7) Inverse Trigonometric Functions

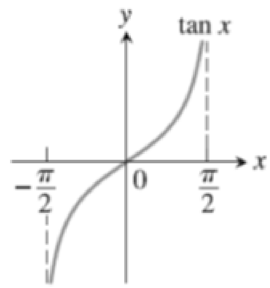
The six basic trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one.



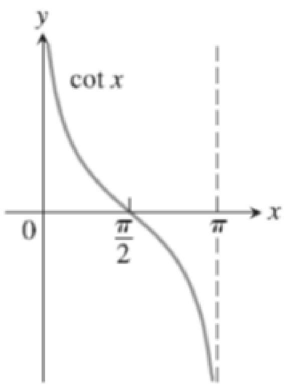
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



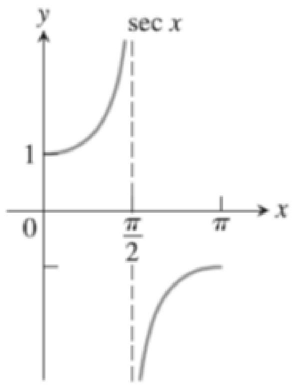
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



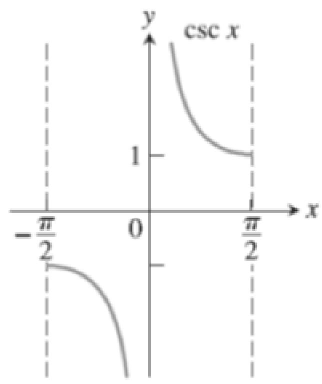
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$

