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- The concept of a *set* appears in all mathematics. The course introduces the notation and terminology of set theory which is basic and used throughout the text.
- SETS AND ELEMENTS, SUBSETS
- A set may be viewed as any well-defined collection of objects, called the *elements* or members of the set.
- One usually uses capital letters, A, B, X, Y,, to denote sets, and lowercase letters, a,b,x,y,...., to denote elements of sets. Synonyms for "set" are "class,""collection" and "family."
- Membership in a set is denoted as follows:
- $a \in S$ denotes that a belongs to a set S
- $a, b \in S$ denotes that a, b belong to a set S.
- Here ∈ is the symbol meaning "is an element of." We use ∉ to mean "is not an element of."



Specifying Sets

 There are essentially two ways to specify a particular set. One way, if possible, is to list its members separated by commas and contained in braces{ }. A second way is to state those properties which characterized the elements in the set. Example illustrating those two ways are:

• $A = \{1,3,5,7,9\}$ and $B = \{x \mid x \text{ is an even integer, } x > 0\}$

• That is, A consists of the numbers 1, 3, 5, 7, 9. the second set, which reads:

• B is the set of x such that x is an even integer and x is greater than 0.

Denotes the set B whose elements are the positive integers. Note that a letter, usually x, is used to denote a typical member of the set: and the vertical line | is read as "such that" and the comma as "and."



• Example 1.1

- (a) The set A above can also be written $A = \{x | x \text{ is an odd positive integer}, x < 10\}$.
- (b) We cannot list all the elements of the above set B although frequently we specify the set by

•
$$B = \{2, 4, 6, \ldots\}$$

- Where we assume that everyone knows what we mean. Observe that $8 \in B$, but $3 \notin B$.
- (c) Let $E = \{x \mid x^2 3x + 2 = 0\}$, $F = \{2,1\}$ and $G = \{1,2,2,1\}$. Then E = F = G.
- We emphasize that a set does not depend on the way in which its elements are displayed. A set remains the same if its elements are repeated or rearranged.
- Even if we can list the elements of a set, it may not be practical to do so. That is, we
 describe a set by listing its elements only if the set contains a few elements: otherwise
 we describe a set by the property which characterizes its elements.



Subset

Suppose every elements in a set A is also an element of a set B, that is, suppose a ∈ A implies a ∈ B. Then A is called a subset of B. We also say that A is contained in B or that B contains A. This relationship is written

• $A \subseteq B$ or $B \supseteq A$

 Two sets are equal if they both have the same elements or, equivalently, if each is contained in the other. That is:

• A = B if and only if $A \subseteq B$ and $B \subseteq A$

If A is not a subset of B, that is, if at least one element of A does not belong to B, we write A ⊈ B.



• **EXAMPLE 1.2** Consider the sets:

• $A = \{1, 3, 4, 7, 8, 9\}, B = \{1, 2, 3, 4, 5\}, C = \{1, 3\}.$

- Then C ⊆ A and C ⊆ B since 1 and 3, the elements of C, are also members of A and B. But B ∉ A since some of the elements of B, e.g., 2 and 5, do not belong to A. Similarly, A ∉ B.
- Property 1: It is common practice in mathematics to put a vertical line "|" or slanted line "/" through a symbol to indicate the opposite or negative meaning of a symbol.
- **Property 2:** The statement $A \subseteq B$ does not exclude the possibility that A = B. In fact, for every set A we have $A \subseteq A$ since, trivially, every element in A belongs to A. However, if $A \subseteq B$ and $A \neq B$, then we say A is a proper subset of B (sometimes written $A \subset B$).



- **Property 3:** Suppose every element of a set A belongs to a set B and every element of B belongs to a set C. Then clearly every element of A also belongs to C. In other words, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
- The above remarks yield the following theorem.
- **Theorem 1.1:** Let A, B, C be any sets. Then:
- (i) $A \subseteq A$
- (ii) If $A \subseteq B$ and $B \subseteq A$, then A = B
- (iii) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$



Special symbols

- Some sets will occur very often in the text, and so we use special symbols for them.
 Some such symbols are:
- **N** = the set of *natural numbers* or positive integers: 1, 2, 3, ...
- **Z** = the set of all integers: ..., -2, -1, 0, 1, 2, ...
- **Q** = the set of rational numbers
- **R** = the set of real numbers
- **C** = the set of complex numbers
- Observe that $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$.



• Universal Set, Empty Set

- All sets under investigation in any application of set theory are assumed to belong to some fixed large set called the *universal set* which we denote by U unless otherwise stated or implied.
- Given a universal set U and a property P, there may not be any elements of U which have property P. For example, the following set has no elements:
- $S = \{x \mid x \text{ is a positive integer, } x^2 = 3\}$
- Such a set with no elements is called the *empty set* or *null set* and is denoted by Ø
 there is only one empty set. That is, if S and T are both empty, then S = T, since they
 have exactly the same elements, namely, none.
- The empty set Ø is also regarded as a subset of every other set. Thus we have the following simple result which we state formally.



• **Theorem 1.2:** For any set *A*, we have $\emptyset \subseteq A \subseteq \mathbf{U}$.

Disjoint Sets

- Two sets A and B are said to be *disjoint* if they have no elements in common. For example, suppose
- *A* = {1, 2}, *B*= {4, 5, 6}, and *C* = {5, 6, 7, 8}
- Then A and B are disjoint, and A and C are disjoint. But B and C are not disjoint since B and C have elements in common, e.g., 5 and 6.We note that if A and B are disjoint, then neither is a subset of the other (unless one is the empty set).





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VENN DIAGRAM

- A Venn diagram is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.
- The universal set **U** is represented by the interior of a rectangle, and the other sets are represented by disks lying within the rectangle.



- If $A \subseteq B$, then the disk representing A will be entirely within the disk representing B
- as in next figure.



VENN DIAGRAM



• If A and B are disjoint, then the disk representing A will be separated from the disk representing B as in the next figure.





VENN DIAGRAM

• However, if A and B are two arbitrary sets, it is possible that some objects are in A but not in B, some are in B but not in A, some are in both A and B, and some are in neither A nor B; hence in general we represent A and B as in the next figure.





ARGUMENTS AND VENN DIAGRAMS

- Many verbal statements are essentially statements about sets and can therefore be described by Venn diagrams. Hence Venn diagrams can sometimes be used to determine whether or not an argument is valid.
- **EXAMPLE 1.3** Show that the following argument (adapted from a book on logic by Lewis Carroll, the author of *Alice in Wonderland*) is valid:
- S1: All my tin objects are saucepans.
- S2: I find all your presents very useful.
- S3: None of my saucepans is of the slightest use.
- S: Your presents to me are not made of tin.
- The statements S1, S2, and S3 above the horizontal line denote the assumptions, and the statement S below the line denotes the conclusion.



ARGUMENTS AND VENN DIAGRAMS

- The argument is valid if the conclusion *S* follows logically from the assumptions *S*1, *S*2, and *S*3.
- By S1 the tin objects are contained in the set of saucepans, and by S3 the set of saucepans and the set of useful things are disjoint. Furthermore, by S2 the set of "your presents" is a subset of the set of useful things. Accordingly, we can draw the Venn diagram in the next figure.
- The conclusion is clearly valid by the Venn diagram because the set of "your presents" is disjoint from the set of tin objects.





SET OPERATIONS

• This section introduces a number of set operations, including the basic operations of union, intersection, and complement.

• Union and Intersection

- The union of two sets A and B, denoted by A ∪ B, is the set of all elements which belong to A or to B; that is,
- $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
- Here "or" is used in the sense of and/or. Figure (a) is a Venn diagram in which A ∪ B is shaded.





(a) $A \cup B$ is shaded

SET OPERATIONS

- The *intersection* of two sets A and B, denoted by $A \cap B$, is the set of elements which belong to both A and B; that is,
- $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
- Figure (b) is a Venn diagram in which $A \cap B$ is shaded.



(b) $A \cap B$ is shaded

- Recall that sets A and B are said to be *disjoint* or *nonintersecting* if they have no elements in common or, using the definition of intersection, if $A \cap B = \emptyset$, the empty set. Suppose
- = $A \cup B$ and $A \cap B = \emptyset$
- Then S is called the *disjoint union* of A and B.



EXAMPLE 1.4

- (a) Let $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6, 7\}, C = \{2, 3, 8, 9\}$. Then
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}, A \cup C = \{1, 2, 3, 4, 8, 9\}, B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9\},\$
- $A \cap B = \{3, 4\}, A \cap C = \{2, 3\}, B \cap C = \{3\}.$
- (b) Let U be the set of students at a university, and let M denote the set of male students and let F denote the set of female students. The U is the disjoint union of M of F; that is,
- $\mathbf{U} = M \cup F$ and $M \cap F = \emptyset$
- This comes from the fact that every student in U is either in M or in F, and clearly no student belongs to both M and F, that is, M and F are disjoint.
- The following properties of union and intersection should be noted.



