Probability and Statistics

The First Lecture

Introduction

Most people become familiar with probability and statistics through radio, television, newspapers, and magazines. For example, the following statements were found in newspapers.

1. Nearly one in seven U.S. families are struggling with bills from medical expenses even though they have health insurance.
2. Eating 10 grams of fiber a day reduces the risk of heart attack by 14%.
3. The average credit card debt per household in 2003 was $9205.
4. About 15% of men in the United States are left handed and 9% of women are left handed.
5. The median age of couples who watch jay leno is 48.1 years.

Statistics is used in almost all fields of human endeavor. In sports, for example, a statistician may keep records of the number of yards a running back gains during a football game, or the number of hits a baseball player gets in a season. In other areas, such as public health, an administrator might be concerned with the number of residents who contract a new strain of flu virus during a certain year. In education, a researcher might want to know if new methods of teaching are bitter than old ones. These are only a few examples of how statistics can be used in various occupations.

Furthermore, statistics is used to analyze the results of surveys and as a tool in scientific research to make decisions based on controlled experiments. Other uses of statistics include operations research, quality control, estimation, and prediction.

Probability as a general concept can be defined as the chance of an event occurring.

Probability experiment is a chance process that leads to well-defined results called outcomes.

Outcome is the result of a single trial of a probability experiment.

Sample space is the set all possible outcomes of probability experiment and denoted by (Ω)**.**

Example

Toss one coin

Ω ={Head, Tail}

Example

Roll a die

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Example

Answer a true, false question

$$\Omega = \{\text{true}, \text{false}\}$$

Example

Toss two coins

$$\Omega = \{\text{head head}, \text{tail tail}, \text{head tail}, \text{tail head}\}$$

Example

Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.

$$\Omega = \{BBB, BBG, BGB, GBB, GGG, GGB, GBG, BGG\}$$

An ***event*** consists of a set of outcomes of a probability experiment.

Equally likely events are events that have the same probability of occurring.

Formula for classical probability

The probability of any event E is $\frac{Number\ of\ outcomes\ in\ E}{Total\ number\ of\ outcomes\ in\ the\ sample\ space}$ this probability is denoted by $P(E) = \frac{n(E)}{n(\Omega)}$ .This probability is called classical probability, and it uses the sample space Ω.

Example

If a family has three children, find the probability that all the children are girls.

Solution

The sample space of the gender of children for a family that has three children is $\Omega = \{BBB, BBG, BGB, GBB, GGG, GGB, GBG, BGG\}$. Since there is one way in eight possibilities for all three children by girls, $P(GGG) = \frac{1}{8}$.

There are four basic probability rules. These rules are helpful in solving probability problems, in understanding the nature of probability, and in deciding if your answers to the problems are correct.

Probability rule1

The probability of any event E is a number (either a fraction or decimal) between and including $0\ and\ 1$. This is denoted by $0 \leq P(E) \leq 1$.

Rule1 states that probability cannot be negative or greater than 1.

Probability rule2

If an event E cannot occur (i.e. the event contains no members in the sample space), its probability is 0.

The second lecture

Example

When a single die is rolled, find the probability of getting a 9.

Solution

Since the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$ it is impossible to get a 9. Hence, the probability is $P(9) = \frac{0}{6} = 0$.

Probability rule3

If an event E is certain, then the probability of E is 1.

Example

When a single die is rolled, what is the probability of getting a number less than 7?

Solution

Since all outcomes $\Omega = \{1, 2, 3, 4, 5, 6\}$ an less than 7, the probability is $p(number\ less\ than\ 7) = \frac{6}{6} = 1$.

The event of getting a number less than 7 is certain.

Probability rule4

The sum of the probability of all the outcomes in the sample space is 1.

Example

In the roll of a fair die, each outcome in the sample space has a probability of $\frac{1}{6}$. Hence, the sum of probabilities of the outcomes is as shown:

Outcomes	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$, $sum = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$.

Complement of an event E is the set of outcomes in the sample space that are not included in the outcomes of event E. The complement of E is denoted by E^C.

Example

Find complement of each event

a. Rolling a die and getting a 4.
b. Selecting a letter of the alphabet and getting a vowel.
c. Selecting a month and getting a month that begins with a j.
d. Selecting a day of the week and getting a weekday.

Solution

a. Getting a $1, 2, 3, 5, or\ 6$.
b. Getting a consonant (assume y is a consonant).
c. Getting February, March, April, May, August, September, October, November, or December.
d. Getting Saturday or Sunday.

Rule for complementary events

$$\mathrm{p(E)} + \mathrm{p}(E^C) = 1$$

Example

If the probability that a person lives in an industrialized country of the world is $\frac{1}{5}$, find the probability that a person does not live in an industrialized country.

Solution

Let E =the event of the person lives in an industrialized country

E^C =the event of the person not lives in an industrialized country

Hence, $p(E) + p(E^C) = 1 \rightarrow p(E^C) = 1 - p(E) = 1 - \frac{1}{5} = \frac{4}{5}$

Mutually exclusive events

Two events are mutually exclusive event if they cannot occur at the same time (i.e. they have no outcomes in common).

