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Computer Mathematics

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Introduction

In this semester, computer mathematics into four chapter will be studied. The first, known as number theory. In this chapter, some of the important concepts of number theory including many of those used in computer science are developed. In the next chapter, two ways that recurrence relations play important roles in the study of algorithms will be discussed. The Fibonacci recurrence and linear recurrences (linear homogeneous and non-homogeneous recurrences) are studied.

In the third chapter, many counting problems in terms of ordered or unordered arrangements of the objects of a set with or without repetitions could be phrased. These arrangements, called permutations and combinations, are used in many counting problems. The last chapter concludes generating functions that can be used to solve many types of counting problems and used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function.

The First Lecturer

Number Theory

The part of mathematics devoted to the study of the set of integers and their properties is known as number theory. In this chapter we will explain some of the important concepts of number theory including many of those used in computer science.

Division

When one integer is divided by a second nonzero integer, the quotient may or may not be an integer. For example, $12/3 = 4$ is an integer, whereas $11/4 = 2.75$ is not.

Definition: If a and b are integers with $a \neq 0$, we say that a divides b (or b is divisible by a) if there is an integer c such that $b = ac$.

If a divides b, we write $a \mid b$, while if a does not divide b, we write $a \nmid b$.

For example: $-5 \mid 30, 7 \nmid 50, 17 \mid 0$.

Example: The divisor of 6 are $\mp 1, \mp 2, \mp 3$ & ∓ 6, the divisors of 17 are ∓ 1 & ∓ 17.

Example: Determine whether $3 \mid 7$ and whether $3 \mid 12$.

We see that $3 \nmid 7$, because $7/3$ is not an integer. On the other hand, $3 \mid 12$ because $12/3 = 4$.

Theorem(1): If a, b and c are integers then the following statements hold:

1. $a \mid o, 1 \mid a, a \mid a$.
2. $a \mid \mp 1$ iff $a = \mp 1$.
3. If $a \mid b$ and $c \mid d$ then $ac \mid bd$.
4. If $a \mid b$ and $b \mid c$ then $a \mid c$.
5. $a \mid b$ and $b \mid a$ iff $a = \mp b$.
6. If $a \mid b$ and $b \neq 0$ then $|a| \leq |b|$.
7. If $a \mid b$ and $a \mid c$ then $a \mid (bx + cy)$ for arbitrary integers x and y.
8. Let $a > 0, b > 0$. If $a \mid b$ then $a \leq b$.

The Division Algorithm

When an integer is divided by a positive integer, there is a quotient and a remainder, as the division algorithm shows.

Theorem (The Division Algorithm)

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \le r < d$, such that $a = dq + r$.

Definition: In the equality given in the division algorithm, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*. This notation is used to express the quotient and remainder:

$$q = a\ \boldsymbol{div}\ d, \quad r = a\ \boldsymbol{mod}\ d.$$

Example: What are the quotient and remainder when 101 is divided by 11?

We have

$101 = 11 . 9 + 2.$

Hence, the quotient when 101 is divided by 11 is $9 = 101\ \boldsymbol{div}\ 11$, and the remainder is $2 = 101\ \boldsymbol{mod}\ 11$.

Example: What are the quotient and remainder when -11 is divided by 3?

We have

$-11 = 3(-4) + 1.$

Hence, the quotient when -11 is divided by 3 is $-4 = -11\ \boldsymbol{div}\ 3$, and the remainder is $1 = -11\ \boldsymbol{mod}\ 3$.

Note that the remainder cannot be negative. Consequently, the remainder is *not* -2, even though

$-11 = 3(-3) - 2,$

because $r = -2$ does not satisfy $0 \le r < 3$.

Modular Arithmetic

Definition: If a and b are integers and m is a positive integer, then a is *congruent to* b *modulo* m if m divides $a - b$. We use the notation $a \equiv b\ (\boldsymbol{mod}\ m)$. If a and b are not congruent modulo m, we write $a \not\equiv b\ (\boldsymbol{mod}\ m)$.

Theorem: Let a and b be integers, and let m be a positive integer. Then $a \equiv b\ (\boldsymbol{mod}\ m)$ if and only if $a\ \boldsymbol{mod}\ m = b\ \boldsymbol{mod}\ m$.

Example: Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Because 6 divides $17 - 5 = 12$, we see that $17 \equiv 5\ (\boldsymbol{mod}\ 6)$. However, because $24 - 14 = 10$ is not divisible by 6, we see that $24 \not\equiv 14\ (\boldsymbol{mod}\ 6)$.

Theorem: Let m be a positive integer. If $a \equiv b \ (\boldsymbol{mod}\ m)$ and $c \equiv d \ (\boldsymbol{mod}\ m)$, then $a + c \equiv b + d \ (\boldsymbol{mod}\ m)$ and $ac \equiv bd \ (\boldsymbol{mod}\ m)$.

Example: Because $7 \equiv 2 \ (\boldsymbol{mod}\ 5)\ and\ 11 \equiv 1 \ (\boldsymbol{mod}\ 5)$, it follows from previous theorem that

$18 = 7 + 11 \equiv 2 + 1 = 3 \ (\boldsymbol{mod}\ 5)$

and that

$77 = 7\,.11 \equiv 2\,.1 = 2 \ (\boldsymbol{mod}\ 5)$.

The Second Lecturer

Arithmetic Modulo m

We can define arithmetic operations on $\boldsymbol{Z_m}$, the set of nonnegative integers less than m, that is, the set $\{0, 1, \ldots, m - 1\}$. In particular, we define addition of these integers, denoted by $+_m$ by $a +_m b = (a + b)\ \boldsymbol{mod}\ m$,

where the addition on the right-hand side of this equation is the ordinary addition of integers, and we define multiplication of these integers, denoted by $._m$ by

$a\ ._m\ b = (a\,.b)\ \boldsymbol{mod}\ m$,

where the multiplication on the right-hand side of this equation is the ordinary multiplication of integers. The operations $+_m$ and $._m$ are called addition and multiplication modulo m and when we use these operations, we are said to be doing **arithmetic modulo** m.

Example: Use the definition of addition and multiplication in $\boldsymbol{Z_m}$ to find $7 +_{11} 9$ and $7\ ._{11}\ 9$.

Using the definition of addition modulo 11, we find that

$7 +_{11} 9 = (7 + 9)\ \boldsymbol{mod}\ 11 = 16\ \boldsymbol{mod}\ 11 = 5$,

and

$7\ ._{11}\ 9 = (7\,.9)\ \boldsymbol{mod}\ 11 = 63\ \boldsymbol{mod}\ 11 = 8$.

Hence $7 +_{11} 9 = 5$ and $7\ ._{11}\ 9 = 8$.

Primes

Every integer greater than 1 is divisible by at least two integers, because a positive integer is divisible by 1 and by itself. Positive integers that have exactly two different positive integer factors are called **primes**.