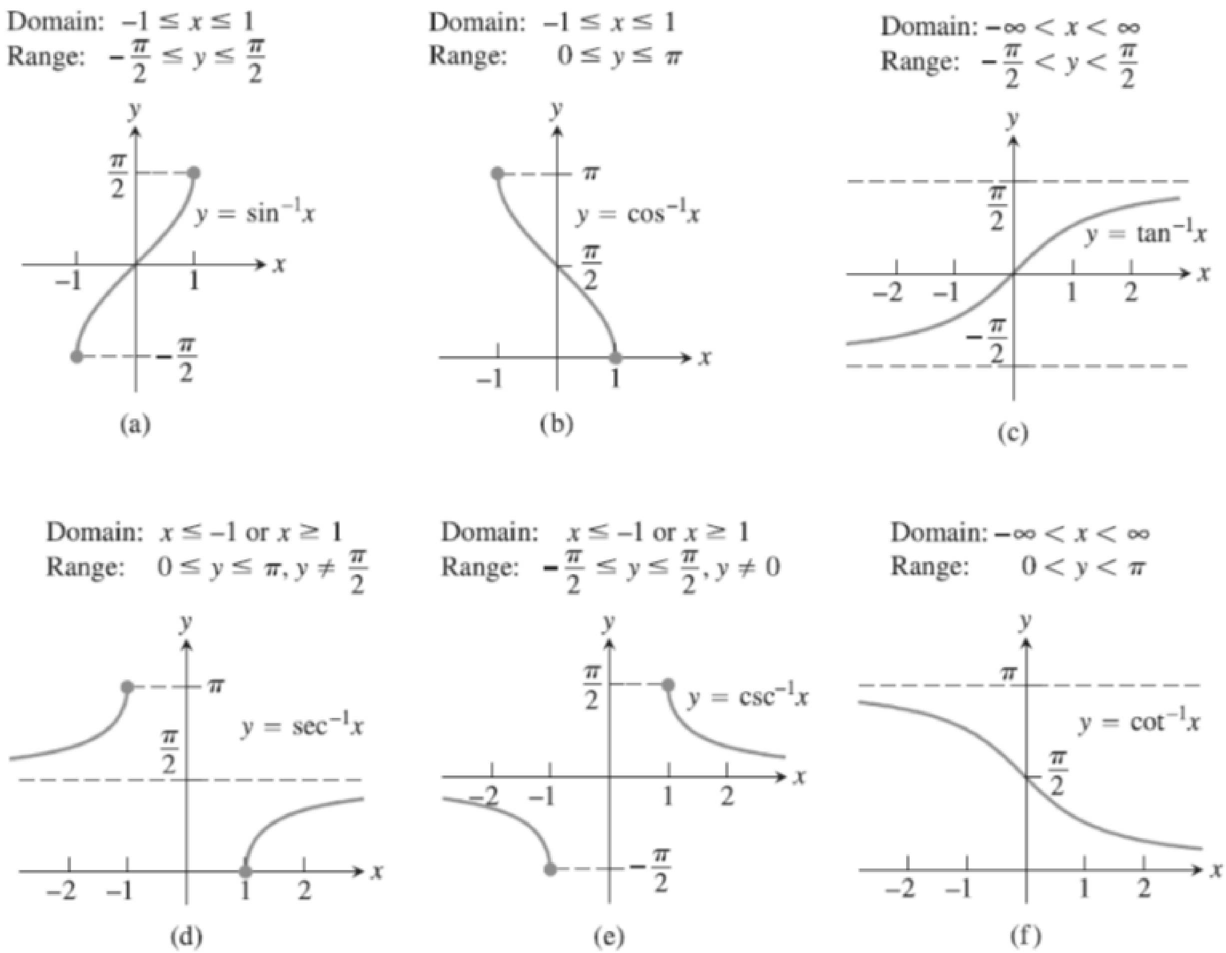


$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

Caution The -1 in the expressions for the inverse means "inverse." It does not mean reciprocal. For example, the *reciprocal* of sin *x* is $(sinx)^{-1} = l/sinx = cscx$.



EXAMPLE 1 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$.

Solution

(a) We see that

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 7.18a.

(b) We have

$$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

It is easy to show

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

The Derivative of $y = \sin^{-1} x$

$y = sin^{-1}x \quad \rightarrow \quad \sin(y) = x \quad \rightarrow \quad \cos y \,.\frac{dy}{dx} = 1 \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{cosy} =$

$\frac{1}{\pm\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-\sin^2 y}}$ since $-\pi/2 < y < \pi/2$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Then $\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}, \quad |u| < 1.$

EXAMPLE 4 Using the Chain Rule, we calculate the derivative

$$\frac{d}{dx}(\sin^{-1} x^2) = \frac{1}{\sqrt{1-(x^2)^2}} \cdot \frac{d}{dx}(x^2) = \frac{2x}{\sqrt{1-x^4}}.$$

The Derivative of $y = \tan^{-1} x$

$y = tan^{-1}x \quad \rightarrow \quad \tan(y) = x \quad \rightarrow \quad \sec^2 y \,.\frac{dy}{dx} = 1 \quad \rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2 y}$

$$= \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}.$$

The Derivative of $y = \sec^{-1} x$

$y = \sec^{-1} x$

$\sec y = x$ — Inverse function relationship

$\frac{d}{dx}(\sec y) = \frac{d}{dx}x$ — Differentiate both sides.

$\sec y \tan y \frac{dy}{dx} = 1$ — Chain Rule

$\frac{dy}{dx} = \frac{1}{\sec y \tan y}$ — Since $|x| > 1$, y lies in $(0, \pi/2) \cup (\pi/2, \pi)$ and $\sec y \tan y \neq 0$

$\sec y = x \quad$ and $\quad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\sec^{-1}x = \begin{cases} +\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2-1}} & \text{if } x < -1. \end{cases}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}} \quad , \quad |x| > 1$$

$$\frac{d}{dx}(\sec^{-1}u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}, \qquad |u| > 1.$$

EXAMPLE 5 Using the Chain Rule and derivative of the arcsecant function, we find

$$\begin{aligned}\frac{d}{dx}\sec^{-1}(5x^4) &= \frac{1}{|5x^4|\sqrt{(5x^4)^2-1}}\frac{d}{dx}(5x^4) \\ &= \frac{1}{5x^4\sqrt{25x^8-1}}(20x^3) \qquad 5x^4 > 1 > 0 \\ &= \frac{4}{x\sqrt{25x^8-1}}.\end{aligned}$$