Example

Determine which events are mutually exclusive and which are not, when a single die is rolled.

a. Getting an odd number and getting an even number.
b. Getting a 3 and getting an odd number.
c. Getting an odd number and getting a number less than 4.
d. Getting a number greater than 4 and getting a number less than 4.

Solution

a. The events are mutually exclusive, since the first event can be 1, 3, or 5 and the second event can be 2, 4, or 6.
b. The events are not mutually exclusive, since the first event is a 3 and the second can be 1, 3, or 5. Hence, 3 is contained in both events.
c. The events are not mutually exclusive, since the first event can be 1, 3, or 5 and the second can be 1, 2, or 3. Hence, 1 and 3 are contained in both events.
d. The events are mutually exclusive, since the first event can be 5 or 6 and the second event can be 1, 2, or 3.

Example

Determine which events are mutually exclusive and which are not, when a single card is drawn from a deck.

Heart red Diamond red Club black Spade black

For each type of these there exist Ace, 2, 3, 4,…, 10, <u>J(Jack), Q(Queen), K(king)</u>
Pictures

a. Getting a 7 and getting a jack.

b. Getting a club and getting a king.
c. Getting a face card and getting an ace.
d. Getting a face card and getting a spade.

Solution

a. The events are mutually exclusive, since the first event can be a 7 and the second event can be a jack.
b. The events are not mutually exclusive, since the first event can be a club and the second event can be a king.
c. The events are mutually exclusive, since the first event can be a face card and the second event can be an ace.
d. The events are not mutually exclusive, since the first event can be a face card and the second event can be a spade.

The Third lecture

Addition rule1

When two events A and B are mutually exclusive, the probability that A or B will occur is

$$p(A\ or\ B) = p(A) + p(B)$$

Example

At a political rally, there are 20 Republicans, 13 Democrats, and 6 Independents. If a person is selected at random, find the probability that he or she is either a Democrat or an Independent.

Solution

Let $A = the\ event\ of\ Republicans$

$B = the\ event\ of\ Democrats$

$C = the\ event\ of\ Independents$

$$p(B\ or\ C) = p(B) + p(C) = \frac{13}{39} + \frac{6}{39} = \frac{19}{39}$$

Example

A day of the week is selected at random. Find the probability that it is a weekend day.

Solution

Let $A = the\ event\ of\ a\ saturday$

$B = the\ event\ of\ a\ sunday$

$$p(A\ or\ B) = p(A) + p(B) = \frac{1}{7} + \frac{1}{7} = \frac{2}{7}$$

Addition rule2

If A and B are not mutually exclusive, then

$$p(A\ or\ B) = p(A) + p(B) - p(A\ and\ B)$$

Note: *This rule can also be used when the events are mutually exclusive, since $p(A\ and\ B)$ will always equal 0.*

Example

In a hospital unit there are 8 nurses and 5 physicians, 7 nurses and 3 physicians are females. If a staff person is selected, find the probability that the subject is a nurse or a male.

Solution

The sample space is shown here

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

The probability is

$$p(nurse\ or\ male) = p(nurse) + p(male) - p(nurse\ and\ male)$$

$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \frac{10}{13}$$

Note: *The probability rule can be extended to three or more events. For three mutually exclusive events* $A, B, and\ C,$

$$p(A\ or\ B\ or\ C) = p(A) + p(B) + p(C)$$

For three events that are not mutually exclusive,

$$p(A\ or\ B\ or\ C) = p(A) + p(B) + p(C) - p(A\ and\ B) - p(A\ and\ C) - p(B\ and\ C) + p(A\ and\ B\ and\ C)$$

Problems

1. Determine whether these events are mutually exclusive.
   a. Roll a die: get an even number, and get number less than 3.
   b. Roll a die: get a prime number (2, 3, 5), and get an odd number.
   c. Roll a die: get a number greater than 3, and get a number less than 3.
   d. Select a student in your class: the student has blond hair, and the student has blue eyes.
2. In a statistics class there are 18 Juniors and 10 Seniors, 6 of the Seniors are females, and 12 of the Juniors are males. If a student is selected at random, find the probability of selecting the following
   a. A Junior or a female.
   b. A Senior or a female.
   c. A junior or Senior.
3. A recent study of 200 nurses found that of 125 female nurses, 56 had bachelor's degrees, and of 75 male nurses, 34 had bachelor's degrees. If a nurse is selected at random, find the probability that the nurse is
   a. A female nurse with a bachelor's degree.
   b. A male nurse.
   c. A male nurse with a bachelor's degree.

d. Based on your answers to parts a, b, and, c, explain which is most likely to occur. Explain why.

Example

Two dice are rolled. Find the probability of getting

a. A sum of 5, 6, or 7.
b. Doubles or a sum of 6 or 8.
c. A sum greater than 8 or less than 3.
d. Based on the answers to parts a, b, and c, which is least likely to occur? Explain why.