PROPERTIES OF UNION AND INTERSECTION

• **Property 1:** Every element x in $A \cap B$ belongs to both A and B; hence x belongs to A and x belongs to B. Thus $A \cap B$ is a subset of A and of B; namely

• $A \cap B \subseteq A$ and $A \cap B \subseteq B$

Property 2: An element x belongs to the union A∪ B if x belongs to A or x belongs to B; hence every element in A belongs to A ∪ B, and every element in B belongs to A ∪ B. That is,

• $A \subseteq A \cup B$ and $B \subseteq A \cup B$

- We state the above results formally:
- **Theorem 1.3:** For any sets *A* and *B*, we have:

• (i) $A \cap B \subseteq A \subseteq A \cup B$ and (ii) $A \cap B \subseteq B \subseteq A \cup B$.



PROPERTIES OF UNION AND INTERSECTION

- The operation of set inclusion is closely related to the operations of union and intersection, as shown by the following theorem.
- **Theorem 1.4:** The following are equivalent: $A \subseteq B, A \cap B = A, A \cup B = B$.
- This theorem is proved in Problem 1.8. Other equivalent conditions to are given in Problem 1.31.

Complements, Differences, Symmetric Differences

 Recall that all sets under consideration at a particular time are subsets of a fixed universal set U. The *absolute complement* or, simply, *complement* of a set A, denoted by A^C, is the set of elements which belong to U but which do not belong to A. That is,

•
$$A^{\mathsf{C}} = \{x \mid x \in \mathbf{U}, x \notin A\}$$



COMPLEMENTS, DIFFERENCES, SYMMETRIC DIFFERENCES

Some texts denote the complement of A by À or A. Fig. (a) is a Venn diagram in which A^C is shaded.



The relative complement of a set B with respect to a set A or, simply, the difference of A and B, denoted by A\B, is the set of elements which belong to A but which do not belong to B; that is

•
$$A \setminus B = \{x \mid x \in A, x \notin B\}$$

The set A\B is read "A minus B." Many texts denote A\B by A - B or A ~ B. Fig. (b) is a Venn diagram in which A\B is shaded.



COMPLEMENTS, DIFFERENCES, SYMMETRIC DIFFERENCES



(b) $A \setminus B$ is shaded

• The symmetric difference of sets A and B, denoted by $A \oplus B$, consists of those elements which belong to A or B but not to both. That is,

• $A \oplus B = (A \cup B) \setminus (A \cap B)$ or $A \oplus B = (A \setminus B) \cup (B \setminus A)$

• Figure (c) is a Venn diagram in which $A \oplus B$ is shaded.





EXAMPLE 1.5

• Suppose $\mathbf{U} = \mathbf{N} = \{1, 2, 3, \ldots\}$ is the universal set. Let

• $A = \{1, 2, 3, 4\}, B = \{3, 4, 5, 6, 7\}, C = \{2, 3, 8, 9\}, E = \{2, 4, 6, \ldots\}$

• (Here *E* is the set of even integers.) Then:

• $AC = \{5, 6, 7, \ldots\}, BC = \{1, 2, 8, 9, 10, \ldots\}, EC = \{1, 3, 5, 7, \ldots\}$

• That is, *EC* is the set of odd positive integers. Also:

• $A \setminus B = \{1, 2\}, A \setminus C = \{1, 4\}, B \setminus C = \{4, 5, 6, 7\}, A \setminus E = \{1, 3\},$

• $B \setminus A = \{5, 6, 7\}, C \setminus A = \{8, 9\}, C \setminus B = \{2, 8, 9\}, E \setminus A = \{6, 8, 10, 12, \ldots\}.$

• Furthermore:

• $A \oplus B = (A \setminus B) \cup (B \setminus A) = \{1, 2, 5, 6, 7\}, B \oplus C = \{2, 4, 5, 6, 7, 8, 9\},\$

• $A \oplus C = (A \setminus C) \cup (C \setminus A) = \{1, 4, 8, 9\}, A \oplus E = \{1, 3, 6, 8, 10, \ldots\}.$



FUNDAMENTAL PRODUCTS

 Consider n distinct sets A1, A2, ..., An. A fundamental product of the sets is a set of the form

•
$$A_1^* \cap A_2^* \cap \ldots \cap A_n^*$$
 where $A_i^* = A$ or $A_i^* = A^C$

- We note that:
- (i) There are $m = 2^n$ such fundamental products.
- (ii) Any two such fundamental products are disjoint.
- (iii) The universal set **U** is the union of all fundamental products.
- Thus U is the disjoint union of the fundamental products. There is a geometrical description of these sets which is illustrated below.



EXAMPLE 1.6

- Figure (a) is the Venn diagram of three sets A, B, C. The following lists the $m = 2^3 = 8$ fundamental products of the sets A, B, C:
- $P1 = A \cap B \cap C$, $P2 = A \cap B \cap C^{C}$, $P3 = A \cap B^{C} \cap C$, $P4 = A \cap B^{C} \cap C^{C}$
- $P5 = A^C \cap B \cap C$, $P6 = A^C \cap B \cap C^C$, $P7 = A^C \cap B^C \cap C$, $P8 = A^C \cap B^C \cap C^C$.
- The eight products correspond precisely to the eight disjoint regions in the Venn diagram of sets A, B, C as indicated by the labeling of the regions in Fig. (b).







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ALGEBRA OF SETS, DUALITY

- Sets under the operations of union, intersection, and complement satisfy various laws (identities) which are listed in Table 1-1. In fact, we formally state this as:
- **Theorem 1.5:** Sets satisfy the laws in Table 1-1.

Idempotent laws:	$(1a) A \cup A = A$	(1b) $A \cap A = A$
Associative laws:	$(2a) (A \cup B) \cup C = A \cup (B \cup C)$	(2b) $(A \cap B) \cap C = A \cap (B \cap C)$
Commutative laws:	$(3a) A \cup B = B \cup A$	$(3b) A \cap B = B \cap A$
Distributive laws:	(4a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	(4b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity laws:	$(5a) A \cup \emptyset = A$	(5b) $A \cap \mathbf{U} = A$
	$(6a) A \cup \mathbf{U} = \mathbf{U}$	(6b) $A \cap \emptyset = \emptyset$
Involution laws:	$(7) (A^{\mathcal{C}})^{\mathcal{C}} = A$	
Complement laws:	$(8a) A \cup A^{C} = \mathbf{U}$	$(8b) A \cap A^{C} = \emptyset$
	(9a) $\mathbf{U}^{\mathbf{C}} = \emptyset$	$(9b) \mathscr{O}^{C} = \mathbf{U}$
DeMorgan's laws:	$(10a) (A \cup B)^{C} = A^{C} \cap B^{C}$	$(10b) (A \cap B)^{C} = A^{C} \cup B^{C}$

Table 1-1Laws of the algebra of sets



ALGEBRA OF SETS, DUALITY

• **Remark:** Each law in Table 1-1 follows from an equivalent logical law. Consider, for example, the proof of DeMorgan's Law 10(a):

• $(A \cup B)^{C} = \{x \mid x \notin (A \text{ or } B)\} = \{x \mid x \notin A \text{ and } x \notin B\} = A^{C} \cap B^{C}$

• Here we use the equivalent (DeMorgan's) logical law:

• $\neg (p \lor q) = \neg p \land \neg q$

where ¬ means "not," ∨ means "or," and ∧ means "and." (Sometimes Venn diagrams are used to illustrate the laws in Table 1-1 as in Problem 1.17.)



ALGEBRA OF SETS, DUALITY

Duality

The identities in Table 1-1 are arranged in pairs, as, for example, (2a) and (2b).We now consider the principle behind this arrangement. Suppose *E* is an equation of set algebra. The dual *E*^{*} of *E* is the equation obtained by replacing each occurrence of ∪, ∩, **U** and Ø in *E* by ∩, ∪, Ø, and **U**, respectively. For example, the dual of

• $(\mathbf{U} \cap A) \cup (B \cap A) = A$ is $(\emptyset \cup A) \cap (B \cup A) = A$

 Observe that the pairs of laws in Table 1-1 are duals of each other. It is a fact of set algebra, called the *principle of duality*, that if any equation E is an identity then its dual E* is also an identity.



FINITE SETS, COUNTING PRINCIPLE

 Sets can be finite or infinite. A set S is said to be *finite* if S is empty or if S contains exactly m elements where m is a positive integer; otherwise S is *infinite*.