Definition: An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called *composite*.

Example: The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3.

Greatest Common Divisors and Least Common Multiples

Definition: Let a *and* b be integers, not both zero. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a *and* b. The greatest common divisor of a and b is denoted by $gcd(a, b)$.

Example: What is the greatest common divisor of 24 and 36?

The positive common divisors of 24 *and* 36 are $1, 2, 3, 4, 6,$ *and* 12. Hence, $gcd(24, 36) = 12$.

Definition: The integers a *and* b are *relatively prime* if their greatest common divisor is 1.

Example: What is the greatest common divisor of 17 *and* 22?

The integers 17 *and* 22 have no positive common divisors other than 1, it follows that the integers 17 and 22 are relatively prime, because $gcd(17, 22) = 1$.

Definition: The integers $a_1,\ a_2, \ldots,\ a_n$ are *pairwise relatively prime* if $gcd(\ a_i,\ a_j\ ) = 1$ whenever $1 \leq i < j \leq n$.

Example: Determine whether the integers 10, 17, and 21 are pairwise relatively prime and whether the integers 10, 19, and 24 are pairwise relatively prime.

Because $gcd(10, 17) = 1$, $gcd(10, 21) = 1$, *and* $gcd(17, 21) = 1$, we conclude that 10, 17, and 21 are pairwise relatively prime.

Because $gcd(10, 24) = 2 > 1$, we see that 10, 19, and 24 are not pairwise relatively prime.

Prime Factorizations

Another way to find the greatest common divisor of two positive integers is to use the prime factorizations of these integers. Suppose that the prime factorizations of the positive integers a *and* b are

$$a = p_1^{a_1} . p_2^{a_2} \ldots p_n^{a_n},\ b = p_1^{b_1} . p_2^{b_2} \ldots p_n^{b_n}$$

where each exponent is a nonnegative integer, and where all primes occurring in the prime factorization of either a *or* b are included in both factorizations, with zero exponents if necessary. Then $gcd(a, b)$ is given by

$$gcd(a, b) = p_1^{\min(a_1, b_1)} . p_2^{\min(a_2, b_2)} \ldots p_n^{\min(a_n, b_n)}$$

Example: Because the prime factorizations of 120 and 500 are $120 = 2^3 . 3 . 5$ and $500 = 2^2 . 5^3$, the greatest common divisor is

$$gcd(120, 500) = 2^{\min(3,2)}. 3^{\min(1,0)}. 5^{\min(1,3)} = 2^2. 3^0. 5^1 = 20$$

Definition: The *least common multiple* of the positive integers a *and* b is the smallest positive integer that is divisible by both a *and* b. The least common multiple of a *and* b is denoted by $lcm(a, b)$.

The least common multiple exists because the set of integers divisible by both a *and* b is nonempty and every nonempty set of positive integers has a least element so the least common multiple of a *and* b is given by

$$lcm(a, b) = p_1^{\max(a_1, b_1)}. p_2^{\max(a_2, b_2)} \dots p_n^{\max(a_n, b_n)}$$

Example: What is the least common multiple of $2^3 3^5 7^2$ and $2^4 3^3$?

We have

$$lcm(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3,4)}. 3^{\max(5,3)}. 7^{\max(2,0)} = 2^4. 3^5. 7^2$$

Theorem: Let a *and* b be positive integers. Then $ab = gcd(a, b). lcm(a, b)$

Example: Find $gcd(1000, 625)$ and $lcm(1000, 625)$ and verify that $gcd(1000, 625). lcm(1000, 625) = 1000. 625$.

We have

$1000 = 5^3. 2^3$ *and* $625 = 5^4$

since, $gcd(1000, 625) = 2^{\min(3,0)}. 5^{\min(3,4)} = 2^0. 5^3 = 125$,

$$lcm(1000, 625) = 2^{\max(3,0)}. 5^{\max(3,4)} = 2^3. 5^4 = 5000,$$

Then, $1000 . 625 = gcd(1000, 625). lcm(1000, 625) = 125 . 5000 = 625000$

The Third Lecturer

The Euclidean Algorithm

Computing the greatest common divisor of two integers directly from the prime factorizations of these integers is inefficient. The reason is that it is time-consuming to find prime factorizations. We will give a more efficient method of finding the greatest common divisor, called the **Euclidean algorithm**.

We will use successive divisions to reduce the problem of finding the greatest common divisor of two positive integers to the same problem with smaller integers, until one of the integers is zero. The Euclidean algorithm is based on the following result about greatest common divisors and the division algorithm.

Theorem: Let $a = bq + r$, where $a, b, q,$ and r are integers. Then $gcd(a, b) = gcd(b, r)$.

Example: Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Successive uses of the division algorithm give:

$662 = 414.1 + 248$

$414 = 248.1 + 166$

$248 = 166.1 + 82$

$166 = 82.2 + 2$

$82 = 2.41$.

Hence, $gcd(414, 662) = 2$, because 2 is the last nonzero remainder.

Problems:

1. Answer of the following with true or false:
   - The prime factorization of 126 is $2 .\ 3^3 .\ 7$.
   - The greatest common divisors of $17, 17^{17}$ is 17.
   - The quotient and remainder when 44 is divided by 8 is $44 = 8.4 + 4$.
   - The quotient of -17 is divided by 2 is $-9 = -17\ div\ 2$ and the remainder is $1 = -17\ mod\ 2$
2. Determine whether the integers in each of these sets are pairwise relatively prime.
   - $14, 15, 21$
   - $12, 17, 31, 37$
3. What are the greatest common divisors and the least common multiple of these pairs of integers?
   - $3^7 . 5^3 . 7^3$, $2^{11} . 3^5 . 5^9$
   - $11 . 13 . 17$, $2^9 . 3^7 . 5^5 . 7^3$
4. Find $gcd(144, 88)$ and $lcm(144, 88)$ and verify that $gcd(144, 88). lcm(144, 88) = 144.88$.
5. If the product of two integers is $2^7 . 3^8 .\ 5^2 .\ 7^{11}$ and their greatest common divisor is $2^3 .\ 3^4 . 5$, what is their least common multiple?

6. Use the Euclidean algorithm to find:
   - $gcd(123, 277)$.
   - $gcd(1001, 1331)$.
7. Suppose that a and b are integers, $a \equiv 4 \ (mod\ 13)$, and $b \equiv 9 \ (mod\ 13)$. Find the integer c with $0 \leq c \leq 12$ such that
   - $c \equiv 9a \ (mod\ 13)$.
   - $c \equiv a + b \ (mod\ 13)$.
   - $c \equiv 2a + 3b \ (mod\ 13)$.
   - $c \equiv a^2 + b^2 \ (mod\ 13)$.
   - $c \equiv a^3 - b^3 \ (mod\ 13)$.