Solution

The sample space of two dice are rolled is

$$\begin{aligned}\Omega = \{&(1,1),(1,2),(1,3),(\underline{1,4}),(1,5),(1,6),\\ &(2,1),(2,2),(\underline{2,3}),(2,4),(2,5),(2,6),\\ &(3,1),(\underline{3,2}),(3,3),(3,4),(3,5),(3,6),\\ &(\underline{4,1}),(4,2),(4,3),(4,4),(4,5),(4,6),\\ &(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),\\ &(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}\end{aligned}$$

a. Let $A = the\ sum\ of\ 5,\quad B = the\ sum\ of\ 6,\quad C = the\ sum\ of\ 7$

$$p(A) = \frac{4}{36} = \frac{1}{9} = 0.11, p(B) = \frac{5}{36} = 0.14, p(C) = \frac{6}{36} = \frac{1}{6} = 0.17$$

b. Let $A = the\ event\ of\ doubles, B = the\ event\ of\ sum\ of\ 6,$

$C = the\ event\ of\ sum\ of\ 8$

$$p(A\ or\ B\ or\ C) = p(A) + p(B) + p(C) - p(AB) - p(AC)$$
$$-p(BC) + p(ABC)$$
$$= \frac{6}{36} + \frac{5}{36} + \frac{5}{36} - \frac{1}{36} - \frac{1}{36} - 0 + 0 = \frac{14}{36} = \frac{7}{18} = 0.38$$

c. Let $A = the\ sum\ greater\ than\ 8,\ B = the\ sum\ less\ than\ 3$

$$p(A\ or\ B) = p(A) + p(B) - p(AB) = \frac{10}{36} + \frac{1}{36} - 0 = \frac{11}{36} = 0.305$$

d. The event in part a is least likely to occur since it has the lowest probability of occurring.

The Fourth Lecture

Example

If one card is drawn from an ordinary deck of cards, find the probability of getting the following.

a. A king or a queen or a jack
b. A club or a heart or a spade
c. A king or a queen or a diamond
d. An ace or a diamond or a heart

Solution

Let $k = the\ event\ of\ king\ drawn,\ q = the\ event\ of\ queen\ drawn,$
$j = the\ event\ of\ jack\ drawn$

a. $p(k\ or\ q\ or\ j) = p(k) + p(q) + p(j) - p(kq) - p(kj) - p(qj) + p(kqj)$

$$= \frac{4}{52} + \frac{4}{52} + \frac{4}{52} - 0 - 0 - 0 + 0 = \frac{3}{13}$$

Let $C = the\ event\ of\ club\ drawn,\ h = the\ event\ of\ heart\ drawn,$

$s = the\ event\ of\ spade\ drawn$

b. $p(C\ or\ h\ or\ s) = p(C) + p(h) + p(s) - p(Ch) - p(Cs) - p(hs) + p(Chs)$

$$= \frac{13}{52} + \frac{13}{52} + \frac{13}{52} - 0 - 0 - 0 + 0 = \frac{3}{4}$$

Let $D = the\ event\ of\ diamond\ drawn$

c. $p(k\ or\ q\ or\ D) = p(k) + p(q) + p(D) - p(kq) - p(kD) - (qD) + p(kqD)$

$$= \frac{4}{52} + \frac{4}{52} + \frac{13}{52} - 0 - \frac{1}{52} - \frac{1}{52} + 0 = \frac{19}{52}$$

Let $a = the\ event\ of\ an\ ace\ drawn,$

d. $p(a\ or\ D\ or\ h) = p(a) + p(D) + p(h) - p(aD) - p(ah) - (Dh) + p(aDh)$

$$= \frac{4}{52} + \frac{13}{52} + \frac{13}{52} - \frac{1}{52} - \frac{1}{52} - 0 + 0 = \frac{28}{52} = \frac{7}{13}$$

Multiplication rule and conditional probability

Multiplication rules

The multiplication rules can be used to find the probability of two or more events that occur in sequence. For example, if a coin is tossed and then a die is rolled, one can find the probability of getting a head on the coin and a 4 on the die. These two events are said to be independent since the outcome of the first event (tossing a coin) does not affect the probability outcome of the second event (rolling a die).

Independent events

Two events $A\ and\ B$ are ***independent events*** if the fact that A occurs does not affect the probability of B occurring.

Multiplication rule1

When two events are independent, the probability of both occurring is

$p(A\ and\ B) = p(A)\ .\ p(B)$

Example

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Solution

Let $A = the\ event\ of\ \ getting\ a\ head\ on\ the\ coin.$

$B = the\ event\ of\ \ getting\ 4\ on\ the\ die.$

$$p(A\ and\ B) = p(A).p(B) = \frac{1}{2}.\frac{1}{6} = \frac{1}{12}$$

Note that the sample space for the coin is H, T, and for the die it is 1, 2, 3, 4, 5, 6.

Example

A card is drawn from a deck and replaced, then a second card is drawn. Find the probability of getting a queen and then an ace.

Solution

The probability of getting a queen is $\frac{4}{52}$, and since the card is replaced, the probability of getting an ace $\frac{4}{52}$. Hence, the probability of getting a queen and an ace is

$$p(queen\ and\ ace) = p(queen).p(ace) = \frac{4}{52}.\frac{4}{52} = \frac{16}{2704} = \frac{1}{169}$$

Example

An urn contains 3 red balls, 2 blue balls, and 5 white balls. A ball is selected and its color noted. Then it is replaced. A second ball is selected and its color noted. Find the probability of each of these.

a. Selecting 2 blue balls.
b. Selecting 1 blue ball and then 1 white ball.
c. Selecting 1 red ball and then 1 blue ball.

