"Functions"

الدوال

Definition: A function $f: X \to Y$ is a relationship between two sets (nonempty sets) such that associates each elements in X with exactly one element in Y.

i.e. $\forall x \in X, \exists y \in Y$ s.t. $f(x) = y$.

$$X \xrightarrow[\text{(Domain)}]{\text{input}} \boxed{f} \xrightarrow[\text{(Range)}]{\text{output}} Y$$

Definition (Domain): The set of all possible elements (input values) is called domain of f (Dom(f) or D_f).

Definition (Range): The set of all corresponding elements in the second set (output values) is called range of f (Range(f) or R_f).

i.e. $f: X \to Y$ be a function, then:

The set X is called domain f (Dom(f) = D_f).
The set Y is called co-domain f (co-Dom(f
$f(x)$ is called range f (Rang(f) = R_f).

"Some Kinds of Functions"

بعض أنواع الدوال

1. Constant Function الدالة الثابتة

A function $f: X \to Y$ is called constant if
$f(x) = c, \forall x \in X, c \in \mathbb{R}$ (set of real numbers).

for example: $f(x) = 1$, $f(x) = -1$, $f(x) = 10$
$f(x) = -100$, $f(x) = 0$ (Zero function).

$D_f = Dom(f) = \mathbb{R} = (-\infty, \infty)$
$R_f = Rang(f) = c$.

2. Identity Function الدالة الذاتية

A function $f: X \to Y$ is called identity if
$f(x) = x, \forall x \in X$.

$D_f = \mathbb{R} = (-\infty, \infty)$
$R_f = \mathbb{R} = (-\infty, \infty)$

3. Polynomial Function (الدالة كثيرة الحدود (متعددة الحدود

A function $f: X \to Y$ is called polynomial if
$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
$0 \neq a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$
$f(x)$ is polynomial of degree (n)
for example: $f(x) = 2x^3 + 5x^2 + 4x + \frac{1}{2}$ (Polynomial of degree 3)
$f(x) = x^7 + 5x^6 + 8x^4 + x + 19$ (Polynomial of degree 7).

③ $f(x) = x^2$

sol.

$D_f = \mathbb{R} = (-\infty, \infty)$

$R_f = [0, \infty) = \mathbb{R}^+ \cup \{0\}$.

④ $f(x) = 5x^2 + 1$

$D_f = \mathbb{R} = (-\infty, \infty)$

$\rightarrow R_f = [1, \infty)$.

6. Square Root Function دالة الجذر التربيعي

A function $f: X \rightarrow Y$ is called square root if
$f(x) = \sqrt{x}$, $x \in X$.

for examples: $f(x) = \sqrt{2x}$, $f(x) = \sqrt{x+9}$,
$f(x) = \sqrt{16 - x^2}$, $f(x) = \sqrt{x^2 - 3x}$.

لإيجاد المجال (Domain) و المدى (Range) لدالة
الجذر التربيعي :

Examples: Find the domain and the range for the following functions.

① $f(x) = \sqrt{x}$

sol.

$x \geq 0 \Rightarrow D_f = [0, \infty) = \mathbb{R}^+ \cup \{0\}$

$R_f = [0, \infty) = \mathbb{R}^+ \cup \{0\}$.

Examples: Find the domain and the range for the following functions.

① $f(x) = x^2 + 2x + 1$

sol.

$D_f = \mathbb{R} = (-\infty, \infty)$

لإيجاد المدى (Range)

$a = 1 > 0$ (موجبة)

$f\left(\frac{-b}{2a}\right) = f\left(\frac{-2}{2}\right)$
$= f(-1)$
$= (-1)^2 + 2(-1) + 1$
$= 0$

$\Rightarrow R_f \geq 0 \quad \left(R_f \geq f\left(\frac{-b}{2a}\right)\right)$

$\therefore R_f = [0, \infty) = \mathbb{R}^+ \cup \{0\}$

② $f(x) = -x^2 + 2x + 1$

sol.

$D_f = \mathbb{R} = (-\infty, \infty)$

$a = -1 < 0$ (سالب)

$f\left(\frac{-b}{2a}\right) = f\left(\frac{-2}{-2}\right)$
$= f(1)$
$= -(1)^2 + 2 + 1 = 2$

$\rightarrow R_f \leq 2 \quad \left(R_f \leq f\left(\frac{-b}{2a}\right)\right)$

$\therefore R_f = (-\infty, 2]$

$D_f = \mathbb{R}$ and $R_f = \mathbb{R}$ (for Polynomial function).

4. Linear Function الدالة الخطية

A function $f : X \to Y$ is called linear if
$f(x) = ax + b$ s.t. $a, b \in \mathbb{R}$ and $a \neq 0$.

for example: $f(x) = x + 1$, $f(x) = 5x + 3$, $f(x) = 2x$.

$D_f = \mathbb{R}$ and $R_f = \mathbb{R}$.

5. Quadratic Function الدالة التربيعية

A function $f : X \to Y$ is called quadratic if
$f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$ and $a \neq 0$.

for example: $f(x) = x^2$, $f(x) = 5x^2 + 1$, $f(x) = x^2 - 2x + 5$

To find the domain and the range:
لايجاد مجال ومدى الدالة التربيعية

$Dom(f) = D_f = \mathbb{R}$ (مجال الدالة التربيعية هو
مجموعة الاعداد الحقيقية $\mathbb{R}$)

Case 1: لايجاد المدى نوجد حالتان :
If $a > 0$ (موجب)

$R_f \geq f\left(\frac{-b}{2a}\right)$

Case 2: If $a < 0$ (سالب)

$R_f \leq f\left(\frac{-b}{2a}\right)$

② $f(x) = \dfrac{7}{x+4}$

sol:

$x+4=0 \Rightarrow x=-4 \Rightarrow D_f = \mathbb{R} - \{-4\}$

$R_f = \mathbb{R} - \{0\}$.

③ $f(x) = \dfrac{1}{\sqrt{x}}$ $\quad \sqrt{x} \quad x \geq 0$

sol: $\quad x > 0 \Rightarrow D_f = (0, \infty) = \mathbb{R}^+$

$R_f = \mathbb{R} - \{0\}$.

④ $f(x) = \dfrac{x^2 - 4}{x - 2}$

sol:

$D_f = \mathbb{R} \cap \mathbb{R} - \{2\} = \mathbb{R} - \{2\}$

لايجاد المدى (Range):

$f(x) = \dfrac{x^2-4}{x-2} = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{x-2}} = x+2$

$f(2) = 2+2 = 4$

$\therefore R_f = \mathbb{R} - \{4\}$

8. Absolute value Function دالة القيمة المطلقة

A function $f: X \to Y$ is called absolute value if $f(x) = |x|, \forall x \in X$.

7. Rational Function الدالة الكسرية

A function $f: X \longrightarrow Y$ is called rational if

$$f(x) = \frac{g(x)}{h(x)} \quad \text{s.t.} \quad h(x) \neq 0, \ \forall x \in X.$$

for example: $f(x) = \frac{1}{x}$, $f(x) = \frac{7}{x-4}$,

$$f(x) = \frac{1}{\sqrt{x}}, \quad f(x) = \frac{x}{\sqrt{x+1}}.$$

لإيجاد مجال الدالة الكسرية:

الحالة الأولى: اذا كان البسط عدد ثابت فان مجال الدالة الكسرية هو جميع الأعداد الحقيقية ماعدا الأعداد التي تجعل المقام يساوي صفر.

الحالة الثانية: اذا كان البسط دالة غير ثابتة (أي نوع من الدوال السابقة).

مجال الدالة الكسرية = مجال البسط $\cap$ مجال المقام ماعدا الأعداد التي تجعل المقام يساوي صفر

Examples: Find the domain and the range for the following functions.

