محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الثاني

ا. م. د. بان جعفر الطائي

المحاضرة ١

Chapter Five

SERIES

5.1 Definition:

A ***sequence*** $\langle z_n \rangle$ is a function whose domain is the set of positive integers and whose range is a subset of the complex numbers $\mathbb{C}$. In other words, to each integer n = 1, 2, 3, ... we assign a single complex number z_n. For example, the sequence $\{1+i^n\}$ is

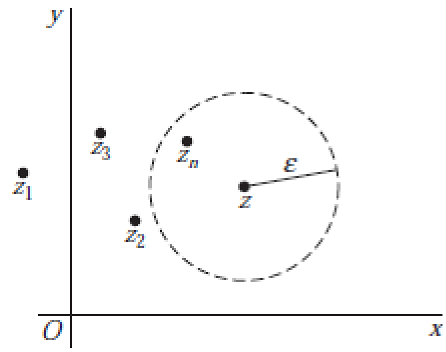
$$\begin{array}{cccccc} 1+i, & 0, & 1-i, & 2, & 1+i, & \cdots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \\ n=1, & n=2, & n=3, & n=4, & n=5, & \cdots \end{array}$$

5.2 Definition:

we say the sequence $\langle z_n \rangle$ is ***convergent***. In other words, $\langle z_n \rangle$ converges to the number L, $(\lim_{n\to\infty} z_n = L)$ if for each positive real number ε an N can be found such that

$$|z_n - L| < \varepsilon \text{ whenever } n > N.$$

If the sequence has no limit, it ***diverges***.



5.3 Example:

Using the definition, prove that the sequence $\langle 1+\frac{z}{n} \rangle$ converge to 1.

Solution:

Given any number $\epsilon > 0$, choose $N = \frac{\epsilon}{n}$ then $\left|1+\frac{z}{n}-1\right| = \left|\frac{z}{n}\right| < \epsilon$ if $n > N$.

5.4 Theorem:

Suppose that $z_n = x_n + iy_n (n = 1,2,\cdots)$ ***and*** $z = x+iy$***. Then*** $\lim_{n\to\infty} z_n = z$ ***if and only if*** $\lim_{n\to\infty} x_n = x$ ***and*** $\lim_{n\to\infty} y_n = y$***.***

5.5 Example:

The sequence $z_n = \frac{1}{n^3} + i$, $(n = 1,2,\cdots)$ converges to i since

$$\lim_{n\to\infty}\left(\frac{1}{n^3}+i\right) = \lim_{n\to\infty}\frac{1}{n^3} + i\lim_{n\to\infty} 1 = 0 + i\cdot 1 = i\,.$$

Note that :-

We can use definition 5.2can also be used to obtain this result. More precisely, for each positive

number ε, choose $N = \frac{1}{\sqrt[3]{\epsilon}}$ then $\left|\frac{1}{n^3}+i-i\right| = \frac{1}{n^3} < \epsilon$ whenever $n > \frac{1}{\sqrt[3]{\epsilon}}$.

5.6 Example:

The sequence $z_n = -2 + i\frac{(-1)^n}{n^2}$, $(n = 1,2,\cdots)$ converges to -2 since

$$\lim_{n\to\infty}\left(-2 + i\frac{(-1)^n}{n^2}\right) = \lim_{n\to\infty} -2 + i \lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2 + i\cdot 0 = -2\ .$$

Note that :-

We can use theorem 5.4 to polar coordinates, as in example 5.6, write $r_n = |z_n|$, $\Theta_n = Arg\, z_n$, $(n = 1,2,\cdots)$ where $Arg\, z_n$ denotes principal arguments $(-\pi < \Theta < \pi)$ of z_n, we find that

$$\lim_{n\to\infty} r_n = \lim_{n\to\infty}\sqrt{4 + \frac{1}{n^4}} = 2\ .$$

But that $\lim_{n\to\infty}\Theta_{2n} = \pi$ and $\lim_{n\to\infty}\Theta_{2n-1} = -\pi$, $(n = 1,2,\cdots)$. Evidently, then, the limit of Θ_n does not exist as n tends to infinity.

5.7 Definition:

An infinite series or series of complex numbers $\sum_{k=1}^{\infty} z_k = z_1 + z_2 + \cdots + z_n + \cdots$ is ***convergent*** if the sequence of partial sums $\{S_n\}$ where $S_n = z_1 + z_2 + \cdots + z_n$ converges. If $S_n \to L$ as $n \to \infty$,we say that the series converges to L or that the ***sum*** of the series is L, *i.e.* $\sum_{k=1}^{\infty} z_k = L$.

Note that :-

Since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it ***diverges***.

5.8 Theorem:

Suppose that $z_n = x_n + iy_n$ $(n = 1,2,\cdots)$ ***and*** $S = X + iY$. ***Then*** $\sum_{n=1}^{\infty} z_n = S$ ***iff*** $\sum_{n=1}^{\infty} x_n = X$ ***and*** $\sum_{n=1}^{\infty} y_n = Y$.

5.9 Definition (Geometric Series):

A ***geometric series*** is any series of the form

$$\sum_{k=1}^{\infty} az^{k-1} = a + az + az^2 + \cdots + az^{n-1} + \cdots \qquad (1)$$

For (1), the nth term of the sequence of partial sums is

$$S_n = a + az + az^2 + \cdots + az^{n-1} = \frac{a(1-z^n)}{1-z}.$$

If $|z| < 1$ then $z^n \to 0$ as $n \to \infty$ and so $S_n \to \frac{a}{1-z}$. If $|z| \geq 1$ then the geometric series is diverges.

5.10 Example:

The infinite series $\sum_{k=1}^{\infty}\frac{(1+2i)^k}{5^k} = \frac{1+2i}{5} + \frac{(1+2i)^2}{5^2} + \frac{(1+2i)^3}{5^3} + \cdots$ is a geometric series. It has the form given in (1) with $a = \frac{1}{5}(1+2i)$ and $z = \frac{1}{5}(1+2i)$. Since $|z| = \frac{\sqrt{5}}{5} < 1$, the series is convergent and its sum is

$$\sum_{k=1}^{\infty} \frac{(1+2i)^k}{5^k} = \frac{a}{1-z} = \frac{\frac{1}{5}(1+2i)}{1-\frac{1}{5}(1+2i)} = \frac{1+2i}{4-2i} = \frac{1+2i}{2(1+2i)} = \frac{1}{2}i.$$

5.11 Theorem (ANecessary Condition for Convergence):

If $\sum_{k=1}^{\infty} z_k$ converges then $\lim_{n\to\infty} z_n = 0$.

5.12 Example:

Does the series $\sum_{k=1}^{\infty} \frac{(ik+5)}{k}$ converge ?

Solution:

Let $z_n = \frac{(in+5)}{n}$ then $\lim_{n\to\infty} \frac{(in+5)}{n} = \lim_{n\to\infty} \left(\frac{in}{n} + \frac{5}{n}\right) = i \neq 0$, so by theorem 5.11 the series $\sum_{k=1}^{\infty} \frac{(ik+5)}{k}$ is diverge.

5.13 Definition (Absolute and Conditional Convergence):

An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be ***absolutely convergent*** if $\sum_{k=1}^{\infty} |z_k|$ converges. An infinite series $\sum_{k=1}^{\infty} z_k$ is said to be ***conditionally convergent*** if it converges but $\sum_{k=1}^{\infty} |z_k|$ diverges.

5.14 Remark:

In elementary calculus a real series of the form $\sum_{k=1}^{\infty} \frac{1}{k^p}$ is called a ***p*-series** and converges for $p > 1$ and diverges for $p \leq 1$.W e use this well-known result in the next example

5.15 Example:

The series $\sum_{k=1}^{\infty} \frac{i^k}{k^2}$ is absolutely convergent since the series $\sum_{k=1}^{\infty} \left|\frac{i^k}{k^2}\right|$ is the same as the real convergent *p*-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Here we identify $p = 2 > 1$.

5.16 Remark:

Two of the most frequently used tests for convergence of infinite series are given in the next theorems.

5.17 Theorem (Ratio Test):

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of nonzero complex terms such that $\lim_{n\to\infty} \left|\frac{z_{n+1}}{z_n}\right| = L$.

1) If $L < 1$, then the series converges absolutely.

2) If $L > 1$ or $L = \infty$, then the series diverges.

3) If $L = 1$, the test is inconclusive.

5.18 Theorem (Root Test):

Suppose $\sum_{k=1}^{\infty} z_k$ is a series of complex terms such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = L$.

1) If $L < 1$, then the series converges absolutely.

2) If $L > 1$ or $L = \infty$, then the series diverges.

3) If $L = 1$, the test is inconclusive.

5.19 Remark:

We are interested primarily in applying the tests in Theorems 5.17 and 5.18 to power series.

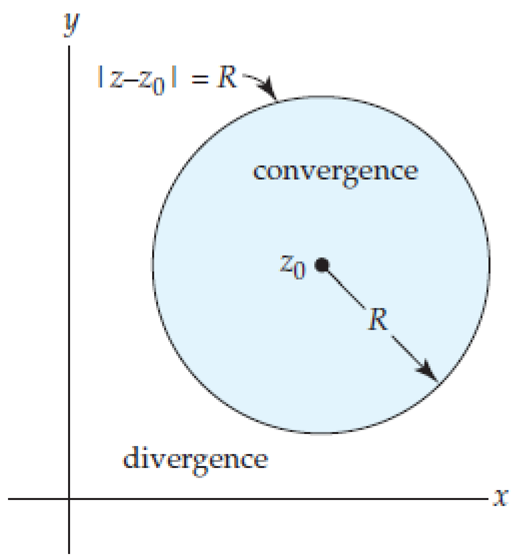
5.20 Definition (Power Series):

The notion of a power series is important in the study of analytic functions. An infinite series of the form

$$\sum_{k=1}^{\infty} a_k (z - z_0)^k = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, \qquad (2)$$

where the coefficients a_k are complex constants, is called a ***power series*** in $z - z_0$. The power series (2) is said to be centered at z_0; the complex point z_0 is referred to as the center of the series. In (2) it is also convenient to define $(z - z_0)^0 = 1$ even when $z = z_0$.

5.20 Definition (Circle of Convergence):



Every complex power series (2) has a ***radius of convergence***. Analogous to the concept of an interval of convergence for real power series a complex power series (2) has a ***circle of convergence***, which is the circle centered at z_0 of largest radius $R > 0$ for which (2) converges at every point within the circle $|z - z_0| = R$.A power series converges absolutely at all points z within its circle of convergence , that is , for all z satisfying $|z - z_0| < R$, and diverges at all points z exterior to the circle, that is, for all z satisfying $|z - z_0| > R$.The radius of convergence can be:

1) $R = 0$ (in which case (2) converges only at its center $z = z_0$),
2) R a finite positive number (in which case (11) converges at all interior points of the circle $|z - z_0| = R$), or
3) R = ∞ (in which case (2) converges for all z).

5.21 Example:

Consider the power series $\sum_{k=1}^{\infty} \frac{z^{k+1}}{k}$. By the ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{z^{n+2}}{n+1}}{\frac{z^{n+1}}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} |z| = |z|.$$

Thus the series converges absolutely for $|z| < 1$.The circle of convergence is $|z| = 1$ and the radius of convergence is $R = 1$.Note that on the circle of convergence $|z| = 1$, the series does not converge absolutely since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the well-known divergent harmonic series.Bear in mind this does not say that the series diverges on the circle of convergence.In fact, at $z = -1$, is the convergent alternating harmonic series.Indeed, it can be shown that the series converges at all points on the circle $|z| = 1$ except at $z = 1$.

5.22 Remark:

It should be clear from Theorem 5.17 and Example 5.21 that for a power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$, the limit depends only on the coefficients a_k.Thus, if

1) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L \neq 0$, the radius of convergence is $R = \frac{1}{L}$;

2) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0$, the radius of convergence is $R = \infty$;

3) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, the radius of convergence is $R = 0$.

Similar conclusions can be made for the root test by utilizing $\lim_{n\to\infty} \sqrt[n]{|a_n|}$. For example if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L \neq 0$ then $R = \frac{1}{L}$.

5.23 Example:

Consider the power series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k!} \cdot (z - 1 - i)^k$. With the identification $a_n = \frac{(-1)^{n+1}}{n!}$ We have $\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+2}}{(n+1)!}}{\frac{(-1)^{n+1}}{n!}} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0$. Hence by remark 5.22(2) the radius of convergence is ∞; the power series with center $z_0 = 1 + i$ converges absolutely for all z, that is, for $|z - 1 - i| < \infty$.

5.24 Example:

Consider the power series $\sum_{k=1}^{\infty} \left(\frac{6k+1}{2k+5}\right)^k \cdot (z - 2i)^k$.With $a_n = \left(\frac{6n+1}{2n+5}\right)^n$, the root test gives $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{6n+1}{2n+5} = 3$. By reasoning similar to that leading to remark 5.22(1), we conclude that the radius of convergence of the series is $R = \frac{1}{3}$. The circle of convergence is $|z - 2i| = \frac{1}{3}$ the power series converges absolutely for $|z - 2i| < \frac{1}{3}$.

EXERCISES:

1. a) Prove that the series $z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$ converges for $|z| < 1$, and find the its sum.

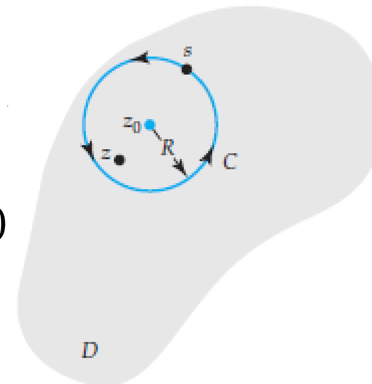
b) Prove that the series is absolutely convergent for $|z| < 1$.

2. Prove that $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges (absolutely) for $|z| \le 1$.

3. Find the region of convergence of the series

a) $\sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{(n+1)^3 4^n}$. **b)** $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^{2n-1}}{(2n-1)!}$. **c)** $\sum_{n=1}^{\infty} n! \cdot z^n$.

5.25 Theorem (Taylor's Theorem):

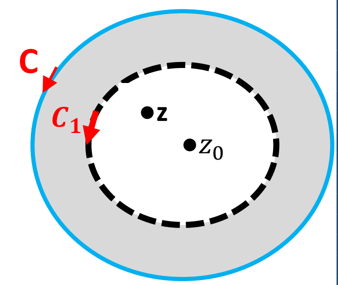


Let f be analytic within a domain D and let $z0$ be a point in has the series representation

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \cdot (z - z_0)^k \qquad (3)$$

valid for the largest circle C with c enter at z_0 and radius R that lies entirely within D (If $z_0 = 0$ in (3) the resulting series is often called a Maclaurin series.

Proof:



Let z be any point inside C. Construct a circle C_1 with center at z_0 and enclosing z. Then, by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z}\, dw. \qquad (3)$$

We have

$$\frac{1}{w-z} = \frac{1}{w-z-z_0+z_0} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0}\left(\frac{1}{1-\frac{z-z_0}{w-z_0}}\right)$$

$$= \frac{1}{w-z_0}\left(1 + \left(\frac{z-z_0}{w-z_0}\right) + \left(\frac{z-z_0}{w-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{w-z_0}\right)^{n-1} + \left(\frac{z-z_0}{w-z_0}\right)^n \frac{1}{1-\frac{z-z_0}{w-z_0}}\right)$$

So $$\frac{1}{w-z} = \frac{1}{w-z_0} + \frac{z-z_0}{(w-z_0)^2} + \frac{(z-z_0)^2}{(w-z_0)^3} + \cdots + \frac{(z-z_0)^{n-1}}{(w-z_0)^n} + \left(\frac{z-z_0}{w-z_0}\right)^n \frac{1}{w-z}. \qquad (4)$$

Multiplying both sides of (4) by $f(w)$ and using (3), we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z_0} dw + \frac{z-z_0}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^2} dw + \cdots + \frac{(z-z_0)^{n-1}}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^n} dw + U_n.$$

Where $U_n = \frac{1}{2\pi i}\int_{C_1} \left(\frac{z-z_0}{w-z_0}\right)^n \frac{f(w)}{w-z} dw$. Using Cauchy's integral formulas $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_1} \frac{f(w)}{(w-z_0)^{n+1}} dw,\ n = 0,1,2,\cdots$ becomes

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + U_n.$$

If we can now show that $\lim_{n\to\infty} U_n = 0$, we will have proved the required result. To do this, we note that since w is on C_1, $\left|\frac{z-z_0}{w-z_0}\right| = \gamma < 1$where γ is a constant. Also, we have $|f(w)| < M$ where M is a constant, and $|w-z| = |(w-z_0)-(z-z_0)| \geq r_1 - |z-z_0|$ where r_1 is the radius of C_1. Hence, from theorem 4.27 we have

$$|U_n| = \frac{1}{2\pi}\left|\int_{C_1} \left(\frac{z-z_0}{w-z_0}\right)^n \frac{j(w)}{w-z} dw\right| \leq \frac{1}{2\pi}\frac{\gamma^n M}{r_1-|z-z_0|} \cdot 2\pi r_1 = \frac{\gamma^n M r_1}{r_1-|z-z_0|} r_1,$$

and we see that $\lim_{n\to\infty} U_n = 0$, completing the proof. □

5.26 Example:

a) Expand $f\ (z) =\ sin\ z$ in a Taylor series about $z = \frac{\pi}{4}$.

b) Determine the region of convergence of this series.

Solution:

a) Since $z_0 = \frac{\pi}{4}$

$$f(z) = \sin z \qquad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$f'(z) = \cos z \qquad f'\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$f''(z) = -\sin z \qquad f''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f'''(z) = -\cos z \qquad f'''\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$f^{(4)}(z) = \sin z \qquad f^{(4)}\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

$$\vdots \qquad \vdots$$

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}\left(\frac{\pi}{4}\right)}{k!} \cdot (z - \frac{\pi}{4})^k$$

$$= f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(z - \frac{\pi}{4}\right) - \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(z - \frac{\pi}{4}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{4}\right)}{4!}(z - \frac{\pi}{4})^4 + \cdots$$

$$= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(z - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(z - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{2 \cdot 3!}\left(z - \frac{\pi}{4}\right)^3 + \frac{\sqrt{2}}{2 \cdot 4!}(z - \frac{\pi}{4})^4 + \cdots$$

$$= \frac{\sqrt{2}}{2}(1 + \left(z - \frac{\pi}{4}\right) - \frac{\left(z - \frac{\pi}{4}\right)^2}{2!} - \frac{\left(z - \frac{\pi}{4}\right)^3}{3!} + \frac{\left(z - \frac{\pi}{4}\right)^4}{4!} + \cdots)$$

b) Since the singularity of $sin\ z$ nearest to $\frac{\pi}{4}$ is at infinity, the series converges for all finite values of z, i.e., $|z| < \infty$.

5.27 Example:

Let $f(z) = \ln(1 + z)$, where we consider the branch that has the zero value when $z = 0$.

a) Expand $f\ (z)$ in a Taylor series about $z = 0$.

b) Determine the region of convergence for the series in (a).

c) Expand $ln\frac{1+z}{1-z}$ in a Taylor series about $z = 0$.

Solution:

a) Since $z_0 = 0$,

$$f(z) = \ln(1+z) \qquad\qquad f(0) = 0$$
$$f'(z) = \frac{1}{1+z} = (1+z)^{-1} \qquad\qquad f'(0) = 1$$
$$f''(z) = -(1+z)^{-2} \qquad\qquad f''^{(0)} = -1$$
$$f'''(z) = (-1)(-2)(1+z)^{-3} \qquad\qquad f'''^{(0)} = 2!$$
$$\vdots \qquad\qquad \vdots$$
$$f^{(n+1)}(z) = (-1)n!\,(1+z)^{-(n+1)} \qquad\qquad f^{(n+1)}(z) = (-1)n!\,.$$

Then

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \cdot z^k = f(0) + f'^{(0)}z - \frac{f''^{(0)}}{2!}z^2 - \frac{f'''^{(0)}}{3!}z^3 + \cdots = z - \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots$$

b) The nth term is $u_n = (-1)^{n-1}\frac{z^n}{n}$. Using the ratio test, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{nz}{n+1}\right| = |z|$ and the series converges for $|z| < 1$. The series can be shown to converge for $|z| = 1$ except for $z = -1$.This result also follows from the fact that the series converges in a circle that extends to the nearest singularity (i.e., $z = -1$) of $f\,(z)$.

c) From the result in (a) we have, on replacing z by $-z$,

$$\ln(1-z) = -z - \frac{z^2}{2!} - \frac{z^3}{3!} - \cdots .$$

EXERCISES:

1. Obtain the Maclaurin series representation $z\cosh(z^2) = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$, $(|z| < \infty)$.

2. Obtain the Taylor series $e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$, $(|z-1| < \infty)$ for the function $f\,(z) = e^z$ by

a) using $f^{(n)}(1), n = 0,1,2,\cdots$. b) writing $e^z = e^{z-1} \cdot e$.

3. Find the Maclaurin series expansion of the function $f(z) = \frac{z}{z^4+9} = \frac{z}{9} \cdot \frac{1}{1+(z^4/9)}$.

4. Show that when $z \neq 0$,

a) $\frac{e^z}{z^2} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots;$ **b)** $\frac{\sin(z^2)}{z^4} = \frac{1}{z^2} - \frac{z^2}{3!} + \frac{z^6}{5!} - \frac{z^{10}}{7!} + \cdots .$

5. Derive the expansions

a) $\frac{\sinh z}{z^2} = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$, $(0 < |z| < \infty)$.

b) $z^3\cosh\left(\frac{1}{z}\right) = \frac{z}{2} + z^3 + \sum_{n=1}^{\infty} \frac{1}{(2n+2)!} \cdot \frac{1}{z^{2n-1}}$, $(0 < |z| < \infty)$.