Note: Let m be a positive integer and let a and b be integers.Then:

$(a + b) mod\ m = ((a\ mod\ m) + (b\ mod\ m)) mod\ m$

and

$a.b\ mod\ m = ((a\ mod\ m)(b\ mod\ m)) mod\ m$

$$a^3 - b^3 \ (mod\ 13) = (4\ mod\ 13)^3 - (9\ mod\ 13)^3\ mod 13$$
$$= (64 - 729) mod\ 13$$
$$= -665\ mod\ 13$$
$$= -665 - 13(-52)$$
$$= 11$$

Note: $mod(x, y) = x - y * n$

$n = floor(\frac{x}{y})$

$= floor\left(\frac{-665}{13}\right)$

$= floor(-51{,}1538)$

$= -52$

So,

$mod(x, y) = x - y * n$

$= -665 - 13(-52)$

$= -665 + 676 = 11$

Recurrences

A recurrence describes a sequence of numbers. Early terms are specified explicitly and later terms are expressed as a function of their predecessors. As a trivial example, this recurrence describes the sequence 1, 2, 3, etc.:

$$T_1 = 1$$

$$T_n = T_{n-1} + 1, \qquad (for\ n \geq 2)$$

Here, the first term is defined to be 1 and each subsequent term is one more than its predecessor.

Applications of Recurrence Relations

The Fibonacci sequence

Fibonacci published in the year 1202 is now famous rabbit puzzle:

A man put a male-female pair of newly born rabbits in a field. Rabbits take a month to mature before mating. One month after mating, females give birth to one male-female pair and then mate again. No rabbits die. How many rabbit pairs are there after one year?

To solve, we construct Table 1.1. At the start of each month, the number of juvenile pairs, adult pairs, and total number of pairs are shown. At the start of January, one pair of juvenile rabbits is introduced into the population. At the start of February, this pair of rabbits has matured. At the start of March, this pair has given birth to a new pair of juvenile rabbits. And so on.

month	**J**	**F**	**M**	**A**	**M**	**J**	**J**	**A**	**S**	**O**	**N**	**D**	**J**
juvenile	1	0	1	1	2	3	5	8	13	21	34	55	89
adult	0	1	1	2	3	5	8	13	21	34	55	89	144
total	1	1	2	3	5	8	13	21	34	55	89	144	233

Table 1.1: Fibonacci's rabbit population

We define the Fibonacci numbers F_n to be the total number of rabbit pairs at the start of the nth month. The number of rabbits pairs at the start of the 13th month, $F_{13} = 233$, can be taken as the solution to Fibonacci's puzzle.

Further examination of the Fibonacci numbers listed in Table 1.1, reveals that these numbers satisfy the recursion relation

$$F_{n+1} = F_n + F_{n-1}. \tag{1.1}$$

This recursion relation gives the next Fibonacci number as the sum of the preceding two numbers. To start the recursion, we need to specify $F_1 and\ F_2$. In Fibonacci's rabbit problem, the initial month starts with only one rabbit pair so that $F_1 = 1$. And this initial rabbit pair is newborn and takes one month to mature before mating so $F_2 = 1$. The first few Fibonacci numbers, read from the table, are given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots$$

and has become one of the most famous sequences in mathematics.

<u>**Example:**</u> The Fibonacci numbers can be extended to zero and negative indices using the relation $F_n = F_{n+2} - F_{n+1}$ with starting values $F_1 = 1\ and\ F_2 = 1$. Determine F_0 and find a general formula for F_{-n} in terms of F_n. Prove your result using mathematical induction.

We calculate the first few terms.

$F_0 = F_2 - F_1 = 0,$
$F_{-1} = F_1 - F_0 = 1,$
$F_{-2} = F_0 - F_{-1} = -1,$
$F_{-3} = F_{-1} - F_{-2} = 2,$
$F_{-4} = F_{-2} - F_{-3} = -3,$
$F_{-5} = F_{-3} - F_{-4} = 5,$
$F_{-6} = F_{-4} - F_{-5} = -8.$

> <u>**Note:**</u> $F_n = F_{n+2} - F_{n+1}$
>
> If $n = 0 \to F_0 = F_2 - F_1 = 0$
>
> If $n = -1 \to F_{-1} = F_1 - F_0 = 1$

The correct relation appears to be

$$F_{-n} = (-1)^{n+1} F_n \qquad (1)$$

Now to prove that by mathematical induction

Base case: Our calculation above already shows that (1) is true for $n = 1\ and\ n = 2$, that is, $F_{-1} = F_1\ and\ F_{-2} = -F_2$.

Induction step: Suppose that (1) is true for positive integers $n = k - 1\ and\ n = k$. Then we have

$If\ n = k + 1$, then

$F_{-(k+1)} = F_{-(k+1)+2} - F_{-(k+1)+1}$

$F_{-(k+1)} = F_{-(k-1)} - F_{-(k)}$

$= (-1)^k F_{(k-1)} - (-1)^{k+1} F_{(k)}$

$= (-1)^2 . (-1)^k F_{(k-1)} + (-1) . (-1)^{k+1} F_{(k)}$

$= (-1)^{k+2} F_{(k-1)} + (-1)^{k+2} F_{(k)}$

$= (-1)^{k+2} (F_{(k-1)} + F_{(k)})$

$= (-1)^{k+2} F_{(k+1)}$

so that (1) is true for n = k + 1. By the principle of induction, (1) is therefore true for all positive integers.

The Fourth Lecturer

Example: The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ *and* $L_2 = 3$. Determine the first 12 Lucas numbers.

$if\ n = 2 \rightarrow L_{2+1} = L_2 + L_{2-1} = 3 + 1 = 4$

$if\ n = 3 \rightarrow L_{3+1} = L_3 + L_{3-1} = 4 + 3 = 7$

By the same way, we found the first 12 Lucas numbers.

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322$$

Example: The generalized Fibonacci sequence satisfies $f_{n+1} = f_n + f_{n-1}$ with starting values $f_1 = p$ *and* $f_2 = q$. Using mathematical induction, prove that

$$f_{n+2} = F_n p + F_{n+1} q. \qquad (2)$$

We now prove (2) by mathematical induction

Base case: To prove that (2) is true for $n = 1$, we write $F_1 p + F_2 q = p + q = f_3$. To prove that (2) is true for $n = 2$, we write $F_2 p + F_3 q = p + 2q = f_3 + f_2 = f_4$.

Induction step: Suppose that (2) is true for positive integers $n = k - 1$ *and* $n = k$. Then we have

$f_{n+2} = f_{n+1} + f_n$

If $n = k + 1$, then

$$\begin{aligned} f_{k+3} &= f_{k+2} + f_{k+1} \\ &= (F_k p + F_{k+1} q) + (F_{k-1} p + F_k q) \\ &= (F_k + F_{k-1}) p + (F_{k+1} + F_k) q \\ &= F_{k+1} p + F_{k+2} q \end{aligned}$$

so that (2) is true for $n = k + 1$. By the principle of induction, (2) is therefore true for all positive integers.