① $f(x) = \frac{1}{x}$

sol.

$$x \neq 0 \Rightarrow D_f = \mathbb{R} - \{0\}$$

$$R_f = \mathbb{R} - \{0\}.$$

② $f(x) = \sqrt{x+9}$

sol.

$x + 9 \geqslant 0 \Rightarrow x \geqslant -9 \Rightarrow D_f = [-9, \infty)$.

$R_f = [0, \infty)$.

③ $f(x) = \sqrt{x-3}$

sol.

$[3, \infty)$

$x - 3 \geqslant 0 \Rightarrow x \geqslant 3 \Rightarrow D_f = [3, \infty)$

$R_f = [0, \infty)$

④ $f(x) = \sqrt{16 - x^2}$

sol.

$16 > x^2$

$16 - x^2 \geqslant 0$

$x^2 \leqslant 16$

$|x| \leqslant 4$

$-4 \leqslant x \leqslant 4$

$\therefore D_f = [-4, 4]$

$R_f = [0, 4]$.

① $f(x) = \dfrac{x^2 - 4}{\sqrt{4x-3}}$

$4x - 3 > 0 \Rightarrow x > \frac{3}{4}$

$(\frac{3}{4}, \infty)$

$D: \mathbb{R} \setminus (\frac{3}{4}, \infty)$

② $f(x) = \dfrac{x}{\sqrt{x+1}}$

$\sqrt{x+1} > 0 \Rightarrow x > -1$

$D: (-1, \infty)$

"Composition of Functions"

تركيب الدوال

Let $f: X \to Y$ and $g: Y \to Z$, then the composition of g with f is a function $g\circ f: X \to Z$ defined by $(g\circ f)(x) = g(f(x))$.

Examples:

① Let $f(x) = x^2$, $g(x) = x-3$. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $(f\circ g)(6)$, $(g\circ f)(\sqrt{3})$.

Sol:

$$(f\circ g)(x) = f(g(x))$$
$$= f(x-3)$$
$$= (x-3)^2 = x^2 - 6x + 9$$

$$(g\circ f)(x) = g(f(x))$$
$$= g(x^2)$$
$$= x^2 - 3$$

$$(f\circ g)(6) = (6)^2 - 6(6) + 9 = 9$$

$$(g\circ f)(\sqrt{3}) = (\sqrt{3})^2 - 3 = 0$$

② Let $f(x) = x^2 + 2$, $g(x) = \sqrt{x-4}$. Find $(f\circ g)(x)$, $(g\circ f)(x)$, $(f\circ g)(-2)$, $(g\circ f)(2)$.

Sol:

$$(f\circ g)(x) = f(g(x))$$
$$= f(\sqrt{x-4})$$
$$= (\sqrt{x-4})^2 + 2$$
$$= x - 4 + 2$$
$$= x - 2.$$

"Operations On Functions" ١٨/١١/٨

العمليات على الدوال

Let f, g be two functions then;

1. $(f \mp g)(x) = f(x) \mp g(x)$.
2. $(f \cdot g)(x) = f(x) \cdot g(x)$.
3. $(f/g)(x) = f(x) / g(x)$, $g(x) \neq 0, \forall x$.
4. $(cf)(x) = cf(x)$, $c \in \mathbb{R}$.

Example: Let $f(x) = x^2$, $g(x) = x-3$, $x \neq 3$.
Find ① $(f+g)(x)$ ② $(f-g)(x)$

③ $(f \cdot g)(x)$ ④ $(f/g)(x)$ ⑤ $(cf)(x)$, $c=5$.
⑥ $(f+g)(2)$.

Sol:

1. $(f+g)(x) = f(x) + g(x)$
$= x^2 + x - 3$.
2. $(f-g)(x) = f(x) - g(x)$
$= x^2 - x + 3$
3. $(f \cdot g)(x) = f(x) \cdot g(x)$
$= x^2(x-3) = x^3 - 3x$
4. $\because x \neq 3 \implies \therefore g(x) \neq 0$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{x^2}{x-3}.$$

5. $(cf)(x) = cf(x)$
$= 5x^2$.

6. $(f+g)(2) = (2)^2 + 2 - 3 = 3$.

9. Trigonometric Functions الدوال المثلثية

$f(x)=\sin x$, $f(x)=\cos x$, $f(x)=\tan x$,
$f(x)=\sec x$, $f(x)=\csc x$, $f(x)=\cot x$.

H.W.

(A) Classify the following functions:

1. $f(x)=3x+7$ 2. $f(x)=2\sqrt{3}$
3. $f(x)=\dfrac{11}{x^2+2x+1}$ 4. $f(x)=5x^2+10x$
5. $f(x)=7x^5+6x^4+x^2+\frac{1}{2}x+\sqrt{5}$
6. $f(x)=|2x+6|$ 7. $f(x)=\sqrt{x^2+4x+3}$ 8. $f(x)=\sin 3x$

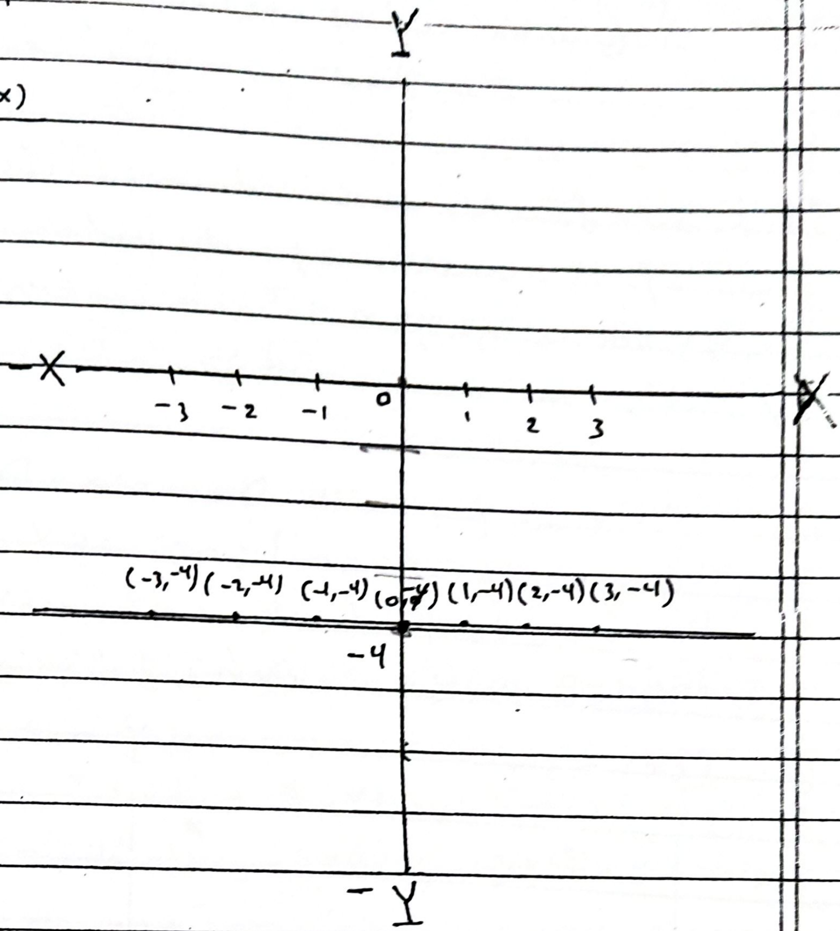
(B) Find the domain and the range for the following functions.