Solution

Let $B = the\ event\ of\ select\ blue\ ball.$
$w = the\ event\ of\ select\ white\ ball.$
$R = the\ event\ of\ select\ red\ ball.$

a. $p(B.B) = p(B).p(B) = \frac{2}{10}.\frac{2}{10} = \frac{4}{100} = \frac{1}{25}$
b. $p(B.w) = p(B).p(w) = \frac{2}{10}.\frac{5}{10} = \frac{10}{100} = \frac{1}{10}$
c. $p(R.B) = p(R).p(B) = \frac{3}{10}.\frac{2}{10} = \frac{6}{100} = \frac{3}{50}$

Note: *multiplication rule1can be extended to three or more independent events by using the formula*

$p(A \text{ and } B \text{ and } C \text{ and } \dots \text{ and } k) = p(A).p(B).p(C).\dots p(k)$

Example

A Harris poll found that 64% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say they suffer great stress at least once a week.

Solution

Let S denote stress. Then

$p(S \text{ and } S \text{ and } S) = p(S).p(S).p(S) = (0.64)(0.64)(0.64) \approx 0.097$

Fifth Lecture

Dependent events

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be dependent events.

Here are some example of dependent events:

1. Drawing a card from a deck, not replacing it, and then drawing a second card.
2. Selecting a ball from an urn, not replacing it, and then selecting a second ball.

To find the probabilities when events are dependent, use the multiplication rule with a modification in notation, for problem just discussed, the probability of getting an ace on the first draw is $\frac{4}{52}$, and the probability of getting a king on the second draw is $\frac{4}{51}$. By the multiplication rule, the probability of both events occurring is $\frac{4}{52} \cdot \frac{4}{51} = \frac{4}{663}$.

The event of getting a king on the second draw given that an ace was drawn the first time is called a conditional probability.

The conditional probability of an event B in relationship to event A is the probability that event B occurs after event A has already occurred. The notation for conditional probability is $p(B\backslash A)$. This notation does not mean that B is divided by A, rather, it mean the probability that event B occurs given that event A has already occurred. In the card example, $p(B\backslash A)$ is the probability that the second card is a king given that the first card is an ace, and it is equal to $\frac{4}{51}$ since the first card was not replaced.

Multiplication rule2

When two events are dependent, the probability of both occurring is

$p(A\ and\ B) = p(A).p(B\backslash A)$

Example

A person owns a collection of 30 CDs, of which 5 are country music. If 2 CDs are selected at random, find the probability that both are country music.

Solution

Since the event are dependent.

$$p(c_1 and\ c_2) = p(c_1).p(c_2\backslash c_1) = \frac{5}{30}.\frac{4}{29} = \frac{20}{870} = \frac{2}{87}$$

Example

Three cards drawn from an ordinary deck not replaced. Find the probability of these.

a. Getting 3 jacks.
b. Getting an ace, a king, and a queen in order.
c. Getting a club, a spade, and a heart in order.
d. Getting 3 clubs.

Solution

a. $p(3\ jacks) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{24}{132{,}600} = \frac{1}{5525}$

b. $p(ace\ and\ king\ and\ queen) = \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{64}{132{,}600} = \frac{8}{16{,}575}$

c. $p(club\ and\ spade\ and\ heart) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} = \frac{2197}{132{,}600} = \frac{169}{10{,}200}$

d. $p(3\ clubs) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{1716}{132{,}600} = \frac{11}{850}$

Conditional probability

The conditional probability of an event B in relationship to an event A was defined as the probability that event B occurs after event A has already occurred. The conditional probability of an event can found by dividing both sides of the equation for multiplication rule2 by $p(A)$, as shown:

$$p(A\ and\ B) = p(A).\,p(B\backslash A)$$

$$\frac{p(A\ and\ B)}{p(A)} = \frac{p(A).\,p(B\backslash A)}{p(A)} = \frac{p(A\ and\ B)}{p(A)} = p(B\backslash A)$$

Example

A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

Find the probability of the following

a. The respondent answered yes, given that the respondent was a female.
b. The respondent was a male, given that the respondent answered no.

Solution

Let $M = respondent\ was\ male.$

$F = respondent\ was\ a\ female.$

$Y = respondent\ answered\ yes.$

$N = respondent\ answerd\ no.$

a. $p(Y \backslash F) = \frac{p(F\ and\ Y)}{p(F)} = \frac{8/100}{50/100} = \frac{4}{25}$

b. $p(M \backslash N) = \frac{p(N\ and\ M)}{p(N)} = \frac{18/100}{60/100} = \frac{3}{10}$

Problems

1. The medal distribution from the 2000 Summer Olympic Games is shown in the table.

	Gold	Silver	Bronze
United States	39	25	33
Russia	32	28	28
China	28	16	15
Australia	16	25	17
others	186	205	235

Choose one medal winner at random.

a. Find the probability that the winner won the gold medal, given that the winner was from the United States.
b. Find the probability that the winner was from the United States, given that she or he won a gold medal.
c. Are the events " $medal\ winner\ is\ from\ United\ States$ " and " $gold\ medal\ was\ won$ " independent? Explain.

2. Eighty students in a school cafeteria were asked if they favored a ban on smoking in the cafeteria. The results of the survey are shown in the table.