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1}x = \pi/2 - \sin^{-1}x$$
$$\cot^{-1}x = \pi/2 - \tan^{-1}x$$
$$\csc^{-1}x = \pi/2 - \sec^{-1}x$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{d}{dx}\left(\frac{\pi}{2} - \sin^{-1}x\right) = -\frac{d}{dx}(\sin^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

Then

$$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2}\frac{du}{dx}$$

$$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \quad |u| > 1$$

Integration Formulas

1. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \qquad \text{(Valid for } u^2 < a^2\text{)}$$

2. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C \qquad \text{(Valid for all } u\text{)}$$

3. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C \qquad \text{(Valid for } |u| > a > 0\text{)}$$

EXAMPLE 6

(a) $$\int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \Big]_{\sqrt{2}/2}^{\sqrt{3}/2} = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$$

(b) $$\int \frac{dx}{\sqrt{3 - 4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C$$

(c) $$\int \frac{dx}{\sqrt{e^{2x} - 6}} = \int \frac{du/u}{\sqrt{u^2 - a^2}}$$

$$= \int \frac{du}{u\sqrt{u^2 - a^2}}$$

$$= \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

$$= \frac{1}{\sqrt{6}}\sec^{-1}\left(\frac{e^x}{\sqrt{6}}\right) + C$$

EXAMPLE 7 Evaluate

(a) $$\int \frac{dx}{\sqrt{4x - x^2}}$$

(b) $$\int \frac{dx}{4x^2 + 4x + 2}$$

Solution

(a) we first rewrite $4x - x^2$ by completing the square:

$$4x - x^2 = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = 4 - (x - 2)^2.$$

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x - 2)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

$$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$$

(b) We complete the square on the binomial $4x^2 + 4x$:

$$4x^2 + 4x + 2 = 4(x^2 + x) + 2 = 4\left(x^2 + x + \frac{1}{4}\right) + 2 - \frac{4}{4}$$

Then,

$$\int \frac{dx}{4x^2 + 4x + 2} = \int \frac{dx}{(2x+1)^2 + 1} = \frac{1}{2}\int \frac{du}{u^2 + a^2}$$

$$= \frac{1}{2} \cdot \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(2x + 1) + C$$

8) Hyperbolic Functions

The hyperbolic sine and hyperbolic cosine functions are defined by:

Hyperbolic sine:
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Hyperbolic cosine:
$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic tangent:
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic secant:
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Hyperbolic cosecant:
$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

<u>Derivatives and Integrals of Hyperbolic Functions</u>

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

proof :

1- $$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$$
$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2}$$
$$= \cosh u \frac{du}{dx}$$

2- $$\frac{d}{dx}(\operatorname{csch} u) = \frac{d}{dx}\left(\frac{1}{\sinh u}\right)$$
$$= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx}$$
$$= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx}$$
$$= -\operatorname{csch} u \coth u \frac{du}{dx}$$

Integrals

$$\int \sinh u\, du = \cosh u + C$$

$$\int \cosh u\, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u\, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\, du = -\operatorname{csch} u + C$$

Example 1

(a) $\frac{d}{dt}\left(\tanh \sqrt{1+t^2}\right) = \operatorname{sech}^2 \sqrt{1+t^2} \cdot \frac{d}{dt}\left(\sqrt{1+t^2}\right)$

$= \frac{t}{\sqrt{1+t^2}} \operatorname{sech}^2 \sqrt{1+t^2}$

(b) $\int \coth 5x\, dx = \int \frac{\cosh 5x}{\sinh 5x} dx = \frac{1}{5}\int \frac{du}{u}$ $u = \sinh 5x$, $du = 5 \cosh 5x\, dx$

$= \frac{1}{5} \ln |u| + C = \frac{1}{5} \ln |\sinh 5x| + C$

(c) $\int_0^1 \sinh^2 x\, dx = \int_0^1 \frac{\cosh 2x - 1}{2} dx$ Table 7.6

$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\, dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$

$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$ Evaluate with a calculator.