6. Show that when $0 < |z| < 4$, $\frac{1}{4z-z^2} = \frac{1}{4z} + \sum_{n=0}^{\infty} \frac{z^n}{4^{n+2}}$.

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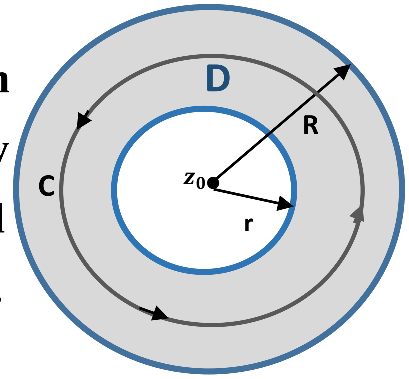
المحاضرة ٢

Chapter Five

SERIES

5.26 Theorem (Laurent Theorem):

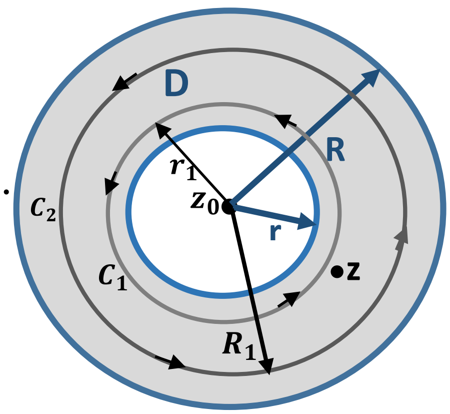


Suppose f is analytic throughout an annular domain $r < |z - z_0| < R$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad (r < |z - z_0| < R). \qquad (1)$$

Where $a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\cdot dz}{(z-z_0)^{n+1}}, (n = 0,1,2,\cdots)$ **and** $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)\cdot dz}{(z-z_0)^{-n+1}}, (n = 1,2,\cdots)$.

Proof:



Let C_1 and C_2 be concentric circles with center z_0 and radii r_1 and R_1, where $r < r_1 < R_1 < R$. Let z be a fixed point in D that also satisfies the inequality $r_1 < |z - z_0| < R_1$. By introducing a crosscut between C_2 and C_1 it follows from Cauchy's integral formula that

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw - \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw. \qquad (2)$$

As in the proof of Theorem 5.25, we can write

$$\frac{1}{2\pi i}\int_{C_1} \frac{f(w)}{w-z} dw = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ where } a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\cdot dz}{(z-z_0)^{n+1}}, (n = 0,1,2,\cdots).$$

We then proceed in a manner similar

$$\frac{-1}{w-z} = \frac{1}{z-w} = \frac{1}{z-w-z_0+z_0} = \frac{1}{(z-z_0)-(w-z_0)} = \frac{1}{z-z_0}\left(\frac{1}{1-\frac{w-z_0}{z-z_0}}\right)$$

$$= \frac{1}{z-z_0}\left(1 + \left(\frac{w-z_0}{z-z_0}\right) + \left(\frac{w-z_0}{z-z_0}\right)^2 + \cdots + \left(\frac{w-z_0}{z-z_0}\right)^{n-1} + \left(\frac{w-z_0}{z-z_0}\right)^n \frac{1}{1-\frac{w-z_0}{z-z_0}}\right)$$

So $$\frac{-1}{w-z} = \frac{1}{z-z_0} + \frac{w-z_0}{(z-z_0)^2} + \frac{(w-z_0)^2}{(z-z_0)^3} + \cdots + \frac{(w-z_0)^{n-1}}{(z-z_0)^n} + \left(\frac{w-z_0}{z-z_0}\right)^n \frac{1}{z-w}. \qquad (3)$$

Multiplying both sides of (3) by $f(w)$ and using (2), we have

$$\frac{-1}{2\pi i}\int_{C_2} \frac{f(w)}{w-z} dw = \frac{1}{2\pi i}\int_{C_2} \frac{f(w)}{z-z_0} dw + \frac{1}{2\pi i}\int_{C_2} \frac{(w-z_0)f(w)}{(z-z_0)^2} dw + \cdots + \frac{1}{2\pi i}\int_{C_2} \frac{(w-z_0)^{n-1}f(w)}{(z-z_0)^n} dw + V_n.$$

Where $V_n = \frac{1}{2\pi i}\int_{C_2}\left(\frac{w-z_0}{z-z_0}\right)^n \frac{f(w)}{z-w}dw$. Using Cauchy's integral formulas $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{C_1}\frac{f(w)}{(w-z_0)^{n+1}}dw, n = 0,1,2,\cdots$ becomes

$$b_1 = \frac{1}{2\pi i}\int_{C_2} f(w)\cdot dw,\ b_2 = \frac{1}{2\pi i}\int_{C_2}(w-z_0)f(w)dw,\cdots,\ b_n = \frac{1}{2\pi i}\int_{C_2}(w-z_0)^{n-1}f(w)dw$$

So

$$-\frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw = \sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}.$$

$$f(z) = \frac{1}{2\pi i}\int_{C_1}\frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_{C_2}\frac{f(w)}{w-z}dw$$

$$= a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_{n-1}(z-z_0)^{n-1} + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + U_n + V_n.$$

The required result follows if we can show that (a) $\lim_{n\to\infty} U_n = 0$ and (b) $\lim_{n\to\infty} V_n = 0$. The proof of (a) follows from Theorem 5.25. To prove (b), we first note that since w is on C_2, $\left|\frac{w-z_0}{z-z_0}\right| = k < 1$, where k is a constant. Also, we have $|f(w)| < M$ where M is a constant and $|z-w| = |(z-z_0)-(w-z_0)| \geq |z-z_0| - r_2$.Hence we have

$$|V_n| = \frac{1}{2\pi}\left|\int_{C_2}\left(\frac{w-z_0}{z-z_0}\right)^n\frac{f(w)}{z-w}dw\right| \leq \frac{1}{2\pi}\cdot\frac{k^nM}{|z-z_0|-r_2}\cdot 2\pi r_2 = \frac{k^nMr_2}{|z-z_0|-r_2}.$$

Then, $\lim_{n\to\infty} V_n = 0$ and the proof is complete.□

5.27 Example:

a) Find the Maclaurin series for the function $f(z) = e^z$.

b) Expand $f(z) = e^{\frac{3}{z}}$ in a Laurent series valid for $0 < |z| < \infty$.

Solution:

a)

$$\begin{aligned} f(z) &= e^z & f(0) &= e^0 = 1\\ f'(z) &= e^z & f'(0) &= e^0 = 1\\ f''(z) &= e^z & f''(0) &= e^0 = 1\\ f'''(z) &= e^z & f'''(0) &= e^0 = 1\\ &\vdots & &\vdots\\ f^{(n+1)}(z) &= e^z & f^{(n+1)}(z) &= e^0 = 1 \end{aligned}$$

$$f(z) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\cdot z^k = f(0) + f'^{(0)}z - \frac{f''^{(0)}}{2!}z^2 - \frac{f'''^{(0)}}{3!}z^3 + \cdots = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$f(z) = e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!},\ (|z| < \infty).$$

b) The Laurent series for f by simply replacing z in (a) by $3/z, z \neq 0$

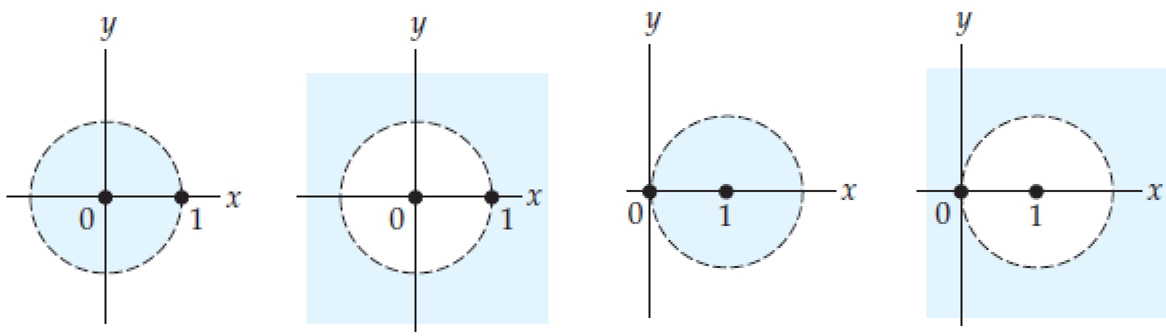
$$f(z) = e^{\frac{3}{z}} = 1 + z + \frac{3}{z} + \frac{3^2}{2!z^2} + \frac{3^3}{3!z^3} + \cdots,\ (0 < |z| < \infty).$$

5.28 Example:

Expand $f(z) = \frac{1}{z(z-1)}$ in a Laurent series valid for the following annular domains.

a) $0 < |z| < 1$, **b)** $1 < |z|$, **c)** $0 < |z-1| < 1$, **d)** $1 < |z-1|$.

Solution:



(a) (b) (c) (d)

The four specified annular domains are shown in above. The black dots in each figure represent the two isolated singularities, $z = 0$ and $z = 1$, of f. In parts (a) and (b) we want to represent f in a series involving only negative and nonnegative integer powers of z, whereas in parts (c) and (d) we want to represent f in a series involving negative and nonnegative integer powers of $z - 1$.

a) By writing $f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} \cdot \frac{1}{1-z}$. We can use geometric series with $a = 1$ and $|z| < 1$ to write $1/(1 - z)$ as a series $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ then

$$f(z) = -\frac{1}{z} \cdot (1 + z + z^2 + z^3 + \cdots) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots .$$

Converges for $0 < |z| < 1$.

b) To obtain a series that converges for $1 < |z|$, we start by constructing a series that converges for $|1/z| < 1$. To this end we write the given function f as

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z} \cdot \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{z^2} \cdot \frac{1}{\left(1-\frac{1}{z}\right)} .$$

Since $\frac{1}{\left(1-\frac{1}{z}\right)} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots$ then the series in the brackets converges for $|1/z| < 1$ or equivalently for $1 < |z|$. Thus the required Laurent series is

$$f(z) = -\frac{1}{z^2} \cdot \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) = -\frac{1}{z^2} - \frac{1}{z^3} - \cdots .$$

c) This is basically the same problem as in part (a), except that we want all powers of $z - 1$.To that end, we add and subtract 1 in the denominator and use geometric series with $a = 1$

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z-1}\cdot\frac{1}{1+(z-1)}=\frac{1}{z-1}\cdot\frac{1}{1-(-(z-1))}$$
$$=\frac{1}{z-1}\cdot(1-(z-1)+(-(z-1))^2+(-(z-1))^3+\cdots$$
$$=\frac{1}{z-1}\cdot(1-\frac{1}{z-1}+(z-1)^2+-(z-1)^3+\cdots$$
$$=\frac{1}{z-1}-1+(z-1)-(z-1)^2+\cdots.$$

The requirement that $z\neq 1$ is equivalent to $0<|z-1|$, and the geometric series in brackets converges for $|z-1|<1$.Th us the last series converges for z satisfying $0<|z-1|$ and $|z-1|<1$, that is, for $0<|z-1|<1$.

d) Proceeding as in part (b), we write

$$f(z)=\frac{1}{z(z-1)}=\frac{1}{z-1}\cdot\frac{1}{1+(z-1)}=\frac{1}{z-1}\cdot\frac{1}{(z-1)(1+\frac{1}{(z-1)})}=\frac{1}{(z-1)^2}\cdot\frac{1}{(1-(-\frac{1}{(z-1)})}$$
$$=\frac{1}{(z-1)^2}\cdot(1-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\cdots$$
$$=\frac{1}{(z-1)^2}-\frac{1}{(z-1)^3}+\frac{1}{(z-1)^4}+\cdots.$$

Because the series within the brackets converges for $|1/(z-1)|<1$, the final series converges for $1<|z-1|$.

EXERCISES:

1. Find the Laurent series that represents the function $f(z)=z^2\sin(\frac{1}{z^2})$ in the domain $0<|z|<\infty$.

2. Derive the Laurent series representation $\frac{e^z}{(z+1)^2}$, $0<|z+1|<\infty$.

3. Represent the function $f(z)=\frac{z+1}{z-1}$

a) by its Maclaurin series, and state where the representation is valid ;

b) by its Laurent series in the domain $1<|z|<\infty$.

4. Show that when $0<|z-1|<2$, $\frac{z}{(z-1)(z-3)}=-3\sum_{n=0}^{\infty}\frac{(z-1)^n}{z^{n+2}}-\frac{1}{2(z-1)}$.

5. Write the two Laurent series in powers of z that represent the function $f(z)=\frac{1}{z(1+z^2)}$ in certain domains, and specify those domains.

6. Find the Laurent series that represents the function $f(z)=\frac{\cos z}{z}$ in the domain $0<|z|$.

Chapter Six

Residues and Poles

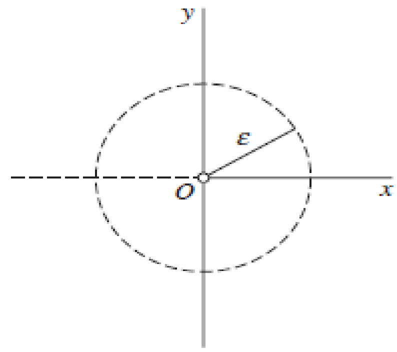
6.1 Definition:

A point z_0 is called a ***singular point*** of a function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0. A singular point z_0 is said to be ***isolated*** if, in addition, there is a deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0 throughout which f is analytic.

6.2 Example:

The function $\frac{z+1}{z^3(z^2+1)}$ has the three isolated singular points $z = 0$ and $z = \pm i$.

6.3 Example:

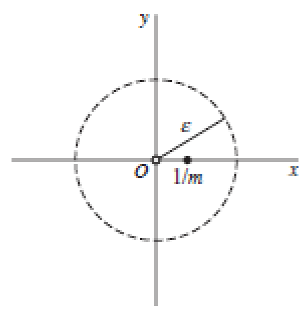


The origin is a singular point of the principal branch

$$\text{Log}\, z = \ln r + i\Theta, (r > 0, -\pi < \Theta < \pi).$$

of the logarithmic function. It is not an isolated singular point since every deleted ε neighborhood of it contains points on the negative real axis and the branch is not even defined there. Similar remarks can be made regarding any branch $\log z = \ln r + i\theta \ (r > 0, \alpha < \theta < \alpha + 2\pi)$, of the logarithmic function.

6.4 Example:



The function $\frac{1}{\sin(\pi/z)}$ has the singular points $z = 0$ and $z = 1/n \ (n = \pm 1, \pm 2, \cdots)$, all lying on the segment of the real axis from $z = -1$ to $z = 1$.Each singular point except The singular point $z = 0$ is not isolated because every deleted ε neighborhood of the origin contains other singular points of the function. More precisely, when a positive number ε is specified and m is any positive integer such that $m > 1/\varepsilon$, the fact that $0 < 1/m < \varepsilon$ means that the point $z = 1/m$ lies in the deleted ε neighborhood $0 < |z| < \varepsilon$.

6.5 Remark:

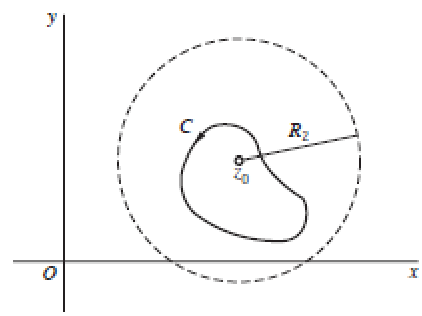
If a function is analytic everywhere inside a simple closed contour C except for a finite number of singular points $z_1, z_2, \cdots, z_n$, those points must all be isolated and the deleted neighborhoods about them can be made small enough to lie entirely inside C. To see that this is so, consider any one of the points z_k. The radius ε of the needed deleted neighborhood can be any positive number that is smaller than the distances to the other singular points and also smaller than the distance from z_k to the closest point on C.

Finally, we mention that it is sometimes convenient to consider the point at infinity as an isolated singular point. To be specific, if there is a positive number R_1 such that f is analytic for $R_1 < |z| < \infty$, then f is said to have an ***isolated singular point*** *at* $z_0 = \infty$.

6.6 Remark:

When z_0 is an isolated singular point of a function f, there is a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$. Consequently, $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}, \quad (0 < |z - z_0| < R_2).$$

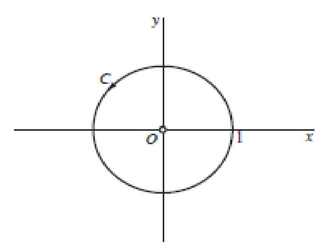


where the coefficients a_n and b_n have certain integral representations. In particular $b_n = \frac{1}{2\pi i} \int_C \frac{f(z) \cdot dz}{(z-z_0)^{-n+1}}$, $(n = 0,1,2,\cdots)$. where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk $0 < |z - z_0| < R_2$. When $n = 1$, this expression for b_n becomes $\int_C f(z) \cdot dz = 2\pi i b_1$. The complex number b_1, which is the coefficient of $1/(z - z_0)$ in expansion (1), is called the ***residue of f*** at the isolated singular point z_0, and we shall often write $b_1 = Res_{z=z_0} f(z)$ then

$$\int_C f(z) \cdot dz = 2\pi i Res_{z=z_0} f(z). \quad (4)$$

This equation provides a powerful method for evaluating certain integrals around simple closed contours.

6.7 Example:



Consider the integral $\int_C z^2 \sin(\frac{1}{z}) \cdot dz$,where C is the positively oriented unit circle $|z| = 1$. Since the integrand is analytic everywhere in the finite plane except at $z = 0$, it has a Laurent series representation that is valid when $0 < |z| < \infty$. Thus the value of integral is $2\pi i$ times the residue of its integrand at $z = 0$.

To determine that residue, we recall the Maclaurin series representation

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \quad (|z| < \infty)$$

and use it to write $z^2 \sin(\frac{1}{z}) = z - \frac{1}{3!} \cdot \frac{1}{z} + \frac{1}{5!} \cdot \frac{1}{z^3} - \frac{1}{7!} \cdot \frac{1}{z^5} + \cdots \quad (0 < |z| < \infty)$.

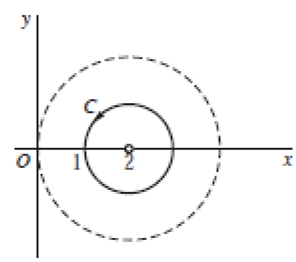
The coefficient of $1/z$ here is the desired residue. Consequently,

$$\int_C z^2 \sin\left(\frac{1}{z}\right) \cdot dz = 2\pi i\left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}.$$

6.8 Example:

Let us show that $\int_C e^{\frac{1}{z^2}} \cdot dz = 0$ when C is the same oriented circle $|z| = 1$ as in Example 6.7. Since $1/z^2$ is analytic everywhere except at the origin, the same is true of the integrand. The isolated singular point $z = 0$ is interior to C. With the aid of the Maclaurin series representation $f(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $(|z| < \infty)$, one can write the Laurent series expansion $e^{\frac{1}{z^2}} = 1 + \frac{1}{1!} \cdot \frac{1}{z^2} + \frac{1}{2!} \cdot \frac{1}{z^4} + \frac{1}{3!} \cdot \frac{1}{z^6} + \cdots, (0 < |z| < \infty)$. The residue of the integrand at its isolated singular point $z = 0$ is, therefore, zero ($b_1 = 0$), and the value of integral is established. We are reminded in this example that although the analyticity of a function within and on a simple closed contour C is a sufficient condition for the value of the integral around C to be zero, it is not a necessary condition.

6.9 Example:



A residue can also be used to evaluate the integral $\int_C \frac{dz}{z(z-2)^4}$ where C is the positively oriented circle $|z - 2| = 1$. Since the integrand is analytic everywhere in the finite plane except at the points $z = 0$ and $z = 2$, it has a Laurent series representation that is valid in the punctured disk $0 < |z - 2| < 2$. Thus the value of integral is $2\pi i$ times the residue of its integrand at $z = 2$. To determine that residue, we recall the Maclaurin series expansion $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots \quad (|z| < 1)$, and use it to write

$$\frac{1}{z(z-2)^4} = \frac{1}{(z-2)^4} \cdot \frac{1}{z} = \frac{1}{(z-2)^4} \cdot \frac{1}{2+(z-2)} = \frac{1}{(z-2)^4} \cdot \frac{1}{2(1-\left(-\frac{z-2}{2}\right))} = \frac{1}{2(z-2)^4} \cdot \frac{1}{(1-\left(-\frac{z-2}{2}\right))}$$

$$= \frac{1}{2(z-2)^4} \cdot \left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \frac{(z-2)^3}{2^3} + \cdots\right)$$

$$= \frac{1}{2(z-2)^4} - \frac{1}{2^2(z-2)^3} + \frac{1}{2^3(z-2)^2} - \frac{1}{2^4(z-2)} + \frac{1}{2^5} - \frac{z-2}{2^6} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-4}, \ (0 < |z-2| < 2).$$

In this Laurent series the coefficient of $1/(z-2)$ is the desired residue, namely $-1/16$. Consequently, $\int_C \frac{dz}{z(z-2)^4} = 2\pi i\left(-\frac{1}{16}\right) = -\frac{\pi i}{8}$.