• EXAMPLE 1.7

- (a) The set A of the letters of the English alphabet and the set D of the days of the week are finite sets. Specifically, A has 26 elements and D has 7 elements.
- (b) Let *E* be the set of even positive integers, and let **I** be the *unit interval*, that is,

•
$$E = \{2, 4, 6, \ldots\}$$
 and $\mathbf{I} = [0, 1] = \{x \mid 0 \le x \le 1\}$

- Then both *E* and **I** are infinite.
- A set **S** is *countable* if **S** is finite or if the elements of **S** can be arranged as a sequence, in which case **S** is said to be *countably infinite*; otherwise **S** is said to be *uncountable*. The above set *E* of even integers is countably infinite, whereas one can prove that the unit interval $\mathbf{I} = [0, 1]$ is uncountable.



- The notation n(S) or |S| will denote the number of elements in a set S. (Some texts use #(S) or card(S) instead of n(S).) Thus n(A) = 26, where A is the letters in the English alphabet, and n(D) = 7, where D is the days of the week. Also $n(\emptyset) = 0$ since the empty set has no elements.
- The following lemma applies.
- Lemma 1.6: Suppose A and B are finite disjoint sets. Then $A \cup B$ is finite and
- $\bullet n(A \cup B) = n(A) + n(B)$
- This lemma may be restated as follows:
- Lemma 1.6: Suppose S is the disjoint union of finite sets A and B. Then S is finite and *Proof.*

$$\bullet n(S) = n(A) + n(B)$$



- In counting the elements of A ∪ B, first count those that are in A. There are n(A) of these. The only other elements of A ∪ B are those that are in B but not in A. But since A and B are disjoint, no element of B is in A, so there are n(B) elements that are in B but not in A. Therefore, n(A ∪ B) = n(A) + n(B).
- For any sets A and B, the set A is the disjoint union of $A \setminus B$ and $A \cap B$. Thus Lemma 1.6 gives us the following useful result.
- Corollary 1.7: Let A and B be finite sets. Then

• $n(A \setminus B) = n(A) - n(A \cap B)$

 For example, suppose an art class A has 25 students and 10 of them are taking a biology class B. Then the number of students in class A which are not in class B is:

• $n(A \setminus B) = n(A) - n(A \cap B) = 25 - 10 = 15$



- Given any set A, recall that the universal set **U** is the disjoint union of A and A^{C} . Accordingly, Lemma 1.6 also gives the following result.
- Corollary 1.8: Let A be a subset of a finite universal set U. Then

• $n(A^{\mathrm{C}}) = n(\mathbf{U}) - n(A)$

- For example, suppose a class U with 30 students has 18 full-time students. Then there are 30–18 = 12 part-time students in the class U.
- Inclusion-Exclusion Principle
- There is a formula for n(A ∪ B) even when they are not disjoint, called the Inclusion–Exclusion Principle. Namely:
- Theorem (Inclusion–Exclusion Principle) 1.9: Suppose A and B are finite sets. Then $A \cup B$ and $A \cap B$ are finite and



 $\bullet n(A \cup B) = n(A) + n(B) - n(A \cap B)$

- That is, we find the number of elements in A or B (or both) by first adding n(A) and n(B) (inclusion) and then subtracting n(A ∩ B) (exclusion) since its elements were counted twice.
- We can apply this result to obtain a similar formula for three sets:
- Corollary 1.10: Suppose A, B, C are finite sets. Then $A \cup B \cup C$ is finite and

■ $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

 Mathematical induction (Section 1.8) may be used to further generalize this result to any number of finite sets.



- EXAMPLE 1.8 Suppose a list A contains the 30 students in a mathematics class, and a list B contains the 35 students in an English class, and suppose there are 20 names on both lists. Find the number of students: (a) only on list A, (b) only on list B, (c) on list A or B (or both), (d) on exactly one list.
- (a) List A has 30 names and 20 are on list B; hence 30 20 = 10 names are only on list A.
- (b) Similarly, 35 20 = 15 are only on list *B*.
- (c) We seek $n(A \cup B)$. By inclusion–exclusion,

• $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 30 + 35 - 20 = 45.$

 In other words, we combine the two lists and then cross out the 20 names which appear twice.


COUNTING ELEMENTS IN FINITE SETS

• (d) By (a) and (b), 10 + 15 = 25 names are only on one list; that is, $n(A \oplus B) = 25$.

I.7 CLASSES OF SETS, POWER SETS, PARTITIONS

- Given a set S, we might wish to talk about some of its subsets. Thus we would be considering a set of sets. Whenever such a situation occurs, to avoid confusion, we will speak of a class of sets or collection of sets rather than a set of sets. If we wish to consider some of the sets in a given class of sets, then we speak of subclass or subcollection.
- **EXAMPLE 1.9** Suppose *S* = {1, 2, 3, 4}.
- (a) Let A be the class of subsets of S which contain exactly three elements of S. Then

• $A = [\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}]$

• That is, the elements of *A* are the sets {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, and {2, 3, 4}.



CLASSES OF SETS, POWER SETS, PARTITIONS

 (b) Let B be the class of subsets of S, each which contains 2 and two other elements of S. Then

• $B = [\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}]$

• The elements of B are the sets {1, 2, 3}, {1, 2, 4}, and {2, 3, 4}. Thus B is a subclass of A, since every element of B is also an element of A. (To avoid confusion, we will sometimes enclose the sets of a class in brackets instead of braces.)

• Power Sets

• For a given set S, we may speak of the class of all subsets of S. This class is called the *power set* of S, and will be denoted by P(S). If S is finite, then so is P(S). In fact, the number of elements in P(S) is 2 raised to the power n(S). That is,

•
$$n(P(S)) = 2^{n(S)}$$

• (For this reason, the power set of S is sometimes denoted by 2^{S} .)



POWER SETS

• **EXAMPLE 1.10** Suppose *S* = {1, 2, 3}. Then

• $P(S) = [\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, S]$

• Note that the empty set \emptyset belongs to P(S) since \emptyset is a subset of S. Similarly, S belongs to P(S). As expected from the above remark, P(S) has $2^3 = 8$ elements.

Partitions

- Let S be a nonempty set. A partition of S is a subdivision of S into nonoverlapping, nonempty subsets. Precisely, a partition of S is a collection {Ai } of nonempty subsets of S such that:
- (i) Each a in S belongs to one of the Ai.
- (ii) The sets of {*Ai* } are mutually disjoint; that is, if

• $A_j \neq A_k$ then $A_j \cap A_k = \emptyset$



PARTITIONS

• The subsets in a partition are called *cells*. Figure 1-6 is a Venn diagram of a partition of the rectangular set S of points into five cells, A1, A2, A3, A4, A5.



- **EXAMPLE 1.11** Consider the following collections of subsets of *S* = {1, 2, ..., 8, 9}:
- (i) [{1,3,5}, {2,6}, {4,8,9}]
- (ii) [{1,3,5}, {2,4,6,8}, {5,7,9}]
- (iii) [{1,3,5}, {2,4,6,8}, {7,9}]
- Then (i) is not a partition of S since 7 in S does not belong to any of the subsets.
 Furthermore, (ii) is not a partition of S since {1, 3, 5} and {5, 7, 9} are not disjoint. On the other hand, (iii) is a partition of S.



GENERALIZED SET OPERATIONS

- The set operations of union and intersection were defined above for two sets. These
 operations can be extended to any number of sets, finite or infinite, as follows.
- Consider first a finite number of sets, say, A1,A2,...,Am. The union and intersection
 of these sets are denoted and defined, respectively, by

 $A_1 \cup A_2 \cup \ldots \cup A_m = \bigcup_{i=1}^m A_i = \{x \mid x \in A_i \text{ for some } A_i\}$ $A_1 \cap A_2 \cap \ldots \cap A_m = \bigcap_{i=1}^m A_i = \{x \mid x \in A_i \text{ for every } A_i\}$

 That is, the union consists of those elements which belong to at least one of the sets, and the intersection consists of those elements which belong to all the sets.

Now let \mathscr{A} be any collection of sets. The union and the intersection of the sets in the collection A is denoted and defined, respectively, by

 $\bigcup (A | A \in \mathscr{A}) = \{x \mid x \in A_i \text{ for some } A_i \in \mathscr{A}\}$ $\bigcap (A | A \in \mathscr{A}) = \{x \mid x \in A_i \text{ for every } A_i \in \mathscr{A}\}$

That is, the union consists of those elements which belong to at least one of the sets in the collection \mathscr{A} and the intersection consists of those elements which belong to every set in the collection A.