1. $f(x)=2x^2-3$
2. $f(x)=x^2-2x+5$
3. $f(x)=\sqrt{3x-4}$ $D=[\frac{4}{3},\infty)$ $R=[0,\infty)$
4. $f(x)=\sqrt{25-x^2}$ $D=[-5,5]$ $R=[0,\infty]$
5. $f(x)=\dfrac{6}{x^2-36}$ $D=R-\{6\}$ $R=R-\{0\}$
6. $f(x)=\dfrac{x^2-9}{x-3}$ $D=R-\{3\}$

$(\cancel{x-3})(x+3)$

② $f(x) = -4$

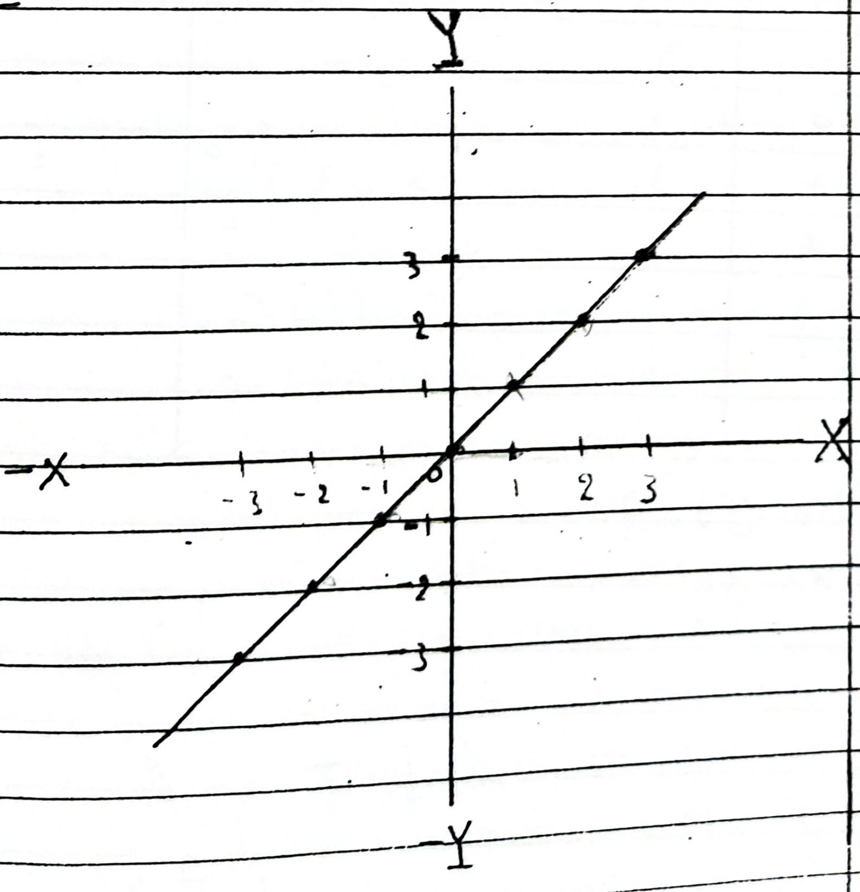
x	$y = f(x)$
⋮	⋮
-3	-4
-2	-4
-1	-4
0	-4
1	-4
2	-4
3	-4
⋮	⋮



$D_f = \mathbb{R}$
$R_f = -4$

③ $f(x) = x$

x	$y = f(x)$
⋮	⋮
-3	-3
-2	-2
-1	-1
0	0
1	1
2	2
3	3
⋮	⋮



$D_f = \mathbb{R}$
$R_f = \mathbb{R}$

"Graphs of Functions"
"رسم الدوال"

If f is a function with domain (D_f), the graph of f consists of all ordered pairs (x, y) where $y = f(x)$ in the cartesian plan (X axis and Y axis).

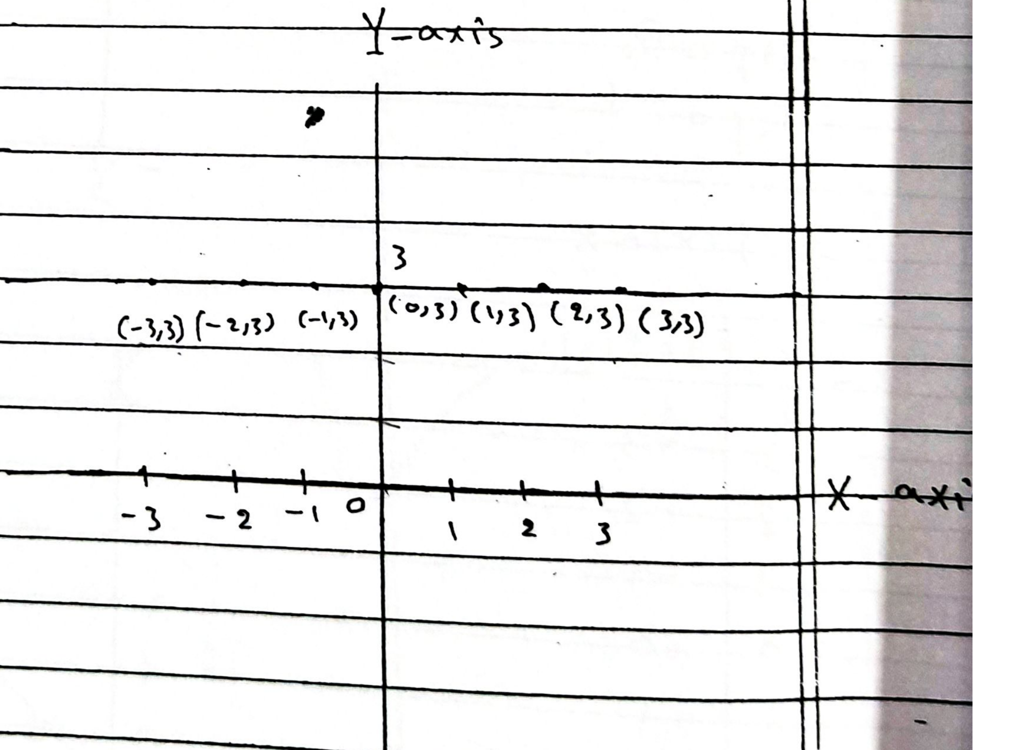
i.e. $f: X \to Y$, with Domain $= D_f$, the graph of f is $\{(x, y) : x \in X,\ y = f(x) \in Y\}$.

Examples: Draw (Graph) the following functions.

① $f(x) = 3$

X	y
(-3, 3)	
(-2, 3)	
(-1, 3)	
(0, 3)	
(1, 3)	
(2, 3)	
(3, 3)	

x	$y = f(x)$
⋮	⋮
-3	3
-2	3
-1	3
0	3
1	3
2	3
3	3
⋮	⋮



Dom$(f) = \mathbb{R}$
Rang$(f) = 3$

$|3x - 5| = \begin{cases} 3x - 5 \\ -(3x - 5) \end{cases}$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$(g \circ f)(x) = g(f(x))$
$= g(x^2+2)$
$= \sqrt{x^2+2-4}$
$= \sqrt{x^2-2}$

$(f \circ g)(-2) = -2-2 = -4$.
$(g \circ f)(2) = \sqrt{(2)^2-2} = \sqrt{2}$.

H.W.

① Let $f(x) = |x|$, $g(x) = 3x-5$. Find $(f \circ g)(x)$, $(f \circ g)(\frac{1}{3})$, $(g \circ f)(x)$, $(g \circ f)(\frac{-1}{3})$.

② Let $f(x) = 2x+1$, $g(x) = 2x^2+3x$. Find $(f \circ g)(x)$, $(f \circ g)(-1)$, $(g \circ f)(x)$, $(g \circ f)(1)$.