6.10 Remark:

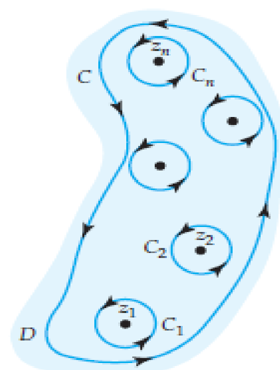
If, except for a *finite* number of singular points, a function *f* is analytic inside a simple closed contour *C*, those singular points must be isolated. The following theorem, which is known as ***Cauchy's residue theorem***, is a precise statement of the fact that if *f* is also analytic on *C* and if *C* is positively oriented, then the value of the integral of *f* around *C* is $2\pi i$ times the sum of the residues of *f* at the singular points inside *C*.

6.11 Theorem:

Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points z_k (k = 1, 2, . . . , n) inside C, then

$$\int_C f(z) \cdot dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \qquad \textbf{(5)}$$

Proof:



Suppose $C_1, C_2, . . . , C_n$ are circles centered at $z_1, z_2, . . . , z_n$, respectively. Suppose further that each circle C_k has a radius r_k small enough so that $C_1, C_2, \cdots, C_n$ are mutually disjoint and are interior to the simple closed curve *C*. Now in (4) we saw that

$$\int_C f(z) \cdot dz = 2\pi i Res_{z=z_k} f(z) = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z). \square$$

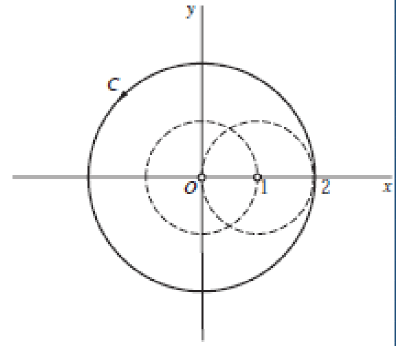
6.12 Example:

Let us use the theorem to evaluate the integral $\int_C \frac{5z-2}{z(z-1)} dz$ where *C* is the circle $|z| = 2$, described counterclockwise. The integrand has the two isolated singularities $z = 0$ and $z = 1$, both of which are interior to *C*. We can find the residues b_1 at $z = 0$ and b_2 at $z = 1$ with the aid of the Maclaurin series $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ $(|z| < 1)$. We observe first that when $0 < |z| < 1$

$$\frac{5z-2}{z(z-1)} = \frac{5z-2}{z} \cdot \frac{-1}{1-z} = \left(5 - \frac{2}{z}\right) \cdot (-1 - z - z^2 - z^3 - \cdots);$$

and, by identifying the coefficient of $1/z$ in the product on the right here, we find that $b_1 = 2$. Also, since

$$\frac{5z-2}{z(z-1)} = \frac{5z-5+3}{z-1} \cdot \frac{1}{z-1+1} = \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)} = \left(5+\frac{3}{z-1}\right) \cdot \frac{1}{1-(-(z-1))}$$
$$= \left(5+\frac{3}{z-1}\right) \cdot \left(1-(z-1)+(z-1)^2-(z-1)^3+\cdots\right)$$



when $0 < |z-1| < 1$, it is clear that $b_2 = 3$.Thus

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(b_1 + b_2) = 10\pi i.$$

6.13 Remark:

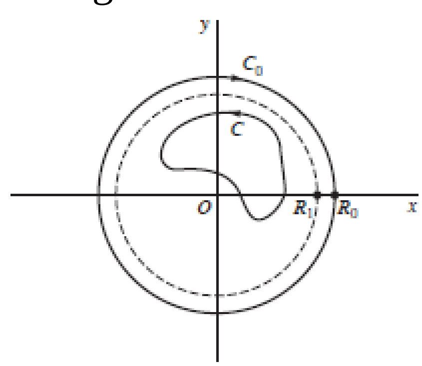
In this example, it is actually simpler to write the integrand as the sum of its partial fractions: $\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$.

Then, since $2/z$ is already a Laurent series when $0 < |z| < 1$ and since $3/(z-1)$ is a Laurent series when $0 < |z-1| < 1$, it follows that

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(2) + 2\pi i(3) = 10\pi i.$$

6.14 Remark:

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Next, let R_1 denote a positive number which is large enough that C lies inside the circle $|z| = R_1$. The function f is evidently analytic throughout the domain $R_1 < |z| < \infty$ and the point at infinity is then said to be an isolated singular point of f. Now let C_0 denote a circle clockwise direction, where $R_0 > R_1$. The ***residue of f at infinity*** is defined by means of the equation



$$\int_{C_0} f(z) \cdot dz = 2\pi i Res_{z=\infty} f(z) \qquad (6)$$

Note that the circle C_0 keeps the point at infinity on the left, just as the singular point in the finite plane is on the left in equation (4). Since f is analytic throughout the closed region bounded by C and C_0, the principle of deformation of paths tells us that

$$\int_C f(z) \cdot dz = \int_{-C_0} f(z) \cdot dz = -\int_{C_0} f(z) \cdot dz.$$

So, in view of equation (6), $\int_C f(z) \cdot dz = -2\pi i Res_{z=\infty} f(z)$. To find this residue, write the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1 < |z| < \infty), \qquad (7)$$

where $c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)dz}{z^{n+1}}$ $(n = 0, \pm 1, \pm 2, \cdots)$. Replacing z by $1/z$ in expansion

(7) and then multiplying through the result by $\frac{1}{z^2}$, we see that

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n} = \quad (0 < |z| < \frac{1}{R_1}) \text{ and } c_{-1} = Res_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

Putting $n = -1$ in expression $c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)dz}{z^{n+1}}$, we now have $c_{-1} = \frac{1}{2\pi i} \int_{-C_0} f(z)dz$ or $\int_{C_0} f(z) \cdot dz = -2\pi i Res_{z=0}[\frac{1}{z^2} f\left(\frac{1}{z}\right)]$. Note how it follows from this and equation (6) that

$$Res_{z=\infty} f(z) = -Res_{z=0}[\frac{1}{z^2} f\left(\frac{1}{z}\right)]. \tag{8}$$

With equations (6) and (9), the following theorem is now established. This theorem is sometimes more efficient to use than Cauchy's residue theorem since it involves only one residue

6.15 Theorem:

If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C \boldsymbol{f(z) \cdot dz = 2Res_{z=0}[\frac{1}{z^2} f\left(\frac{1}{z}\right)]}.$$

6.16 Example:

In the example 6.12, we evaluated the integral of $f(z) = \frac{5z-2}{z(z-1)}$ around the circle $|z| = 2$, described counterclockwise, by finding the residues of $f(z)$ at $z = 0$ and $z = 1$. Since

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \cdot \frac{5\frac{1}{z}-2}{\frac{1}{z}\left(\frac{1}{z}-1\right)} = \frac{1}{z^2} \cdot \frac{\frac{5-2z}{z}}{\frac{1-z}{z^2}} = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z} = \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \cdots)$$

$$= \frac{5}{z} + 5 + 5z + 5z^2 + \cdots - 2 - 2z - 2z^2 - 2z^3 + \cdots$$

$$= \frac{5}{z} + 3 + 3z + \cdots \quad (0 < |z| < 1),$$

we see that the theorem here can also be used, where the desired residue is 5. More precisely, $\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(5) = 10\pi i$, where C is the circle in question. This is, of course, the result obtained in the example 6.12.

EXERCISES:

1. Find the residue at $z = 0$ of the function

a) $\frac{1}{z+z^2}$; **b)** $z \cdot cos\left(\frac{1}{z}\right)$; **c)** $\frac{z-\sin z}{z}$; **d)** $\frac{\cot z}{z^4}$; **e)** $\frac{\sinh z}{z^4(1-z^2)}$.

2. Use Cauchy's residue theorem to evaluate the integral of each of these functions around the circle $|z| = 3$ in the positive sense:

a) $\frac{e^{-z}}{z^2}$; **b)** $\frac{e^{-z}}{(z-1)^2}$; **c)** $z^2 e^{\frac{1}{z}}$; **d)** $\frac{z+1}{z^2-2z}$.

3. Use the theorem, involving a single residue, to evaluate the integral of each of these functions around the circle $|z| = 2$ in the positive sense:

a) $\frac{z^5}{1-z^3}$; **b)** $\frac{1}{1+z^2}$; **c)** $\frac{1}{z}$.

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Chapter Six

Residues and Poles

6.17 Remark:

We saw that the theory of residues is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots,$$

in a punctured disk $0 < |z - z_0| < R_2$. The portion $\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$ of the series, involving negative powers of $z - z_0$, is called the ***principal part*** of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory that appears in following later.

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer m ($m \geq 1$) such that $b_m \neq 0$ and $b_{m+1} = b_{m+2} = \cdots = 0$. That is, a Laurent series takes the form

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}, \quad (0 < |z - z_0| < R_2),$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called a ***pole of order m***. A pole of order $m = 1$ is usually referred to as a ***simple pole***.

6.18 Example:

Observe that the function

$$\frac{z^2-2z+3}{z-2} = \frac{z(z-2)+3}{z-2} = z + \frac{3}{z-2} = 2 + (z-2) + \frac{3}{z-2}, \qquad (0 < |z - 2| < \infty)$$

has a simple pole ($m = 1$) at $z_0 = 2$. Its residue b_1 there is 3.When a Laurent series representation is written in the form $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$,$(0 < |z| < R_2)$ the residue of f at z_0 is, of course, the coefficient c_{-1}.

6.19 Example:

From the representation

$$f(z) = \frac{1}{z^2(1+z)} = \frac{1}{z^2}\cdot\frac{1}{1-(-z)} = \frac{1}{z^2}(1 - z + z^2 - z^3 + \cdots)$$

$$= \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 + \cdots, (0 < |z| < 1),$$

one can see that f has a pole of order $m = 2$ at the origin and that $Res_{z=z_0} f(z) = -1$.

6.20 Example:

The function

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots\right) = \frac{1}{z^3} + \frac{1}{3!}\cdot\frac{1}{z} + \frac{z}{5!}\cdot\frac{z^3}{7!} + \cdots, (0 < |z| < \infty);$$

has a pole of order $m = 3$ at $z_0 = 0$, with residue $b = 1/6$.

6.21 Remark:

There remain two extremes, the case in which every coefficient in the principal part a Laurent series is zero and the one in which an infinite number of them are nonzero. When every b_n is zero, so that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots, (0 < |z - z_0| < R_2),$$

z_0 is known as a ***removable singular point***. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0) = a_0$, expansion becomes valid throughout the entire disk $|z - z_0| < R_2$. Since a power series always represents an analytic function interior to its circle of convergence, it follows that f is analytic at z_0 when it is assigned the value a_0 there. The singularity z_0 is, therefore, ***removed***.

6.22 Example:

The point $z_0 = 0$ is a removable singular point of the function $f(z) = \frac{1-\cos z}{z^2}$ because $f(z) = \frac{1}{z^2}(1 - \cos z) = \frac{1}{z^2}\left(1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right)\right)$

$$= \frac{1}{z^2}\left(\frac{z^2}{2!} - \frac{z^4}{4!} + \frac{z^6}{6!} - \cdots\right) = \frac{1}{2!} - \frac{z^2}{4!} + \frac{z^4}{6!} - \cdots, (0 < |z| < \infty).$$

When the value $f(0) = 1/2$ is assigned, f becomes entire.

6.23 Remark:

If an infinite number of the coefficients b_n in the principal part of a Laurent series are nonzero, z_0 is said to be an ***essential singular point*** of f.

6.24 Example:

Since $e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\frac{1}{z^n} = 1 + \frac{1}{1!}\cdot\frac{1}{z} + \frac{1}{2!}\cdot\frac{1}{z^2} + \cdots, (0 < |z| < \infty)$ then $e^{\frac{1}{z}}$ has an essential singular point at $z_0 = 0$, where the residue b_1 is unity.

6.25 Remark:

This example can be used to illustrate an important result known as ***Picard's theorem***. It concerns the behavior of a function near an essential singular point and states that ***in each neighborhood of an essential singular point, a function***

assumes every finite value, with one possible exception, an infinite number of times.

EXERCISES:

1. In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

a) $ze^{\frac{1}{z}}$; **b)** $\frac{z^2}{1+z}$; **c)** $\frac{\sin z}{z}$; **d)** $\frac{\cos z}{z}$; **e)** $\frac{1}{(2-z)^3}$.

2. Show that the singular point of each of the following functions is a pole. Determine the order *m* of that pole and the corresponding residue *b*.

a) $\frac{1-\cosh z}{z^3}$; **b)** $\frac{1-e^{2z}}{z^4}$; **c)** $\frac{e^{2z}}{(z-1)^2}$.

3. Suppose that a function f is analytic at z_0, and write $g(z) = f(z)/(z - z_0)$. Show that

a) if $f(z_0) \neq 0$, then z_0 is a simple pole of g, with residue $f(z_0)$;

b) if $f(z_0) = 0$, then z_0 is a removable singular point of g.

4. Write the function $f(z) = \frac{8a^3z^2}{(z^2+a^2)^3}$ $(a > 0)$ as $f(z) = \frac{\emptyset(z)}{(z-ai)^3}$ where $\emptyset(z) = \frac{8a^3z^2}{(z+ai)^3}$ Point out why $\varphi(z)$ has a Taylor series representation about $z = ai$, and then use it to show that the principal part of *f* at that point is

$$\frac{\emptyset''(ai)/2}{z-ai} + \frac{\emptyset'(ai)}{(z-ai)^2} + \frac{\emptyset(ai)}{(z-ai)^3} = -\frac{i/2}{z-ai} - \frac{a/2}{(z-ai)^2} - \frac{a^2 i}{(z-ai)^3} .$$

6.26 Remark:

When a function f has an isolated singularity at a point z_0 , the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $1/(z - z_0)$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

6.27 Theorem:

An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}, \tag{1}$$

where $\varphi(z)$ is analytic and nonzero at z_0 . Moreover,

$$Res_{z=z_0} f(z) = \varphi(z_0) \quad if\ m = 1 \tag{2}$$

and

$$Res_{z=z_0} f(z) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!} \quad if\ m \geq 2. \tag{3}$$

6.28 Remark:

Observe that expression (2) need not have been written separately since, with the convention that $\varphi^{(0)}(z_0) = \varphi(z_0)$ and $0! = 1$, expression (3) reduces to it when $m = 1$. The following examples serve to illustrate the use of the theorem:

6.29 Example:

The function $f(z) = \frac{z+1}{z^2+9}$ has an isolated singular point at $z = 3i$ and can be written $f(z) = \frac{\varphi(z)}{z-3i}$ where $\varphi(z) = \frac{z+1}{z+3i}$. Since $\varphi(z)$ is analytic at $z = 3i$ and $\varphi(3i) \neq 0$, that point is a simple pole of the function f; and the residue there is

$$B_1 = \varphi(3i) = \frac{3i+1}{6i} \cdot \frac{-i}{-i} = \frac{3-i}{6}.$$

The point $z = -3i$ is also a simple pole of f , with residue $B_2 = \frac{3+i}{6}$.

6.30 Example:

If $f(z) = \frac{z^3+2z}{(z-i)^3}$ then $f(z) = \frac{\varphi(z)}{(z-i)^3}$ where $\varphi(z) = z^3 + 2z$. The function $\varphi(z)$ is entire, and $\varphi(i) = i \neq 0$. Hence f has a pole of order 3 at $z = i$, with residue

$$B = \frac{\varphi''(i)}{2!} = \frac{6i}{2} = 3i\ .$$

6.31 Remark:

The theorem can, of course, be used when branches of multiple-valued functions are involved.

6.32 Example:

Suppose that $f(z) = \frac{(\log z)^3}{z^2+1}$ where the branch $\log z = \ln r + i\theta$ ($r > 0, 0 < \theta < 2\pi$) of the logarithmic function is to be used. To find the residue of f at the singularity $z = i$, we write $f(z) = \frac{\varphi(z)}{z-i}$ where $\varphi(z) = \frac{(\log z)^3}{z+i}$. The function $\varphi(z)$ is clearly analytic at $z = i$; and, since $\varphi(i) = \frac{(\log i)^3}{2i} = \frac{\left(\ln 1 + i\frac{\pi}{2}\right)^3}{2i} = -\frac{\pi^3}{16} \neq 0$, f has a simple pole there. The residue is $B = \varphi(i) = -\frac{\pi^3}{16}$.

6.33 Remark:

While the theorem 6.27 can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

6.34 Example:

If, for instance, the residue of the function $f(z) = \frac{\sinh z}{z^4}$ is needed at the singularity $z = 0$, it would be incorrect to write $f(z) = \frac{\varphi(z)}{z^4}$ where $\varphi(z) = \sinh z$, and to attempt an application of formula (3) with $m = 4$. For it is necessary that $\varphi(z_0) \neq 0$ if that formula is to be used. In this case, the simplest way to find the residue is to write out a few terms of the Laurent series for $f(z)$, as was done in Example 6.20. There it was shown that $z = 0$ is a pole of the ***third*** order, with residue $B = 1/6$.

6.35 Remark:

In some cases, the series approach can be effectively combined with the theorem 6.27.

6.36 Example:

Suppose that $f(z) = \frac{1}{z(e^z-1)}$. Since $z(e^z - 1)$ is entire and its zeros are $z = 2n\pi i$, $(n = 0, \pm 1, \pm 2, \cdots)$, the point $z = 0$ is clearly an isolated singular point of the function. From the Maclaurin series $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$; $(|z| < \infty)$, we see that

$$z(e^z - 1) = z\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots - 1\right) = z^2(1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + \cdots); (|z| < \infty).$$

Thus $f(z) = \frac{\varphi(z)}{z^2}$ where $\varphi(z) = \frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots}$. Since $\varphi(z)$ is analytic at $z = 0$ and $\varphi(0) = 1 \neq 0$, the point $z = 0$ is a pole of the ***second*** order; and, according to formula (3), the residue is $B = \varphi'(i)$. Because

$$\varphi'(z) = \frac{-(\frac{1}{2!}+\frac{2z}{3!}+\frac{3z}{4!}+\cdots)}{\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+\cdots\right)^2},$$

in a neighborhood of the origin, then, $B = -1/2$.

EXERCISES:

1. In each case, show that any singular point of the function is a pole. Determine the order m of each pole, and find the corresponding residue B.

a) $\frac{z^2+2}{z-1}$; **b)** $\left(\frac{z}{2z+1}\right)^3$; **c)** $\frac{e^z}{z^2+\pi^2}$.

2. Show that:

a) $Res_{z=-1}\frac{z^{1/4}}{z+1} = \frac{1+i}{\sqrt{2}}$, $(|z| > 0, 0 < \arg z < 2\pi)$;

b) $Res_{z=i}\frac{Log\, z}{(z^2+1)^2} = \frac{\pi+2i}{8}$;

c) $Res_{z=i}\frac{z^{1/2}}{(z^2+1)^2} = \frac{1-i}{8\sqrt{2}}$, $(|z| > 0, 0 < \arg z < 2\pi)$.

3. Find the value of the integral $\int_C \frac{3z^3+2}{(z-1)(z^2+9)}dz$ taken counterclockwise around the circle

a) $|z - 2| = 2$; **b)** $|z| = 4$.

4. Find the value of the integral $\int_C \frac{dz}{z^3(z=4)}$ taken counterclockwise around the circle

a) $|z| = 2$; **b)** $|z + 2| = 3$.

5. Evaluate the integral $\int_C \frac{\cosh \pi z}{z(z^2+1)}dz$ when C is the circle $|z| = 2$, described in the positive sense.

6. Use the theorem 6.15, involving a single residue, to evaluate the integral of $f(z)$ around the positively oriented circle $|z| = 3$ when

a) $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$; **b)** $f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$; **c)** $f(z) = \frac{z^3 e^{1/z}}{1+z^3}$.

6.37 Definition:

Suppose that a function f is analytic at a point z_0 then all of the derivatives $f^{(n)}(z)$ $(n = 1, 2, \cdots)$ exist at z_0. If $f(z_0) = 0$ and if there is a positive integer m such that $f^{(m)}(z_0) \neq 0$ and each derivative of lower order vanishes at z_0, then f is said to have a ***zero of order m*** at z_0.

6.38 Remark:

Our first theorem here provides a useful alternative characterization of zeros of order m.

6.38 Theorem:

Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that

$$\boldsymbol{f(z) = (z - z_0)^m g(z)} \qquad \textbf{(4)}$$

6.39 Example:

The polynomial $f(z) = z^3 - 8 = (z - 2)(z^2 + 2z + 4)$ has a zero of order $m = 1$ at $z_0 = 2$ since

$$f(z) = (z - 2)g(z),$$

where $g(z) = z^2 + 2z + 4$, and because f and g are entire and $g(2) = 12 \neq 0$. Note how the fact that $z_0 = 2$ is a zero of order $m = 1$ of f also follows from the observations that f is entire and that $f(2) = 0$ and $f'(2) = 12 \neq 0$.

6.40 Example:

The entire function $f(z) = z(e^z - 1)$ has a zero of order $m = 2$ at the point $z_0 = 0$ since $f(0) = f'(0) = 0$ and $f''(0) = 2 \neq 0$.In this case, expression (4) becomes $f(z) = (z - 0)^2 g(z)$, where g is the entire function defined by means of the equations $g(x) = \begin{cases} \frac{e^z - 1}{z} & when\ z \neq 0, \\ 1 & when\ z = 0. \end{cases}$

6.41 Remark:

Our next theorem tells us that the zeros of an analytic function are isolated when the function is not identically equal to zero.