GENERALIZED SET OPERATIONS

• EXAMPLE 1.12 Consider the sets

 $A_1 = \{1, 2, 3, \ldots\} = \mathbb{N}, \quad A_2 = \{2, 3, 4, \ldots\}, \quad A_3 = \{3, 4, 5, \ldots\}, \quad A_n = \{n, n+1, n+2, \ldots\}.$

Then the union and intersection of the sets are as follows:

$$\bigcup (A_k \mid k \in \mathbf{N}) = \mathbf{N} \text{ and } \bigcap (A_k \mid k \in \mathbf{N}) = \emptyset$$

DeMorgan's laws also hold for the above generalized operations. That is:

Theorem 1.11: Let \mathscr{A} be a collection of sets. Then:

- (i) $\left[\bigcup (A \mid A \in \mathscr{A}) \right]^{C} = \bigcap (A^{C} \mid A \in \mathscr{A})$
- (ii) $\left[\bigcap (A \mid A \in \mathscr{A}) \right]^{C} = \bigcup (A^{C} \mid A \in \mathscr{A})$



MATHEMATICAL INDUCTION

- An essential property of the set $\mathbf{N} = \{1, 2, 3, ...\}$ of positive integers follows:
- Principle of Mathematical Induction I: Let P be a proposition defined on the positive integers N; that is, P(n)
- is either true or false for each $n \in \mathbf{N}$. Suppose *P* has the following two properties:
- (i) *P*(1) is true.
- (ii) P(k + 1) is true whenever P(k) is true.
- Then *P* is true for every positive integer $n \in \mathbf{N}$.
- We shall not prove this principle. In fact, this principle is usually given as one of the axioms when N is developed axiomatically



MATHEMATICAL INDUCTION

• **EXAMPLE 1.13** Let P be the proposition that the sum of the first n odd numbers is n^2 ; that is,

• $P(n): 1 + 3 + 5 + \cdot \cdot \cdot + (2n - 1) = n^2$

(The kth odd number is 2k-1, and the next odd number is 2k +1.) Observe that P(n) is true for n = 1; namely,

•
$$P(1) = 1^2$$

- Assuming P(k) is true, we add 2k + 1 to both sides of P(k), obtaining
- $1 + 3 + 5 + \cdot \cdot \cdot + (2k 1) + (2k + 1) k^2 + (2k + 1) = (k + 1)^2$
- which is P(k + 1). In other words, P(k + 1) is true whenever P(k) is true. By the principle of mathematical
- induction, *P* is true for all *n*.
- There is a form of the principle of mathematical induction which is sometimes more convenient to use.
- Although it appears different, it is really equivalent to the above principle of induction.



MATHEMATICAL INDUCTION

- Principle of Mathematical Induction II: Let P be a proposition defined on the positive integers N such that:
- (i) *P*(1) is true.
- (ii) P(k) is true whenever P(j) is true for all $1 \le j \le k$.
- Then *P* is true for every positive integer $n \in \mathbf{N}$.
- Remark: Sometimes one wants to prove that a proposition P is true for the set of integers
- {*a*, *a* + 1, *a* + 2, *a* + 3, . . .}
- where a is any integer, possibly zero. This can be done by simply replacing 1 by a in either of the above Principles of Mathematical Induction.





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- The reader is familiar with many relations such as "less than," "is parallel to," "is a subset of," and so on. In a certain sense, these relations consider the existence or nonexistence of a certain connection between pairs of objects taken in a definite order. Formally, we define a relation in terms of these "ordered pairs."
- An ordered pair of elements a and b, where a is designated as the first element and b as the second element, is denoted by (a, b). In particular,

$$\bullet (a,b) = (c,d)$$

• if and only if a = c and b = d. Thus $(a, b) \neq (b, a)$ unless a = b. This contrasts with sets where the order of elements is irrelevant; for example, $\{3, 5\} = \{5, 3\}$.



PRODUCT SETS

• Consider two arbitrary sets A and B. The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is called the *product*, or *Cartesian product*, of A and B. A short designation of this product is $A \times B$, which is read "A cross B." By definition.

• $A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$

- One frequently writes A^2 instead of $A \times A$.
- EXAMPLE 2.1 R denotes the set of real numbers and so R² = R×R is the set of ordered pairs of real numbers. The reader is familiar with the geometrical representation of R² as points in the plane as in Fig. 2-1. Here each point *P* represents an ordered pair (*a*, *b*) of real numbers and vice versa; the vertical line through *P* meets the *x*-axis at *a*, and the horizontal line through *P* meets the *y*-axis at *b*. R² is frequently called the *Cartesian plane*.



PRODUCT SETS



• **EXAMPLE 2.2** Let *A* = {1, 2} and *B* = {*a*, *b*, *c*}. Then

• $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Fig. 2-1

- Also, $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
- There are two things worth noting in the above examples. First of all $A \times B = B \times A$. The Cartesian product deals with ordered pairs, so naturally the order in which the sets are considered is important. Secondly, using n(S) for the number of elements in a set *S*, we have:

• $B \times A = \{(a, 1), (b, 1), (c, 1), (a, 2), (b, 2), (c, 2)\}$

•
$$n(A \times B) = 6 = 2(3) = n(A)n(B)$$



- In fact, n(A×B) = n(A)n(B) for any finite sets A and B. This follows from the observation that, for an ordered pair (a, b) in A × B, there are n(A) possibilities for a, and for each of these there are n(B) possibilities for b.
- The idea of a product of sets can be extended to any finite number of sets. For any sets A1,A2,...,An, the set of all ordered *n*-tuples (a1, a2,...,an) where a1 ∈ A1, a2 ∈ A2,..., an ∈ An is called the *product* of the sets A1,...,An and is denoted by

$$A_1 \times A_2 \times \cdots \times A_n$$
 or $\prod_{i=1}^n A_1$

• Just as we write A^2 instead of $A \times A$, so we write A^n instead of $A \times A \times \cdots \times A$, where there are *n* factors all equal to *A*. For example, $\mathbf{R}^3 = \mathbf{R} \times \mathbf{R} \times \mathbf{R}$ denotes the usual three-dimensional space.



- We begin with a definition.
- Definition 2.1: Let A and B be sets. A binary relation or, simply, relation from A to B is a subset of A × B.
- Suppose R is a relation from A to B. Then R is a set of ordered pairs where each first element comes from A and each second element comes from B. That is, for each pair a ∈ A and b ∈ B, exactly one of the following is true:
 - (i) $(a, b) \in R$; we then say "*a* is *R*-related to *b*", written *aRb*.
 - (ii) $(a, b) \notin R$; we then say "*a* is not *R*-related to *b*", written $a \not R b$.
- If R is a relation from a set A to itself, that is, if R is a subset of $A^2 = A \times A$, then we say that R is a relation on A.



• The *domain* of a relation *R* is the set of all first elements of the ordered pairs which belong to *R*, and the *range* is the set of second elements.

• EXAMPLE 2.3

• (a) A = (1, 2, 3) and $B = \{x, y, z\}$, and let $R = \{(1, y), (1, z), (3, y)\}$. Then R is a relation from A to B since R is a subset of $A \times B$. With respect to this relation,

1Ry, 1Rz, 3Ry, but 1Rx, 2Rx, 2Ry, 2Rz, 3Rx, 3Rz

- The domain of R is $\{1, 3\}$ and the range is $\{y, z\}$.
- (b) Set inclusion ⊆ is a relation on any collection of sets. For, given any pair of set A and B, either A ⊆ B or A ⊄ B.
 - (c) A familiar relation on the set Z of integers is "*m* divides *n*." A common notation for this relation is to write $m \mid n$ when *m* divides *n*. Thus 6 30 but 7 \neq 25.



• (d) Let A be any set. An important relation on A is that of equality,

• $\{(a, a) \mid a \in A\}$

- which is usually denoted by "=." This relation is also called the *identity* or *diagonal* relation on A and it will also be denoted by Δ_A or simply Δ .
- (e) Let A be any set. Then A × A and Ø are subsets of A × A and hence are relations on A called the *universal relation* and *empty relation*, respectively.

Inverse Relation

Let R be any relation from a set A to a set B. The *inverse* of R, denoted by R⁻¹, is the relation from B to A which consists of those ordered pairs which, when reversed, belong to R; that is,

•
$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$



• For example, let $A = \{1, 2, 3\}$ and $B = \{x, y, z\}$. Then the inverse of

• $R = \{(1, y), (1, z), (3, y)\}$ is $R^{-1} = \{(y, 1), (z, 1), (y, 3)\}$

• Clearly, if R is any relation, then $(R^{-1})^{-1} = R$. Also, the domain and range of R^{-1} are equal, respectively, to the range and domain of R. Moreover, if R is a relation on A, then R^{-1} is also a relation on A.

• PICTORIAL REPRESENTATIVES OF RELATIONS

There are various ways of picturing relations.