(d) $\int_0^{\ln 2} 4e^x \sinh x\, dx = \int_0^{\ln 2} 4e^x \frac{e^x - e^{-x}}{2} dx = \int_0^{\ln 2} (2e^{2x} - 2)\, dx$

$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2 \ln 2} - 2 \ln 2) - (1 - 0)$

$= 4 - 2 \ln 2 - 1 \approx 1.6137$ ■

Inverse Hyperbolic Functions

Derevatives

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\text{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\text{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad u \neq 0$$

Integrals

1. $$\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$$

2. $$\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$$

3. $$\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$$

4. $$\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\text{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$$

5. $$\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\text{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$$

EXAMPLE 2: find the derivative of y

a) $y = \cosh^{-1} 2\sqrt{x+1}$ b) $y = \text{csch}^{-1}\left(\frac{1}{2}\right)^{\theta}$ c) $y = \sinh^{-1}(\tan x)$

sol:

a) $y = \cosh^{-1} 2\sqrt{x+1} = \cosh^{-1}\left(2(x+1)^{1/2}\right) \Rightarrow \frac{dy}{dx} = \frac{(2)\left(\frac{1}{2}\right)(x+1)^{-1/2}}{\sqrt{[2(x+1)^{1/2}]^2-1}} = \frac{1}{\sqrt{x+1}\sqrt{4x+3}} = \frac{1}{\sqrt{4x^2+7x+3}}$

b) $y = \text{csch}^{-1}\left(\frac{1}{2}\right)^{\theta} \Rightarrow \frac{dy}{d\theta} = -\frac{\left[\ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^{\theta}}{\left(\frac{1}{2}\right)^{\theta}\sqrt{1+\left[\left(\frac{1}{2}\right)^{\theta}\right]^2}} = -\frac{\ln(1)-\ln(2)}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}} = \frac{\ln 2}{\sqrt{1+\left(\frac{1}{2}\right)^{2\theta}}}$

c) $y = \sinh^{-1}(\tan x) \Rightarrow \frac{dy}{dx} = \frac{\sec^2 x}{\sqrt{1+(\tan x)^2}} = \frac{\sec^2 x}{\sqrt{\sec^2 x}} = \frac{\sec^2 x}{|\sec x|} = \frac{|\sec x|\,|\sec x|}{|\sec x|} = |\sec x|$

EXAMPLE 3: Evalua $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}}$

a) $\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}$. b) c) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}}$

Sol:

a)
$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$
$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.11}$$
$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,
$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Bigg]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665.$$

b) $\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\,dx$, $a = 1$

$= \left[-\operatorname{sech}^{-1} u\right]_{4/5}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{13} + \operatorname{sech}^{-1}\frac{4}{5}$

c) $\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\,dx$, $a = 1$

$= \left[\sinh^{-1} u\right]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$

9). TECHNIQUES OF INTEGRATION

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \quad$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

10) Integration by Parts

Integration by parts is a technique for simplifying integrals of tl $\int f(x)g(x)\,dx$.

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du$$

EXAMPLE 1 Find

$$\int x \cos x\,dx.$$

Solution We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x\,dx = x \sin x - \int \sin x\,dx = x \sin x + \cos x + C.$$

EXAMPLE 2 Find

$$\int \ln x\,dx.$$

Solution Since $\int \ln x\,dx$ can be written as $\int \ln x \cdot 1\,dx$, we use the formula $\int u\,dv = uv - \int v\,du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x}\,dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x\,dx = x \ln x - \int x \cdot \frac{1}{x}\,dx = x \ln x - \int dx = x \ln x - x + C.$$

Remark: Sometimes we have to use integration by parts more than once as follows:

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

■

EXAMPLE 4 Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

$$2\int e^{x}\cos x\,dx = e^{x}\sin x + e^{x}\cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^{x}\cos x\,dx = \frac{e^{x}\sin x + e^{x}\cos x}{2} + C.$$

Evaluating Definite Integrals by Parts:

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx$$

EXAMPLE 6 Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^{0}) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

■

11) Tabular Integration

EXAMPLE 7 Evaluate

$$\int x^2 e^{x}\,dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^{x}$, we list:

d		$\int$
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

Then

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + C.$$

EXAMPLE 8 Evaluate

$$\int x^3 \sin x\,dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x\,dx = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C.$$

12) Trigonometric Integrals

$$\int \sec^2 x\,dx = \tan x + C.$$

Products of Powers of Sines and Cosines

We begin with integrals of the form: $\int \sin^m x \cos^n x\,dx,$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and *n* being odd or even.