6.42 Theorem:

Given a function f and a point z_0, suppose that

a) f is analytic at z_0;

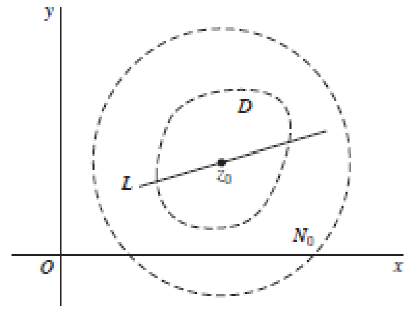
b) $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0.

Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \varepsilon$ of z_0.

6.43 Remark:

Our final theorem here concerns functions with zeros that are not all isolated.

6.44 Theorem:



Given a function f and a point z_0, suppose that

a) f is analytic throughout a neighborhood N_0 of z_0;

b) $f(z) = 0$ at each point z of a domain D or line segment L containing z_0.

Then $f(z) \equiv 0$ in N_0; that is, $f(z)$ is identically equal to zero throughout N_0.

6.45 Remark:

The following theorem shows how zeros of order *m* can create poles of order *m*.

6.46 Theorem:

Suppose that

a) two functions p and q are analytic at a point z_0;

b) $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0.

6.47 Example:

The two functions $p(z) = 1$ and $q(z) = z(e^z - 1)$ are entire; and we know from Example 6.40 that q has a zero of order $m = 2$ at the point $z_0 = 0$. Hence it follows from Theorem 6.46 that the quotient has a pole of order 2 at that point. This was demonstrated in another way in example 6.36.

6.48 Remark:

Theorem 6.46 leads us to another method for identifying simple poles and finding the corresponding residues. This method, stated just below as Theorem 6.49, is sometimes easier to use than the theorem 6.27.

6.49 Theorem:

Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$, and $q'(z_0) \neq 0$ then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\boldsymbol{Res_{z=z_0} f(z) = \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}} \qquad \textbf{(6)}$$

6.50 Example:

Consider the function $f(z) = cot\, z = \frac{cos\, z}{\sin z}$ which is a quotient of the entire functions $p(z) = cos\, z$ and $q(z) = sin\, z$. Its singularities occur at the zeros of q, or at the points $z = n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$. Since $p(n\pi) = (-1)^n \neq 0$,

$q(n\pi) = 0$ and $q'(n\pi) = (-1)^n \neq 0$, each singular point $z = n\pi$ of f is a simple pole, with residue $B_n = \frac{p(n\pi)}{q'(n\pi)} = \frac{(-1)^n}{(-1)^n} = 1$.

6.51 Example:

The residue of the function $f(z) = \frac{\tanh z}{z^2} = \frac{\sinh z}{z^2 \cosh z}$ at the zero $z = \frac{\pi i}{2}$ of $\cosh z$ is realy found by writing $p(z) = \sinh z$ and $q(z) = z^2 \cosh z$. Since $p\left(\frac{\pi i}{2}\right) = \sinh\left(\frac{\pi i}{2}\right) = i \sin\frac{\pi i}{2} = i \neq 0$ and $q\left(\frac{\pi i}{2}\right) = 0$, $q'(\frac{\pi i}{2}) = \left(\frac{\pi i}{2}\right)^2 \sinh\left(\frac{\pi i}{2}\right) = -\frac{\pi^2}{4} i \neq 0$, we find $z = \frac{\pi i}{2}$ is a simple pole of f and that the residue there is $B = \frac{p\left(\frac{\pi i}{2}\right)}{q'(\frac{\pi i}{2})} = \frac{4}{\pi^2}$.

6.52 Example:

Since the point $z_0 = \sqrt{2} e^{i\pi/4} = 1 + i$ is a zero of the polynomial $z^4 + 4$, it is also an isolated singularity of the function $f(z) = \frac{z}{z^4 + 4}$. Writing $p(z) = z$ and $q(z) = z^4 + 4$, we find that $p(z_0) = z_0 \neq 0$, $q(z_0) = 0$ and $q'(z_0) = 4z_0^3 \neq 0$ and hence that z_0 is a simple pole of f . The residue there is, moreover

$$B_0 = \frac{p(z_0)}{q'(z_0)} = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8} .$$

EXERCISES:

1. Show that the point $z = 0$ is a simple pole of the function $f(z) = csc\, z = \frac{1}{\sin z}$ and that the residue there is unity by appealing to

a) Theorem 6.49;

b) the Laurent series for $csc\, z$.

2. Show that

a) $Res_{z=\pi i} \frac{z - \sinh z}{z^2 \sinh z} = \frac{i}{\pi}$;

b) $Res_{z=\pi i} \frac{e^{zt}}{\sinh z} + Res_{z=-\pi i} \frac{e^{zt}}{\sinh z} = -2\cos(\pi t)$.

3. Show that

a) $Res_{z=z_n}(z \sec z) = (-1)^{n+1} z_n$ where $z_n = \frac{\pi}{2} + n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$;

b) $Res_{z=z_n}(\tanh z) = 1$ where $z_n = \left(\frac{\pi}{2} + n\pi\right) i$,$(n = 0, \pm 1, \pm 2, \cdots)$.

4. Let C denote the positively oriented circle $|z| = 2$ and evaluate the integral

a) $\int_C \tan z \, dz$; **b)** $\int_C \frac{dz}{\sinh 2z}$.

$z - z_0$	**Laurent Series for** $0 < \lvert z - z_0 \rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order *n*	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n}$
Simple pole	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0}$
Essential singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots$

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الثاني

ا. م. د. بان جعفر الطائي

المحاضرة ٤

Chapter Seven

APPLICATIONS OF RESIDUES

7.1 Remark:

In calculus, the improper integral of a continuous function $f\,(x)$ over the semi-infinite interval $0 \leq x < \infty$ is defined by means of the equation

$$\int_0^\infty f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx. \qquad (1)$$

When the limit on the right exists, the improper integral is said to ***converge*** to that limit. If $f\,(x)$ is continuous for all x, its improper integral over the infinite interval $-\infty < x < \infty$ is defined by writing

$$\int_{-\infty}^\infty f(x)dx = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)dx; \qquad (2)$$

and when both of the limits here exist, we say that integral (2) converges to their sum. Another value that is assigned to integral (2) is often useful. Namely, the ***Cauchy principal value*** (**P.V.**) of integral (2) is the number

$$\text{P.V.} \int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty} \int_{-R}^R f(x)dx; \qquad (3)$$

provided this single limit exists. If integral (2) converges, its Cauchy principal value (3) exists; and that value is the number to which integral (2) converges. This is because

$$\lim_{R\to\infty} \int_{-R}^R f(x)dx = \lim_{R\to\infty} \left[\int_{-R}^0 f(x)dx + \int_0^R f(x)dx\right]$$

$$= \lim_{R\to\infty} \int_{-R}^0 f(x)dx + \lim_{R\to\infty} \int_0^R f(x)dx.$$

and these last two limits are the same as the limits on the right in equation (2). It is not, however, always true that integral (2) converges when its Cauchy principal value exists, as the following example shows.

7.2 Example:

Observe that

$$\text{P.V.} \int_{-\infty}^\infty x dx = \lim_{R\to\infty} \int_{-R}^R x dx = \lim_{R\to\infty} \left[\frac{x^2}{2}\right]_{-R}^R = \lim_{R\to\infty} 0 = 0.$$

On the other hand,

$$\int_{-\infty}^\infty x dx = \lim_{R_1\to\infty} \int_{R_1}^0 x dx + \lim_{R_2\to\infty} \int_0^{R_2} x dx = \lim_{R_1\to\infty} \left[\frac{x^2}{2}\right]_{-R_1}^0 + \lim_{R_2\to\infty} \left[\frac{x^2}{2}\right]_0^{R_2} = -\lim_{R_1\to\infty} \frac{R_1^2}{2} + \lim_{R_2\to\infty} \frac{R_2^2}{2};$$

and since these last two limits do not exist, we find that the improper integral fails to exist.

7.3 Remark:

But suppose that $f(x)\ (-\infty < x < \infty)$ is an ***even*** function, one where $f(-x) = f(x)$ for all x, and assume that the Cauchy principal value (3) exists. The symmetry of the graph of $y = f(x)$ with respect to the y axis tells us that

$$\int_{-R_1}^{0} f(x)dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)dx \text{, and } \int_{0}^{R_2} f(x)dx = \frac{1}{2}\int_{-R_2}^{R_2} f(x)dx.$$

Thus $$\int_{-R_1}^{0} f(x)dx + \int_{0}^{R_2} f(x)dx = \frac{1}{2}\int_{-R_1}^{R_1} f(x)dx + \frac{1}{2}\int_{-R_2}^{R_2} f(x)dx.$$

If we let R_1 and R_2 tend to ∞ on each side here, the fact that the limits on the right exist means that the limits on the left do too. In fact,

$$\int_{-\infty}^{\infty} f(x)dx = \text{P.V.}\int_{-\infty}^{\infty} f(x)dx. \tag{4}$$

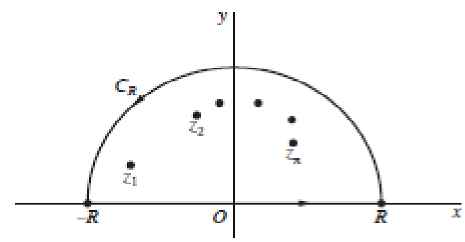
Moreover, since $\int_{0}^{R} f(x)dx = \frac{1}{2}\int_{-R}^{R} f(x)dx$ it is also true that

$$\int_{0}^{\infty} f(x)dx = \frac{1}{2}\left[\text{P.V.}\int_{-\infty}^{\infty} f(x)dx\right]. \tag{5}$$

7.4 Remark:

We now describe a method involving sums of residues that is often used to evaluate improper integrals of rational functions $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. We agree that $q(z)$ has no real zeros but has at least one zero above the real axis.

The method begins with the identification of all the distinct zeros of the polynomial $q(z)$ that lie above the real axis. They are finite in number and may be labeled $z_1, z_2, \cdots, z_n$ where n is less than or equal to the degree of $q(z)$. We then integrate the quotient $f(x) = p(x)/q(x)$ around the positively oriented boundary of the semicircular region.



That simple closed contour consists of the segment of the real axis from $z = -R$ to $z = R$ and the top half of the circle $|z| = R$, described counterclockwise and denoted by C_R. It is understood that the positive number R is large enough so that the points $z_1, z_2, \cdots, z_n$ all lie inside the closed path.

The parametric representation $z = x, (-R \le x \le R)$ of the segment of the real axis just mentioned and Cauchy's residue theorem can be used to write

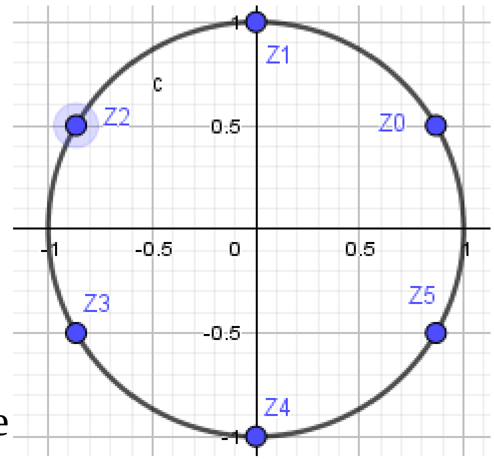
$$\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \tag{6}$$

or $$\int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) - \int_{C_R} f(z)dz$$

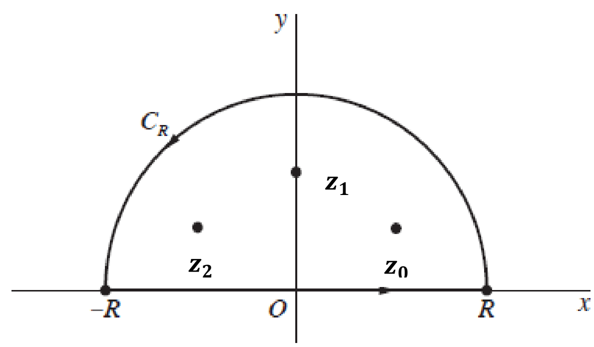
If $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$ it then follows that P.V. $\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z)$; and if $f\ (x)$ is even, equations (4) and (5) tell us that

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \text{ and } \int_{0}^{\infty} f(x)dx = \pi i \sum_{k=1}^{n} Res_{z=z_k} f(z) \tag{7}$$

7.5 Example:



In order to evaluate the integral $\int_{0}^{\infty} \frac{x^2}{x^6+1} dx$, we start with the observation that the function $f(z) = \frac{z^2}{z^6+1}$ has isolated singularities at the zeros of $z^6 + 1$, which are the sixth roots of -1 and is analytic everywhere else. The sixth roots of -1 are $z_k = e^{i(\frac{\pi}{6}+\frac{2k\pi}{6})}$, $(k = 0,1,2,3,4,5)$ and it is clear that none of them lies on the real axis.The first three roots, $z_0 = e^{i\frac{\pi}{6}}$, $z_1 = i$ and $z_2 = e^{i\frac{5\pi}{6}}$ lie in the upper half plane and the other three lie in the lower one. When $R > 1$, the points z_k , $(k = 0,1,2)$ lie in the interior of the semicircular region bounded by the segment $z = x$, $(-R \leq x \leq R)$ of the real axis and the upper half C_R of the circle $|z| = R$ from $z = R$ to $z = -R$. Integrating $f\ (z)$ counterclockwise around the boundary of this semicircular region, we see that $\int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i(B_0 + B_1 + B_2)$ where B_k is the residue of $f\ (z)$ at z_k, $(k = 0,1,2)$.



With the aid of Theorem 6.49 we have $p(z) = z^2$ and $q(z) = z^6 + 1$ are entire, q has a zero of order $m = 1$ at the point z_k, $(k = 0,1,2)$. and $q'(z_k) = 6z_k^5 \neq 0$,$(k = 0,1,2)$ then z_k, $(k = 0,1,2)$ is a simple pole of the quotient $f(z) = p(z)/q(z)$ and that

$$B_k = Res_{z=z_k} \frac{z^2}{z^6+1} = \frac{p(z_0)}{q\prime(z_0)} = \frac{z_k^2}{6z_k^5} = \frac{1}{6z_k^3} = \frac{1}{6\left(e^{i(\frac{\pi}{6}+\frac{2k\pi}{6})}\right)^3}, (k = 0,1,2).$$

$$k = 0 \Longrightarrow B_0 = \frac{1}{6\left(e^{i\left(\frac{\pi}{6}\right)}\right)^3} = \frac{1}{6e^{i\frac{\pi}{2}}} == \frac{1}{6\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)} = \frac{1}{6i}$$

$$k = 1 \Longrightarrow B_1 = \frac{1}{6\left(e^{i\left(\frac{\pi}{6}+\frac{2\pi}{6}\right)}\right)^3} = \frac{1}{6e^{i\frac{3\pi}{2}}} = \frac{1}{6\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)} = -\frac{1}{6i}$$

$$k = 2 \Longrightarrow B_2 = \frac{1}{6\left(e^{i\left(\frac{\pi}{6}+\frac{4\pi}{6}\right)}\right)^3} = \frac{1}{6e^{i\frac{5\pi}{2}}} = \frac{1}{6\left(\cos\frac{5\pi}{2}+i\sin\frac{5\pi}{2}\right)} = \frac{1}{6i}$$

Thus $$2\pi i(B_0 + B_1 + B_2) = 2\pi i\left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3};$$

so $\int_{-R}^{R} f(x)dx = \frac{\pi}{3} - \int_{C_R} f(z)dz$ which is valid for all values of R greater than 1. Next, we show that the value of the integral on the right in equation tends to 0 as R tends to ∞. To do this, we observe that when $|z| = R$, $|z^2| = |z|^2 = R^2$ and $|z^6 + 1| \geq ||z|^6 - 1| = R^6 - 1$. So, if z is any point on C_R,

$$|f(z)| = \frac{|z^2|}{|z^6+1|} \leq M_R \text{ where } M_R = \frac{R^2}{R^6-1};$$

and this means that $$\left|\int_{C_R} f(z)dz\right| \leq M_R\pi R,$$

πR being the length of the semicircle C_R. Since the number $M_R\pi R = \frac{\pi R^3}{R^6-1}$ is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to ∞. More precisely, if we divide both numerator and denominator by R^6 and write

$$M_R\pi R = \frac{\frac{\pi}{R^3}}{1-\frac{1}{R^6}} ,$$

it is evident that $M_R\pi R$ tends to zero. Consequently, in view of inequality $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$. It now follows that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1} dx = \frac{\pi}{3}, \quad \text{or} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{3},$$

Since the integrand here is even, we know from equation (8) that $\int_0^{\infty} \frac{x^2}{x^6+1} dx = \frac{\pi}{6}$.

EXERCISES:

Use residues to evaluate the improper integrals in Exercises 1 through 5.

1. $\int_0^\infty \frac{dx}{x^2+1}$.

2. $\int_0^\infty \frac{dx}{(x^2+1)^2}$.

3. $\int_0^\infty \frac{dx}{x^4+1}$.

4. $\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)}$.

5. $\int_0^\infty \frac{x^2dx}{(x^2+9)(x^2+4)}$.

6. Use residues to find the Cauchy principal values of the integrals $\int_{-\infty}^{\infty} \frac{xdx}{(x^2+1)(x^2+2x+2)}$.

7. Let m and n be integers, where $0 \le m < n$. Follow the steps below to derive the integration formula $\int_0^\infty \frac{x^{2m}dx}{x^{2n}+1} = \frac{\pi}{2n} \cdot csc\left(\frac{2m+1}{2n}\pi\right)$.

a) Show that the zeros of the polynomial $z^{2n}+1$ lying above the real axis are $c_k = e^{i\frac{(2k+1)\pi}{2n}}, (k = 0,1,2,\cdots,n-1)$ and that there are none on that axis.

b) With the aid of Theorem 6.49, show that $Res_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = -\frac{1}{2n}e^{i(2k+1)\alpha}$, $(k = 0,1,2,\cdots,n-1)$where c_k are the zeros found in part (a) and $\alpha = \frac{2m+1}{2n}\pi$. Then use the summation formula $\sum_{k=0}^{n-1} z^k = \frac{1-z^n}{1-z}$,$(z \neq 1)$ to obtain the expression $2\pi i \sum_{k=0}^{n-1} Res_{z=c_k} \frac{z^{2m}}{z^{2n}+1} = \frac{\pi}{n \sin \alpha}$.

c) Use the final result in part (b) to complete the derivation of the integration formula.

7.6 Remark:

Residue theory can be useful in evaluating convergent improper integrals of the Form

$$\int_{-\infty}^{\infty} f(x) \sin ax \, dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x) \cos ax \, dx, \tag{8}$$

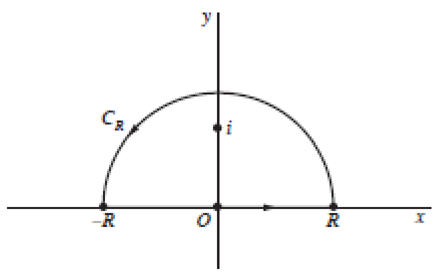
where a denotes a positive constant. These integrals are encountered in applications of Fourier analysis, they often are referred to as ***Fourier integrals*****.** Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x)e^{iax}dx$. In view of Euler's formula $e^{iax} = cos\, ax + i\, sin\, ax$, where a is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = \int_{-\infty}^{\infty} f(x) \cos ax \, dx + i \int_{-\infty}^{\infty} f(x) \sin ax \, dx.$$

Whenever both integrals on the right-hand side converge. Suppose $f(x) = p(x)/q(x)$ is a rational function that is continuous on $(-\infty, \infty)$,where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. Also, $q(x)$ has no zeros on the real axis and at least one zero above it.

7.7 Example:

Show that $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}$.



Solution:

Since the integrand is even, it is sufficient to show that the Cauchy principal value of the integral exists and to find that value. We introduce the function $f(z) = \frac{1}{(z^2+1)^2}$ and observe that the product $f(z)e^{i3z}$ is analytic everywhere on and above the real axis except at the point $z = i$. The singularity $z = i$ lies in the interior of the semicircular region whose boundary consists of the segment $-R \le x \le R$ of the real axis and the upper half C_R of the circle $|z| = R\ (R > 1)$ from $z = R$ to $z = -R$. Integration of $f(z)e^{i3z}$ around that boundary yields the equation

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2} dx = 2\pi i B_1 - \int_{C_R} f(z)e^{i3z}dz, \text{ where } B_1 = Res_{z=i} f(z)e^{i3z}. \tag{9}$$

Since $f(z)e^{i3z} = \frac{e^{i3z}}{(z^2+1)^2} = \frac{e^{i3z}}{((z-i)(z+i))^2} = \frac{e^{i3z}}{(z-i)^2(z+i)^2} = \frac{\varphi(z)}{(z-i)^2}$ where $\varphi(z) = \frac{e^{i3z}}{(z+i)^2}$ the point $z = i$ is evidently a pole of order $m = 2$ of $f(z)e^{i3z}$; and

$$\varphi'(z) = \frac{3i(z+i)^2 e^{i3z} - 2(z+i)e^{i3z}}{(z+i)^4} = \frac{3i(z^2+2zi-1)e^{i3z} - 2(z+i)e^{i3z}}{(z+i)^4} = \frac{(3iz^2-8z-5i)e^{i3z}}{(z+i)^4}.$$

$$B_1 = \varphi'(i) = \frac{(3i\cdot i^2 - 8i - 5i)e^{3i^2}}{(i+i)^4} = \frac{-16ie^{-3}}{(2i)^4} = \frac{-16ie^{-3}}{16}\cdot\frac{i}{i} = \frac{1}{ie^3}.$$

By equating the real parts on each side of equation (9), then, we find that

$$\int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2}dx = \frac{2\pi}{e^3} - Re\int_{C_R} f(z)e^{i3z}dz.$$

Finally, we observe that when z is a point on C_R, $|f(z)| = \left|\frac{1}{(z^2+1)^2}\right| \leq M_R$ where $M_R = \frac{1}{(R^2-1)^2}$ and that $|e^{i3z}| = e^{-3y} \leq 1$ for such a point. Consequently,

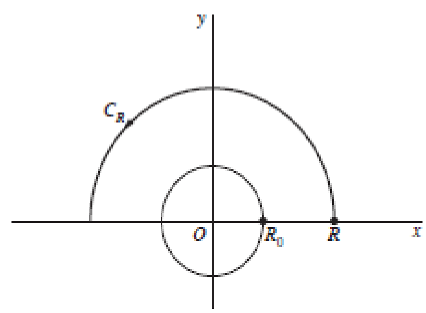
$$\left|Re\int_{C_R} f(z)e^{i3z}dz\right| \leq \left|\int_{C_R} f(z)e^{i3z}dz\right| \leq M_R\pi R. \qquad (10)$$

Since the quantity $M_R\pi R = \frac{\pi R}{(R^2-1)^2}\cdot\frac{\frac{1}{R^4}}{\frac{1}{R^4}} = \frac{\frac{\pi}{R^3}}{\left(1-\frac{1}{R^2}\right)^2}$ tends to 0 as R tends to ∞ and because of inequalities (10), we need only let R tend to ∞ in equation (9) to arrive at the desired result , i.e. $\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2}dx = \frac{2\pi}{e^3}$.