Solving Linear Recurrence Relations

Linear recurrences

1. Linear homogeneous recurrences

2. linear non-homogeneous recurrences

<u>Definition:</u> A **linear homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

a_n is expressed in terms of the previous k terms of the sequence, so its degree is k. This recurrence includes k initial conditions

$a_0 = C_0$ $\quad a_1 = C_1$ $\quad \ldots$ $\quad a_k = C_k$.

<u>Example:</u> Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$

  a linear homogeneous recurrence relation of degree one.
- $a_n = a_{n-1} + a_{n-2}^2$

  not linear
- $f_n = f_{n-1} + f_{n-2}$

  a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1} + 1$

  not homogeneous because $f(x) = 1$.
- $a_n = a_{n-6}$

  a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$

  does not have constant coefficien

<u>Theorem:</u> Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume the sequence a_n satisfies the recurrence and the sequence g_n also satisfies the recurrence. So, $b_n = a_n + g_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence. (α is any constant)

<u>Note:</u> Geometric sequences come up a lot when solving linear homogeneous recurrences. So, try to find any solution of the form $a_n = r^n$ that satisfies the recurrence relation.

Recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$$

Try to find a solution of form r^n

$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}$

$r^n - c_1r^{n-1} - c_2r^{n-2} - \cdots - c_kr^{n-k} = 0$

$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0$ (dividing both sides by r^{n-k})

This equation is called the **characteristic equation**.

Example: The Fibonacci recurrence is $F_n = F_{n-1} + F_{n-2}$. Its characteristic equation is $r^2 - r - 1 = 0$.

Theorem: r is a solution of $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0$ if and only if r^n is a solution of $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$.

Example: Consider the characteristic equation $r^2 - 4r + 4 = 0$.

$r^2 - 4r + 4 = (r-2)^2 = 0$

So, $r = 2$, then 2^n satisfies the recurrence $F_n = 4F_{n-1} - 4F_{n-2}$

$2^n = 4.2^{n-1} - 42^{n-2}$

$2^n - 4.2^{n-1} + 42^{n-2} = 0$

$2^{n-2}(2^2 - 4.2 + 4) = 0$

$2^{n-2}(4 - 8 + 4) = 0$

Theorem: Consider the characteristic equation $r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_k = 0$ and the recurrence $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$. Assume $r_1, r_2, \dots, r_m$ all satisfy the equation. Let $\alpha_1, \alpha_2, \dots, \alpha_m$ by any constants. So, $a_n = \alpha_1r_1^n + \alpha_2r_2^n + \cdots + \alpha_mr_m^n$ satisfies the recurrence.

Example: What is the solution of the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with $f_0 = 0 \; and \; f_1 = 1$?

Since it is linear homogeneous recurrence, first find its characteristic equation

$r^2 - r - 1 = 0, r_1 = \frac{1+\sqrt{5}}{2} \; and \; r_2 = \frac{1-\sqrt{5}}{2}$

So, by theorem $f_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$ is a solution. Now, we should find α_1 and α_2 using initial conditions

$f_0 = \alpha_1 + \alpha_2 = 0$

$f_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 1$

So, $\alpha_1 = \frac{1}{\sqrt{5}}$ *and* $\alpha_2 = -\frac{1}{\sqrt{5}}$.

$a_n = \frac{1}{\sqrt{5}}.(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}.(\frac{1-\sqrt{5}}{2})^n$ is a solution.

Fifth lecture

Example: What is the solution of the recurrence relation $a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3}$ with $a_0 = 8, a_1 = 6$ *and* $a_2 = 26$?

Since it is linear homogeneous recurrence, first find its characteristic equation

$r^3 + r^2 - 4r - 4 = 0$

$(r+1)(r+2)(r-2) = 0\ r_1 = -1, r_2 = -2$ *and* $r_3 = 2$

So, by theorem $a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_3(2)^n$ is a solution. Now, we should find α_1, α_2 and α_3 using initial conditions.

$a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8$

$a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6$

$a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26$

So, $\alpha_1 = 2, \alpha_2 = 1$ *and* $\alpha_3 = 5$.

$a_n = 2(-1)^n + (-2)^n + 5(2)^n$ is a solution.

Theorem: Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$. Assume the characteristic equation has $t \le k$ distinct solutions. Let $\forall i (1 \le i \le t) r_i$ with multiplicity m_i be a solution of the equation and let $\forall i,j (1 \le i \le t \text{ and } 0 \le j \le m_i - 1) \alpha_{ij}$ be a constant. So,

$$a_n = (\alpha_{10} + \alpha_{11} n + \cdots + \alpha_{1m_1-1} n^{m_1-1})\, r_1^n + (\alpha_{20} + \alpha_{21} n + \cdots + \alpha_{2m_2-1} n^{m_2-1})\, r_2^n + \ldots + (\alpha_{t0} + \alpha_{t1} n + \cdots + \alpha_{tm_t-1} n^{m_t-1})\, r_t^n$$

satisfies the recurrence.

Example: What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$with $a_0 = 1$ *and* $a_1 = 6$?

First find its characteristic equation

$r^2 - 6r + 9 = 0 \quad \rightarrow \quad (r-3)^2 \quad \rightarrow \quad r_1 = 3$ (Its multiplicity is 2)

So, by theorem $a_n = (\alpha_{10} + \alpha_{11}n)(3)^n$ is a solution. Now, we should find α_{10} and α_{11} using initial conditions.

$a_0 = \alpha_{10} = 1$

$a_1 = 3\alpha_{10} + 3\alpha_{11} = 6$

Hence, $\alpha_{10} = 1 and\ \alpha_{11} = 1$.

$a_n = (3)^n + n(3)^n$ is a solution.

Linear non-homogeneous recurrences

Definition: A **linear non-homogenous recurrence relation of degree k** with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$$

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $f(n)$ is a function depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$$

is called the associated homogeneous recurrence relation.

This recurrence includes k initial conditions

$a_0 = C_0 \qquad a_1 = C_1 \quad \ldots \quad a_k = C_k.$

Example: The following recurrence relations are linear nonhomogeneous recurrence relations.

- $a_n = a_{n-1} + 2^n$
- $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$
- $a_n = a_{n-1} + a_{n-2} + n!$
- $a_n = a_{n-6} + n2^n$

Theorem: Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$ be a linear nonhomogeneous recurrence. Assume the sequence b_n satisfies the recurrence and another sequence a_n also satisfies the non-homogeneous recurrence if and only if $h_n = a_n - b_n$ is also sequences that satisfies the associated homogeneous recurrence.

Example: What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} + 3n + 1$ for $n \geq 2$ with $a_0 = 2\ and\ a_1 = 3$?

Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn + d$

$b_n = b_{n-1} + b_{n-2} + 3n + 1$

$cn + d = c(n-1) + d + c(n-2) + d + 3n + 1$

$cn + d = cn - c + d + cn - 2c + d + 3n + 1$

$(c-2c)n + (d-2d) = -3c + 3n + 1$

$-cn - d = 3n - 3c + 1$

$c = -3 \qquad d = -10$

So, $b_n = -3n - 10$

(b_n only satisfies the recurrence, it does not satisfy the initial conditions.)

We are looking for an that satisfies both recurrence and initial conditions. $a_n = b_n + h_n$ where h_n is a solution for the associated homogeneous recurrence: $h_n = h_{n-1} + h_{n-2}$

By previous example, we know $h_n = \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$

$a_n = b_n + h_n$

$= -3n - 10 + \alpha_1(\frac{1+\sqrt{5}}{2})^n + \alpha_2(\frac{1-\sqrt{5}}{2})^n$

Now we should find constants using initial conditions

$a_0 = -10 + \alpha_1 + \alpha_2 = 2$

$a_1 = -13 + \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 3$

Hence, $\alpha_1 = 6 + 2\sqrt{5}\ and\ \alpha_2 = 6 - 2\sqrt{5}$.

So, $a_n = -3n - 10 + (6 + 2\sqrt{5})\left(\frac{1+\sqrt{5}}{2}\right)^n + (6 - 2\sqrt{5})\left(\frac{1-\sqrt{5}}{2}\right)^n$.

Sixth lecture

Problems:

1. Prove that $L_n = F_{n-1} + F_{n+1}$ by using the Lucas sequence $L_{n+1} = L_n + L_{n-1}$ and this relation $f_{n+2} = F_n p + F_{n+1} q$ with values $p = 1\ and\ q = 3$?
2. Determine values of the constants A and B such that $a_n = An + B$ is a solution of recurrence relation $a_n = 2a_{n-1} + n + 5$?
3. Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with the initial conditions $a_0 = 2, a_1 = 5, and\ a_2 = 15$?
4. What is the solution of the recurrence relation $a_n = 8a_{n-2} - 16a_{n-4}\ for\ n \geq 4$ with $a_0 = 1, a_1 = 4, a_2 = 28\ and\ a_3 = 32$?

5. What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?
6. What is the solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2} + 2^n \ for\ n \geq 2$ with $a_0 = 1\ and\ a_1 = 2$?
7. Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?
8. What is the solution of the recurrence relation $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$ with $a_0 = 1$ and $a_1 = 2$?

$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$

Since it is linear non-homogeneous recurrence, b_n is similar to $f(n)$

Guess: $b_n = cn^2 + en + d$

$b_n = b_{n-1} + b_{n-2} + n^2 + n + 1$

$cn^2 + en + d = c(n-1)^2 + e(n-1) + d + c(n-2)^2 + e(n-2) + d + n^2 + n + 1$

$= cn^2 - 2cn + c + en - e + d + cn^2 - 4cn + 4c + en - 2e + d + n^2 + n + 1$

$= (cn^2 + cn^2 + n^2) + (-2cn + en - 4cn + en + n) + (-e + d + c + 4c - 2e + d + 1)$

$c = 2c + 1 \rightarrow c = -1$

$e = -6c + 2e + 1 \rightarrow e = -7$

$d = -3e + 5c + 2d + 1 \rightarrow d = -17$

$b_n = -n^2 - 7n - 17$

$a_n = b_n + h_n \rightarrow h_n = h_{n-1} + h_{n-2}$

$r^2 - r - 1 = 0 \rightarrow r = \frac{1 \pm \sqrt{5}}{2}$

$h_n = \alpha_1 (\frac{1+\sqrt{5}}{2})^n + \alpha_2 (\frac{1-\sqrt{5}}{2})^n$

$a_n = -n^2 - 7n - 17 + \alpha_1 (\frac{1+\sqrt{5}}{2})^n + \alpha_2 (\frac{1-\sqrt{5}}{2})^n$

$a_0 = -17 + \alpha_1 + \alpha_2 = 1$

$a_1 = -1 - 7 - 17 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 2$

$= -25 + \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) = 2$

$\alpha_1 = 9 + \frac{17}{\sqrt{5}}$, $\alpha_2 = 9 - \frac{17}{\sqrt{5}}$

$$\therefore a_n = -n^2 - 7n - 17 + (9 + \frac{17}{\sqrt{5}})(\frac{1+\sqrt{5}}{2})^n + (9 - \frac{17}{\sqrt{5}})(\frac{1-\sqrt{5}}{2})^n$$

Counting

Counting is the task of finding the number of elements (also called the **cardinality**) of a given set.

When the set is small, we can count its elements "by hand". When sets are larger we need a more systematic way to count.

Say you have a six-sided die and a two-sided coin it comes out heads (H) or tails (T). What is the number of possible outcomes when both the die and the coin are tossed? There are 6 possible outcomes for the die and 2 for the coin, so the total number of outcomes is 6 . 2 = 12.

It will be useful to describe this kind of problem using the language of sets. The set **S** of possible outcomes of the die is **S** = {1, 2, 3, 4, 5, 6}, so **|S|** = 6. The set **T** of possible outcomes of the coin is **T** = {H, T}, so **|T|** = 2. The set of possible outcomes of the die and the coin is the product set $\boldsymbol{S} \times \boldsymbol{T}$:

$$\boldsymbol{S} \times \boldsymbol{T} = \{(1, \mathrm{H}), (1, \mathrm{T}), (2, \mathrm{H}), (2, \mathrm{T}), \ldots, (6, \mathrm{H}), (6, \mathrm{T})\}.$$

The number of elements of $\boldsymbol{S} \times \boldsymbol{T}$ is **|S|**.**|T|** =6 . 2 = 12.

In general, given any two finite sets **S** and **T** the product set $\boldsymbol{S} \times \boldsymbol{T}$ consists of all ordered pairs of elements (**s**, **t**) such that **s** is in **S** and **t** is in **T**:

$$\boldsymbol{S} \times \boldsymbol{T} = \{(s, t); s \in \boldsymbol{S} \; and \; t \in \boldsymbol{T}\}$$

Example

Let R and B be the sets of outcomes of a toss of a red and a blue six-sided die, respectively. Then R = {1, 2, 3, 4, 5, 6} and B = {1, 2, 3, 4, 5, 6}. When both dies are tossed, the set of outcomes is $R \times B = \{(1, 1), (1, 2), \ldots, (6, 6)\}$

and the number of outcomes is $\boldsymbol{R} \times \boldsymbol{B} = |\boldsymbol{R}|.|\boldsymbol{B}| = \mathbf{36}$

In cases like this when S = T we can denote the set $S \times T$ by $S^{\mathbf{2}}$**.** (This is the square of a set, not the square of a number). The set $S^{\mathbf{2}}$ has $|S|^2$ elements.