Relations on R

• Let S be a relation on the set **R** of real numbers; that is, S is a subset of $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Frequently, S consists of all ordered pairs of real numbers which satisfy some given equation E(x, y) = 0 (such as $x^2 + y^2 = 25$).



PICTORIAL REPRESENTATIVES OF RELATIONS

• Since \mathbb{R}^2 can be represented by the set of points in the plane, we can picture S by emphasizing those points in the plane which belong to S. The pictorial representation of the relation is sometimes called the *graph* of the relation. For example, the graph of the relation $x^2+y^2 = 25$ is a circle having its center at the origin and radius 5. See Fig. 2-2(a).





DIRECTED GRAPHS OF RELATIONS ON SETS

- Suppose A and B are finite sets. There are two ways of picturing a relation R from A to B.
- (i) Form a rectangular array (matrix) whose rows are labeled by the elements of A and whose columns are labeled by the elements of B. Put a 1 or 0 in each position of the array according as $a \in A$ is or is not related to $b \in B$. This array is called the *matrix of the relation*.
- (ii) Write down the elements of A and the elements of B in two disjoint disks, and then draw an arrow from a ∈ A to b ∈ B whenever a is related to b. This picture will be called the arrow diagram of the relation.
- Figure 2-3 pictures the relation *R* in Example 2.3(a) by the above two ways.



DIRECTED GRAPHS OF RELATIONS ON SETS







COMPOSITION OF RELATIONS

 Let A, B and C be sets, and let R be a relation from A to B and let S be a relation from B to C. That is, R is a subset of A × B and S is a subset of B × C. Then R and S give rise to a relation from A to C denoted by R ∘ S and defined by:

• $a(R \circ S) c$ if for some $b \in B$ we have aRb and bSc.

• That is ,

• $R \circ S = \{(a, c) \mid \text{there exists } b \in B \text{ for which } (a, b) \in R \text{ and } (b, c) \in S\}$

- The relation R
 S is called the composition of R and S; it is sometimes denoted simply by RS.
- Suppose *R* is a relation on a set *A*, that is, *R* is a relation from a set *A* to itself. Then $R \circ R$, the composition of *R* with itself, is always defined. Also, $R \circ R$ is sometimes denoted by R^2 . Similarly, $R^3 = R^2 \circ R = R \circ R \circ R$, and so on. Thus R^n is defined for all positive *n*.



COMPOSITION OF RELATIONS

• Let $A = \{1, 2, 3, 4\}, B = \{a, b, c, d\}, C = \{x, y, z\}$ and let

• $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$

Consider the arrow diagrams of R and S as in Fig. 2-4. Observe that there is an arrow from 2 to d which is followed by an arrow from d to z. We can view these two arrows as a "path" which "connects" the element 2 ∈ A to the element z ∈ C. Thus:

• $2(R \circ S)z$ since 2Rd and dSz

• Similarly there is a path from 3 to x and a path from 3 to z. Hence

• $3(R \circ S)x$ and $3(R \circ S)z$

• No other element of A is connected to an element of C. Accordingly,

• $R \circ S = \{(2, z), (3, x), (3, z)\}$

Our first theorem tells us that composition of relations is associative.



COMPOSITION OF RELATIONS

• **Theorem 2.1:** Let A, B, C and D be sets. Suppose R is a relation from A to B, S is a relation from B to C, and T is a relation from C to D. Then

•
$$(R \circ S) \circ T = R \circ (S \circ T)$$







COMPOSITION OF RELATIONS AND MATRICES

• There is another way of finding $R \circ S$. Let M_R and M_S denote respectively the matrix representations of the relations R and S. Then

$$M_{R} = \begin{array}{cccc} a & b & c & d \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{array} \begin{array}{c} a & X & y & z \\ a & 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}$$
and
$$M_{S} = \begin{array}{c} a \\ b \\ c \\ d \\ 0 & 0 & 1 \end{array}$$

Multiplying M_R and M_S we obtain the matrix

$$M = M_R M_S = \begin{bmatrix} x & y & z \\ 0 & 0 & 0 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} x & y & z \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

• The nonzero entries in this matrix tell us which elements are related by $R \circ S$. Thus $M = M_R M_S$ and $M_{R \circ S}$ have the same nonzero entries.





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- Relations:
- Let A and B are two sets, every subset of A×B is called relation from A to B, denoted by R.
- Remark:
- 1. We say R relation on A if $R \subseteq A \times A$.
- 2. If $(a,b) \in \mathbb{R}$, then denoted by a \mathbb{R} b.
- 3. If $(a,b) \notin \mathbb{R}$, then denoted by $a \not R b$.
- Example:
- 1. Let A={1,2,3} and B={a,b}. Then R={(1,a),(1,b),(3,a)} is a relation from A to B.
 Furthermore, 1 R a, 1 R b, 3 R a.



- 2. Let w={a,b,c}. then R={(a,b),(a,c),(c,c),(c,b)} is a relation in w. Moreover, a R a,b R a,c R c,a R b.
- **Reflexive relation:** Let R be a relation in a set A, i.e. Let R be a subset of $A \times A$. then R is called a reflexive relation if for every $a \in A$, $(a, a) \in R$. In other words, R is reflexive if every element in A is related to its self.
- Example:
- Let V={1,2,3,4} and R={(1,1),(2,4),(3,3),(4,1),(4,4)}. then R is not reflexive since (2,2) dose not belong to R. Notice that all ordered pairs (a,a) must belong to R in order for R to be reflexive.
- 2. Let R be the relation in the real numbers defined by the open sentence " x<y". Then R is not reflexive since a < a for every real number a.



- 3. Let Q be a family of sets, and let R be the relation in Q defined by
 "x is a subset of y". Then R is a reflexive relation since every set is a subset of
 itself.
- Symmetric relation: Let R be a relation in a set A. Then R is called a symmetric relation if $(a, b) \in R$ implies $(b, a) \in R$ that is, if a related to b, then b is also related to a.

• Example:

- 1. Let $S = \{1, 2, 3\}$ and let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$. Then R is not a symmetric relation since $(2, 3) \in R$ but $(3, 2) \notin R$.
- 2. Let R be the relation in the natural numbers N which is defined by "x divides y". Then R is not symmetric since 2 divides 4 but 4 does not divides 2. i.e. (2, 4) ∈ R but (4, 2) ∉ R.



• **Transitive relation:** A relation R in a set A is called a transitive relation if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.

• Example:

- 1. Let *R* be the relation in the real numbers defined by "x < y ". Then if a < b and b < c implies a < c. Thus *R* is a transitive relation.
- 2. Let $w = \{a, b, c\}$, and let $R = \{(a, b), (c, b), (b, a), (a, c)\}$. Then R is not a transitive relation since $(c, b) \in R$ and $(b, a) \in R$ but $(c, a) \notin R$.
- **Equivalence relation:** A relation R in a set A is an equivalence relation if
- 1. *R* is reflexive, that is, for every $a \in A$, $(a, a) \in R$.
- 2. *R* is symmetric, that is, $(a, b) \in R$ implies $(b, a) \in R$.
- 3. *R* is transitive, that is, $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$.



- **Example:** Let $A = \{1, 2, 3\}$. Determine whether the relation is reflexive, symmetric or transitive.
- 1. $R = \{(1,1), (1,2), (2,1), (2,2), (3,3)\}.$
- **2.** $R = \{(1,3), (1,1), (3,1), (1,2), (3,3)\}.$
- **3**. $R = \{(1,2), (1,3), (3,1), (1,1), (3,3), (3,2)\}.$
- 4. $R = \{(1,2), (1,3), (2,3), (1,1), (3,1), (3,3), (2,1), (2,2)\}.$
- 5. $R = \{(1,3), (3,1), (2,2)\}.$
- Solution:
- 1. Reflexive, symmetric, transitive.
- 2. Not reflexive, not symmetric, not transitive.
- 3. Not reflexive, not symmetric, transitive.
- 4. Reflexive, not symmetric, not transitive.
- 5. Not reflexive, symmetric, not transitive.



• Example:

• Consider the relation R that is define on Z by $a R b \leftrightarrow \exists k \in Z, a - b = 3k$.

Solution:

- $(3,0) \in R$ since 3 0 = 3 = 3(1), k = 1
- $(0,3) \in \mathbb{R}$ since 0 3 = -3 = 3(-1), k = -1
- $(15,9) \in R$ since 15 9 = 6 = 3(2), k = 2
- $(-1, -2) \notin R$ since $-1 (-2) = 1 = 3k, k = \frac{1}{3} \notin Z$
- Now,
- I. R is reflexive?
- Let $a \in Z \rightarrow a a = 0 = 3(0)$, $k = 0 \rightarrow a R a$. So that, R is reflexive.