7.8 Remark:

In the evaluation of integrals of the type treated example 7.8, it is sometimes necessary to use ***Jordan's lemma***, which is stated just below as a theorem.

7.9 Theorem:



Suppose that

a) A function $f(z)$ is analytic at all points in the upper half plane $y \geq 0$ that are exterior to a circle $|z| = R_0$;

b) C_R denotes a semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$, where $R > R_0$;

b) for all points z on C_R, there is a positive constant M_R such that $|f(z)| \leq M_R$ and $\lim_{R\to\infty} M_R = 0$.

Then, for every positive constant a, $\lim_{R\to\infty}\int_{C_R} f(z)e^{iaz}dz = 0$.

7.10 Example:

Find the Cauchy principal value of the integral $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+2x+2}dx$.

Solution:

$$z^2+2z+2=0 \Longrightarrow z=\frac{-2\pm\sqrt{4-4(2)}}{2}=\frac{-2\pm\sqrt{-4}}{2}=\frac{-2\pm 2i}{2}=\frac{2(-1\pm i)}{2}=-1\pm i$$

Write $f(z)=\frac{z}{z^2+2z+2}=\frac{z}{(z-z_1)(z-\overline{z_1})}$, where $z_1=-1+i$. The point z_1, which lies above the *x* axis, is a simple pole of the function

$$f(z)e^{iz}=\frac{ze^{iz}}{(z-z_1)(z-\overline{z_1})}=\frac{\varphi(z)}{z-z_1}, \text{ where } \varphi(z)=\frac{ze^{iz}}{(z-\overline{z_1})},$$

with residue $B_1=\varphi(z_1)=\frac{z_1e^{iz_1}}{z_1-\overline{z_1}}$. Hence, when $R>\sqrt{2}$ and C_R denotes the upper half of the positively oriented circle $|z|=R$,

$$\int_{-R}^{R}\frac{xe^{ix}}{x^2+2x+2}dx=2\pi iB_1-\int_{C_R}f(z)e^{iz}dz;$$

and this means that $\int_{-R}^{R}\frac{x\sin x}{x^2+2x+2}dx=Im(2\pi iB_1)-Im\int_{C_R}f(z)e^{iz}dz$. Now

$$\left|Im\int_{C_R}f(z)e^{iz}dz\right|\le\left|\int_{C_R}f(z)e^{iz}dz\right|;$$

and we note that when *z* is a point on C_R, $|f(z)|\le M_R$ where $M_R=\frac{R}{\left(R-\sqrt{2}\right)^2}$ and that $\left|e^{iz}\right|=e^{-y}\le 1$ for such a point. The inequality $\left|Im\int_{C_R}f(z)e^{iz}dz\right|$ tends to zero as *R* tends to infinity. For the quantity $M_R\pi R=\frac{\pi R^2}{\left(R-\sqrt{2}\right)^2}=\frac{\pi}{\left(1-\frac{\sqrt{2}}{R}\right)^2}$ does not tend to zero. The theorem 7.9 provide the desired limit namely $\lim_{R\to\infty}\int_{C_R}f(z)e^{iz}dz=0$.

Since $M_R=\frac{\frac{1}{R}}{\left(1-\frac{\sqrt{2}}{R}\right)^2}\to 0$ as $R\to\infty$. So it does, indeed, follow from inequality (5) that the left-hand side there tends to zero as *R* tends to infinity. Consequently, equation (4), together with expression (3) for the residue *B*1, tells us that

$$\text{P.V.}\int_{-\infty}^{\infty}f(x)dx=\lim_{R\to\infty}\int_{-R}^{R}f(x)dx,$$

$$z_1-\overline{z_1}=-1+i-\overline{-1+i}=-1+i-(-1-i)=2i,$$

$$iz_1=i(-1+i)=-1-i,$$

$$e^{iz_1}=e^{-1-i}=e^{-1}\cdot e^{-i}=\frac{1}{e}(cos1-i\sin 1).$$

$$z_1e^{iz_1}=\frac{1}{e}(-1+i)(cos1-i\sin 1)=\frac{1}{e}((\sin 1-\cos 1)+i(\sin 1+\cos 1))$$

$$2\pi iB_1=2\pi i\frac{z_1e^{iz_1}}{z_1-\overline{z_1}}=\frac{2\pi i}{e}\cdot\frac{((\sin 1-\cos 1)+i(\sin 1+\cos 1))}{2i}$$

$$= \frac{\pi}{e} \cdot ((\sin 1 - \cos 1) + i(\sin 1 + \cos 1))$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2+2x+2} dx = Im(2\pi i B_1) = \frac{\pi}{e}(\sin 1 + \cos 1) .$$

EXERCISES:

Use residues to evaluate the improper integrals in exercises 1 through 5.

1. $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{(x^2+a^2)(x^2+b^2)}, \quad (a > b > 0).$
2. $\int_{-\infty}^{\infty} \frac{\cos ax \, dx}{x^2+1}, \quad (a > 0).$
3. $\int_{0}^{\infty} \frac{x \sin 2x \, dx}{x^2+3}$.
4. $\int_{-\infty}^{\infty} \frac{x^3 \sin ax \, dx}{x^4+4}, \quad (a > 0).$
5. $\int_{0}^{\infty} \frac{x^3 \sin x \, dx}{(x^2+1)(x^2+9)}$.

Use residues to find the Cauchy principal values of the improper integrals in exercises 1 and 2.

1. $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5}$ ‘
2. $\int_{-\infty}^{\infty} \frac{(x+1) \cos x \, dx}{x^2+4x+5}$.

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ا. م. د. بان جعفر الطائي

المحاضرة ٥

Chapter Seven

APPLICATIONS OF RESIDUES

7.11 Remark:

Now we illustrate the use of indented paths. We begin with an important limit that will be used in the example 7.13.

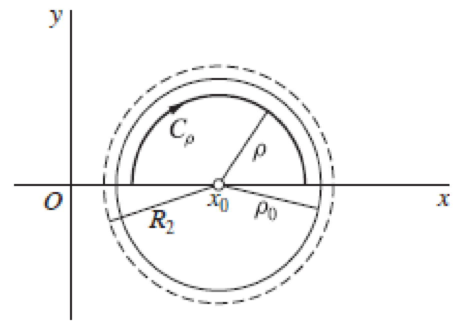
7.12 Theorem:

Suppose that

a) a function $f\ (z)$ has a simple pole at a point $z = x_0$ on the real axis,with a Laurent series representation in a punctured disk $0 < |z - x_0| < R_2$ and with residue B_0 ;

b) C_ρ denotes the upper half of a circle $|z - x_0| = \rho$, where $\rho < R_2$ and the clockwise direction is taken.

Then $\lim\limits_{\rho\to 0} \int_{C_\rho} f(z)dz = -B_0\pi i$.

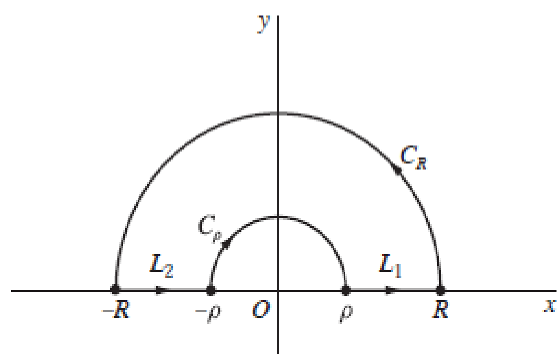


7.13 Example:

Show that $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

Solution:

Integrate $\frac{e^{iz}}{z}$ around the simple closed contour. Denote ρ and R be positive real numbers, where $\rho < R$; and L_1 and L_2 represent the intervals $\rho \le x \le R$ and $-R \le x \le -\rho$, respectively on the real axis. While the semicircle C_R of the circle $|z| = R$ from $z = R$ to $z = -R$, the semicircle C_ρ is introduced here in order to avoid passing through the singularity $z = 0$ of the quotient $\frac{e^{iz}}{z}$.



The Cauchy –Goursat theorem tells us that

$$\int_{L_1} \frac{e^{iz}}{z} dz + \int_{C_R} \frac{e^{iz}}{z} dz + \int_{L_2} \frac{e^{iz}}{z} dz + \int_{C_\rho} \frac{e^{iz}}{z} dz = 0,$$

or
$$\int_{L_1} \frac{e^{iz}}{z} dz + \int_{L_2} \frac{e^{iz}}{z} dz = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz. \qquad (1)$$

Moreover, since the legs L_1 and $-L_2$ have parametric representations

$z = re^{i0} = r(\cos 0 + i\sin 0) = r \ (\rho \le r \le R)$ and $z = re^{i\pi} = r(\cos\pi + i\sin\pi) = -r \ (\rho \le r \le R)$, respectively, the left-hand side of equation (1) can be written

$$\int_{L_1} \frac{e^{iz}}{z} dz - \int_{-L_2} \frac{e^{iz}}{z} dz = \int_\rho^R \frac{e^{ir}}{r} dr - \int_\rho^R \frac{e^{-ir}}{r} dr$$
$$= \int_\rho^R \frac{\cos r}{r} dr + i\int_\rho^R \frac{\sin r}{r} dr - \left(\int_\rho^R \frac{\cos r}{r} dr - i\int_\rho^R \frac{\sin r}{r} dr\right)$$
$$= 2i\int_\rho^R \frac{\sin r}{r} dr.$$

Consequently, equation (1) becomes

$$2i\int_\rho^R \frac{\sin r}{r} dr = -\int_{C_\rho} \frac{e^{iz}}{z} dz - \int_{C_R} \frac{e^{iz}}{z} dz. \quad (2)$$

Now, from the Laurent series representation

$$\frac{e^{iz}}{z} = \frac{1}{z}\left[1 + \frac{(iz)}{1!} + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \cdots\right] = \frac{1}{z} + \frac{i}{1!} + \frac{i^2}{2!}z + \frac{i^3}{3!}z^2 + \cdots \ (0 < |z| < \infty),$$

it is clear that $\frac{e^{iz}}{z}$ has a simple pole at the origin, with residue unity. So, according to the theorem 7.12 $\lim_{\rho\to 0}\int_{C_\rho} \frac{e^{iz}}{z} dz = -B_0\pi i = -\pi i.$

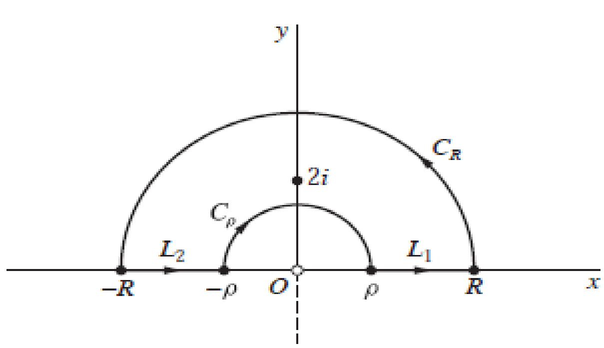
Also since $\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{R}$ when z is a point on C_R, we know from Jordan's lemma 7.9 that $\lim_{R\to\infty}\int_{C_R} \frac{e^{iz}}{z} dz = 0$. Thus, by letting ρ tend to 0 in equation (2) and then letting R tend to ∞, we arrive at the result $\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$.

7.14 Remark:

The example here involves the same indented path that was used in the example 7.13. The indentation is, however, due to a branch point, rather than an isolated singularity

7.15 Example:

Show that $\int_0^\infty \frac{\ln x}{(x^2+4)^2} dx = \frac{\pi}{32}(\ln 2 - 1)$.



Solution

Consider the branch

$$f(z) = \frac{\log z}{(z^2+4)^2} \quad \left(|z| > 0, \ -\frac{\pi}{2} < arg\ z < \frac{3\pi}{2}\right)$$

of the multiple-valued function $(log\ z)/(z^2+4)^2$. This branch, whose branch cut consists of the origin and the negative imaginary axis, is analytic everywhere in the stated domain except at the point $z = 2i$, where the same indented path and the same labels L_1, L_2, C_ρ, and C_R are used. In order that the isolated singularity $z = 2i$ be inside the closed path, we require that $\rho < 2 < R$.

According to Cauchy's residue theorem,

$$\int_{L_1} f(z)dz + \int_{C_R} f(z)dz + \int_{L_2} f(z)dz + \int_{C_\rho} f(z)dz = 2\pi i Res_{z=2i} f(z).$$

That is

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz = 2\pi i Res_{z=2i} f(z) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz. \quad (3)$$

Since $f(z) = \frac{\ln r + i\theta}{\left(r^2 e^{i2\theta}+4\right)^2}$, $\left(z = re^{i\theta}\right)$, the parametric representations.

$$z = re^{i0} = r \ (\rho \le r \le R) \text{ and } z = re^{i\pi} = -r \ (\rho \le r \le R),$$

For the legs L_1 and $-L_2$ respectively can be used to write the left-hand side of equation (3) as

$$\int_{L_1} f(z)dz - \int_{-L_2} f(z)dz = \int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + \int_\rho^R \frac{\ln r + i\pi}{(r^2+4)^2} dr$$

$$= \int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + \int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + \int_\rho^R \frac{i\pi}{(r^2+4)^2} dr$$

$$= 2\int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + i\pi \int_\rho^R \frac{dr}{(r^2+4)^2}.$$

Also since $f(z) = \frac{\log z}{(z^2+4)^2} = \frac{\log z}{(z+2i)^2(z-2i)^2} = \frac{\emptyset(z)}{(z-2i)^2}$ where $\emptyset(z) = \frac{\log z}{(z+2i)^2}$, the singularity $z = 2i$ of $f(z)$ is a pole of order 2, with residue

$$\emptyset'(z) = \frac{(z+2i)^2 \cdot \frac{1}{z} - 2(z+2i)\log z}{(z+2i)^4} = \frac{\frac{(z+2i)}{z} - 2\log z}{(z+2i)^3} = \frac{(z+2i) - 2z\log z}{z(z+2i)^3}$$

$$\emptyset'(2i) = \frac{(2i+2i) - 2(2i)\log(2i)}{(2i)(2i+2i)^3} = \frac{4i - 4i(ln2 + i\frac{\pi}{2})}{128} = \frac{i - i(ln2 + i\frac{\pi}{2})}{32} = \frac{\pi}{64} + i\frac{(1-\ln 2)}{32}.$$

Equation (3) thus becomes

$$2\int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + i\pi \int_\rho^R \frac{dr}{(r^2+4)^2} = 2\pi i\left(\frac{\pi}{64} + i\frac{(1-\ln 2)}{32}\right) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

$$2\int_\rho^R \frac{\ln r}{(r^2+4)^2} dr + i\pi \int_\rho^R \frac{dr}{(r^2+4)^2} = \frac{\pi}{16}(\ln 2 - 1) + i\frac{\pi^2}{32} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz; \quad (4)$$

and, by equating the real parts on each side here, we find that

$$2\int_\rho^R \frac{\ln r}{(r^2+4)^2} dr = \frac{\pi}{16}(\ln 2 - 1) - Re\int_{C_\rho} f(z)dz - Re\int_{C_R} f(z)dz. \quad (5)$$

It remains only to show that $\lim_{\rho\to 0} Re\int_{C_\rho} f(z)dz = 0$ and $\lim_{R\to\infty} Re\int_{C_R} f(z)dz = 0$.

For, by letting ρ and R tend to 0 and ∞, respectively, in equation (5), we then arrive at $\int_0^\infty \frac{\ln r}{(r^2+4)^2} dr = \frac{\pi}{32}(\ln 2 - 1)$ which is solve the example.

First, we note that if $\rho < 1$ and $z = \rho e^{i\theta}$ is a point on C_ρ, then $|log\, z| = |ln\, \rho + i\theta| \le |ln\, \rho| + |i\theta| \le -ln\, \rho + \pi$ and $|z^2+4| \ge ||z|^2 - 4| = 4 - \rho^2$,

$$\left|f(z) = \frac{\log z}{(z^2+4)^2}\right| = \frac{|log\, z|}{|z^2+4|^2} \le \frac{-ln\, \rho + \pi}{(4-\rho^2)^2}$$

As a consequence,

$$\left|Re\int_{C_\rho} f(z)dz\right| \le \left|\int_{C_\rho} f(z)dz\right| \le \int_{C_\rho} |f(z)|dz \le \int_{C_\rho} \frac{-ln\, \rho + \pi}{(4-\rho^2)^2} dz = \frac{-ln\, \rho + \pi}{(4-\rho^2)^2}\pi\rho = \frac{\pi\rho - \rho \ln \rho}{(4-\rho^2)^2}\pi;$$

and, by Hospital's rule, the product $\rho\, ln\, \rho$ in the numerator on the far right here tends to 0 as ρ tends to 0. So $\lim_{\rho\to 0} Re\int_{C_\rho} f(z)dz = 0$. Likewise, by writing

$$\left|Re\int_{C_R} f(z)dz\right| \le \left|\int_{C_R} f(z)dz\right| \le \int_{C_R} |f(z)|dz \le \frac{\pi - ln\, R}{(R^2-4)^2}\pi R$$

$$= \frac{\pi - ln\, R}{\left(R(R-\frac{4}{R})\right)^2}\pi R = \frac{\pi - ln\, R}{R^2(R-\frac{4}{R})^2}\pi R = \frac{\frac{\pi}{R} - \frac{\ln R}{R}}{(R-\frac{4}{R})^2}\pi;$$

and using Hospital's rule to show that the quotient $(\ln R)/R$ tends to 0 as R tends to ∞, we obtain $\lim_{R\to\infty} Re\int_{C_R} f(z)dz = 0$.

7.16 Remark:

The integration formula $\int_0^\infty \frac{dx}{(x^2+4)^2} = \frac{\pi}{32}$ follows by equating imaginary, rather than real, parts on each side of equation (4) :

$$\pi \int_\rho^R \frac{dr}{(r^2+4)^2} = \frac{\pi^2}{32} - Im\int_{C_\rho} f(z)dz - Im\int_{C_R} f(z)dz.$$

Since $\left|Im\int_{C_\rho} f(z)dz\right| \le \left|\int_{C_\rho} f(z)dz\right|$ and $\left|Im\int_{C_R} f(z)dz\right| \le \left|\int_{C_R} f(z)dz\right|$ then by letting ρ and R tend to 0 and ∞,we obtained $\int_0^\infty \frac{dx}{(x^2+4)^2} = \frac{\pi}{32}$.

7.17 Remark:

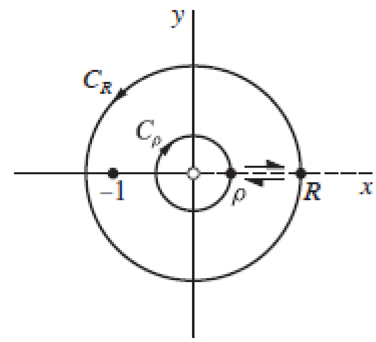
Cauchy's residue theorem can be useful in evaluating a real integral when part of the path of integration of the function $f\,(z)$ to which the theorem is applied lies along a branch cut of that function.

7.18 Example:

Let x^{-a}, where $x > 0$ and $0 < a < 1$, denote the principal value of the indicated power of x; that is, x^{-a} is the positive real number $e^{\ln x^{-a}} = e^{-a\ln x}$.We shall evaluate here the improper real integral

$$\int_0^{\infty} \frac{x^{-a}}{x+1}dx\,, \quad (0 < a < 1),$$

which is important in the study of the ***gamma function***.



Let C_ρ and C_R denote the circles $|z| = \rho$ and $|z| = R$, respectively, where $\rho < 1 < R$; and we assign them the orientations. We then integrate the branch

$$f(z) = \frac{z^{-a}}{z+1} \quad (|z| > 0, 0 < \arg z < 2\pi);$$

of the multiple-valued function $\frac{z^{-a}}{z+1}$ with branch cut $arg\ z = 0$, around the simple closed contour. That contour is traced out by a point moving from ρ to R along the top of the branch cut for $f(z)$, next around C_R and back to R, then along the bottom of the cut to ρ, and finally around C_ρ back to ρ.