Case 1 If ***m* is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution This is an example of Case 1.

$$\begin{aligned}
\int \sin^3 x \cos^2 x\, dx &= \int \sin^2 x \cos^2 x \sin x\, dx && m \text{ is odd.} \\
&= \int (1 - \cos^2 x) \cos^2 x\, (-d\,(\cos x)) && \sin x\, dx = -d(\cos x) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2)\, du && \text{Multiply terms.} \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

■

EXAMPLE 2 Evaluate

$$\int \cos^5 x\, dx.$$

Solution This is an example of Case 2, where $m = 0$ is even and $n = 5$ is odd.

$$\int \cos^5 x\, dx = \int \cos^4 x \cos x\, dx = \int (1 - \sin^2 x)^2\, d(\sin x) \qquad \cos x\, dx = d(\sin x)$$

$$= \int (1 - u^2)^2\, du \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du \qquad \text{Square } 1 - u^2.$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$ ■

EXAMPLE 3 Evaluate

$$\int \sin^2 x \cos^4 x\, dx.$$

Solution This is an example of Case 3.

$$\int \sin^2 x \cos^4 x\, dx = \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \qquad m \text{ and } n \text{ both even}$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$, we use

$$\int \cos^2 2x\, dx = \frac{1}{2}\int (1 + \cos 4x)\, dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right). \qquad \text{Omitting the constant of integration until the final result}$$

For the $\cos^3 2x$ term, we have

$$\int \cos^3 2x \, dx = \int (1 - \sin^2 2x) \cos 2x \, dx \qquad \begin{aligned} u &= \sin 2x, \\ du &= 2\cos 2x \, dx \end{aligned}$$

$$= \frac{1}{2} \int (1 - u^2) \, du = \frac{1}{2} \left(\sin 2x - \frac{1}{3} \sin^3 2x \right). \qquad \text{Again omitting } C$$

Combining everything and simplifying, we get

$$\int \sin^2 x \cos^4 x \, dx = \frac{1}{16} \left(x - \frac{1}{4} \sin 4x + \frac{1}{3} \sin^3 2x \right) + C. \qquad \blacksquare$$

Eliminating Square Roots

In the next example, we use the identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx.$$

Solution To eliminate the square root, we use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2 \theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} \, dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x} \, dx = \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} \, dx$$

$$= \sqrt{2} \int_0^{\pi/4} |\cos 2x| \, dx = \sqrt{2} \int_0^{\pi/4} \cos 2x \, dx \qquad \begin{aligned} &\cos 2x \geq 0 \\ &\text{on } [0, \pi/4] \end{aligned}$$

$$= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} = \frac{\sqrt{2}}{2} [1 - 0] = \frac{\sqrt{2}}{2}. \qquad \blacksquare$$

Integrals of Powers of tan x and sec x

We use $\mathbf{\tan^2 x = \sec^2 x - 1}$ and $\mathbf{\sec^2 x = \tan^2 x + 1}$

EXAMPLE 5 Evaluate

$$\int \tan^4 x\, dx.$$

Solution

$$\begin{aligned}\int \tan^4 x\, dx &= \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x \cdot (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \tan^2 x\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int (\sec^2 x - 1)\, dx \\ &= \int \tan^2 x \sec^2 x\, dx - \int \sec^2 x\, dx + \int dx.\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\, dx$$

and have

$$\int u^2\, du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\, dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

EXAMPLE 6 Evaluate

$$\int \sec^3 x\, dx.$$

Solution We integrate by parts using

$$u = \sec x, \qquad dv = \sec^2 x\, dx, \qquad v = \tan x, \qquad du = \sec x \tan x\, dx.$$