- 2. *R* is symmetric?
- Let a R b, we have to show b R a?
- Since $a \ R \ b \rightarrow \exists \ k \in Z; a b = 3k$
- Now,
- $b a = -(a b) = -3k = 3(-k) = 3k_1$ s.t. $k_1 = -k$
- $\therefore b a = 3k_1 \rightarrow b R a$ so that *R* is symmetric.
- 3. *R* is transitive?
- Let $a R b \wedge b R c$, we have to show a R c?
- $a b = 3k_1 \land b c = 3k_2; k_1, k_2 \in Z$
- $a-c = a b + b c = 3k_1 + 3k_2 = 3(k_1 + k_2) = 3k$; where $k = k_1 + k_2$
- ∴ a R c
- So that from (1), (2), and (3) *R* is equivalence relation.



- Problem:
- Let $A = \{1, 2, 3, 4, 5\}$ and R is defined on A by $R = \{(x, y), x + y = 5\}$. Is R an equivalence relation.
- Example:
- Let $A = Z \times Z \{0\}$, we define the relation R on A by $(a, b) R (c, d) \leftrightarrow ad = bc$.
- Solution:
- (0,1) R (0,2)
- (2,4) R (10,20)
- (5,15) *R* (25,10)
- 1. *R* is reflexive relation?
- Let $(a, b) \in A$, it is clear that (a, b) R (a, b) since ab = ba.



- 2. R is symmetric relation?
- Let $(a, b) R (c, d) \rightarrow ad = bc \rightarrow bc = ad \rightarrow cb = da \rightarrow (c, d) R (a, b)$.
- 3. *R* is transitive relation?
- Let $(a, b) R(c, d) \land (c, d) R(e, f)$, we have to show that (a, b) R(e, f)?
- Since $ad = bc \land cf = de \rightarrow c = \frac{ad}{b} \rightarrow cf = \frac{ad}{b}f = de \rightarrow adf = bde \rightarrow$
- af = be, \therefore (a, b) R (e, f) so that by (1), (2), and (3) R is equivalence relation.
- Functions
- **Definition:** Let f be a relation defined from a set A to set B, then f is called a function " denoted by $f: A \rightarrow B$ " if the following holds:
- 1. $\forall a \in A, \exists b \in B \text{ such that } (a, b) \in f$.
- 2. If $(a, b_1) \in f$ and $(a, b_2) \in f$ then $b_1 = b_2$.



• The set A is called the domain of the function f, and B is called the co-domain or range of f. Further, if $a \in A$ then the element in B which is assigned to a is called image of a and is denoted by f(a).

• Example:

- Let A = {a, b, c, d} and B = {a, b, c}. Define a function f of A into B be the correspondence f(a) = b, f(b) = c, f(c) = c and f(d) = b. By this definition, the image for example are b and c.
- 2. Let A = {1, 2, 3}, g: A → A defined by g = {(1, 2), (3, 1)} is not function because not for each a ∈ A, i.e. 2 ∈ A, ∄b ∈ s.t (2, b) ∈ g.
- 3. $h: A \to A$ define by $h: \{(1,3), (2,1), (1,2), (3,1)\}$ is not function since $(1,2) \in h$ and $(1,3) \in h$ but $2 \neq 3$.


TYPES OF RELATIONS

- **Example:** Let $A = \{m, n, o, p\}$ and $B = \{1, 2\}$. Determine whether the relation R from A to B is a function. If it is function, give its range.
- 1. $R = \{(m, 1), (n, 1), (o, 1), (p, 1)\}.$
- 2. $R = \{(m, 1), (n, 2), (m, 2), (o, 1), (p, 2)\}.$
- **3.** $R = \{(m, 2), (p, 1), (n, 1), (o, 1), (m, 1)\}.$
- Solution:
- 1.Yes function, $Range(R) = \{1\}$.
- 2.No.
- 3.No.



- **Problem:** Let $A = \{w, x, y, z\}$ and $B = \{1, 2\}$. Determine whether the relation R from A to B is a function. If it is function, give its range.
- 1. $R = \{(w, 2), (x, 2), (y, 2), (z, 2)\}.$
- **2.** $R = \{(w, 1), (x, 2), (w, 2), (y, 1), (z, 2)\}.$
- **3.** $R = \{(w, 2), (z, 1), (x, 1), (y, 1), (w, 1)\}.$

Equal function

• If f and g are two functions defined on the same domain D and if f(a) = g(a) for every $a \in D$, then the functions f and g are equal and we write f = g.

• Example:

• Let $f(x) = x^2$ where x is a real number. Let $g = x^2$ where x is a complex number, then the function f is not equal to g since they have different domains.



• **Example:** Let the function f be defined by the diagram



• Let a function g be defined by the formula $g(x) = x^2$ where the domain of g is the set {1,2}, then f = g since they both have the same domain and since f and g assign the same image to each element in the domain.





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RANGE OF A FUNCTION

Let *f* be a mapping of *A* into *B*, that is, *f*: *A* → *B* we define the range of *f* to consist of those elements in *B* which appear as the image of at least one element in *A*. We denote the range of *f*: *A* → *B* by *f*(*A*); i.e. *f*(*A*) = {*b* ∈ *B*; (*a*, *b*) ∈ *f*}.

Identity function

• Let *A* be any set. Let the function $f: A \to A$ be defined by the formula f(x) = x for each element in *A* then *f* is called the identity function and denote by *I* or I_A .

Constant function

• A function f of A into B is called a constant function if the same element $b \in B$ is assigned to every element in A. In other words, $f: A \to B$ is a constant function if the range of f consists of only one element.

• Example:

• Let the function *f* be defined by the diagram





• Then f is a constant function since 3 is assigned to every element in A.

• Example:

• Let the function be defined by the diagram



- Then f is not a constant function since the range of f consists of both 1 and 2.
- **Definition:** Let f be a function of A into B. Then f is called
- I. One one [injective]
- If $x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$
- or

• If
$$f(x_1) = f(x_2) \to x_1 = x_2$$

- 2. Onto [surjective]
- If for each $b \in B$, there exists $a \in A$ such that f(a) = b. i.e. f(A) = B.
- 3. Bijective
- If *f* is both injective and surjective.



- 1. The identity function is injective since $I(x_1) = I(x_2) \rightarrow x_1 = x_2$.
- 2. The identity function is surjective since for each $b \in B$, $\exists a \in A$ such that f(A) = B.
- 3. Let $A = \{a, b, c, d\}, B = \{a, b, c\}$. Define by f(a) = b, f(b) = c, f(c) = c and f(d) = bthen $f(A) = \{b, c\} \neq B$ so that f is not onto.
- 4. Let $A = \{a, b, c, d\}, B = \{x, y, z\}, Let f : A \to B$ define by



- f is not injective because f(a) = f(d) but $a \neq d$. f is surjective, notice that $f(A) = \{x, y, z\} = B$, thus f is onto.
- Problem:
- Let *A* and *B* are two sets and the function from *A* to *B* are given. Determine whether the function is one to one or onto (or both or neither).
- 1. $A = \{0.2, 0.07, 5\}, B = \{r, s, t\},\$
- $f = \{(5, t), (0.2, s), (0.07, r)\}$
- **2.** $A = \{m, n, o\}, B = \{2, 1, 5\},$
- $f = \{(m, 5), (o, 1), (n, 1)\}$



Composition function

• Let $f: A \to B$ and $g: B \to C$ be two functions then we define a function $gof: A \to C$ by $(gof)(a) = g(f(a)) \forall a \in A$. This new function is called the composition function of f and g.



- Let $f: A \rightarrow B$ and $g: B \rightarrow C$ be defined by the diagram
- We compute $(gof): A \to C$ by its definition

•
$$(gof)(a) = g(f(a)) = g(y) = t$$

•
$$(gof)(b) = g(f(b)) = g(z) = r$$

•
$$(gof)(c) = g(f(c)) = g(y) = t$$





• Example:

- Let $f: N \to N$, $g: N \to N$ such that $f(x) = 3x^2$ and $g(x) = 4x^2 + 2$. Now,
- $(gof)(x) = g(f(x)) = g(3x^2) = 4(3x^2)^2 + 2 = 36x^4 + 2.$
- $(fog)(x) = f(g(x)) = f(4x^2 + 2) = 3(4x^2 + 2)^2 = 3(16x^4 + 16x^2 + 4)$. It is clear that $gof \neq fog$.

Inverse of function

• Let f be a function of A into B, and let $b \in B$. Then the inverse of b, denoted by $f^{-1}(b)$ consists of those elements in A which are mapped onto b, that is, those elements in A which have be as their image. i.e. if $f: A \to B$ then $f^{-1}(b) = \{x; x \in A, f(x) = b\}$. Notice that $f^{-1}(b)$ is always a subset of A.