Now $\theta = 0$ and $\theta = 2\pi$ along the upper and lower "edges" respectively, of the cut annulus that is formed. Since $f(z) = \frac{e^{-a\log z}}{z+1} = \frac{e^{-a\,(\ln r + i\theta)}}{re^{i\theta}+1}$ where $z = re^{i\theta}$, it follows that $f(z) = \frac{e^{-a\,(\ln r + i0)}}{re^{i0}+1} = \frac{r^{-a}}{r+1}$ on the upper edge, where $z = re^{i0}$, and that $f(z) = \frac{e^{-a\,(\ln r + i2\pi)}}{re^{i2\pi}+1} = \frac{r^{-a}e^{-i2a\pi}}{r+1}$ on the lower edge, where $z = re^{i2\pi}$. The residue theorem thus suggests that

$$\int_\rho^R \frac{r^{-a}}{r+1}dr + \int_{C_R} f(z)dz - \int_\rho^R \frac{r^{-a}e^{-i2a\pi}}{r+1}dr + \int_{C_\rho} f(z)dz = 2\pi i Res_{z=-1} f(z). \quad (6)$$

$$\int_\rho^R \frac{r^{-a}}{r+1}dr - e^{-i2a\pi}\int_\rho^R \frac{r^{-a}}{r+1}dr = 2\pi i Res_{z=-1} f(z) - \int_{C_R} f(z)dz - \int_{C_\rho} f(z)dz.$$

$$(1 - e^{-i2a\pi})\int_\rho^R \frac{r^{-a}}{r+1}dr = 2\pi i Res_{z=-1} f(z) - \int_{C_R} f(z)dz - \int_{C_\rho} f(z)dz. \quad (7)$$

The residue in equation (6) can be found by noting that the function

$$f(z) = \frac{z^{-a}}{z+1} = \frac{\varphi(z)}{z+1},\ \varphi(z) = z^{-a} = e^{\log z^{-a}} = e^{-a\log z} = e^{-a(\ln r + i\theta)},\ (r > 0, 0 < \theta < 2\pi)$$

is analytic at $z = -1$ and that

$$\varphi(-1) = e^{-a(\ln 1 + i\pi)} = e^{-i\pi a} \neq 0\,.$$

This shows that the point $z = -1$ is a simple pole of the function $f(z)$ and that $Res_{z=-1}f(z) = e^{-i\pi a}$. Equation (7) can, therefore, be written as

$$(1 - e^{-i2a\pi})\int_\rho^R \frac{r^{-a}}{r+1}dr = 2\pi i e^{-i\pi a} - \int_{C_R} f(z)dz - \int_{C_\rho} f(z)dz. \qquad (8)$$

According to definition of $f\ (z)$,

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\rho^{-a}}{1-\rho}\cdot 2\pi\rho = \frac{2\pi}{1-\rho}\cdot \rho^{1-a} \text{and} \left|\int_{C_R} f(z)dz\right| \le \frac{R^{-a}}{R-1}\cdot 2\pi R = \frac{2\pi R}{R-1}\cdot\frac{1}{R^a}.$$

Since $0 < a < 1$, the values of these two integrals evidently tend to 0 as ρ and R tend to 0 and ∞, respectively. Hence, if we let ρ tend to 0 and then R tend to ∞ in equation (8), we arrive at the result $(1 - e^{-i2a\pi})\int_0^\infty \frac{r^{-a}}{r+1}dr = 2\pi i e^{-i\pi a}$

Or $\int_0^\infty \frac{r^{-a}}{r+1}dr = 2\pi i \frac{e^{-i\pi a}}{1-e^{-i2a\pi}}\cdot\frac{e^{i\pi a}}{e^{i\pi a}} = \pi\frac{2i}{e^{i\pi a}-e^{-i\pi a}}$.

This is, of course, the same as $\int_0^\infty \frac{x^{-a}}{x+1}dx = \frac{\pi}{\sin a\pi}$, $(0 < a < 1)$.

EXERCISES:

1. Derive the integration formula $\int_0^\infty \frac{\cos ax - \cos bx}{x^2}dx = \frac{\pi}{2}(b-a)$,$(a > 0, b > 0)$. Then, with the aid of the trigonometric identity $1 - cos(2x) = 2\,sin^2 x$, point out how $\int_0^\infty \frac{sin^2 x}{x^2}dx = \frac{\pi}{2}$.
2. Evaluate the improper integral $\int_0^\infty \frac{x^a}{(x^2+1)^2}dx$ where $(-1 < a < 3)$ and $x^a = e^{a\ln x}$.
3. Use the function $f(z) = \frac{z^{\frac{1}{3}}\log z}{z^2+1} = \frac{e^{\frac{1}{3}\log z}\log z}{z^2+1}$,$(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2})$ to derive this pair of integration formulas: $\int_0^\infty \frac{\sqrt[3]{x}\ln x}{x^2+1}dx = \frac{\pi^2}{6}$, $\int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}dx = \frac{\pi}{\sqrt{3}}$.
4. Use the function $f(z) = \frac{(\log z)^2}{z^2+1}$,$(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2})$ to show that $\int_0^\infty \frac{(\ln x)^2}{x^2+1}dx = \frac{\pi^3}{8}$, $\int_0^\infty \frac{\ln x}{x^2+1}dx = 0$.
5. Use the function $f(z) = \frac{z^{\frac{1}{3}}}{(z+a)(z+b)} = \frac{e^{\frac{1}{3}\log z}}{(z+a)(z+b)}$,$(|z| > 0, 0 < \arg z < 2\pi)$ to show formally that $\int_0^\infty \frac{x^{\frac{1}{3}}}{(x+a)(x+b)}dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{a^{\frac{1}{3}}-b^{\frac{1}{3}}}{a-b}$, $(a.b > 0$.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الثاني

ا. م. د. بان جعفر الطائي

المحاضرة ٦

Chapter Seven

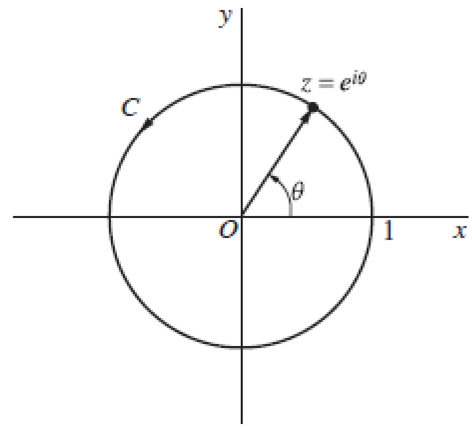
APPLICATIONS OF RESIDUES

7.19 Remark:

The method of residues is also useful in evaluating certain definite integrals of the

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta. \qquad (1)$$

The fact that θ varies from 0 to 2π leads us to consider θ as an argument of a point z on a positively oriented circle C centered at the origin. Taking the radius to be unity we use the parametric representation type $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$ to describe C . We then refer to the differentiation formula $\frac{dz}{d\theta} = ie^{i\theta} = iz$ and since $sin\ \theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and $cos\ \theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ .These relations suggest that we make the substitutions



$$sin\ \theta = \frac{z-z^{-1}}{2i}\ ,\quad cos\ \theta = \frac{z+z^{-1}}{2}\, , d\theta = \frac{dz}{iz}\, , \qquad (2)$$

which transform integral (1) into the contour integral

$$\int_0^{2\pi} F\left(\frac{z-z^{-1}}{2i}, \frac{z+z^{-1}}{2}\right) \frac{dz}{iz}\, , \qquad (3)$$

of a function of z around the circle C. The original integral (1) is simply a parametric form of integral (3). When the integrand in integral (3) reduces to a rational function of z , we can evaluate that integral by means of Cauchy's residue theorem once the zeros in the denominator have been located and provided that none lie on C.

7.20 Example:

Show that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$,$(-1 < a < 1)$.

Solution:

Since $z = e^{i\theta}$, $(0 \le \theta \le 2\pi)$, $d\theta = \frac{dz}{iz}$ and $sin\ \theta = \frac{z-z^{-1}}{2i}$,

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{\frac{dz}{iz}}{1+a\frac{z-z^{-1}}{2i}} = \int_C \frac{dz}{iz\left(1+a\frac{z-z^{-1}}{2i}\right)} = \int_C \frac{dz}{iz+az\frac{z-z^{-1}}{2}} = \int_C \frac{2\cdot dz}{i2z+az^2-a} = \int_C \frac{2/a\cdot dz}{z^2+(2i/a)z-1}\, .$$

Where C is the positively oriented circle $|z| = 1$. The quadratic formula reveals that the denominator of the integrand here has the pure imaginary zeros $z_1 = \left(\frac{-1+\sqrt{1-a^2}}{a}\right)i,\ z_2 = \left(\frac{-1-\sqrt{1-a^2}}{a}\right)i$.So if $f(z)$ denotes the integrand in integral $\int_C \frac{2/a \cdot dz}{z^2(2i/a)z-1}$, then $f(z) = \frac{2/a}{(z-z_1)(z-z_2)}$.

Note that because $|a| < 1$, $|z_2| = \frac{1+\sqrt{1-a^2}}{|a|} > 1$. Also, since $|z_1 \cdot z_2| = 1$, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B_1 is found by writing $f(z) = \frac{\emptyset(z)}{(z-z_1)}$ where $\emptyset(z) = \frac{2/a}{(z-z_2)}$. This shows that z_1 is a simple pole and that

$$z_1 - z_2 = \frac{-1+\sqrt{1-a^2}}{a} i - \frac{-1-\sqrt{1-a^2}}{a} i = \frac{2\sqrt{1-a^2}}{a} i$$

$$B_1 = \emptyset(z_1) = \frac{2/a}{(z_1-z_2)} = \frac{2/a}{\frac{2\sqrt{1-a^2}}{a} i} = \frac{1}{i\sqrt{1-a^2}} .$$

Consequently $\int_C \frac{2/a \cdot dz}{z^2(2i/a)z-1} = 2\pi i B_1 = 2\pi i \cdot \frac{1}{i\sqrt{1-a^2}} = \frac{2\pi}{\sqrt{1-a^2}}$.

EXERCISES:

Use residues to evaluate the definite integrals in Exercises 1 through 3

1. $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

2. $\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta}$.

3. $\int_0^{\pi} \sin^{2n}\theta \cdot d\theta$, $(n = 1,2, \cdots)$.

7.21 Theorem (Argument Principle):

Let C be a simple closed contour lying entirely within a domain D.Suppose f is analytic in D except at a finite number of poles inside C, and that $f(z) \neq 0$ on C.Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0 - N_p\,,$$

where N_0 is the total number of zeros of f inside C and N_p is the total number of poles of f inside C. In determining N_0 and N_p, zeros and poles are counted according to their order or multiplicities.

7.22 Example:

Find the zeros and poles of the function $f(z) = \frac{(z-1)(z-9)^4(z+i)^2}{(z^2-2z+2)^2(z-i)^6(z+6i)^7}$ in a simple closed contour $C: |z| = 2$ then evaluate $\int_C \frac{f'(z)}{f(z)}dz$.

Solution:

The numerator of f reveals that the zeros inside C are $z = 1$ (a simple zero) and $z = -i$ (a zero of order or multiplicity 2).Therefore, the number N_0 of zeros inside C is taken to be $N_0 = 1 + 2 = 3$.Similarly , inspection of the denominator of f shows, after factoring $z^2 - 2z + 2$, that the poles inside C are $z = 1 - i$ (pole of order 2), $z = 1 + i$ (pole of order 2), and $z = i$ (pole of order 6).The number N_p of poles inside C is taken to be $N_p = 2 + 2 + 6 = 10$. By theorem 7.21 we have

$$\int_C \frac{f'(z)}{f(z)}dz = 2\pi i\left(\overbrace{N_0}^{zeros\ of\ f} - \overbrace{N_p}^{poles\ of\ f}\right) = 2\pi i(3-10) = -14\pi i.$$

7.23 Theorem(Rouch´e's theorem):

Let C denote a simple closed contour, and suppose that
a) two functions $f(z)$ and $g(z)$ are analytic inside and on C;
b) $|f(z)| > |g(z)|$ at each point on C.

Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

7.24 Example:

Determine the number of roots of the equation $z^7 - 4z^3 + z - 1 = 0$ inside the circle $|z| = 1$.

Solution:

Write $f(z) = -4z^3$ and $g(z) = z^7 + z - 1$.Then observe that $|f(z)| = 4|z|^3 = 4$ and $|g(z)| \leq |z|^7 + |z| + 1 = 3$ when $|z| = 1$, then $|f(z)| > |g(z)|$. The conditions in Roche's theorem are thus satisfied. Consequently, since $f(z)$ has three zeros, counting multiplicities, inside the circle $|z| = 1$, so does $f(z) + g(z)$. That is, equation $z^7 - 4z^3 + z - 1 = 0$ has three roots there.

7.25 Example:

Use the Roche's theorem to prove the fundamental theorem of algebra "***Any polynomial*** $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ $(a_n \neq 0)$ ***of degree*** n $(n \geq 1)$ ***has at least one zero. That is, there exists at least one point*** z_0 ***such that*** $P(z_0) = 0$".

Solution:

Write $f(z) = a_n z^n$ and $g(z) \leq a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$.
and let z be any point on a circle $|z| = R$, where $R > 1$. When such a point is taken, we see that

$|f(z)| = |a_n| R^n$. Also $|g(z)| = |a_0| + |a_1| R + |a_2| R^2 + \cdots + |a_{n-1}| R^{n-1}$.

Consequently, since $R > 1$,

$$|g(z)| = |a_0| R^{n-1} + |a_1| R^{n-1} + |a_2| R^{n-1} + \cdots + |a_{n-1}| R^{n-1};$$

and it follows that $\frac{|g(z)|}{|f(z)|} \leq \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n| R} < 1$; if, in addition to being greater than unity, $R > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}$. That is, $|f(z)| > |g(z)|$ when $R > 1$ which satisfied. Roche's theorem then tells us that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, namely n, inside C. Hence we may conclude that $P(z)$ has precisely n zeros, counting multiplicities, in the plane.

Note how Lowville's theorem 4.77 only ensured the existence of at least one zero of a polynomial; but Roche's theorem actually ensures the existence of n zeros, counting multiplicities.

EXERCISES:

1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Determine the value of $\int_C \frac{f\prime(z)}{f(z)} dz$ when

a) $f(z) = z^2$; **b)** $f(z) = \frac{z^3 + 2}{z}$; **c)** $f(z) = \frac{(2z - 1)^7}{z^3}$.

2. Determine the number of zeros, counting multiplicities, of the polynomial

a) $z^6 - 5z^4 + z^3 - 2z$; **b)** $2z^4 - 2z^3 + 2z^2 - 2z + 9$.

Inside the circle $|z| = 1$.

3. Determine the number of zeros, counting multiplicities, of the polynomial

a) $z^4 + 3z^3 + 6$; **b)** $z^4 - 2z^3 + 9z^2 + z - 1$; **c)** $z^5 + 3z^3 + z^2 + 1$.

Inside the circle $|z| = 2$.

4. Determine the number of roots, counting multiplicities, of the equation

$2z^5 - 6z^2 + z + 1 = 0$ in the annulus $1 \leq |z| < 2$.

7.26 Remark:

Suppose that $F(s)$ has a pole of order m at a point $s = s_0$ and that its Laurent series representation in a punctured disk $0 < |s - s_0| < R_2$ has principal part

$$F(z) = \sum_{n=0}^{\infty} a_n (s - s_0)^n + \frac{b_1}{s-s_0} + \frac{b_2}{(s-s_0)^2} + \cdots + \frac{b_m}{(s-s_0)^m}, \quad b_m \neq 0,$$

note that $(s - s_0)^m F(s)$ is represented
in that domain by the power series

$$b_m + b_{m-1}(s - s_0) + \cdots + b_2(s - s_0)^{m-2} + b_1(s - s_0)^{m-1} + \sum_{n=0}^{\infty} a_n (s - s_0)^{m+n}.$$

By collecting the terms that make up the coefficient of $(s - s_0)^{m-1}$ in the product of this power series and the Taylor series expansion

$$e^{st} = e^{s_0 t}\left(1 + \frac{t}{1!}(s - s_0) + \cdots + \frac{t^{m-2}}{(m-2)!}(s - s_0)^{m-2} + \frac{t^{m-1}}{(m-1!}(s - s_0)^{m-1} + \cdots\right),$$

of the entire function $e^{st} = e^{s_0 t} e^{(s-s_0)t}$, show that

$$Res_{s=s_0}\left(e^{st}F(s)\right) = e^{s_0 t}\left(b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_{m-1}}{(m-2)!}t^{m-2} + \frac{b_m}{(m-1)!}t^{m-1}\right). \quad (4)$$

When the pole s_0 is of the form $s_0 = \alpha + i\beta \ (\beta \neq 0)$ and $\overline{F(s)} = F(\bar{s})$ at points of analyticity of $F(s)$ the conjugate $\bar{s_0} = \alpha - i\beta$ is also a pole of order m, Moreover,

$$Res_{s=s_0}\left(e^{st}F(s)\right) + Res_{s=\bar{s_0}}\left(e^{st}F(s)\right) = 2e^{\alpha t} Re\left(e^{i\beta t}\left(b_1 + \frac{b_2}{1!}t + \cdots + \frac{b_m}{(m-1)!}t^{m-1}\right). \quad (5)$$

When t is real. Note that if s_0 is a simple pole $(m = 1)$, expressions (4) and (5) become

$$Res_{s=s_0}\left(e^{st}F(s)\right) = e^{s_0 t} Res_{s=s_0}\left(F(s)\right), \text{ and} \quad (6)$$

$$Res_{s=s_0}\left(e^{st}F(s)\right) + Res_{s=\bar{s_0}}\left(e^{st}F(s)\right) = 2e^{\alpha t} Re\left(e^{i\beta t} Res_{s=s_0}\left(F(s)\right)\right) \quad (7)$$

respectively.

if $F(s)$ is the ***Laplace transform*** of $f\ (t)$, defined by means of the equation $F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$ then we can use the residue of $F(s)$ to define the function $f\ (t)$, i.e.

$$f\ (t) = \sum_{n=1}^{N} Res_{s=s_n}\left(e^{st}F(s)\right), \quad (t > 0).$$

7.27 Example:

Find the function $f(t)$ corresponding to the given function

$$F(s) = \frac{s}{(s^2+a^2)^2}, \ (a > 0).$$

Solution:

The singularities of $F(s)$ are the conjugate points $s_0 = ai$ and $s_0 = -ai$. Upon writing $F(s) = \frac{\emptyset(s)}{(s-ai)^2}$ where $\emptyset(s) = \frac{s}{(s+ai)^2}$, we see that $\varphi(s)$ is analytic and nonzero at $s_0 = ai$. Hence s_0 is a pole of order $m = 2$ of $F(s)$. Furthermore, $\overline{F(s)} = F(\bar{s})$ at points where $F(s)$ is analytic. Consequently, $\bar{s_0}$ is also a pole of order 2 of $F(s)$; and we know from expression (5) that

$$Res_{s=s_0}\left(e^{st}F(s)\right) + Res_{s=\overline{s_0}}\left(e^{st}F(s)\right) = 2Re(e^{i\alpha t}(b_1 + b_2 t) ,$$

where b_1 and b_2 are the coefficients in the principal part $\frac{b_1}{s-ai} + \frac{b_2}{(s-ai)^2}$ of $F(s)$ at ai. These coefficients are readily found with the aid of the first two terms in the Taylor series for $\varphi(s)$ about $s_0 = ai$:

$$F(s) = \frac{1}{(s-ai)^2}\emptyset(s) = \frac{1}{(s-ai)^2}(\emptyset(ai) + \frac{\emptyset'(ai)}{1!}(s-ai) + \cdots)$$

$$= \frac{\emptyset(ai)}{(s-ai)^2} + \frac{\emptyset'(ai)}{(s-ai)} + \cdots \ \ (0 < |s-ai| < 2a).$$

It is straightforward to show that $\emptyset(ai) = -i/(4a)$ and $\emptyset'(ai) = 0$, and we find that $b_1 = 0$ and $b_2 = -i/(4a)$. Hence expression (6) becomes

$$Res_{s=s_0}\left(e^{st}F(s)\right) + Res_{s=\overline{s_0}}\left(e^{st}F(s)\right) = 2Re(e^{iat}\left(-\frac{i}{4a}t\right) = \frac{1}{2a}t\sin at.$$

We can, then, conclude that

$$f(t) = \frac{1}{2a}\ t\sin at\,, (t > 0),$$

provided that *F(s)* satisfies the boundedness condition. To verify that boundedness, we let s be any point on the semicircle $s = \gamma + Re^{i\theta}$, $(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2})$, where $\gamma > 0$ and $R > a + \gamma$; and we note that

$$|s| = \left|\gamma + Re^{i\theta}\right| \le \gamma + R \quad \text{and} \quad |s| = \left|\gamma + Re^{i\theta}\right| \ge |\gamma - R| = R - \gamma > a.$$

Since

$$|s^2 + a^2| \ge ||s|^2 - a^2| \ge (R-\gamma)^2 - a^2 > 0,$$

it follows that

$$|F(s)| = \frac{|s|}{|s^2+a^2|^2} \le M_R \quad \text{where} \quad M_R = \frac{\gamma+R}{((R-\gamma)^2-a^2)^2}\,.$$

The desired boundedness is now established, since $M_R \to 0$ as $R \to \infty$.