Then

$$\begin{aligned}\int \sec^3 x\, dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\, dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1) \sec x\, dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\, dx - \int \sec^3 x\, dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \text{and} \qquad \int \cos mx \cos nx\,dx$$

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x].$$

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x\,dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$, we get

$$\begin{aligned}\int \sin 3x \cos 5x\,dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\,dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\,dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

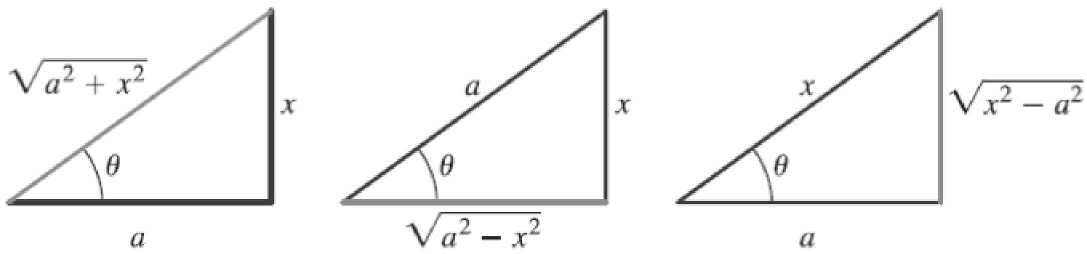
15) Trigonometric Substitutions

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. The most common substitutions are:

If $\sqrt{a^2 + x^2}$ then we use $x = a\tan\theta$, $a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta$.

If $\sqrt{a^2 - x^2}$ then we use $x = a\sin\theta$ $a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta$

If $\sqrt{x^2 - a^2}$ then we use $x = a\sec\theta$ $x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta$



$x = a \tan \theta$ $\qquad$ $x = a \sin \theta$ $\qquad$ $x = a \sec \theta$

$\sqrt{a^2 + x^2} = a|\sec \theta|$ $\qquad$ $\sqrt{a^2 - x^2} = a|\cos \theta|$ $\qquad$ $\sqrt{x^2 - a^2} = a|\tan \theta|$

Remark : In order to get θ we use the invers of trigonometric functions then we suppose that:

$$x = a \tan \theta, \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a \sin \theta \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a \sec \theta \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

EXAMPLE 1 Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

Solution We set

$$x = 2 \tan \theta, \qquad dx = 2 \sec^2 \theta \, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4 \tan^2 \theta = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta.$$

Then

$$\int \frac{dx}{\sqrt{4 + x^2}} = \int \frac{2 \sec^2 \theta \, d\theta}{\sqrt{4 \sec^2 \theta}} = \int \frac{\sec^2 \theta \, d\theta}{|\sec \theta|} \qquad \sqrt{\sec^2 \theta} = |\sec \theta|$$

$$= \int \sec \theta \, d\theta \qquad \sec \theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln |\sec \theta + \tan \theta| + C$$

$$= \ln \left| \frac{\sqrt{4 + x^2}}{2} + \frac{x}{2} \right| + C. \qquad \text{From Fig. 8.4}$$

EXAMPLE 2 Evaluate

$$\int \frac{x^2\,dx}{\sqrt{9 - x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

Then

$$\begin{aligned}
\int \frac{x^2\,dx}{\sqrt{9 - x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\,d\theta && \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= 9\int \frac{1 - \cos 2\theta}{2}\,d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9 - x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9 - x^2} + C.
\end{aligned}$$

■

EXAMPLE 3 Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\begin{aligned}
\sqrt{25x^2 - 4} &= \sqrt{25\left(x^2 - \frac{4}{25}\right)} \\
&= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}
\end{aligned}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$

$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for $0 < \theta < \pi/2$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \frac{dx}{5\sqrt{x^2-(4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5\cdot(2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C.$$

Fig:

16) Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called partial fractions, which are easily integrated.

Writing a rational function *f(x)/g(x)* as a sum of partial fractions depends on two things:

• *The degree of f(x) must be less than the degree of g(x).* That is, the fraction must be proper. If it isn't, divide *f(x)* by *g(x)* and work with the remainder term.