① $F \circ g(x) = F(g(x))$
$= F(3x-5)$
$F \circ g(x) = |3x-5|$

$F \circ g(\frac{1}{3}) = |3\frac{1}{3}-5| = |-2| = 2$

$g \circ f(x) = g\{f(x)\}$
$= g(|x|)$
$= 3|x|-5$

$g \circ f(\frac{-1}{3}) = 3|\frac{-1}{3}|-5$
$= 1-5$
$= -4$

-1
$-3-5 = -8$
58

$3|-1|$
-5
$3-5$
$5?$

"Shifting (Moving) Graph of Functions"

تزحيف رسم الدوال

Let $f: \mathbb{R} \to \mathbb{R}$ be a function; $c > 0$, then:

1. $y = f(x) + c$, the graph shif (UP) c units
2. $y = f(x) - c$, the graph shift (down) c units
3. $y = f(x+c)$, the graph shift to the (left) c units

x^- $\Leftarrow c$

4. $y = f(x-c)$, the graph shift to the (right) c units.

$\Rightarrow x^+$

Examples: Graph (Draw) the following functions:

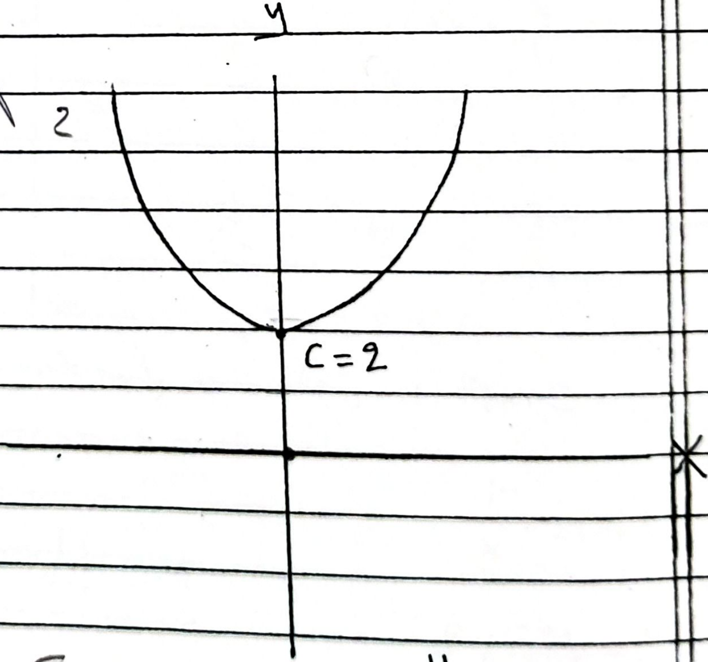
$y = f(x) + c$

① $y = x^2 + 2$ $\uparrow 2$

Sol.

$y = x^2 + 2$

i.e. $y = f(x) + c$



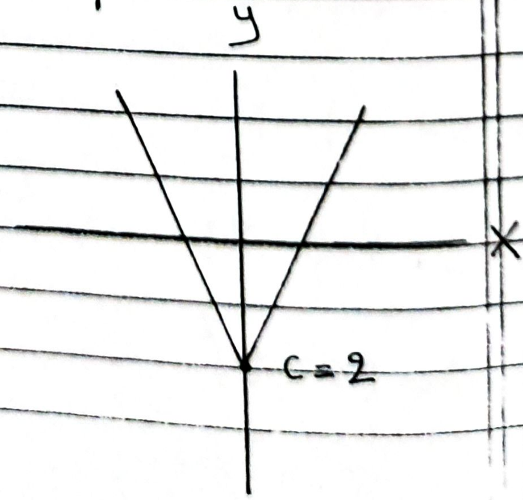
$y = f(x) - c$

② $y = |x| - 2$ $\downarrow 2$

Sol.