7.28 Example:

Find the function $f(t)$ corresponding to the given function

$$F(s) = \frac{\tanh s}{s^2} = \frac{\sinh s}{s^2 \cdot \cosh s}, \ (a > 0).$$

Solution:

$F(s)$ has isolated singularities at $s = 0$ and at the zeros $s = \left(\frac{\pi}{2} + n\pi\right) i, (n = 0, \pm 1, \pm 2, \cdots)$ of $cosh\ s$. We list those singularities as

$$s_0 = 0 \text{ and } s_n = \left(n\pi - \frac{\pi}{2}\right) i \ , \ \overline{s_n} = -\left(n\pi - \frac{\pi}{2}\right) i, (n = 1, 2, \cdots).$$

Then, formally,

$$f(t) = Res_{s=s_0}\left(e^{st}F(s)\right) + \sum_{n=1}^{\infty}\left(Res_{s=s_n}\left(e^{st}F(s)\right) + Res_{s=\overline{s_n}}\left(e^{st}F(s)\right)\right).$$

$g(s) = \tanh s$ $\qquad g(0) = \tanh 0 = \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \mathbf{0}$

$g'(s) = sech^2\, s$ $\qquad g'(0) = sech^2\, 0 = \left(\frac{2}{e^0 + e^{-0}}\right)^2 = \mathbf{1}$

$g''(s) = -2sech^2 s \cdot \tanh s$ $\qquad g''(0) = -2sech^2 0 \cdot \tanh 0 = -2\left(\frac{2}{e^0 + e^{-0}}\right)^2 \frac{e^0 - e^{-0}}{e^0 + e^{-0}} = \mathbf{0}$

$g'''(s) = -2sech^4 s + 4sech^2 s \cdot tanh^2 s$ $\qquad g'''(0) = -2sech^4 0 + 4sech^2 0 \cdot tanh^2 0 = -2$

$\vdots$ $\qquad \vdots$

$$g(s) = g(0) + \frac{g'(0)}{1!}s + \frac{g''(0)}{2!}s^2 + \frac{g'''(0)}{3!}s^3 + \cdots, (0 < |s| < \frac{\pi}{2}).$$

$$g(s) = s - \frac{1}{3}s^3 + \cdots \Longrightarrow F(s) = \frac{1}{s^2} \cdot g(s) = \frac{1}{s} - \frac{1}{3}s + \cdots, (0 < |s| < \frac{\pi}{2}).$$

Division of Maclaurin series yields the Laurent series representation

$$F(s) = \frac{1}{s^2} \cdot \frac{\sinh s}{\cosh s} = \frac{1}{s} - \frac{1}{3}s + \cdots, (0 < |s| < \frac{\pi}{2}),$$

which tells us that $s_0 = 0$ is a simple pole of $F(s)$, with residue unity. Thus

$$Res_{s=s_0}\left(e^{st}F(s)\right) = Res_{s=s_0}\left(F(s)\right) = 1, \qquad (8)$$

according to expression (3).

The residues of $F(s)$ at the points s_n $(n = 1, 2, \cdots)$ are readily found by applying the method of Theorem 6.49 for identifying simple poles and determining the residues at such points. To be specific we write $F(s) = \frac{p(s)}{q(s)}$ where $p(s) = \sinh s$ and $q(s) = s^2 \cosh s$, $q'(s) = s^2 \sinh s + 2s \cdot cosh\ s$ and observe that

$$\sinh s_n = \sinh\left(\left(n\pi - \frac{\pi}{2}\right) i\right) = i \sin\left(n\pi - \frac{\pi}{2}\right) = -i cos\ n\pi = (-1)^{n+1} i \neq 0.$$

$$\cosh s_n = \cosh\left(\left(n\pi - \frac{\pi}{2}\right) i\right) = \cos\left(n\pi - \frac{\pi}{2}\right) = 0$$

Then since $p(s_n) = \sinh s_n = (-1)^{n+1} i \neq 0$. $q(s_n) = s_n^2 \cosh s_n = 0$ and $q'(s_n) = s_n^2 \sinh s_n + 2s_n \cdot cosh\ s_n = s_n^2 \cdot (-1)^{n+1} i \neq 0$, we find that

$$Res_{s=s_n}(F(s)) = \frac{p(s_n)}{q'(s_n)} = \frac{(-1)^{n+1}i}{s_n^2\cdot(-1)^{n+1}i} = \frac{1}{s_n^2} = \frac{1}{\left(\left(n\pi-\frac{\pi}{2}\right)i\right)^2} = -\frac{1}{\left(\frac{2n-1}{2}\right)^2\pi^2} = -\frac{4}{\pi^2}\cdot\frac{1}{(2n-1)^2}$$

$,(n = 1,2,\cdots)$.

The identities $\overline{\sinh s} = \sinh\bar{s}$ and $\overline{\cosh s} = \cosh\bar{s}$ ensure that

$$\overline{F(s)} = \overline{\frac{\sinh s}{s^2\cdot\cosh s}} = \frac{\overline{\sinh s}}{\overline{s^2}\cdot\overline{\cosh s}} = \frac{\sinh \bar{s}}{\bar{s}^2\cdot\cosh \bar{s}} = F(\bar{s}),$$

at points of analyticity of $F(s)$. Hence $\bar{s}_n$ is also a simple pole of $F(s)$, $(s_n = \alpha + i\beta = \left(n\pi - \frac{\pi}{2}\right)i, (n = 1, 2, \cdots)$, so expression (7) can be used to write

$$Res_{s=s_n}(e^{st}F(s)) + Res_{s=\bar{s}_n}(e^{st}F(s)) = 2e^{\alpha t}Re(e^{i\beta t}Res_{s=s_n}(F(s)))$$

$$= 2Re\left(-\frac{4}{\pi^2}\cdot\frac{1}{(2n-1)^2}e^{i\left(n\pi-\frac{\pi}{2}\right)t}\right)$$

$$= 2Re\left(-\frac{4}{\pi^2}\cdot\frac{1}{(2n-1)^2}\left(\cos\left(n\pi-\frac{\pi}{2}\right)t + i\sin\left(n\pi-\frac{\pi}{2}\right)t\right)\right)$$

$$= -\frac{8}{\pi^2}\cdot\frac{1}{(2n-1)^2}\left(\cos\left(n\pi-\frac{\pi}{2}\right)t\right), (n = 1, 2, \cdots). \qquad (9)$$

Finally, by substituting expressions (8) and (9), we arrive at the desired result:

$$f(t) = Res_{s=s_0}(e^{st}F(s)) + \sum_{n=1}^{\infty}(Res_{s=s_n}(e^{st}F(s)) + Res_{s=\bar{s}_n}(e^{st}F(s)))$$

$$= 1 - \frac{8}{\pi^2}\cdot\frac{1}{(2n-1)^2}\left(\cos\left(n\pi-\frac{\pi}{2}\right)t\right), (n = 1, 2, \cdots), (t > 0).$$

7.29 Example:

Find the function $f(t)$ corresponding to the given function

$F(s) = \frac{\sinh\left(xs^{\frac{1}{2}}\right)}{s\cdot\sinh\left(s^{\frac{1}{2}}\right)}$, $(0 < x < 1)$, where $s^{\frac{1}{2}}$ denotes any branch of this double-valued function.

Solution:

$\sinh s = \sum_{s=0}^{\infty}\frac{s^{2n+1}}{(2n+1)} = s + \frac{s^3}{3!} + \frac{s^5}{5!} + \cdots$. Since $s^{\frac{1}{2}}$ denotes any branch of this double-valued function. We use the same branch in the numerator and denominator, so that

$$\sinh xs^{\frac{1}{2}} = xs^{1/2} + \frac{\left(xs^{1/2}\right)^3}{3!} + \frac{\left(xs^{1/2}\right)^5}{5!} + \cdots = s^{1/2}\left(x + \frac{x^3 s}{6} + \frac{x^5 s^2}{120} + \cdots\right),$$

$$\sinh s^{\frac{1}{2}} = s^{1/2} + \frac{\left(s^{1/2}\right)^3}{3!} + \frac{\left(s^{1/2}\right)^5}{5!} + \cdots = s^{1/2}\left(1 + \frac{s}{6} + \frac{s^2}{120} + \cdots\right),$$

$$s\cdot\sinh s^{\frac{1}{2}} = s^{3/2}\left(1 + \frac{s}{6} + \frac{s^2}{120} + \cdots\right).$$

$$F(s)=\frac{\sinh\left(xs^{\frac{1}{2}}\right)}{s\cdot\sinh\left(s^{\frac{1}{2}}\right)}=\frac{s^{1/2}(x+\frac{x^3s}{6}+\frac{x^5s^2}{120}+\cdots}{s^{3/2}(1+\frac{s}{6}+\frac{s^2}{120}+\cdots}=\frac{x+\frac{x^3s}{6}+\frac{x^5s^2}{120}+\cdots}{s(1+\frac{s}{6}+\frac{s^2}{120}+\cdots)}. \quad (10)$$

when s is not a singular point of $F(s)$. One such singular point is clearly $s_0=0$.The branch cut of $s^{1/2}$ does not lie along the negative real axis, so that $sinh(s^{1/2})$ is well defined along that axis, the other singular points occur if $s^{1/2}=\pm n\pi i$ $(n=1,2,\cdots)$. The points $s_0=0$ and $s_n=-n^2\pi^2$,$(n=1,2,\cdots)$, thus constitute the set of singular points of *F(s)*. The problem is now to evaluate the residues in the formal series representation

$$f(t)=Res_{s=s_0}\left(e^{st}F(s)\right)+\sum_{n=1}^{\infty}Res_{s=s_n}\left(e^{st}F(s)\right).$$

Now $F(s)=\frac{\emptyset(s)}{s}=\frac{x+\frac{x^3s}{6}+\frac{x^5s^2}{120}+\cdots}{s(1+\frac{s}{6}+\frac{s^2}{120}+\cdots)}$where $\emptyset(s)=\frac{x+\frac{x^3s}{6}+\frac{x^5s^2}{120}+\cdots}{(1+\frac{s}{6}+\frac{s^2}{120}+\cdots)}$ then

$$Res_{s=s_0}\left(F(s)\right)=\emptyset(0)=x$$

which tells us that $s_0=0$ is a simple pole of $F(s)$, with residue *x*. Thus

$$Res_{s=s_0}\left(e^{st}F(s)\right)=Res_{s=s_0}\left(F(s)\right)=x$$

The residues of $F(s)$ at the singular points $s_n=-n^2\pi^2$,$(n=1,2,\cdots)$,we write $F(s)=\frac{p(s)}{q(s)}$ where $p(s)=\sinh\left(xs^{\frac{1}{2}}\right)$ and $q(s)=s\cdot\sinh\left(s^{\frac{1}{2}}\right)$.Now

$$p(s_n)=\sinh\left(xs_n^{1/2}\right)=x+\frac{x^3s_n}{6}+\frac{x^5s_n^2}{120}+\cdots\neq 0,$$

$$p(s_n)=\sinh\left(xs_n^{1/2}\right)=\sinh xn\pi i=-i\sin xn\pi\neq 0,(n=1,2,\cdots),(0<x<1).$$

$$q(s_n)=s\cdot\sinh\left(s_n^{1/2}\right)=s\cdot\sinh n\pi i=-is\cdot\sin n\pi=0,(n=1,2,\cdots).$$

$$q'(s)=\frac{1}{2}s^{\frac{1}{2}}\cdot\cosh\left(s^{\frac{1}{2}}\right)+\sinh\left(s^{\frac{1}{2}}\right)\Longrightarrow q'(s_n)=\frac{1}{2}s_n^{1/2}\cdot\cosh\left(s_n^{1/2}\right)+\overbrace{\sinh\left(s_n^{1/2}\right)}^{=0}$$
$$=\frac{1}{2}n\pi i\cdot(-1)^{n+1}\neq 0,$$

and this tells us that each s_n is a simple pole of $F(s)$, with residue

$$Res_{s=s_n}\left(F(s)\right)=\frac{p(s_n)}{q'(s_n)}=\frac{-i\sin xn\pi}{\frac{1}{2}n\pi i\cdot(-1)^{n+1}}=\frac{2\cdot(-1)^n}{n\pi}\sin xn\pi,(n=1,2,\cdots),(0<x<1).$$

So, $Res_{s=s_n}\left(e^{st}F(s)\right)=2e^{s_nt}Res_{s=s_n}\left(F(s)\right)=\frac{2\cdot(-1)^n}{n\pi}e^{-n^2\pi^2}\sin xn\pi$ .Then

$$f(t)=Res_{s=s_0}\left(e^{st}F(s)\right)+\sum_{n=1}^{\infty}(Res_{s=s_n}\left(e^{st}F(s)\right)=x+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cdot e^{-n^2\pi^2}}{n}\sin xn\pi,(t>0).$$

EXERCISES:

In Exercises 1 through 8 find the function $f(t)$ corresponding to the given function $F(s)$.

1. $F(s) = \frac{2s^3}{s^4-4}$.
2. $F(s) = \frac{2s-2}{(s+1)(s^2+2s+5)}$.
3. $F(s) = \frac{8a^3s^2}{(s^2+a^2)^3}, (a > 0)$.
4. $F(s) = \frac{\sinh(xs)}{s^2 \cosh s}, (0 < x < 1)$.
5. $F(s) = \frac{1}{s \cdot \cosh(s^{1/2})}$.
6. $F(s) = \frac{\coth(\pi s/2)}{s^2+1}$.
7. $F(s) = \frac{\sinh(xs^{1/2})}{s^2 \cdot \sinh(s^{1/2})}, \ (0 < x < 1)$.
8. $F(s) = \frac{1}{s^2} - \frac{1}{s \cdot \sinh s}$.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الثاني

ا. م. د. بان جعفر الطائي

المحاضرة ٧

Chapter Eight

MAPPING BY ELEMENTARY FUNCTIONS

8.1 Remark:

In first course the geometric interpretation of a function of a complex variable as a mapping. We saw there how the nature of such a function can be displayed graphically, to some extent, by the manner in which it maps certain curves and regions. Now we shall see further examples of how various curves and regions are mapped by elementary analytic functions.

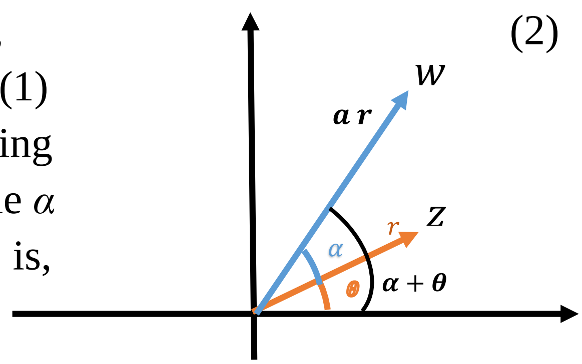
8.2 Linear Transformations:

To study the mapping

$$w = Az, \tag{1}$$

where A is a nonzero complex constant and $z \neq 0$, we write A and z in exponential form $A = ae^{i\alpha}, z = re^{i\theta}$ then

$$w = (a\,r)e^{i(\alpha+\theta)}, \tag{2}$$



and we see from equation (2) that transformation (1) expands or contracts the radius vector representing z by the factor a and rotates it through the angle α about the origin. The image of a given region is, therefore, geometrically similar to that region.

The mapping

$$w = z + B, \tag{3}$$

where B is any complex constant, is a translation by means of the vector representing B. That is, if $w = u + iv$, $z = x + iy$, and $B = b_1 + ib_2$, then the image of any point (x, y) in the z- plane is the point

$$(u, v) = (x + b_1, y + b_2), \tag{4}$$

in the w- plane. Since each point in any given region of the z- plane is mapped into the w- plane in this manner, the image region is geometrically congruent to the original one.

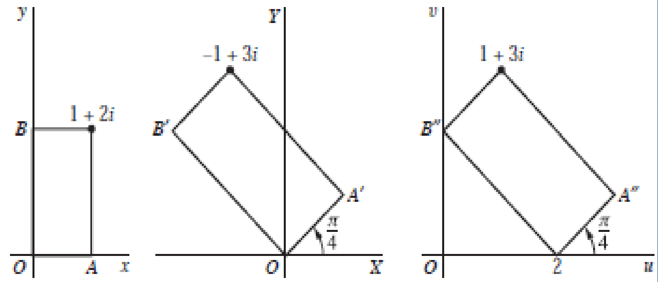
The general (no constant) linear transformation

$$w = Az + B, \ (A \neq 0), \tag{5}$$

a composition of the transformations $Z = Az$ $(A \neq 0)$ and $w = Z + B$. When $z \neq 0$, it is evidently an expansion or contraction and a rotation, followed by a translation.

8.3 Example:

The mapping $w = (1+i)z + 2$ transforms the rectangular $[0, 1] \times [0, 2]$ region in the $z = (x, y)$ plane into the rectangular region shown in the $w = (u, v)$ plane there. This is seen by expressing it as a composition of the transformations



$$Z = (1+i)z \text{ and } w = Z + 2 . \quad (6)$$

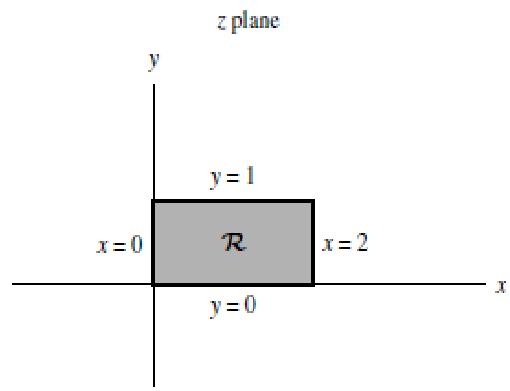
Writing $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $z = re^{i\theta}$, so

$$Z = (1+i)z = \sqrt{2}e^{i\frac{\pi}{4}} \cdot re^{i\theta} = \sqrt{2} \cdot re^{i(\theta+\frac{\pi}{4})}.$$

This first transformation thus expands the radius vector for a nonzero point z by the factor $\sqrt{2}$ and rotates it counterclockwise $\pi/4$ radians about the origin. The second of transformations (6) is a translation two units to the right.

8.4 Example:

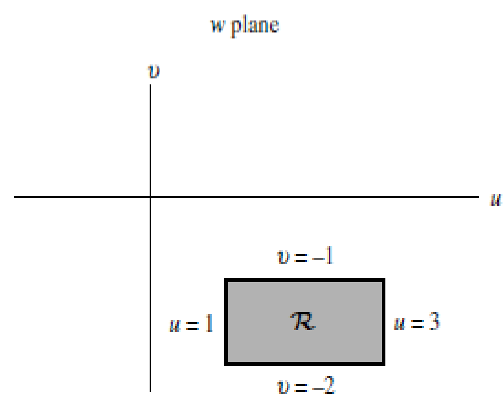
Let the rectangular region R in the z - plane be bounded by $x = 0, y = 0, x = 2, y = 1$.Determine the region R' of the w plane into which R is mapped under the transformations:



a) $w = z + (1 - 2i)$, **b)** $w = \sqrt{2}e^{i\frac{\pi}{4}}z$,

c) $w = \sqrt{2}e^{i\frac{\pi}{4}}z + (1 - 2i)$.

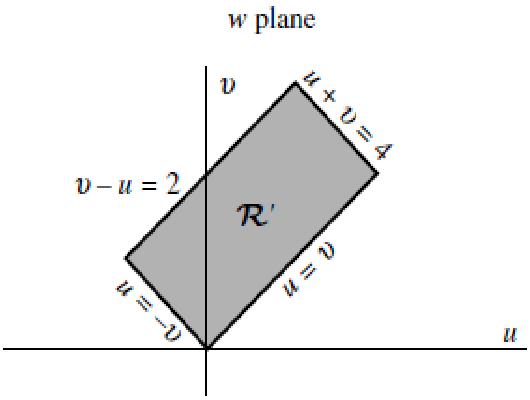
Solution:



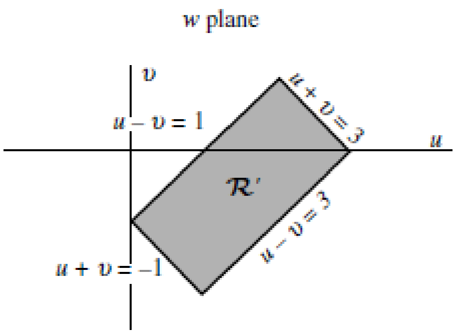
a) Given $w = z + (1 - 2i)$. Then $u + iv = x + iy + 1 - 2i = (x + 1) + i(y - 2)$ and $u = x + 1$, $v = y - 2$. Line $x = 0$ is mapped into $u = 1$; $y = 0$ into $v = -2$; $x = 2$ into $u = 3$; $y = 1$ into $v = -1$.Similarly, we can show that each point of $\mathcal{R}$ is mapped into one and only one point of $\mathcal{R}'$and conversely. The transformation or mapping accomplishes a translation of the rectangle. In general,$w = z + \beta$ accomplishes a translation of any region.

b) Given $w = \sqrt{2}e^{i\frac{\pi}{4}}z$. Then $u + iv = (1 + i)(x + iy) = (x - y) + i(x + y)$ and $u = x - y, v = x + y$.Line $x = 0$ is mapped into $u = -y, v = y$ or $u = -v$;

$y = 0$ into $u = x, v = x$ or $u = v$; $x = 2$ into $u = 2 - y$, $v = 2 + y$ or $u + v = 4$; $y = 1$ into $u = x - 1, v = x + 1$ or $v - u = 2$. The mapping accomplishes a rotation of $\mathcal{R}$ (through angle $\pi/4$ or $45°$) and a stretching of lengths (of magnitude $\sqrt{2}$).In general the transformation $w = az$ accomplishes a rotation and stretching of a region.



c) Given $w = \sqrt{2}e^{\pi/4}z + (1 - 2i)$. Then $u + iv = (1 + i)(x + iy) + 1 - 2i$ and $u = x - y + 1$, $v = x + y - 2$.The lines $x = 0, y = 0, x = 2$, $y = 1$ are mapped respectively into $u + v = -1$, $u - v = 3$,$u + v = 3$, $u - v = 1$. The mapping accomplishes a rotation and stretching as in (b) and a subsequent translation. In general, the transformation $w = az + \beta$ accomplishes a rotation, stretching, and translation. This can be considered as two successive mappings $w = az_1$ (rotation and stretching) and $z_1 = z + \beta/\alpha$ (translation).