Class	Favor	Oppose	No opinion
Freshman	15	27	8
Sophomore	23	5	2

If a student is selected at random, find these probabilities.

a. Given that the student is a freshman, he or she opposes the ban.

b. Given that the student favors the ban, the student is a sophomore.

Sixth Lecture

Counting rules

Many times one wishes to know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used, the fundamental counting rule, the permutation rule, and the combination rule.

The fundamental counting rule

In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \,.\, k_2 \,.\, k_3 \,.\, \ldots \, k_n$$

Example

A coin is tossed and a die is rolled. Find the number of outcomes for the sequence of events.

Solution

Since the coin can land either heads up or tails up and since the die can land with any one of six numbers showing face up, there are $2.6 = 12\ possibilities.$

Example

A paint manufacturer wishes to manufacture several different paints. The categories include

Color	red, blue, white, black, green, brawn, yellow
Type	latex, oil
Texture	flat, semi gloss, high gloss

Use outdoor, indoor

How many different kinds of paint can be made if a person can select one color, one type, one texture, and one use?

Solution

A person can choose one color and one type and one texture and one use. Since there are 7 color choices, 2 type choices, 3 texture choices, and 2 use choices, the total number of possible different paints is

Color	Type	Texture	Use	
7 .	2 .	3 .	2	= 84

Example (H.W)

There are four blood types, $A, B, AB, and\ O$. Blood can also be $Rh_{+}\ and\ Rh_{-}$. Finally a blood donor can be classified as either male or female. How many different mays can donor have his or her blood labeled?

Factorial notation

The factorial notation uses the exclamation point.

$$5! = 5.4.3.2.1$$

$$9! = 9.8.7.6.5.4.3.2.1$$

To use the formulas in the permutation and combination rules, a special definition of 0! is needed $0! = 1$

Factorial formulas

For any counting n

$n! = n\,(n-1)(n-2)\ldots 1$

$0! = 1$

Permutations

A permutation is an arrangement of n objects in a specific order.

Example

Suppose a business owner has a choice of five locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the five locations?

Solution

There are

$5! = 5 . 4 . 3 . 2 . 1 = 120$

different possible rankings. The reason is that she has 5 choices for the first location, 4 choices for second location, 3 choices for the third location, etc. in this example all objects were used up. But what happens when not all objects are used up?

Example

Suppose the business owner in the previous example wishes to rank only the top three of the five locations. How many different ways can she rank them?

Solution

Using the fundamental counting rule, she can select any one of the five for first choice, then any one of the remaining four locations for her second choices and finally, any one of the remaining locations for her third choice, as shown

First choice	Second choice	Third choice	
5	4	3	= 60

Seventh Lecture

Permutation rule

The arrangement of n objects in a specific order using r objects at time is called a permutation of n objects taking r objects at time. It is written as $n\,p_r$, and the formula is

$$n\,p_r = \frac{n!}{(n-r)!}$$

Note: The notation $n\,p_r$ is used for permutations.

$$6\,p_4 \ means \ \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6.5.4.3.2.1}{2.1} = 360$$

$$5\,p_5 \ means \ \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5.4.3.2.1}{1} = 120$$

Example

A television news director wishes to use three news stories on an evening show. One story will be the lead story, one will be the second story, and the last will be a closing story. If the director has a total eight stories to choose form, how many possible ways can the program be set up?

Solution

Since order is important, the solution is

$$8\,p_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8.7.6.5!}{5!} = 336$$

Hence, there would be 336 ways to set up the program.

Example

How many different ways can a chairperson and an assistant chairperson be selected for a research project if there are seven scientists available?

Solution

$$7\,p_2 = \frac{7!}{(7-2)!} = \frac{7!}{5!} = \frac{7.6.5!}{5!} = 42$$

Combinations

Suppose a dress designer wishes to select two colors of material to design a new dress, and she has no hand for colors. How many different possibilities can there be in this situation?

This type of problem differs from previous, ones in that the order of selection is not important. That is, if the designer selects yellow and red, this selection is the same as the selection red and yellow. This type of selection is called a combination. The difference between a permutation and a combination is that in a combination, the order or arrangement of the objects is not important, by contrast, order is important in a permutation.

Combination

A selection of distinct objects without regard to order is called a combination.

Example

Given the letters $A, B, C, and\ D,$ list the permutation and combinations for selecting two letters.

Solution

The permutation are

AB BA CA DA

AC BC CB DB

AD BD CD DC

In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations.

Therefore, if duplicates are removed from a list of permutation, what is left is a list of combinations, as shown.

AB	$\cancel{BA}$	$\cancel{CA}$	$\cancel{DA}$
AC	BC	$\cancel{CB}$	$\cancel{DB}$
AD	BD	CD	$\cancel{DC}$

Hence, the combinations of $A, B, C, and\ D$ are $AB, AC, AD, BC, BD, and\ CD$.

Combination rule

The number of combinations of r objects selected from n objects is denoted by $n\ C_r$ and is given by the formula

$$n\ C_r = \frac{n!}{(n-r)!\, r!}$$

Example

How many combinations of 4 objects are there, taken 2 at a time?