Problems

1. How many elements of the set of outcomes when 9 different six-sided dies are tossed?

2. How many elements of the product set of $S \times T \times R$ are there? Where $S = \{1,1,2\}$,$T = \{2, 1,1\}$, and $R = \{1, 2, 2\}$.

The sum rule

The sum rule says that if $A_1,\ A_2, \dots,\ A_n$ are disjoint sets then

$$|A_1\ \cup\ A_2 \cup \dots \cup\ A_n| = |A_1| + |A_2| + \cdots + |A_n|$$

Example

If you have 10 white balls, 7 blue balls, and 4 red balls, then the total number of balls you have is $10\ +\ 7\ +\ 4\ =\ 21$.

Example

Bart allocates his little sister Lisa a quota of 20 bad days, 40 irritable days, and 60 generally surly days. On how many days can Lisa be out-of-sorts one way or another?

Let set B be her bad days, I be her irritable days, and S be the generally surly. In these terms, the answer to the question is $|B\ \cup I \cup S|$. Now assuming that she is permitted at most one bad quality each day, the size of this union of sets is given by

$$|B\ \cup I \cup S| = |B| + |I| + |S| = 20 + 40 + 60 = 120 \text{ days.}$$

Few counting problems can be solved with a single rule. More often, a solution is a flurry of sums, products, and other methods.

Seventh lecture

Example

For solving problems involving passwords, telephone numbers, and license plates, the sum and product rules are useful together. For example, on a certain computer system, a valid password is a sequence of between six and eight symbols. The first symbol must be a letter (which can be lowercase or uppercase), and the remaining symbols must be either letters or digits. How many different passwords are possible?

Let's define two sets, corresponding to valid symbols in the first and subsequent positions in the password.

$$F = \{a, b, \dots, z, A, B, \dots, Z\}$$

$$S = \{a, b, \dots, z, A, B, \dots, Z, 0,1, 2, \dots ,9\ \}$$

In these terms, the set of all possible passwords is:

$$(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)$$

Thus, the length-six passwords are in the set $F \times S^5$, the length-seven passwords are in $F \times S^6$, and the length-eight passwords are in $F \times S^7$. Since these sets are disjoint, we can apply the sum rule and count the total number of possible passwords as follows:

$$\begin{aligned} |(F \times S^5) \cup (F \times S^6) \cup (F \times S^7)| &= |F \times S^5| + |F \times S^6| + |F \times S^7| \\ &= |F|.|S|^5 + |F|.|S|^6 + |F|.|S|^7 \\ &= 52 .\ 62^5 + 52 .\ 62^6 + 52 .\ 62^7 \\ &\approx 1.8.10^{14} \text{ different passwords} \end{aligned}$$

Product Rule

Let S be a set of length-k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- n_3 possible third entries for each first entry,

  ⋮
- n_k possible kth entries for each sequence of first $k - 1$ entries,

Then $|S| = n_1 . n_2 . n_3 \dots n_k$

Example

In how many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column?

The position of the three pieces is specified by a six numbers $(r_P , C_P , r_N , C_N, r_B , C_B)$ where, r_P, r_N and r_B are distinct rows and C_P, C_N and C_B are distinct columns. In particular, r_P is the pawn's row C_P is the pawn's column r_N is the knight's row, etc. Now we can count the number of such sequences using the product rule:

- r_P is one of 8 rows
- C_P is one of 8 columns
- r_N is one of 7 rows (anyone but r_P)
- C_N is one of 7 columns (anyone but C_P)
- r_B is one of 6 rows (anyone but r_P or r_N)
- C_B is one of 6 columns (anyone but C_P or C_N)

Thus, the total number of configurations is $8.8.7.7.6.6 = (8.7.6)^2$

Example

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Example

In how many ways can we select three students from a group of five students to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture? First, note that the order in which we select the student's matters. There are five ways to select the first student to stand at the start of the line. Once this student has been selected, there are four ways to select the second student in the line. After the first and second students have been selected, there are three ways to select the third student in the line. By the product rule, there are $5.4.3 = 60$ ways to select three students from a group of five students to stand in line for a picture.

To arrange all five students in a line for a picture, we select the first student in five ways, the second in four ways, the third in three ways, the fourth in two ways, and the fifth in one way. Consequently, there are $5.4.3.2.1 = 120$ ways to arrange all five students in a line for a picture.

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an ***r*-permutation**.

Example

Let $S = \{1, 2, 3\}$.The ordered arrangement 3, 1, 2 is a permutation of S. The ordered arrangement 3, 2 is a 2-permutation of S.

The number of r-permutation of a set with n elements is denoted by $p(n, r)$ We can find $P(n, r)$ using the product rule.

Theorem

If n and r are integers with $0 \leq r \leq n$, then $P(n, r) = \frac{n!}{(n-r)!}$

Example

Let $S = \{a, b, c\}$.The 2-permutation of S are the ordered arrangements a, b; a, c; b, a; b, c; c, a; and c, b. Consequently, there are six 2-permutation of this set with three elements, it follows that $P(n,r) = \frac{n!}{(n-r)!} = \frac{3!}{(3-2)!} = 3 \cdot 2 = 6$.

Example

How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

Because it matters which person wins which prize, the number of ways to pick the three prize winners is the number of ordered selections of three elements from a set of 100 elements, that is, the number of 3-permutations of a set of 100 elements. Consequently, the answer is $P(n,r) = \frac{n!}{(n-r)!} = \frac{100!}{(100-3)!} = 100 \cdot 99 \cdot 98 = 970{,}200$.

The eighth lecture

Example

Suppose that there are eight runners in a race. The winner receives a gold medal, the second-place finisher receives a silver medal, and the third-place finisher receives a bronze medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur and there are no ties?

The number of different ways to award the medals is the number of 3-permutations of a set with eight elements. Hence, there are $P(n,r) = \frac{n!}{(n-r)!} = \frac{8!}{(8-3)!} = 8 \cdot 7 \cdot 6 = 336$ possible ways to award the medals.

Example

How many permutations of the letters $ABCDEFGH$ contain the string ABC?

Because the letters ABC must occur as a block, we can find the answer by finding the number of permutations of six objects, namely, the block ABC and the individual letters $D, E, F, G, and\ H$. Because these six objects can occur in any order, there are $P(n,r) = \frac{n!}{(n-r)!} = \frac{6!}{(6-6)!} = 6! = 720$ permutations of the letters $ABCDEFGH$ in which ABC occurs as a block.

Combinations

An $\boldsymbol{r}$**-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Example

Let S be the set {1, 2, 3, 4}. Then {1, 3, 4} is a 3-combination from S. (Note that {4, 1, 3} is the same 3-combination as {1, 3, 4}, because the order in which the elements of a set are listed does not matter.)