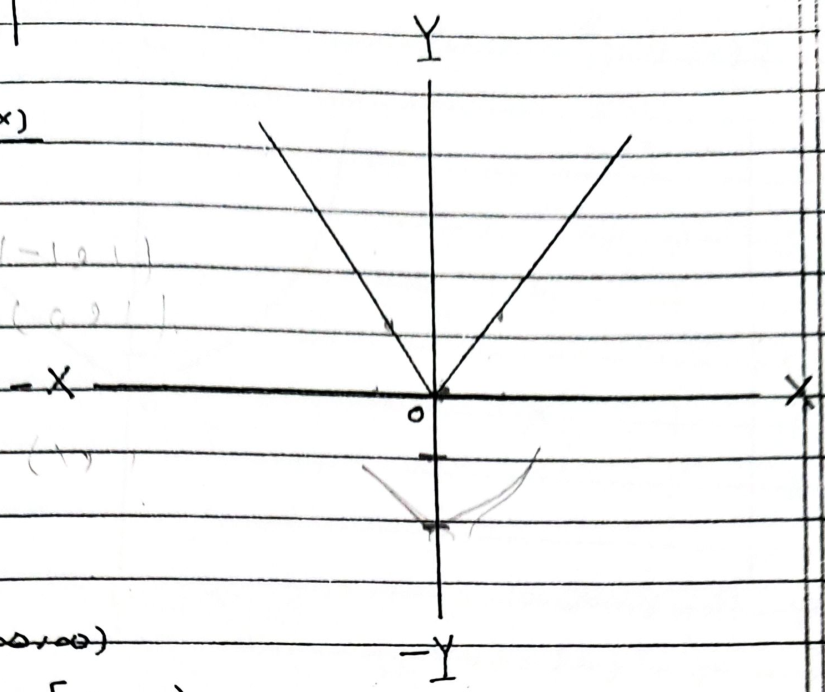
$y = |x| - 2$

i.e. $y = f(x) - c$



⑥ $f(x) = |x|$

x	$y = f(x)$
⋮	⋮
-2	2
-1	1
0	0
1	1
2	2
⋮	⋮

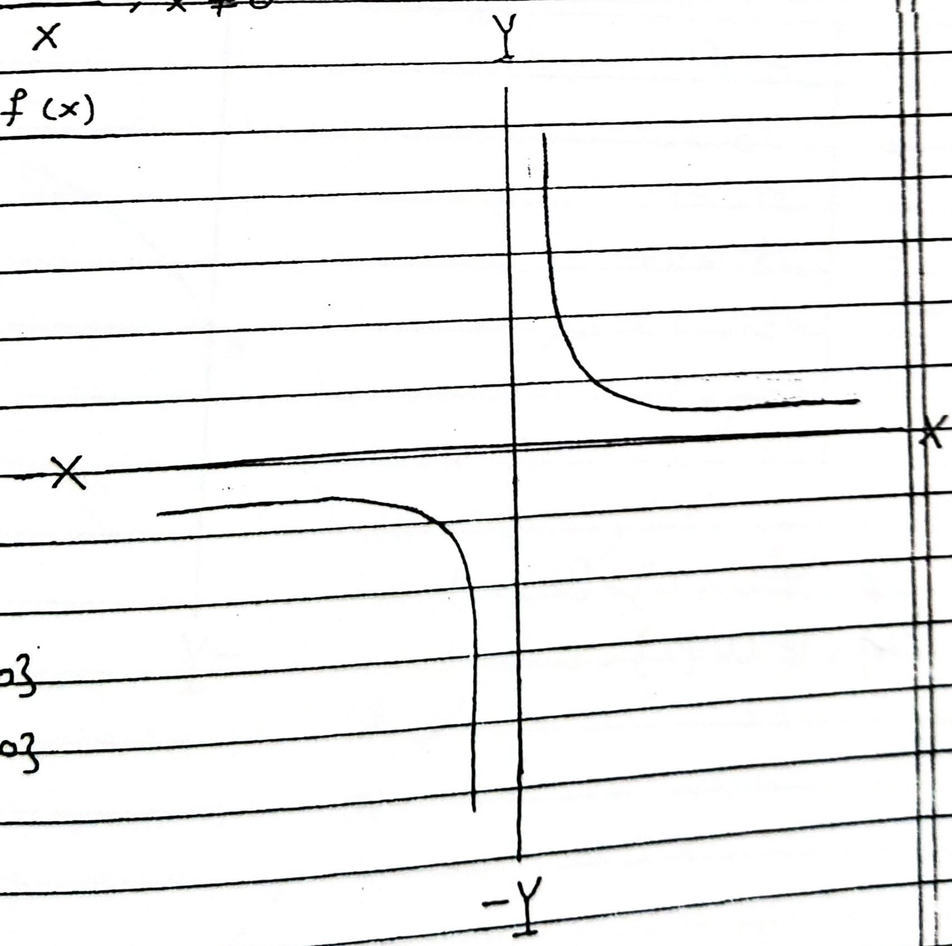


$D_f = \mathbb{R} = (-\infty, \infty)$

$R_f = \mathbb{R}^+ \cup \{0\} = [0, \infty)$

⑦ $f(x) = \dfrac{1}{x}$, $x \neq 0$

x	$y = f(x)$
⋮	⋮
-2	$\frac{-1}{2}$
-1	-1
1	1
2	$\frac{1}{2}$
⋮	⋮



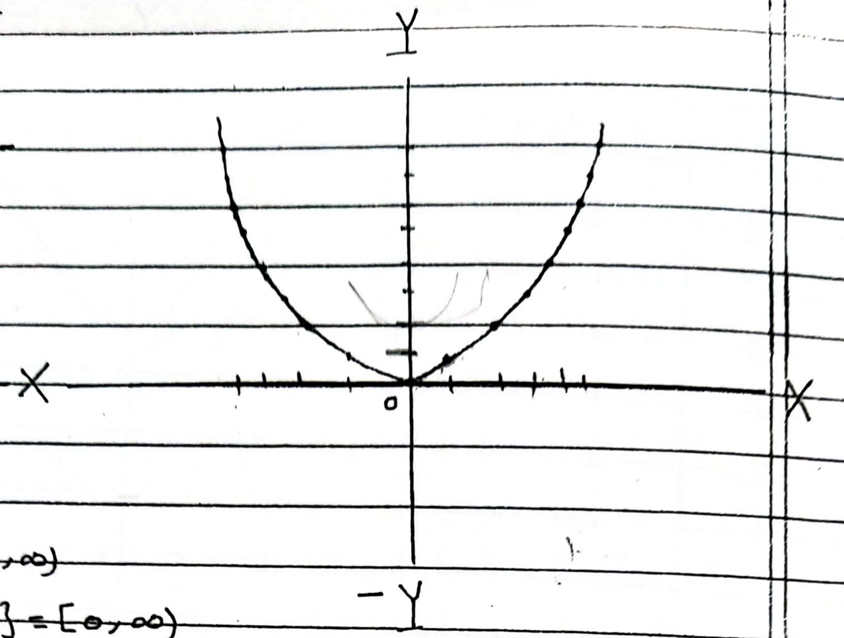
$D_f = \mathbb{R} - \{0\}$

$R_f = \mathbb{R} - \{0\}$

$x^2 + 2$

④ $f(x) = x^2$

x	$y = f(x)$
⋮	⋮
-2	4
-1	1
0	0
1	1
2	4
⋮	⋮

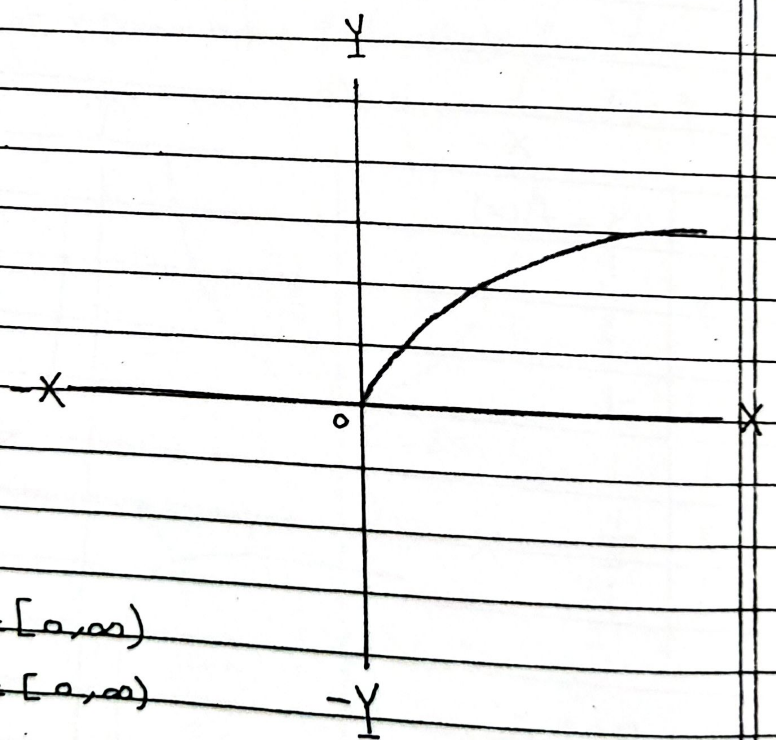


$D_f = R = (-\infty, \infty)$

$R = R^+ \cup \{0\} = [0, \infty)$

⑤ $f(x) = \sqrt{x}$

x	$y = f(x)$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} = 1.4$
3	$\sqrt{3} = 1.7$
4	$\sqrt{4} = 2$
⋮	⋮



$D_f = R^+ \cup \{0\} = [0, \infty)$

$R_f = R^+ \cup \{0\} = [0, \infty)$

Examples: Compute the limits for the following functions.

1. $\lim\limits_{x \to 3} \sqrt[3]{2(x+1)} = \sqrt[3]{2(3+1)} = \sqrt[3]{8} = 2$

2. $\lim\limits_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x} = \dfrac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = \dfrac{-8+8-1}{5+6} = \dfrac{-1}{11}$

3. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9} = \dfrac{\sqrt{9} - 3}{9 - 9} = \dfrac{0}{0}$ كمية غير معرفة

$\Rightarrow \lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x - 9} = \lim\limits_{x \to 9} \dfrac{\cancel{\sqrt{x} - 3}}{(\cancel{\sqrt{x} - 3})(\sqrt{x} + 3)}$

$= \lim\limits_{x \to 9} \dfrac{1}{\sqrt{x} + 3} = \dfrac{1}{\sqrt{9} + 3} = \dfrac{1}{6}$

4. $\lim\limits_{x \to 0} \dfrac{(4+x)^2 - 16}{x} = \dfrac{0}{0}$ كمية غير معرفة

$\Rightarrow \lim\limits_{x \to 0} \dfrac{(4+x)^2 - 16}{x} = \lim\limits_{x \to 0} \dfrac{\cancel{16} + 8x + x^2 - \cancel{16}}{x}$

$= \lim\limits_{x \to 0} \dfrac{\cancel{x}(8 + x)}{\cancel{x}}$

$= \lim\limits_{x \to 0} (8 + x) = 8 + 0 = 8.$

Hw:

Find the limits for the following functions.

1. $\lim\limits_{x \to 0} \dfrac{\sqrt{5x+4}}{x^2 - 2} = \dfrac{\sqrt{4}}{-2} = -1$

2. $\lim\limits_{x \to -3} \dfrac{x^2 + 4x + 3}{x + 3} = \dfrac{(x+3)(x+1)}{(x+3)} = x+1$

$= -2$

"Limits of Functions"

غايات الدوال

Definition: Let $f: X \to Y$ be a function, then

$$\lim_{x\to a} f(x) = L$$

is read (the limit of $f(x)$ as x approaches to a).
It means $f(x)$ gets closer and closer to L as x gets closer and closer to a.

Properties of Limits: خواص الغايات

1. $\lim_{x\to a} c = c$

2. $\lim_{x\to a} x = a$

3. $\lim_{x\to a} (c f(x)) = c \lim_{x\to a} f(x)$, $c \in \mathbb{R}$

4. $\lim_{x\to a} (f(x) \mp g(x)) = \lim_{x\to a} f(x) \mp \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, $\lim_{x\to a} g(x) \neq 0$

6. $\lim_{x\to a} (f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, n is any positive integer number and $\lim_{x\to a} f(x) \geq 0$.