- Let the function $f : A \rightarrow B$ be defined by the diagram
- Then $f^{-1}(x) = \{b, c\}, f^{-1}(y) = \{a\}, f^{-1}(z) = \emptyset.$





Graph Theory

- Graph theory began in 1936 when Leonhard Euler solved a problem that had been puzzling the good citizens of the town Kaliningrad in Russia. The river Pregl divides the town into four sections, and in The Euler's days seven bridges connected these sections. The people wanted to know if it were possible to start at any location in town, cross every bridge exactly once, and return to the starting location. Euler showed that it is impossible to take such a walk.
- A problem in graph theory that attracted considerably more attention is the fourcolor map problem. Frank Guthrie, a former student of Augustus De Morgan, observed that he was able to color the map England so that no two adjacent counties have the same color by using four different colors. He asked his brother, who in 1852 was a student of De Morgan, to ask De Morgan whether his conjecture that four colors will suffice to color every map so that no two adjacent counties have the same color was true.
- In 1879, Alfred Bray Kempe, a lawyer who had studied mathematics at Cambridge University, published a proof of the four-color conjecture that was highly acclaimed. Unfortunately, his proof had a fatal error. The theorem was finally proved by the American mathematicians Kenneth Appel and wolfgang Haken at the University of Illinois.



• Graph

 A graph is a nonempty finite set of vertices V along with a set E of 2-element subsets of V. The elements of V are called vertices, and the elements of E are edges. Mathematically, G=(V, E). where, E={(v_i, v_j), v_i, v_j ∈ V}.

G:

- Let us consider $V = \{P, Q, T, S, R\}$ and $E = \{(P,Q), (P,T), (T,S), (Q,S), (Q,R), (S,R)\}.$
- Hence, the graph G=(V, E) becomes as G



- Example:
- Let $V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_4, v_5), (v_5, v_6)\}$.
- Hence, the graph becomes as G





Order and Size

- The number of vertices in a graph G=(V, E) is called its order, and the number of edges its size. That is the order of G is |V| and its size |E|
- Consider the following graph G
- The order of G = |V| = 5 and the size of G = |E| = 8
- Example:
- Consider the graph G as
- The order of G = |V| = 4 and the size of G = |E| = 5

• Adjacent Vertices

 Two vertices are adjacent if they are joined by an edge. In other words, there is an edge in the graph that has both vertices as endpoints.





- Consider the graph G as
- Here the vertices a and b are adjacent, b and c are adjacent,
- a and c are adjacent, c and d are adjacent. Vertices a and d
- are not adjacent.

• Parallel Edges

• Two or more edges are called parallel edges when they have same end vertices.

 e_1

 e_4

G:

- Consider the graph G as
- Here the edges p and q are parallel

• Loop

- A graph that contain an edge from a vertex to itself;
- such an edge is referred to as loop.
- Consider the graph G as
- It is clear that the edge e_1 is a loop.



b





Simple and Multi Graph

- If a graph has no loops and parallel edges then it's known as a simple graph, otherwise a multi graph.
- Consider the graphs G and D as
- The graph in D is simple graph
- because there exists no loop and
- parallel edges between vertices,
- while the graph in G is a multi- graph because there exists parallel edges between vertices A and B.

• Pseudo Graph

- A graph G is known as a pseudo graph if we allow both parallel edges and loops.
- Consider the graph G as







• Digraph

- A directed graph (or digraph) is a graph that is a set of vertices connected by edges, where the edges have a direction associated with them.
- Consider the graph G as a simple directed graph
- Example:
- Let $V = \{1,2,3,4,5\}$ and $E = \{(1,2), (1,3), (2,3), (2,4), (3,4), (4,5)\}$
- Hence, the digraph G becomes as

- In many applications, each edge of a graph has an associated numerical value, called a weight.
- Consider the graph G as

Weighted Graph







- Let us consider V={1, 2, 3, 4, 5} and E={ e_1, e_2, e_3, e_4, e_5 }, where, e_1 =(1, 2), e_2 =(1, 3), e_3 =(2, 4), e_4 =(3, 4), e_5 =(4, 5) and w(e_1)=3, w(e_2)=5, w(e_3)=2, w(e_4)=4, w(e_5)=1. Find the graph G.
- Degree of Vertex
- The degree of vertex is the number of edges having that vertex as an end point.
- Isolated Vertex
- A vertex with degree 0 is called an isolated vertex.
- Example:
- Consider the following graphs G and D
- In the graph in Figure G, the vertex A has
- degree 2, vertex B has degree 4, and vertex
- E has degree 3. In Figure D, vertex a has
- degree 4, vertex e (isolated degree) has
- degree 0, and vertex b has degree 2.





• Path

- A path in a graph is a sequence of v_1, v_2,..., v_k of vertices each adjacent to the next, and a choice of an edge between each v_i and v_(i+1) so that no edge is chosen more than once.
- Consider the graph G as
- Path is a, b, c, d or can use a, c, d



• *Circuit* is a path that begins and ends at the same vertex.

Complete Graph

- The graph G is called complete if every vertex in G is connected to every other vertex.
- Consider the graph G as
- The graph **G** is complete





• Regular Graph

- If each vertex of a graph has the same degree as every other vertex, the graph is called regular.
- Consider the graph G as
- In graph G, degree $(v_1) = degree (v_2) = degree (v_3) = 2$
- Also the graph is complete.







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REGULAR GRAPH

- If each vertex of a graph has the same degree as every other vertex, the graph is called regular. v_1
- Consider the graph G as
- In graph G, degree (v_1) = degree (v_2) = degree (v_3) = 2
- Also the graph is complete.
- Example:
- Consider the graph G as
- Here the degree of every vertex is 3, but the graph is not
- complete.







• Problems

1. In exercises **a** through **d**, give **V**, the set of vertices, and **E**, the set of edges for the graph G.



- 2. Draw a picture of the graph G=(V, E), where $V=\{a, b, c, d, e\}$, $E=\{e_1, e_2, e_3, e_4, e_5, e_6\}$, and $e_1 = e_5 = \{a, c\}$, $e_2 = \{a, d\}$, $e_3 = \{e, c\}$, $e_4 = \{b, c\}$, and $e_6 = \{e, d\}$.
- 3. Give the degree of each vertex in Figure *a* in 1.
- 4. Give the degree of each vertex in Figure *c* in 1.
- 5. List all paths that begin at a in Figure **b** in 1.
- 6. List three circuits that begin at 5 in Figure *d* in 1.
- 7. Which of the graphs in 1(a, b, or c) are regular?



CYCLE

- If there is a path containing one or more edges which starts from a vertex v and terminates into the same vertex, then the path is known as a cycle.
- Example
- Consider the graph G as
- In the graph G, one cycle is 1,2,3,1, and another cycle is 1
- Pendant Vertex
- A vertex v of a graph of G is said to be a pendant vertex if and only if it has degree

 .

G:

G:

- Consider the graph G as
- In graph G degree of g, f, e, d are equal 1. Therefore, these
- vertices are pendant vertices.





ADJACENCY MATRIX

• The most useful way of representing any graph is the matrix representation. It is a square matrix of order $(n \times n)$ where, n is the number of vertices in the graph G. Generally denoted by $A = [a_{ij}]$ where a_{ij} is the *i*th row and *j*th column element. The general form of adjacency matrix is given as

•
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & & a_{3n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$$
 Where, $a_{ij} = \begin{cases} 1; & if there is an edge from v_i to v_j \\ 0; & Otherwise \end{cases}$

This matrix is called as adjacency matrix.

• Notes

- In the adjacency matrix if the main diagonal elements are zero, then the graph is said to be a simple graph.
- In case of a multi graph the adjacency matrix can be found out with the relation. $a_{ij} = \begin{cases} n; & n \text{ be the number of edges from } v_i \text{ to } v_j \\ 0; & Otherwise \end{cases}$



 In case of a weighted graph the adjacency matrix can be found out with the relation.

 $a_{ij} = \begin{cases} w; & w \text{ is the weight of the edges from } v_i \text{ to } v_j \\ 0; & Otherwise \end{cases}$

- Consider the graph G as
- Hence, the adjacency matrix is given as

 $\mathbf{A} = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ v_1 & 0 & 1 & 1 & 0 & 0 & 1 \\ v_2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$





- Example:
- Consider the graph G as
- The adjacency matrix of the above graph with respect to

• the ordering A, B, C, and D is given as
$$A = \begin{bmatrix} 0 & 0 & 3 & 0 \\ 5 & 0 & 1 & 7 \\ 2 & 0 & 0 & 4 \\ 0 & 6 & 8 & 0 \end{bmatrix}$$

- Example:
- Consider the graph G as
- Find the adjacency matrix of the graph with respect to the
- ordering A, B, C and D.