Solution

Since this is a combination problem, the answer is

$$4\ C_2 = \frac{4!}{(4-2)!\, 2!} = \frac{4!}{2!\, 2!} = \frac{4.3.\cancel{2!}}{2.1.\cancel{2!}} = 6$$

Example

A bicycle shop owner has 12 mountain bicycles in the showroom. The owner wishes to select 5 of them to display at a bicycle show. How many different ways can a group of 5 be selected?

Solution

$$12C_5 = \frac{12!}{(12-5)!\, 5!} = \frac{12!}{7!\, 5!} = \frac{12.11.10.9.8.\cancel{7!}}{\cancel{7!}\, 5.4.3.2.1} = 792$$

Example

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to the chosen. How many different possibilities are there?

Solution

Here, one must select 3 women from 7 women, which can be done in $7C_3, or\ 35,$ ways. Next, 2 men must be selected from 5 men, which can be done in $5C_2, or\ 10,$ ways. Finally, by the fundamental counting rule, the total number of different ways is $35.10 = 350,$ since one is choosing both men and women. Using the formula gives

$$7C_3.5C_2 = \frac{7!}{(7-3)!\ 3!} \cdot \frac{5!}{(5-2)!\ 2!} = 350$$

Problems

1. A store manager wishes to display 8 different brands of shampoo in a row. How many ways can this be done.
2. There are 8 different statistics books, 6 different geometry books, and 3 different trigonometry books. A student select one book of each type. How many different ways can this be done?

The eighth lecture

Probability and Counting Rules

The counting rules can be combined with the probability rules to solve many type of probability problems. By using the fundamental counting rule, the permutation rules, and the combination rule, one can compute the probability of outcomes of many experiments, such as getting a full house when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

Example

Find the probability of getting 4 aces when 5 cards are drawn from an ordinary deck of cards.

Solution

There are $52\ C_5$ ways to draw 5 cards from a deck. There is only one way to get 4 aces (i.e, $4\ C_4$), but there are 48 possibilities to get the fifth card. Therefore, there are 48 ways to get 4 aces and 1 other card. Hence

$$p(4\ aces) = \frac{4C_4.48}{52\ C_5} = \frac{1\ .\ 48}{2,598,960} = \frac{48}{2,598,960} = \frac{1}{54,145}$$

Example

A box contains 24 transistors, 4 of which are defective. If 4 are sold at random, find the following probabilities.

a. Exactly 2 are defective.
b. None is defective.
c. All are defective.

Solution

There are $24\ C_4$ ways to sell 4 transistors, so the denominator in each case will be 10,626

a. Two defective transistors can be selected as $4\ C_2$ and 2 non defective ones as $20\ C_2$. Hence,

$$p(exactly\ 2\ defectives) = \frac{4\ C_2.\ 20\ C_2}{24\ C_4} = \frac{1140}{10,626} = \frac{190}{1771}$$

b. The number of ways to choose no defectives is $20\ C_4$. Hence,

$$p(no\ defectives) = \frac{20\ C_4}{24\ C_4} = \frac{4845}{10,626} = \frac{1615}{3542}$$

c. The number of ways to choose 4 defectives from 4 is $4\ C_4$, or 1. Hence,

$$p(all\ defective) = \frac{1}{24\ C_4} = \frac{1}{10,626}$$

Example (WH)

A store has 6 Tv graphic magazines and 8 news time magazines on the counter. If two customers purchased a magazine, find the probability that one of each magazine was purchased.

A ***Random Variables*** is a variable whose values are determined by chance.

Discrete Random Variable has a countable number of possible values.

Example

If a die is rolled. Find the discrete random variable?

Solution

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Let x be represent the discrete random variable then x takes the value of outcomes

$\therefore x = 1, 2, 3, 4, 5, or\ 6$

Probability Distribution

If X is a discrete random variable with distinct values $x_1,\ x_2, \dots, x_n$, ..., and S is any countable set of real numbers, then the function given by

$$f_x(x) = \begin{cases} p(X = x) & if \quad x \in S \\ 0 & if \quad x \notin S \end{cases}$$

is called the probability distribution of X.

A function can serve as the probability distribution of a discrete random variable X if its values, $f_x(x)$, satisfy the following conditions:

1. $f_x(x) > 0 \quad \forall\ x \in S$
2. $f_x(x) = 0 \quad \forall\ x \notin S$
3. $\sum_{\forall\ x \in S} f_x(x) = 1$

Example

Let $f_x(x) = \begin{cases} \frac{x}{10} & x = 1, 2, 3, 4 \\ 0 & Otherwise \end{cases}$

Is $f_x(x)$ a probability distribution?

Solution

1. $p(X = 1) = \frac{1}{10}, \ p(X = 2) = \frac{2}{10}, \ p(X = 3) = \frac{3}{10}, \ p(X = 4) = \frac{4}{10}$
2. $\sum_{i=1}^{4} f_x(x_i) = \frac{1}{10} + \frac{2}{10} + \frac{3}{10} + \frac{4}{10} = 1$

$\therefore f_x(x) \ is \ a \ probability \ distribution$

Problem:

Check whether the function:

$f_x(x) = \frac{x+2}{25} \quad for \ x = 1, 2, 3, 4, 5$

can serve as the probability distribution of a discrete random variable.