Note:- The number of r-combinations of a set with n distinct elements is denoted by $C(n, r)$.

Example

We see that $C(4,2) = 6$, because the 2-combinations of $\{a, b, c, d\}$ are the six subsets $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, and\ \{c, d\}$.

Theorem

The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals $C(n,r) = \frac{n!}{(n-r)!.r!}$

Example

How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Because the order in which the five cards are dealt from a deck of 52 cards does not matter, there are $C(52, 5) = \frac{52!}{47!.5!} = 26 \cdot 17 \cdot 10 \cdot 49 \cdot 12 = 2{,}598{,}960$ different poker hands of five cards that can be dealt from a standard deck of 52 cards.

Note that there are $C(52,47) = \frac{52!}{47!.5!} = 2{,}598{,}960$ different ways to select 47 cards from a standard deck of 52 cards.

Example

How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?

The answer is given by the number of 5-combinations of a set with 10 elements. the number of such combinations is $C(10,5) = \frac{10!}{5!.5!} = 252$.

Example

Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Problems

1. Answer the following with true or false:
   - There are 700 permutations of $\{a, b, c, d, e, f, g\}$ end with a.
   - The value of $C(8,4)$ is 70.
   - There are $23 + 15 = 38$ ways for a student can choose a computer project from one of three lists contain $23, 15,$ and 19 possible projects, respectively. No project is on more than one list.
   - **There are $\mathbf{5 \times 6 \times 21^5}$ =122,523,030 strings have exactly one vowel** of six lowercase letters of the English alphabet, where an alphabet has 21 consonants and 5 vowels.
2. Suppose that either a member of the mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 37 members of the mathematics faculty and 83 mathematics majors and no one is both a faculty member and a student?
3. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?
4. List all the permutations of $\{a, b, c\}$.
5. Find the number of 5-permutations of a set with nine elements.
6. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?

7. How many permutations of the letters $ABCDEFG$ contain
   - The string BCD?
   - The string $CFGA$?
   - The strings ABC and DE?
8. A club has 25 members.

- How many ways are there to choose four members of the club to serve on an executive committee?
- How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

9. The English alphabet contains 21 consonants and five vowels. How many strings of six lowercase letters of the English alphabet contain exactly two vowels?

The ninth lecture

Generating Functions

Generating functions are used to represent sequences efficiently by coding the terms of a sequence as coefficients of powers of a variable x in a formal power series. Generating functions can be used to solve many types of counting problems, such as the number of ways to select or distribute objects of different kinds, subject to a variety of constraints. Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating function. From this closed form, the coefficients of the power series for the generating function can be found, solving the original recurrence relation.

Definition The generating function for the sequence $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series $G(x) = a_0 + a_1 x + \cdots + a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$

Example

The generating functions for the sequences $\{a_k\}$ with $a_k = 3$, $a_k = k + 1$, $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$, $\sum_{k=0}^{\infty} (k+1)x^k$, *and* $\sum_{k=0}^{\infty} 2^k x^k$, respectively.

Example

What is the generating function for the sequence $1, 1, 1, 1, 1, 1$?

The generating function of $1, 1, 1, 1, 1, 1$ is

$$1 + x + x^2 + x^3 + x^4 + x^5$$

By **theorem (*)** we have

$$\frac{(x^6 - 1)}{(x - 1)} = 1 + x + x^2 + x^3 + x^4 + x^5$$

when $x \neq 1$. Consequently, $G(x) = \frac{(x^6-1)}{(x-1)}$ is the generating function of the sequence $1, 1, 1, 1, 1, 1$.

Theorem (*) If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & if \ r \neq 0 \\ (n+1)a & if \ r = 1 \end{cases}$$

Example

Let m be a positive integer. Let $a_k = C(m, k)$, for $k = 0, 1, 2, \ldots, m$. What is the generating function for the sequence $a_0, a_1, \ldots, a_m$?

The generating function for this sequence is

$$G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \cdots + C(m,m)x^m$$

The binomial theorem let x and y be variables, and let n be a nonnegative integer. then $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$

The **binomial theorem** shows that $G(x) = (1 + x)^m$.

Counting Problems and Generating Functions

Generating functions can be used to solve a wide variety of counting problems. In particular, they can be used to count the number of combinations of various types. Such problems are equivalent to counting the solutions to equations of the form

$$e_1 + e_2 + \cdots + e_n = C$$

where C is a constant and each e_i is a nonnegative integer that may be subject to a specified constraint.

Example

Find the number of solutions of $e_1 + e_2 + e_3 = 17$,

Where $e_1, e_2, and\ e_3$ are nonnegative integers with_$2 \leq e_1 \leq 5,\ \ 3 \leq e_2 \leq 6,\ \ 4 \leq e_3 \leq 7$.

The number of solutions with the indicated constraints is the coefficient of x^{17} in the expansion of

$$(x^2 + x^3 + x^4 + x^5)(x^3 + x^4 + x^5 + x^6)(x^4 + x^5 + x^5 + x^6)$$

This follows because we obtain a term equal to x^{17} in the product by picking a term in the first sum x^{e_1} , a term in the second sum x^{e_2} , and a term in the third sum x^{e_3}, where the exponents $e_1, e_2,\ and e_3$ satisfy the equation $e_1 + e_2 + e_3 = 17$ and the given constraints. It is not hard to see that the coefficient of x^{17} in this product is 3.

Example

In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies?

Because each child receives at least two but no more than four cookies, for each child there is a factor equal to

$$x^2 + x^3 + x^4$$

in the generating function for the sequence $\{c_n\}$, where c_n is the number of ways to distribute the number of ways to distribute n cookies. Because there are three children, this generating function is

$$(x^2 + x^3 + x^4)^3$$

We need the coefficient of x^8 in this product. The reason is that the x^8 terms in the expansion correspond to the ways that three terms can be selected, with one from each factor, that have exponents adding up to 8. Furthermore, the exponents of the term from the first, second, and third factors are the numbers of cookies the first, second, and third children receive, respectively. Computation shows that this coefficient equals 6. Hence, there are six ways to distribute the cookies so that each child receives at least two, but no more than four, cookies.

Tenth lecture

Using Generating Functions to Solve Recurrence Relations

We can find the solution to a recurrence relation and its initial conditions by finding an explicit formula for the associated generating function.

Example

Solve the recurrence relation $a_k = 3a_{k-1}\ \ for\ k = 1, 2, 3, ...$ and initial condition $a_0 = 2$.