Remarks:

1. $\lim_{x\to a} f(x) = \frac{0}{0}$ imposible (undefined)

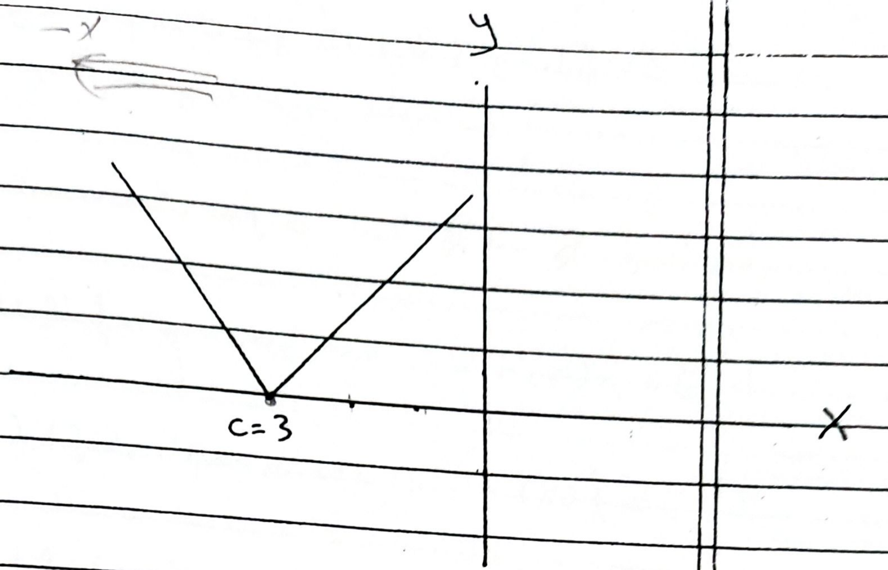
2. $\lim_{x\to a} f(x) = \frac{k}{0}$ imposible, $k \in \mathbb{R}$. (undefine

③ $y = |x+3|$

sol.

$y = |x+3|$

i.e. $y = f(x+c)$

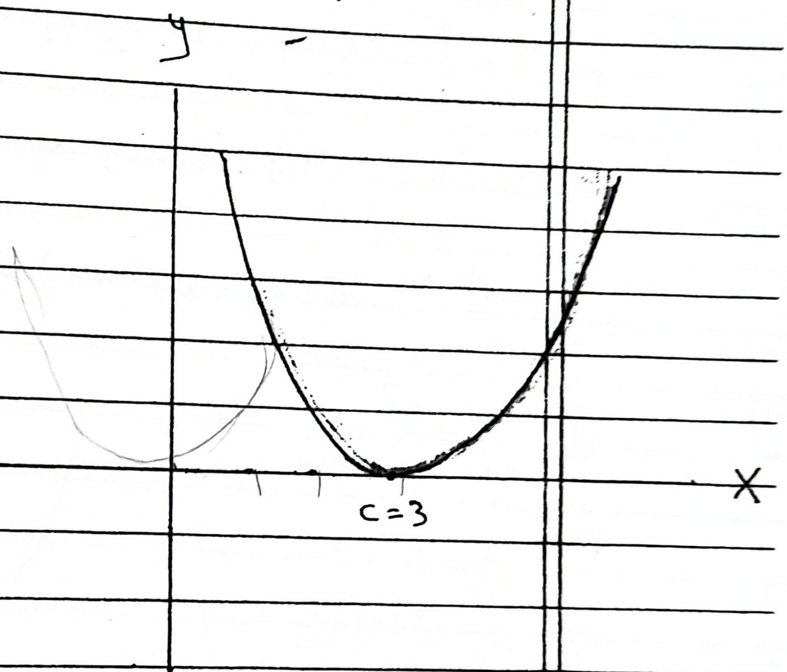


④ $y = (x-3)^2$

sol.

$y = (x-3)^2$

i.e. $y = f(x-c)$



H.W. Graph the following functions.

1. $f(x) = \dfrac{-1}{x}$

2. $y = \sqrt{x} + 2$

3. $y = |x| - 1$

4. $y = |x-2|$

5. $y = (x+2)^2$.

"One-Side Limits"

الغايات من جهة واحدة

① **Right-hand side Limit:** الغاية من جهة اليمين

Let f be a function, then $\lim\limits_{\substack{x \to a^+ \\ (x>a)}} f(x) = L_1$

is called right-hand limit.

② **Left-hand side Limit:** الغاية من جهة اليسار

Let f be a function, then $\lim\limits_{\substack{x \to a^- \\ (x<a)}} f(x) = L_2$

is called left-hand limit.

Remark:

If $\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x)$, then we say that $f(x)$ has a limit as $x \to a$ ($\lim\limits_{x \to a} f(x) = L$ exists).

Example:

Let $f(x) = \begin{cases} 2x+5 & , \ x > 3 \\ x^2+2 & , \ x \leq 3 \end{cases}$ find the limit.

Sol:

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (2x+5) = 2(3)+5 = 11$$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2+2) = (3)^2+2 = 11$$

$$\because \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} f(x) \Rightarrow \lim_{x \to 3} f(x) = 11$$

5. $$\lim_{x\to\infty} \frac{\sqrt{x^3-3x^2}}{\sqrt[4]{x^6+2x}}$$

نلاحظ أن أعلى قوة للمتغير x في المقام هي $x^{\frac{6}{4}} = x^{\frac{3}{2}}$

(الجذر الرابع للمقدار x^6) بقسمة كل من البسط والمقام

على $x^{\frac{3}{2}} = \sqrt{x^3}$ نحصل على:

$$\lim_{x\to\infty} \frac{\sqrt{\frac{x^3}{x^3}-\frac{3x^2}{x^3}}}{\sqrt[4]{\frac{x^6}{x^6}+\frac{2x}{x^6}}} = \lim_{x\to\infty} \frac{\sqrt{1-\frac{3}{x}}}{\sqrt[4]{1+\frac{2}{x^5}}}$$

$$= \frac{\sqrt{1-\frac{3}{\infty}}}{\sqrt[4]{1+\frac{2}{\infty}}} = \frac{\sqrt{1-0}}{\sqrt[4]{1+0}} = 1$$

Hw:

Compute the limits for the following functions.

① $$\lim_{x\to\infty} 2-\frac{1}{\sqrt{x^2+1}}$$

② $$\lim_{x\to\infty} \frac{3x^2+1}{5x^2+7x-39}$$

③ $$\lim_{x\to\infty} \frac{3x+7}{\sqrt{4x^2+5}}$$

"Limits at Infinity"

الغايات اللانهائية

Remarks:

1. $\lim_{x \to \infty} f(x) = \frac{0}{1} = 0$

2. $\lim_{x \to \infty} f(x) = \frac{0}{k} = 0, \; k \in \mathbb{R}$

3. $\lim_{x \to \infty} f(x) = \frac{1}{\infty} = 0$

4. $\lim_{x \to \infty} f(x) = \frac{k}{\infty} = 0, \; k \in \mathbb{R}$

5. $\lim_{x \to \infty} f(x) = \frac{0}{\infty}$ imposible (undefined)

6. $\lim_{x \to \infty} f(x) = \frac{\infty}{\infty}$ imposible (undefined)

١- اذا كانت درجة المقام اكبر من البسط = ٠

٢- درجة البسط اكبر = ∞

٣- متساوية = معامل اكبر اس بالبسط / معامل اكبر اس مقام

Examples: Find the limits for the following functions.