• We must know the factors *of g(x).* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \qquad (1)$$

To find A and B, we first clear Equation (1) of fractions and regroup in powers of x, obtaining

$$\frac{5x-3}{x^2-2x-3} = \frac{2}{x+1} + \frac{3}{x-3}.$$

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

EXAMPLE 1 Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx.$$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x+1)(x+3) + B(x-1)(x+3) + C(x-1)(x+1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\, dx &= \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx \\ &= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K, \end{aligned}$$

EXAMPLE 2 Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x+2)^2}\, dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermine coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$

$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$

$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$

$$= 6\ln|x+2| + 5(x+2)^{-1} + C.$$

EXAMPLE 3 Use partial fractions to evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r@{}l} & 2x \\ x^2 - 2x - 3 \,\big)\! & \overline{2x^3 - 4x^2 - x - 3} \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \qquad\qquad 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx = \int 2x\,dx + \int \frac{5x-3}{x^2 - 2x - 3}\,dx$$

$$= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$

$$= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.$$ ■

EXAMPLE 4 Use partial fractions to evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx.$$

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned} -2x+4 &= (Ax+B)(x-1)^2 + C(x-1)(x^2+1) + D(x^2+1) \\ &= (A+C)x^3 + (-2A+B-C+D)x^2 \\ &\quad + (A-2B+C)x + (B-C+D). \end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$
Coefficients of x^2:	$0 = -2A + B - C + D$
Coefficients of x^1:	$-2 = A - 2B + C$
Coefficients of x^0:	$4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$	Subtract fourth equation from second.
$C = -A = -2$	From the first equation
$B = (A + C + 2)/2 = 1$	From the third equation and $C = -A$
$D = 4 - B + C = 1.$	From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned} \int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \end{aligned}$$

EXAMPLE 5 Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned} 1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A \end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \quad C=0, \quad 2A+B+D=0, \quad C+E=0, \quad A=1.$$

Solving this system gives $A=1, B=-1, C=0, D=-1$, and $E=0$. Thus,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx$$

$$\begin{aligned} &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} && u = x^2+1,\ du = 2x\,dx \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \end{aligned}$$

■

The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

there is a quick way to expand f*(x)/ g(x)* by partial fractions.

EXAMPLE 6 Find A, B, and C in the partial fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \qquad (3)$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} \qquad (4)$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\boxed{(x - 1)}\,(1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.$$

$\Uparrow$ Cover

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Expression (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2 + 1}{(2 - 1)\,\boxed{(x - 2)}\,(2 - 3)} = \frac{5}{(1)(-1)} = -5.$$

$\Uparrow$ Cover

Finally, C is found by covering the $(x - 3)$ in Expression (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3-1)(3-2)\boxed{(x-3)}} = \frac{10}{(2)(1)} = 5.$$

$\Uparrow$ Cover

■

EXAMPLE 7 Use the Heaviside Method to evaluate

$$\int \frac{x+4}{x^3 + 3x^2 - 10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of the cubic polynomial $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x+4}{x^3 + 3x^2 - 10x} = \frac{x+4}{x(x-2)(x+5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0+4}{\boxed{x}\,(0-2)(0+5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$\Uparrow$ Cover

$$A_2 = \frac{2+4}{2\,\boxed{(x-2)}\,(2+5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$\Uparrow$ Cover

$$A_3 = \frac{-5+4}{(-5)(-5-2)\boxed{(x+5)}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

$\Uparrow$ Cover

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C.$$

Other Ways to Determine the Coefficients

EXAMPLE 8 Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}.$$ ■

EXAMPLE 9 Find A, B, and C in the expression

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$$

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \quad & (1)^2 + 1 = A(-1)(-2) + B(0) + C(0) \\
& 2 = 2A \\
& A = 1 \\
x = 2: \quad & (2)^2 + 1 = A(0) + B(1)(-1) + C(0) \\
& 5 = -B \\
& B = -5 \\
x = 3: \quad & (3)^2 + 1 = A(0) + B(0) + C(2)(1) \\
& 10 = 2C \\
& C = 5.
\end{aligned}$$

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$