The ninth lecture

Binomial Distribution

Many types of probability problems have only two outcomes or can be reduced in two outcomes. For example, when a coin is tossed, it can land heads or tails. When a baby is born, it will be either male or female. In a basketball game, a team either wins or loses. A true/false item can be answered in only two ways, true or false. Other situations can be reduced to two outcomes.

Binomial experiment

A binomial experiment is a probability experiment that satisfies the following four requirements:

1. There must be a fixed number of trials.
2. Each trial can have only two outcomes or outcomes that be reduced to two outcomes. These outcomes can be considered as either success or failure.
3. The outcomes of each trial must be independent of each other.
4. The probability of a success must remain the same for each trial.

Notation for the Binomial distribution

$p(s)$ The symbol for the probability of success.

$p(F)$ The symbol for the probability of failure.

p The numerical probability of a success.

q The numerical probability of a failure

$$p(s) = p \quad and \quad p(F) = 1 - p = q$$

n The number of trials.

x The number of successes in n trials.

Note that $0 \leq x \leq n \quad and \quad x = 0, 1, 2, 3, \dots, n$

Binomial probability formula

In a binomial experiment, the probability of exactly x successes in n trials is

$$p(x) = \frac{n!}{(n-x)!.x!} \, . p^x \, . q^{n-x}$$

Example

Public Opinion reported that 0.3 of Americans are afraid of being alone in a hose at night. If a random sample of 10 Americans is selected, find these probabilities.

a. There are exactly 3 people in the sample who are afraid of being alone at night.

b. There are at most 2 people in the sample who are afraid of being alone at night.

c. There are at least 2 people in the sample who are afraid of being alone at night.

Solution

a. $n = 10,\ \ p = 0.3,\quad q = 0.7,\quad x = 3$

$p(x) = \frac{n!}{(n-x)!.x!}\,p^x\,.\,q^{n-x} \rightarrow p(3) = \frac{10!}{(10-3)!\,.3!}\ (0.3)^3.\,(0.7)^{10-3}$

$p(3) = 0.26$

b. $n = 10,\ \ p = 0.3,\quad q = 0.7$ "At most 2 people" means 0, or 1, or 2.

Hence , the solution is $p(0) + p(1) + p(2)$

$$p(x) = \frac{n!}{(n-x)!.\,x!}\,p^x\,.\,q^{n-x}$$

$p(0) = \frac{10!}{(10-0)!.0!}\ (0.3)^0\,.\,(0.7)^{10-0} = 0.028 = 0.03,$ ***note:-*** $0! = 1$

$p(1) = \frac{10!}{(10-1)!.1!}\ (0.3)^1\,.\,(0.7)^{10-1} = 0.12,$ ***note:-*** $1! = 1$

$$p(2) = \frac{10!}{(10-2)!\,.2!}\ (0.3)^2\,.\,(0.7)^{10-2} = 0.23$$

$\therefore$ The solution is $p(0) + p(1) + p(2) = 0.03 + 0.12 + 0.23 = 0.38$

c. $n = 10, p = 0.3,\ q = 0.7$ "At least 2 people" means 2, 3, 4…, and 10. This problem can best be solved by finding $p(0) + p(1)$ and subtracting from 1

$p(0) + p(1) = 0.03 + 0.12 = 0.15 \rightarrow \therefore 1 - 0.15 = 0.85$

Tenth lecture

Problem:

A survey found that one out of five Americans say he or she visited a doctor in any given month. If 10 people are selected at random, find the probability that exactly 3 will have visited a doctor last month.

Geometric Distribution

Another useful distribution is called the *geometric distribution.* This distribution can be used when we have an experiment that has two outcomes and is repeated until a successful outcome is obtained. For example, we could flip a coin until a head is obtained, or we could roll a die until we get a 6. In these cases, our successes would come on the nth trial. The geometric probability distribution tells us when the success is likely to occur.

A ***geometric experiment*** is a probability experiment if it satisfies the following requirements:

1. Each trial has two outcomes that can be either success or failure.
2. The outcomes are independent of each other.
3. The probability of a success is the same for each trial.
4. The experiment continues until a success is obtained.

Formula for the Geometric Distribution

If p is the probability of a success on each trial of a binomial experiment and n is the number of the trial at which the first success occurs, then the probability of getting the first success on the nth trial is $P(n) = p\,(1-p)^{n-1},\ \ when \quad n = 1, 2, 3, 4, \ldots.$

Example

A coin is tossed. Find the probability of getting the first head on the third toss.

Solution

$\Omega = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$

The objective for tossing a coin and getting a head on the third toss is TTH.

Now by using the formula and $n = 3,\ p = \frac{1}{2}$

$$P(n) = p\,(1-p)^{n-1} \rightarrow p(3) = \frac{1}{2}(1-\frac{1}{2})^{3-1} = \frac{1}{2}\,.\,(\frac{1}{2})^{2} = \frac{1}{8}$$

Hence, there is a 1 out of 8 chance or 0.125 probability of getting the first head on the third toss of a coin.

Problem:

In the United States, approximately 42% of people have type ***A*** blood. If 4 people are selected at random, find the probability that the fourth person is the first one selected with type ***A*** blood.

Continuous Random Variable continuous variable can assume all values between any two given values of the variables. Examples of continuous variables are the height of adult men, body temperature of rats, and cholesterol level of adults and many continuous variables.