1. $\lim_{x \to \infty} \left(-4 + \frac{1}{x}\right) = -4 + \frac{1}{\infty} = -4$

2. $\lim_{x \to \infty} \frac{3}{x^2} = \frac{3}{\infty} = 0$

3. $\lim_{x \to \infty} \frac{x}{7x+4} = \frac{\infty}{7(\infty)+4} = \frac{\infty}{\infty}$ (كمية غير معرفة دينامكية)

$$\Rightarrow \lim_{x \to \infty} \frac{x}{7x+4} = \lim_{x \to \infty} \frac{\frac{x}{x}}{\frac{7x}{x} + \frac{4}{x}} = \frac{1}{7 + \frac{4}{\infty}} = \boxed{\frac{1}{7}}$$

4. $\lim_{x \to \infty} \frac{5x+2}{2x^2+1} = \frac{\infty}{\infty}$ لاينكر

$$\Rightarrow \lim_{x \to \infty} \frac{5x+2}{2x^2+1} = \lim_{x \to \infty} \frac{\frac{5x}{x^2} + \frac{2}{x^2}}{\frac{2x^2}{x^2} + \frac{1}{x^2}} = \lim_{x \to \infty} \frac{\frac{5}{x} + \frac{2}{x^2}}{2 + \frac{1}{x^2}}$$

$$= \frac{\frac{5}{\infty} + \frac{2}{(\infty)^2}}{2 + \frac{1}{(\infty)^2}} = \frac{0+0}{2+0} = \frac{0}{2} = 0$$

③ Is the function $f(x) = \begin{cases} \frac{x^2-4}{x-2} & , x \neq 2 \\ 4 & , x = 2 \end{cases}$

continuous? Explain.

Sol: at $(a = 2)$

1. $f(2) = 4$

2. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{x^2-4}{x-2}$
$x \to 2^-$

$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{x-2}}$

$= \lim_{x \to 2} (x+2) = 2+2 = 4$

$\therefore \lim_{x \to 2} f(x) = 4$

3. $f(2) = \lim_{x \to 2} f(x)$

$\therefore f(x)$ is continuous at $a = 2$.

H.W: Is the following functions continuous or discontinuous? Explain your answer.

① $f(x) = \begin{cases} x^2+5 & , x \geq -1 \\ 6x & , x < -1 \end{cases}$

② $f(x) = \begin{cases} 2x+5 & , x > 3 \\ x^2+2 & ; x \leq 3 \end{cases}$

② Is the function $f(x) = \begin{cases} 4x-2 & , x > 2 \\ 2 & , x = 2 \\ 3x & , x < 2 \end{cases}$

continuous? Explain.

Sol: at $(a = 2)$

1. $f(2) = 2$

2. $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (4x-2) = 4(2)-2 = 6$

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3x) = 3(2) = 6$

$\therefore \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x) \Rightarrow \lim\limits_{x \to 2} f(x) = 6$ exists

3. $f(2) \neq \lim\limits_{x \to 2} f(x)$

$2 \neq 6$

$\therefore f(x)$ is not continuous at $a = 2$.
(discontinuous)

"Countinuous Functions"

الدوال المتصلة

Definition: A function f is said to be continuous at a if the following conditions are satisfied:

1. $f(a)$ defined
2. $\lim\limits_{x \to a} f(x)$ exists
3. $f(a) = \lim\limits_{x \to a} f(x)$.

Con. $\left\{ \begin{array}{l} \text{① موجود} \\ \text{② غاية} \\ \text{③ موجود = غاية} \end{array} \right.$

Examples:

① Is the function $f(x) = \begin{cases} 2x+3 & , x \leq 4 \\ 7 + \frac{16}{x} & , x > 4 \end{cases}$ continuous? Explain.

Sol. at $(a = 4)$

1. $f(4) = 2(4) + 3 = 8 + 3 = 11$

2. $\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^+} \left(7 + \frac{16}{x}\right) = 7 + \frac{16}{4} = 7 + 4 = 11$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4^-} (2x+3) = 2(4) + 3 = 8 + 3 = 11$

$\therefore \lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4^-} f(x) \Rightarrow \lim\limits_{x \to 4} f(x) = 11$ exits

3. $f(4) = \lim\limits_{x \to 4} f(x)$

$\therefore f(x)$ is continuous at $a = 4$.

② Find the equation of the line passing through the point $P(-2, 3)$, with slope 9.

Sol:

$$M = \frac{y - y_1}{x - x_1}$$

$$9 = \frac{y - 3}{x + 2}$$

$$y - 3 = 9(x + 2)$$

$$y - 3 = 9x + 18$$

$$y = 9x + 18 + 3$$

$$y = 9x + 21$$

$$\boxed{y = mx + c}$$

$$y = 9x + c$$

$$3 = 9(-2) + c$$

$$\Rightarrow \boxed{c = 21}$$

$$y = 9x + 21$$

③ $P(3, 4)$, slope $\frac{1}{2}$.

Sol:

$$M = \frac{y - y_1}{x - x_1}$$

$$\frac{1}{2} = \frac{y - 4}{x - 3}$$

$$2(y - 4) = x - 3$$

$$2y - 8 = x - 3$$

$$2y = x - 3 + 8$$

$$2y = x + 5$$

$$y = \frac{1}{2}x + \frac{5}{2}.$$

$$y = \frac{1}{2}x + b$$

$$4 = \frac{1}{2}(3) + b$$

$$b =$$

Equation of Line: معادلة الخط المستقيم

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y - y_1}{x - x_1}$$

$$M = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\boxed{M = \frac{y - y_1}{x - x_1}} \longrightarrow (1)$$

$$\boxed{y = mx + c}$$

$$y - y_1 = M(x - x_1)$$

$$y - y_1 = Mx - Mx_1$$

$$y - Mx + (Mx_1 - y_1) = 0$$

ميل

نقطة

$$\boxed{y = Mx + b} \longrightarrow (2)$$

Examples: ① Find the equation of the line passing through the point $P_1(2,5)$ with slope $\frac{3}{4}$.

Sol:

$$M = \frac{y - y_1}{x - x_1}$$

$$M = \frac{3}{4}$$

$$\frac{3}{4} = \frac{y - 5}{x - 2}$$

$$P(\overset{x}{2}, \overset{y}{5})$$

$$4(y - 5) = 3(x - 2)$$

$$4y - 20 = 3x - 6$$

$$4y = 3x - 6 + 20$$

$$4y = 3x + 14$$

$$y = \frac{3}{4}x + \frac{14}{4}$$

$$y = \frac{3}{4}x + \frac{7}{2}$$

$$y = mx + c$$

$$\boxed{y = \frac{3}{4}x + c}$$

$$5 = \frac{3}{4}(2) + c$$

$$5 = \frac{3}{2} + c$$

$$c = 5 - \frac{3}{2} = \boxed{\frac{7}{2}}$$

"Slope of Line"

ميل الخط

Def: If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are the end points of nonvertical line segment, then the slope M of the line segment defined by:

$$\boxed{M = \frac{y_2 - y_1}{x_2 - x_1}}$$

Examples: Find the slope of the line passing through the points:

$x_1\ y_1 \qquad x_2\ y_2$

① $P_1(11, 3)$ and $P_2(14, 7)$

② $P_1(3, 6)$ and $P_2(-5, -9)$ $\qquad M = \dfrac{y_2 - y_1}{x_2 - x_1}$

③ $P_1(4, 5)$ and $P_2(8, 13)$

Sol.

1. $\boxed{M = \dfrac{y_2 - y_1}{x_2 - x_1}} = \dfrac{7 - 3}{14 - 11} = \dfrac{4}{3} = 1.3$

2. $M = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-9 - 6}{-5 - 3} = \dfrac{-15}{-8} = \dfrac{15}{8} = 1.87$

3. $M = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{13 - 5}{8 - 4} = \dfrac{8}{4} = 2$

"Higher Derivatives"

$\overset{=}{y}$

1. $y' = \frac{dy}{dx}$ first derivative
2. $y'' = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ second derivative
3. $y''' = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ third derivative

In general to find n-derivative

$$y^{(n)} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n}.$$

Examples:

Let $y = 3x^4 - 2x^3 + x^2 - 4x + 2$. Find $y^{(5)}$ (or $\frac{d^5y}{dx^5}$).