Let $G(x)$ be the generating function for the sequence $\{a_k\}$, that is, $G(x) = \sum_{k=0}^{\infty} a_k x^k$. First note that

$$xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k.$$

Using the recurrence relation, we see that

$$G(x) - 3xG(x) = \sum_{k=0}^{\infty} a_k x^k - 3\sum_{k=1}^{\infty} a_{k-1} x^k$$
$$= a_0 + \sum_{k=1}^{\infty} (a_k - 3a_{k-1}) x^k$$
$$= 2,$$

because $a_0 = 2$ and $a_k = 3a_{k-1}$. Thus,

$$G(x) - 3xG(x) = (1 - 3x)G(x) = 2$$

Solving for $G(x)$ shows that $G(x) = \frac{2}{(1-3x)}$. Using the identity $\frac{1}{(1-ax)} = \sum_{k=0}^{\infty} a^k x^k$.

We have

$$G(x) = 2\sum_{k=0}^{\infty} 3^k x^k = \sum_{k=0}^{\infty} 2.3^k x^k.$$

Consequently, $a_k = 2.3^k$

Note:-

1. $\frac{1}{(1-ax)} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$
2. $\frac{1}{(1-x)} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$

Example

Suppose that a valid codeword is an n-digit number in decimal notation containing an even number of $0s$. Let a_n denote the number of valid codewords of length n. the sequence $\{a_n\}$ satisfies the recurrence relation

$$a_n = 8a_{n-1} + 10^{n-1}$$

and the initial condition $a_0 = 1$. Use generating functions to find an explicit formula for a_n.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

$xG(x) = x\sum_{n=1}^{\infty} a_{n-1}x^{n-1} = \sum_{n=1}^{\infty} a_{n-1}x^n$

Using the recurrence relation, we see that

$$G(x) - 8xG(x) = \sum_{n=0}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1}x^n = a_0 + \sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1}x^n$$

$$= 1 + \sum_{n=1}^{\infty}(a_n - 8\, a_{n-1})x^n$$

$$= 1 + \sum_{n=1}^{\infty}(8a_{n-1} - 8a_{n-1} + 10^{n-1})x^n$$

$$= 1 + \sum_{n=1}^{\infty} 10^{n-1}\, x^n$$

$$= 1 + \sum_{n=0}^{\infty} 10^{n-1+1}\, x^{n+1}$$

$$= 1 + x\sum_{n=0}^{\infty} 10^n\, x^n$$

$$(1-8x)G(x) = 1 + \frac{x}{1-10x}$$

Solving for $G(x)$ shows that

$$G(x) = \frac{1-9x}{(1-8x)(1-10x)}$$

Expanding the right-hand side of this equation into partial fractions (as is done in the integration of rational functions studied in calculus) gives

$$G(x) = \frac{1}{2}\left(\frac{1}{1-8x} + \frac{1}{1-10x}\right)$$

$$G(x) = \frac{1}{2}\left(\sum_{n=0}^{\infty} 8^n x^n + \sum_{n=0}^{\infty} 10^n x^n\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{2}(8^n + 10^n)x^n$$

Consequently, we have shown that

$$a_n = \frac{1}{2}(8^n + 10^n)$$

Problems

1. Answer of the following with true or false:
   - The generating function for the finite sequence 2, 2, 2, 2, 2 is $G(x) = \frac{2(x^4-1)}{(x-1)}$.
   - The expansion of $(x+y)^4$ is $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
   - There are 9 different ways for generating functions of the number of different ways 10 identical balloons can be given to four children if each child receives at least two balloons.

- The generating function of the recurrence relation $a_k = 3a_{k-1} + 2$ with the initial condition $a_0 = 1$ is $a_k = 2.3^k - 1$.

2. Find the generating function for the finite sequence $1, 4, 16, 64, 256$.
3. Use generating functions to determine the number of different ways 15 identical stuffed animals can be given to six children so that each child receives at least one but no more than three stuffed animals.
4. Use generating functions to solve the following recurrence relation:
   - $a_k = 5a_{k-1} - 6a_{k-2}$ with initial conditions $a_0 = 6$ and $a_1 = 30$.
   - $a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6$ with initial conditions $a_0 = 20$ and $a_1 = 60$.

$a_k = 2a_{k-1} + 3a_{k-2} + 4^k + 6, \qquad a_0 = 20$ and $a_1 = 60$

Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$

$xG(x) = x\sum_{k=1}^{\infty} a_{k-1}x^{k-1} = \sum_{k=1}^{\infty} a_{k-1}x^k$

$x^2G(x) = x^2\sum_{k=2}^{\infty} a_{k-2}x^{k-2} = \sum_{k=2}^{\infty} a_{k-2}x^k$

$G(x) - 2xG(x) - 3x^2G(x) = \sum_{k=0}^{\infty} a_k x^k - 2\sum_{k=1}^{\infty} a_{k-1}x^k - 3\sum_{k=2}^{\infty} a_{k-2}x^k$

$(1 - 2x - 3x^2)G(x) = a_0 + a_1x + \sum_{k=2}^{\infty} a_k x^k - 2a_0x - 2\sum_{k=2}^{\infty} a_{k-1}x^k - 3\sum_{k=2}^{\infty} a_{k-2}x^k$

$(1 - 2x - 3x^2)G(x) = 20 + 60x - 40x + \sum_{k=2}^{\infty}(a_k - 2a_{k-1} - 3a_{k-2})x^k$

$(1 - 2x - 3x^2)G(x) = 20 + 20x + \sum_{k=2}^{\infty}(2a_{k-1} + 3a_{k-2} + 4^k + 6 - 2a_{k-1} - 3a_{k-2})x^k$

$(1 - 2x - 3x^2)G(x) = 20 + 20x + \sum_{k=2}^{\infty} 4^k x^k + 6\sum_{k=2}^{\infty} x^k$

$(1 - 2x - 3x^2)G(x) = 20 + 20x - 1 - 4x + \sum_{k=0}^{\infty} 4^k x^k - 6 - 6x + 6\sum_{k=0}^{\infty} x^k$

$(1 - 2x - 3x^2)G(x) = 13 + 10x + \frac{1}{1-4x} + \frac{6}{1-x}$

$G(x) = \frac{13+10x}{(1+x)(1-3x)} + \frac{1}{(1+x)(1-3x)(1-4x)} + \frac{6}{(1+x)(1-3x)(1-x)}$

Expanding the right-hand side of this equation into partial fractions gives

$G(x) = \frac{\frac{31}{20}}{(1+x)} + \frac{-\frac{3}{2}}{(1-x)} + \frac{\frac{16}{5}}{(1-4x)} + \frac{\frac{67}{4}}{(1-3x)}$

$G(x) = \frac{31}{20}\sum_{k=0}^{\infty}(-1)^k x^k - \frac{3}{2}\sum_{k=0}^{\infty} x^k + \frac{16}{5}\sum_{k=0}^{\infty} 4^k x^k + \frac{67}{4}\sum_{k=0}^{\infty} 3^k x^k$

$a_k = \frac{31}{20}.(-1)^k - \frac{3}{2} + \frac{16}{5}.4^k + \frac{67}{4}.3^k$