Probability Density Function

A function with values $f_x(x)$, defined over the set of all real numbers, is called a probability density function of the continuous random variable x if and only if:

$$p(a < x \leq b) = \int_a^b f_x(x)\,dx \qquad for\ any\ real\ constants\ \ a < b$$

Note: A function can serve as a probability density of continuous random variable x if $f_x(x)$ satisfy the following conditions:

1. $f_x(x) \geq 0 \qquad \forall\, x \in R$
2. $\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

Example:

If x has the probability density

$$f_x(x) = \begin{cases} \frac{2}{x^3}, & x \geq 1 \\ 0, & Otherwise \end{cases}$$

Is $f_x(x)$ a probability density function?

Solution:

$f_x(x)$ is a probability density function if it satisfies:

1. $f_x(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_x(x)\,dx = \int_1^{\infty} \frac{2}{x^3}\,dx = \int_1^{\infty} 2\ x^{-3} dx = 2\ \frac{x^{-2}}{-2}|_1^{\infty} = 1$

$\therefore\ from\ 1\ and\ 2\ \ f_x(x)$ is a $\mathrm{P.d.f.}$

Problem:

If x has the probability density function:

$$f_x(x) = \begin{cases} ke^{-3x}, & x > 0 \\ 0\,, & Otherwise \end{cases}$$

Find the value of k and $p(0.5 \le x \le 1)$

Eleventh lecture

Probability Distribution Function

If X is a continuous random variable and the value of its probability density at t is $f_x(t)$, then the function given by

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f_x(t)\,dt \quad for \; -\infty < x < \infty$$

is called the distribution function of X.

The values $F_X(x)$ of the distribution function of a continuous random variable X satisfies the conditions: $F_X(-\infty) = 0$, $F_X(\infty) = 1$ and $F_X(a) \le F_X(b)$ *when* $a < b$.

Theorem: If $f_x(x)$ and $F_X(x)$ are the values of the probability density and the distribution function of X at x, then $P(a \le X \le b) = F_X(b) - F_X(a)$ for any real constants a and b with $a \le b$, and $f_x(x) = \frac{dF_X(x)}{dx}$. Where the derivative exists.

Example:

Find the distribution function of the random variable X If the probability density function is given by the following:

$$f_x(x) = \begin{cases} 3e^{-3x}, & x > 0 \\ 0\,, & Otherwise \end{cases}$$

And then evaluate $p(0.5 \le x \le 1)$

Solution:

For $x > 0$,

$$F_X(x) = \int_{-\infty}^{x} f_x(t)\,dt = \int_0^x 3e^{-3t}\,dt = -e^{-3t}|_0^x = 1 - e^{-3x}$$

and since $F_X(x) = 0$ for $x \le 0$, we can write

$$F_X(x) = \begin{cases} 1 - e^{-3x}, & x > 0 \\ 0\,, & x \le 0 \end{cases}$$

To determine the probability $\mathrm{p}(0.5 \le \mathrm{x} \le 1)$ using the previous theorem we get,

$$p(0.5 \le x \le 1) = F_X(1) - F_X(0.5) = (1 - e^{-3}) - (1 - e^{-1.5}) = -e^{-3} + e^{-1.5}$$

$$= -(0.0498) + 0.2231 \approx 0.173$$

Example:

The random variable X has probability density function

$$f_x(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0\,, & Otherwise \end{cases}$$

Use the $\mathrm{P.d.f.}$ to find

1. The constant c.
2. $p(0 \le x \le 1)$
3. $p\left(-\frac{1}{2} \le x \le \frac{1}{2}\right)$
4. The DF $F_X(x)$

Solution:

$$f_x(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0\,, & Otherwise \end{cases}$$

1. From the above P.d.f. we can determine the value of c by integrating the P.d.f. and setting it equal to 1.

   $\int_0^2 f_x(x)\,dx = \int_0^2 cx\,dx = c\,\frac{x^2}{2}\Big|_0^2 = c\left[\frac{2^2}{2} - \frac{0^2}{2}\right] = 2c = 1 \to c = \frac{1}{2}$
2. $p(0 \le x \le 1) = \int_0^1 \frac{1}{2} x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^1 = \frac{1}{4}$
3. $p\left(-\frac{1}{2} \le x \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{1}{2} x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]\Big|_0^{\frac{1}{2}} = \frac{1}{16}$
4. For $0 \le x \le 2$, $\quad F_X(x) = \int_{-\infty}^{x} f_x(t)\,dt = \int_0^x \frac{1}{2} t\,dt = \frac{1}{2}\left[\frac{t^2}{2}\right]\Big|_0^x = \frac{1}{4}x^2$,

   $F_X(x) = 0, \quad for\ x < 0\ \ and\ \ F_X(x) = 1,\ \ for\ x > 2$

   $$\therefore the\ DF\ is\ given\ by \quad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2\,, & 0 \le x \le 2 \\ 1, & x > 2 \end{cases}$$

Problem:

Find the distribution function of the random variable X whose probability density is given by

$$f_x(x) = \begin{cases} \frac{1}{4} \ , & -2 < x < 2 \\ 0 \ , & Otherwise \end{cases}$$