Sol:

$y' = 12x^3 - 6x^2 + 2x - 4$

$y'' = 36x^2 - 12x + 2$

$y''' = 72x - 12$

$y^{(4)} = 72$

$y^{(5)} = 0$

y'

y'''

Show that $y = x^3 + 3x + 1$ satisfies the equation $y''' + xy'' - 2y' = 0$.

Sol:

$y = x^3 + 3x + 1$

$y' = 3x^2 + 3$

$y'' = 6x$

$y''' = 6$

1 $y' = 3x^2 + 3$

2 $y'' = 6x$

3 $y''' = 6$

نعوض عن y ، y' ، y'' ، y''' نحصل على

$\boxed{y''' + xy'' - 2y'} = 6 + x(6x) - 2(3x^2 + 3)$

$= \cancel{6} + \cancel{6x^2} - \cancel{6x^2} - \cancel{6} = 0.$

Examples: Find $\frac{dy}{dx}$ for the following function

① $y = 2x^3 + \sqrt{5x^2+2}$

Sol.

$y' = 6x^2 + \frac{1}{2}(5x^2+2)^{-\frac{1}{2}} \cdot 10x$

$= 6x^2 + \frac{10x}{2\sqrt{5x^2+2}}$

$= 6x^2 + \frac{5x}{\sqrt{5x^2+2}}$

② $y = (4x^2-1)(7x^3+x)$

Sol.

$y' = (4x^2-1)\cdot(21x^2+1) + (7x^3+x)\cdot(8x)$

$= 140x^4 - 9x^2 - 1$

③ $y = \frac{x^2-1}{x^4+1}$

Sol.

$y' = \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$

$= \frac{2x^5 + 2x - 4x^5 + 4x^3}{(x^4+1)^2}$

$= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2} = \frac{-2x(x^4 - 4x^2 - 1)}{(x^4+1)^2}$

④ $y = (3x^5 + 2x + \frac{1}{x})^3$

Sol.

$y' = 3(3x^5 + 2x + \frac{1}{x})^2 \cdot (15x^4 + 2 - \frac{1}{x^2})$

$+ x^{-1}$

"Derivatives"

Def: The function f' defined by the formula

$$f'(x) = \lim_{a \to 0} \frac{f(x+a) - f(x)}{a}$$

is called the derivative of f with respect to x. when the limit exist, f is said to be differentiable.

Remark: The derivative of the function f (or y) w.r.t. x is denoted by

$$f'(x) = \frac{d f(x)}{dx} , \quad y' = \frac{dy}{dx}$$

Differentiation Rules: Let f, g be a differentiable functions at x, and c be any constant, then:

1. $\frac{dx}{dx} = 1$

2. $\frac{d}{dx} c = 0$

3. $\frac{d}{dx} (x^n) = n x^{n-1}$

4. $\frac{d}{dx} (f(x))^n = n (f(x))^{n-1} \cdot f'(x)$.

5. $\frac{d}{dx} [c f(x)] = c f'(x)$

6. $\frac{d}{dx} [f(x) \mp g(x)] = f'(x) \mp g'(x)$

7. $\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

8. $\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$

"Trigonometric Functions"

الدوال المثلثية

① $y = \sin\theta$ ② $y = \cos\theta$ ③ $y = \tan\theta$

④ $y = \sec\theta$ ⑤ $y = \csc\theta$ ⑥ $y = \cot\theta$.

Some Relation:

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

2. $\sec\theta = \dfrac{1}{\cos\theta}$

3. $\csc\theta = \dfrac{1}{\sin\theta}$

4. $\sin^2\theta + \cos^2\theta = 1$

5. $1 + \cot^2\theta = \csc^2\theta$ (by devided (1) by $\sin^2\theta$).

6. $\tan^2\theta + 1 = \sec^2\theta$

7. $\cos(A \mp B) = \cos A \cos B \pm \sin A \sin B$

8. $\sin(A - B) = \sin A \cos B - \sin B \cos A$

9. $\cos(A+B) = \cos A \cos B - \sin A \sin B$

10. $\sin(A+B) = \sin A \cos B + \sin B \cos A$

11. $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

12. $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

6. $\log_a(bc) = \log_a b + \log_a c$
7. $\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$
8. $\log_a\left(\frac{1}{c}\right) = -\log_a c$
9. $\log_a b^n = n \log_a b$
10. $\log_a 1 = 0$
11. $\log_a a = 1$

ملاحظة: عندما لا يكتب الأساس a في $\log u(x)$ فذلك يعني أن $a = 10$.

"Derivative of Exponential and Logarithum Functions"

$$\frac{d}{dx}\left[a^{u(x)}\right] = a^{u(x)} \cdot \ln a \cdot \frac{du}{dx}$$

الدالة نفسها   اللوغاريتم   مشتقة الأس

If $a = e$

$$\frac{d}{dx}\left[e^{u(x)}\right] = e^{u(x)} \cdot \frac{du}{dx} \quad (\text{Since } \ln e = 1)$$

$$\frac{d}{dx}\left[\ln u(x)\right] = \frac{1}{u(x)} \cdot \frac{du}{dx}, \left(\text{i.e. } \frac{u'(x)}{u(x)}\right).$$

Examples: Find $\frac{dy}{dx}$, $(= y')$.

$y = 5^{x^2} \Rightarrow y' = 5^{x^2} \cdot \ln 5 \cdot 2x = 2x\, 5^{x^2} \ln 5.$

$y = e^{4x^2} \Rightarrow y' = e^{4x^2} \cdot 8x = 8x\, e^{4x^2}.$

$y = \ln 2x^3 \Rightarrow y' = \frac{1}{2x^3} \cdot 6x^2 = \frac{3}{x}.$

$y = e^{2x} + \ln(5x^2 + 7x + 1)$

$y' = e^{2x} \cdot 2 + \frac{1}{5x^2 + 7x + 1} \cdot (10x + 7)$

$= 2e^{2x} + \frac{10x + 7}{5x^2 + 7x + 1}.$

H.W.
$y = e^{3x^3} \ln(x^4 + 4).$

Idea

"Exponential Functions"

الدوال الأسية

Def. (1): A function of the form $\boxed{y = a^{u(x)}}$ $0 < a \neq 1$ is called an exponential function with bassis a.

$y = e^{u(x)}$

Def. (2): (The Natural Exponential Function) الدالة الأسية الطبيعية

If $a = e$ in Def (1) $\Rightarrow y = e^{u(x)}$ is called natural exponential function.

من الدوال الأسية المهمة هي الدالة e وتسمى الدالة الأسية الطبيعية

$e = 2.7182828 \simeq 2.7$ وهي قيمة ثابتة

$\log_a u(x)$ $\quad \log_e u(x) = \ln u(x)$

Def. (3): (Logarithum Function) الدالة اللوغاريتمية

A function of the form $y = \log_a u(x)$ is called logarithum function of $u(x)$ with bassis a whire $0 < a \neq 1$.

Def. (4): (The Natural Logarithum Function)

If $a = e$ in Def. (3) $\Rightarrow y = \log_e u(x) = \ln u$ is called natural logarithum function.

Remarks:

1. $\ln e = 1$
2. $\ln e^{u(x)} = u(x)$
3. $e^{\ln u(x)} = u(x)$
4. $\ln 1 = 0$
5. $e^0 = 1$

الدالة الأسية $\quad y = a^{u(x)}$

الدالة الطبيعية $\quad y = e^{u(x)}$

الدالة اللوغاريتمية $\quad \log_a u(x)$

$a = e \quad \ln u$