G:



INCIDENCE WATRIX

- Suppose that G be a simple undirected graph with m vertices and n edges, then the incidence matrix I=[a_{ij}] is a matrix of order (m × n) where the element a_{ij} is defined as
- $a_{ij} = \begin{cases} 1; & if \ vertex \ i \ belongs \ to \ edges \ j \\ 0; & Otherwise \end{cases}$
- Consider the graph G as
- Hence, the incidence matrix of the graph G is of order (4×5)
- G: e_1 e_2 e_3 e_2 e_3 e_4 e_3 e_2 e_3 e_4 e_3 e_5 e_6 e_7 e_8 e_8 e
- The incidence matrix relative to the ordering a, b, c, d and
- e_1, e_2, e_3, e_4, e_5 is given as

•
$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$



PATH MATRIX

- Suppose that G be simple graph with n vertices. Then the $(n \times n)$ matrix $P=[p_{ij}]_{(n \times n)}$ defined by
- $p_{ij} = \begin{cases} 1; & if there is a path from v_i to v_j \\ 0; & Otherwise \end{cases}$
- is known as the path matrix of the graph G.
- Consider the graph G as
- The path matrix of the graph relative to the ordering
- v_1, v_2, v_3, v_4, v_5 is given as

$$\bullet P = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$





CONNECTED GRAPH

- A graph is called connected if there is a path from any vertex to any other vertex in graph. Otherwise, the graph is disconnected.
- Consider the graphs G and D as
- The graph in G is connected while,
- in graph D is disconnected.
- Statements



Statements is a verbal sentence helpful will be denoted by the letters , q, r, The fundamental property of a statement is that it is either true or false but not both. The truth fullness or falsity of a statement is called its truth value. Some statements are composite, that is, composed of sub statements are various connectives which will be discussed subsequently.



• Example:

- 1. "*Hiba is a nice and a clever girl* " is a composite statement with sub statements "*Hiba is a nice* " and "*Hiba is a clever girl* ".
- 2. "*Where are you going*?" is not a statement since it is neither true nor false.
- 3. " John is sick or old " is a composite statement with sub statements " John is sick " or " John is old ".
- A fundamental property of a composite statement is that, its truth value is completely determined by the truth value of its sub statements and the way they are connected to form the composite statement.

Conjunction

 Any two statements can be combined by the word " and " to form a composite statement which is called the conjunction of the original statements. The conjunction of the two statements p and q is denoted by p ∧ q.



• Example:

- Let p be "It is rainig " and let q be "The sun is shining ". Then p ∧ q denotes the statement "It is rainig and the sun is shining ". The truth value of the composite statement p ∧ q satisfies the following property:
- If p is true and q is true, then $p \land q$ is true, otherwise $p \land q$ is false. In other words, the conjunction of two statement is true only if each component is true.

- Consider the following four statements
- 1. Paris is in France and 2 + 2 = 5.
- 2. Paris is in England and 2 + 2 = 4.
- 3. Paris is in England and 2 + 2 = 5.
- 4. Paris is in France and 2 + 2 = 4.
- It is clear that only (4) is true. Each of the other statements is false since at least one
 of its components is false.



• Truth table of " $p \land q$ " can be written in the form



Disjunction

• Any two statements can be combined by the word " or " to form a new statement which is called the disjunction of the original two statements. The disjunction of statements p and q is denoted by $p \lor q$.

- Let p be " He studied French at the university ", and let q be " He lived in France " then p \ q is the statement " He studied French at the university or he lived in France ".
- The truth value of the composite statement $p \lor q$ satisfies the following property:
- If p is true or q is true or both p and q are true, then p ∨ q is true, otherwise, p ∨ q is false. In other words, the disjunction of two statements is false only if p and q are false. The truth table of " p ∨ q " can be written in the form



р	q	$p \lor q$
Т	Т	Т
Т	F	Т
F	Т	Т
F	F	F

• Example:

- Consider the following four statements
- 1. Paris is in France or 2 + 2 = 5.
- 2. Paris is in England or 2 + 2 = 4.
- 3. Paris is in France or 2 + 2 = 4.
- 4. Paris is in England or 2 + 2 = 5.
- It is clear that only (4) is false. Each of the other statements is true since at least one
 of its components is true.

Negation

 Given any statement p, another statement, called the negation of p, can be formed by writing "It is false that ... " before p or, if possible by inserting in p the word "not". The negation of p is denoted by ~p.



- Consider the following three statements
- 1. Paris is in France.
- 2. It is false that Paris is in France.
- 3. Paris is not in France.
- Then (2) and (3) are each the negation of (1).
- Example:
- Consider the following statements
- 1.2 + 2 = 5
- 2. It is false that 2 + 2 = 5
- **3.** 2 + 2 ≠ 5
- Then (2) and (3) are each the negation of (1).
- The truth value of the negation of a statement satisfies the following property:
- If p is true, then $\sim p$ is false; if p is false, then $\sim p$ is true.





Conditional

- Many statements, especially in mathematics are of the form " if p then q " such statements are called conditional statements and are denoted by p → q. The truth value of the conditional statement p → q satisfies the following property:
- The conditional $p \rightarrow q$ is true unless p is true and q is false.
- The truth table of " $p \rightarrow q$ " can be written in the form

р	q	p ightarrow q
Т	Т	Т
Т	F	F
F	Т	Т
F	F	Т





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• Remark:

 Consider the conditional proposition p → q and other simple conditional proposition which contain p and q, i.e. p → q, q → p, ~p → ~q and ~q → ~p, called respectively, the converse, inverse, and contra positive propositions. The truth table of these four propositions are as follows:

р	P	$\sim p$	$\sim q$	p ightarrow q	q ightarrow p	$\sim p ightarrow \sim q$	$\sim q ightarrow \sim p$
т	Т	F	F	Т	Т	Т	Т
т	F	F	Т	F	Т	Т	F
F	Т	Т	F	Т	F	F	Т
F	F	Т	Т	Т	Т	Т	Т

- Let *p*: Noor at home.
- *q*: Noor answers the phone.
- $p \rightarrow q$: If Noor at home then she will answer the phone.
- $q \rightarrow p$: If Noor answered to the phone then she is at home.
- $\sim p \rightarrow \sim q$: If Noor is not at home then she is not answer the phone.
- $\sim q \rightarrow \sim p$: If Noor is not answer the phone then she is not at home.


Biconditional

- Another common statement is of the form " p if and only if q " or simply, " p iff q ". Such statements are called biconditional statements and denoted by $p \leftrightarrow q$.
- The truth value of the biconditional statement $p \leftrightarrow q$ satisfies property:
- If p and q have the same truth value, then " $p \leftrightarrow q$ " is true,
- If p and q have opposite truth value, then " $p \leftrightarrow q$ " is false.
- Example:
- Consider the following statements
- 1. Paris is in France iff 2 + 2 = 5.
- 2. Paris is in England iff 2 + 2 = 4.
- 3. Paris is in France iff 2 + 2 = 4.
- 4. Paris is in England iff 2 + 2 = 5.
- According, (3) and (4) are true while (1) and (2) are false.



• The truth table written as follows

p	q	$p \leftrightarrow q$
Т	Т	Т
Т	F	F
F	Т	F
F	F	Т

- Logical Equivalence
- Two statements are said to be logically equivalent if their truth table are identical. We denote the logical equivalent of p and q by " \equiv ".

• Example:

• The truth tables of $(p \rightarrow q) \land (q \rightarrow p)$ and $p \leftrightarrow q$ are as follows:

р	q	$oldsymbol{p} ightarrow oldsymbol{q}$	$oldsymbol{q} o oldsymbol{p}$	$(\boldsymbol{p} o \boldsymbol{q}) \wedge (\boldsymbol{q} o \boldsymbol{p})$
т	Т	Т	Т	Т
т	F	F	Т	F
F	Т	Т	F	F
F	F	Т	Т	Т
р	q	$p \leftrightarrow q$		
Т	Т	Т		
Т	F	F		
F	Т	F		
F	F	Т		



• Hence, $(p \to q) \land (q \to p) \equiv p \leftrightarrow q$

• Example:

• The truth tables below show that $p \to q$ and $\sim p \lor q$ are logically equivalent, i.e. $p \to q \equiv \sim p \lor q$



р	q	~ <i>p</i>	$\sim p \lor q$
Т	Т	F	Т
Т	F	F	F
F	Т	Т	Т
F	F	Т	Т

• Problems

- Show that
- 1. $\sim (p \land q) \equiv \sim p \land \sim q$
- 2. $\sim (p \rightarrow q) \equiv p \land \sim q$