EXERCISES:

1. State why the transformation $w = iz$ is a rotation in the z plane through the angle $\pi/2$. Then find the image of the infinite strip $0 < x < 1$.
2. Show that the transformation $w = iz + i$ maps the half plane $x > 0$ onto the half plane $v > 1$.
3. Find and sketch the region onto which the half plane $y > 0$ is mapped by the transformation $w = (1 + i)z$.
4. Find the image of the half plane $y > 1$ under the transformation $w = (1 - i)z$.
5. Find the image of the semi-infinite strip $x > 0$, $0 < y < 2$ when $w = iz + 1$. Sketch the strip and its image.
6. Give a geometric description of the transformation $w = A(z + B)$, where A and B are complex constants and $A \neq 0$.

8.5 THE TRANSFORMATION *w* = 1/*z*:

The equation

$$w = \frac{1}{z}, \tag{7}$$

establishes a one to one correspondence between the nonzero points of the *z* and the *w* - planes. When a point $w = u + iv$ is the image of a nonzero point $z = x + iy$ in the finite plane under the transformation *w* = 1/*z* , writing

$$w = \frac{1}{z}\cdot\frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i$$

reveals that

$$u = \frac{x}{x^2+y^2} \quad , \ v = \frac{-y}{x^2+y^2} \tag{8}$$

Also since $w = \frac{1}{z} \Longrightarrow z = \frac{1}{w}\cdot\frac{\bar{w}}{\bar{w}} = \frac{w}{|w|^2} = \frac{u}{u^2+v^2} - \frac{v}{u^2+v^2}i$ one can see that

$$x = \frac{u}{u^2+v^2} \quad , \ y = \frac{-v}{u^2+v^2} \tag{9}$$

8.6 Remark:

The following argument, based on these relations between coordinates, shows that the mapping *w* = 1/*z* transforms circles and lines into circles and lines. When *A*, *B*, *C*, and *D* are all real numbers satisfying the condition $B^2 + C^2 > 4AD$, the equation

$$A(x^2+y^2) + Bx + Cy + D = 0, \tag{10}$$

represents an arbitrary circle or line, where $A \neq 0$ for a circle and $A = 0$ for a line. The need for the condition $B^2 + C^2 > 4AD$ when $A \neq 0$ is evident if, by the method of completing the squares, we rewrite equation (10) as

$$(Ax^2 + Bx + \frac{B^2}{4A}) - \frac{B^2}{4A} + (Ay^2 + Cy + \frac{C^2}{4A}) - \frac{C^2}{4A} + D = 0;$$

$$A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}\right) + A\left(y^2 + \frac{C}{A}y + \frac{C^2}{4A^2}\right) = \frac{B^2}{4A} + \frac{C^2}{4A} - D;$$

$$\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}\right) + \left(y^2 + \frac{C}{A}y + \frac{C^2}{4A^2}\right) = \frac{B^2}{4A^2} + \frac{C^2}{4A^2} - \frac{D}{A};$$

$$\left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \frac{B^2+C^2-4AD}{4A^2} \Rightarrow \left(x + \frac{B}{2A}\right)^2 + \left(y + \frac{C}{2A}\right)^2 = \left(\frac{\sqrt{B^2+C^2-4AD}}{2A}\right)^2.$$

When $A = 0$, the condition becomes $B^2 + C^2 > 0$, which means that *B* and *C* are not both zero. Returning to the *w* = 1/*z* we observe that if *x* and *y* satisfy equation (10), we can use relations (9) to substitute for those variables. After some simplifications, we find that *u* and *v* satisfy the equation

$$A(x^2+y^2)+Bx+Cy+D=0$$

$$\Longrightarrow A\left(\left(\frac{u}{u^2+v^2}\right)^2+\left(\frac{-v}{u^2+v^2}\right)^2\right)+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0,$$

$$\Longrightarrow A\left(\frac{u^2}{(u^2+v^2)^2}+\frac{v^2}{(u^2+v^2)^2}\right)+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0;$$

$$\Longrightarrow A\left(\frac{u^2+v^2}{(u^2+v^2)^2}\right)+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0;$$

$$\Longrightarrow A\left(\frac{1}{u^2+v^2}\right)+B\frac{u}{u^2+v^2}-C\frac{v}{u^2+v^2}+D=0;$$

$$\Longrightarrow \boxed{D(u^2+v^2)+Bu-Cv+A=0,} \tag{11}$$

which also represents a circle or line. Conversely, if *u* and *v* satisfy equation (11), it follows from relations (8) that *x* and *y* satisfy equation (10).It is now clear from equations (10) and (11) that :

a) A circle $(A\neq 0)$ not passing through the origin $(D\neq 0)$ in the *z*-plane is transformed into a circle not passing through the origin in the *w*- plane;

b) A circle $(A\neq 0)$ through the origin $(D=0)$ in the *z*- plane is transformed into a line that does not pass through the origin in the *w*- plane;

c) A line $(A=0)$ not passing through the origin $(D\neq 0)$ in the *z*-plane is transformed into a circle through the origin in the *w* -plane;

d) A line $(A=0)$ through the origin $(D=0)$ in the *z*- plane is transformed into a line through the origin in the *w*- plane.

8.7 Example:

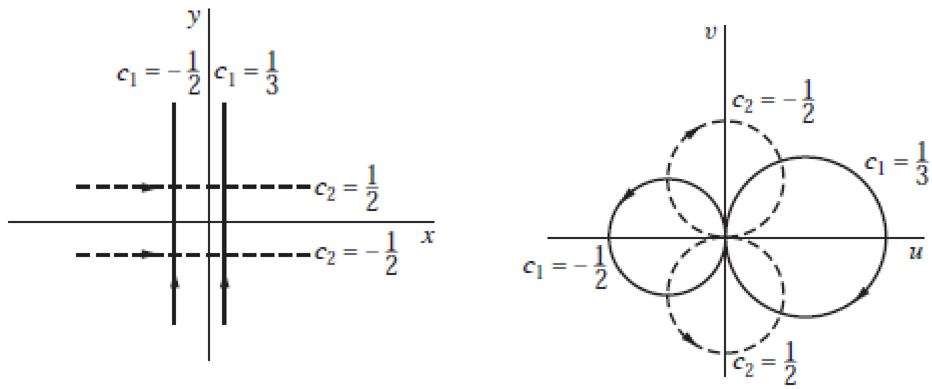
A vertical line $x=c_1,(c_1\neq 0)$ $\{A=C=0, B=1, D=-c_1\}$ is transformed by $w=1/z$ into the circle $-c_1(u^2+v^2)+u=0$ or $\left(u-\frac{1}{2c_1}\right)^2+v^2=\left(\frac{1}{2c_1}\right)^2$, where

$$D(u^2+v^2)+Bu-Cv+A=0\Longrightarrow \boxed{-c_1(u^2+v^2)+u=0,}$$

Or $$-c_1(u^2+v^2)+u=0\Longrightarrow(u^2+v^2)-\frac{u}{c_1}+\frac{1}{4c_1^2}-\frac{1}{4c_1^2}=0$$

$$\Longrightarrow \overbrace{u^2-\frac{u}{c_1}+\frac{1}{4c_1^2}}^{\text{كامل مربع}}+v^2=\frac{1}{4c_1^2}\Longrightarrow \boxed{\left(u-\frac{1}{2c_1}\right)^2+v^2=\left(\frac{1}{2c_1}\right)^2,}$$

which is centered on the *u* axis and tangent to the *v* axis. The image of a typical point (c_1, y) on the line is $\left(u=\frac{x}{x^2+y^2}=\frac{c_1}{c_1^2+y^2}\;,\; v=\frac{-y}{x^2+y^2}=\frac{-y}{c_1^2+y^2}\right).$



If $c_1 > 0$, the circle $\left(u - \frac{1}{2c_1}\right)^2 + v^2 = \left(\frac{1}{2c_1}\right)^2$ is evidently to the right of the *v* axis. As the point (c_1, y) moves up the entire line, its image traverses the circle once in the clockwise direction, the point at infinity in the extended *z*-plane corresponding to the origin in the *w*- plane. (For example if we take $c_1 = 1/3$). Note that $v > 0$ if $y < 0$; and as *y* increases through negative values to 0, one can see that *u* increases from 0 to $1/c_1$. Then, as *y* increases through positive values, *v* is negative and *u* decreases to 0.

If, on the other hand, $c_1 < 0$, the circle lies to the left of the *v* axis. As the point (c_1, y)moves upward, its image still makes one cycle, but in the counterclockwise direction. (For example if we take $c_1 = -1/2$).

8.8 Example:

A horizontal line $y = c_2, (c_2 \neq 0)\{A = B = 0, C = 1, D = -c_2\}$ is mapped by $w = 1/z$ onto the circle $-c_2(u^2 + v^2) - v = 0$ or $u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$, where

$$D(u^2 + v^2) + Bu - Cv + A = 0 \Longrightarrow -c_2(u^2 + v^2) - v = 0,$$

Or $$-c_2(u^2 + v^2) - v = 0 \Longrightarrow (u^2 + v^2) + \frac{v}{c_2} + \frac{1}{4c_2^2} - \frac{1}{4c_2^2} = 0$$

$$\Longrightarrow u^2 + \overbrace{v^2 + \frac{v}{c_2} + \frac{1}{4c_2^2}}^{\text{كامل مربع}} = \frac{1}{4c_2^2} \Longrightarrow \boxed{u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2,}$$

which is centered on the *v* axis and tangent to the *u* axis. . The image of a typical point (x, c_2) on the line is $\left(u = \frac{x}{x^2+y^2} = \frac{x}{x^2+c_2^2} \;,\; v = \frac{-y}{x^2+y^2} = \frac{-c_2}{x^2+c_2^2}\right)$.

If $c_2 > 0$,the circle $u^2 + \left(v + \frac{1}{2c_2}\right)^2 = \left(\frac{1}{2c_2}\right)^2$ is evidently down of the *u* axis. As the point (x, c_2) moves right the entire line, its image traverses the circle once in the counterclockwise direction, the point at infinity in the extended *z*-plane corresponding to the origin in the *w*- plane. (For example if we take $c_2 = 1/2$). Note that *u* > 0 if *x*> 0 ; and as *x* increases through 0 to positive values, one can see that *v* decreases from $1/c_2$ to 0. Then, as *x* decreases through negative values, *u* is negative and *v* increses to $1/c_2$.

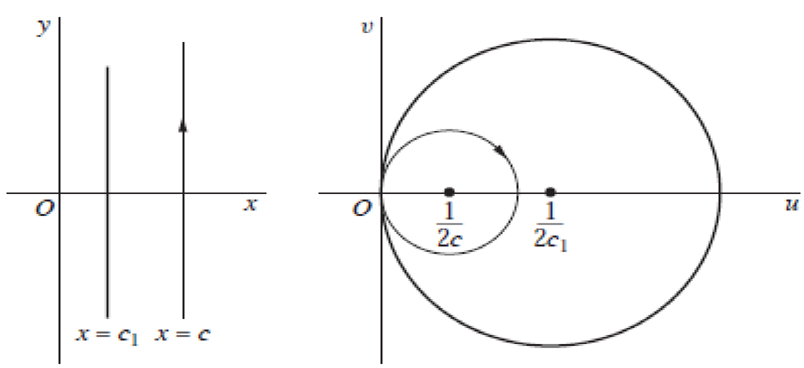
If, on the other hand, $c_2 < 0$, the circle lies above of the *v* axis. As the point (x, c_2) moves upward, its image still makes one cycle, but in the clockwise direction. (For example if we take $c_2 = -1/2$).

8.9 Example:

A half plane $x \geq c_1, (c_1 > 0)$ is mapped by *w* = 1/*z* onto the disk $\left(u - \frac{1}{2c_1}\right)^2 + v^2 \leq \left(\frac{1}{2c_1}\right)^2$.

Solution:

According to Example 8.7, any line $x = c$ $(c \geq c_1)$ is transformed into the circle $\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$. Furthermore, as *c* increases through all values greater than c_1, the lines *x* = *c* move to the right and the image circles $\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$ shrink in size. Since the lines $x = c$ pass through all points in the half plane $x \geq c_1$ and the circles $\left(u - \frac{1}{2c}\right)^2 + v^2 = \left(\frac{1}{2c}\right)^2$ pass through all points in the disk $\left(u - \frac{1}{2c_1}\right)^2 + v^2 \leq \left(\frac{1}{2c_1}\right)^2$, the mapping is established.

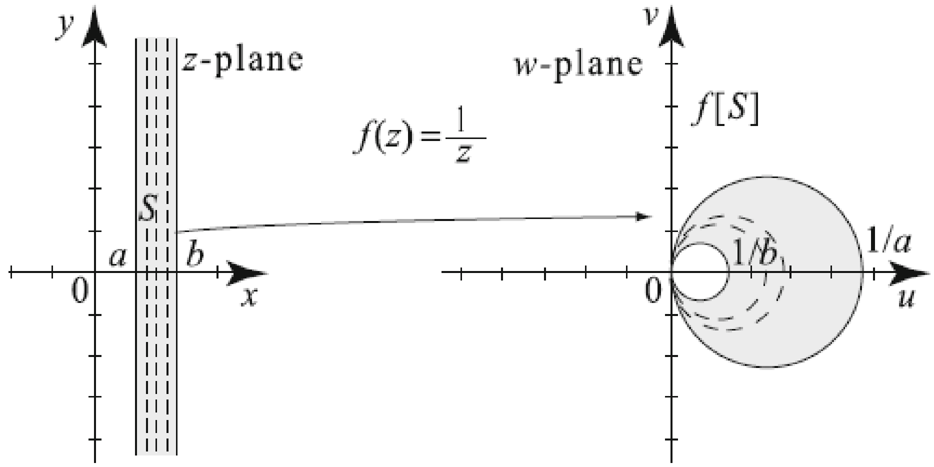


8.10 Example:

Let $0 < a < b$ be real numbers. Determine the image of the vertical strip $S = \{z = x + iy : a \leq x \leq b\}$ under the mapping $f(z) = 1/z$.

Solution:

Notice that as x_0 varies from a to b, the line $x = x_0$ sweeps the vertical strip S, and the image of the line $x = x_0$ sweeps the image of S. So the image of S is the annular region bounded by the outer circle with radius $1/2a$ centered at $(1/2a, 0)$ and the inner circle with radius $1/2b$ centered at $(1/2b, 0)$.



The inversion $f(z) = 1/z$ maps the line $x = a$ onto a circle.

8.11 Example:

Find the image of the following sets under the mapping $f(z) = 1/z$.

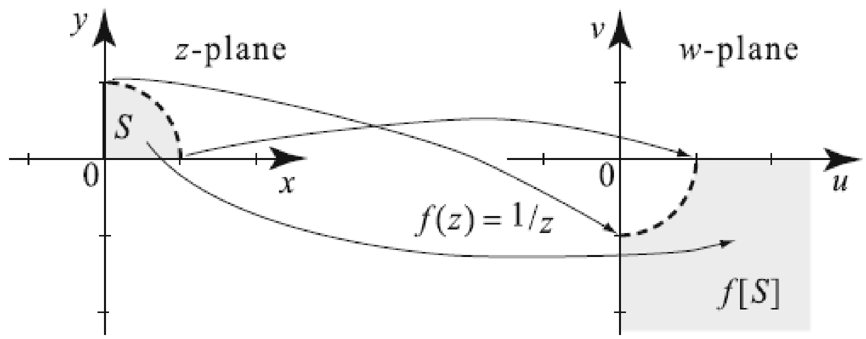
(a) $S = \{z : 0 < |z| < 1, 0 \leq arg z \leq \pi/2\}$.

(b) $S = \{z : 2 \leq |z|, 0 \leq arg z \leq \pi\}$.

Solution:

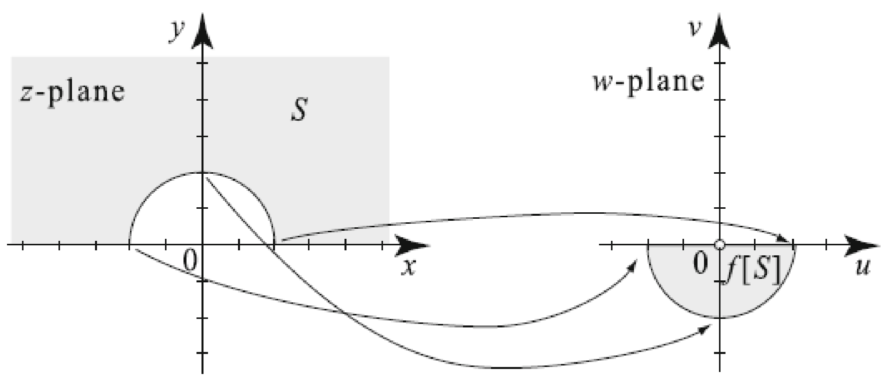
(a) Let $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$ then $\frac{1}{z} = re^{-i\theta} = r(\cos\theta - i\sin\theta)$. According to this formula, the modulus of the number $1/z$ is the reciprocal of the modulus of z and the argument of $f(z)$ is the negative of the argument of z. Consequently, numbers inside the unit circle $(|z| \leq 1)$ get mapped to numbers outside the unit circle $(1/|z| \geq 1)$, and numbers in the upper half-plane get mapped to numbers in the lower half-plane. Looking at S, as the modulus of z goes from 1 down to 0, the modulus of $f(z)$ goes from 1 up to infinity. As the argument of z goes from 0 up to $\pi/2$, the argument of $1/z$ goes from 0 down to $-\pi/2$. Hence $f[S]$ is the set of all points in the fourth quadrant, including the border axes, that lie outside the unit circle

$$f\ (S)\ =\ \{w:\ 1\ <\ |w|,\ -\pi/2\ \leq\ argz\ \leq\ 0\}.$$



The function $w = 1/z$ has the effect of inverting the modulus and changing the sign of the argument, i.e., $|w|\ =\ \frac{1}{|z|}$ and $Arg\ w\ =\ -Arg\ z$.

(b) As the modulus of z increases from 2 up to infinity, the modulus of $1/z$ decreases from $1/2$ down to zero (but never equals zero). As the argument of z goes from 0 up to π, the argument of $1/z$ goes from 0 down to $-\pi$.



Under the inversion $f(z) = 1/z$, points outside the circle of radius 2, $|z|\ \geq\ 2$, get mapped to points inside the circle of radius $\frac{1}{2}$, $|w|\ \leq\ \frac{1}{2}$.

Hence $f(S)$ is the set of nonzero points in the lower half-plane, including the real axis, with $0\ <\ |w|\ <\ 1/2$:

$$f(S)\ =\ \{w:\ 0\ <\ |w|\ <\ 1/2, -\pi\ \leq\ arg\ z\ \leq\ 0\}.$$

EXERCISES:

1. Show that when $c_1 < 0$, the image of the half plane $x < c_1$ under the transformation $w = 1/z$ is the interior of a circle. What is the image when $c_1 = 0$?
2. Show that the image of the half plane $y > c_2$ under the transformation $w = 1/z$ is the interior of a circle when $c_2 > 0$. Find the image when $c_2 < 0$ and when $c_2 = 0$.
3. Find the image of the infinite strip $0 < y < 1/(2c)$ under the transformation $w = 1/z$. Sketch the strip and its image.
4. Find the image of the region $x > 1$, $y > 0$ under the transformation $w = 1/z$.
5. Describe geometrically the transformation $w = i/z$. State why it transforms circles and lines into circles and lines.
6. Find the image of the semi-infinite strip $x > 0$, $0 < y < 1$ when $w = i/z$. Sketch the strip and its image.

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Chapter Eight

MAPPING BY ELEMENTARY FUNCTIONS

8.12 LINEAR FRACTIONAL TRANSFORMATIONS:

The transformation

$$w = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0); \tag{1}$$

where a, b, c, and d are complex constants, is called a ***linear fractional transformation***, or ***Möbius transformation***. Observe that equation (1) can be written in the form

$$Awz + Bz + Cw + D = 0,\ (AD - BC \neq 0); \tag{2}$$

$cwz + dw = az + b \Longrightarrow cwz - az + dw - b = 0; A = c, B = -a, C = d, D = -b$

$ad - bc = -BC - (-D)A = AD - BC \neq 0$

and, conversely, any equation of type (2) can be put in the form (1). Since this alternative form is linear in z and linear in w, another name for a linear fractional transformation is ***bilinear transformation.***

8.13 Remark:

If c is zero or nonzero, ***any linear fractional transformation transforms circles and lines into circles and lines*** since

When $\boldsymbol{c = 0}$, the condition $ad - bc \neq 0$ with equation (1) becomes $ad \neq 0$; and we see that the transformation reduces to a nonconstant linear function, i.e. $w = \frac{a}{d}z + \frac{b}{d}$.

When $\boldsymbol{c \neq 0}$, equation (1) can be written

$$w = \frac{az+b}{cz+d}\cdot\frac{c}{c} = \frac{azc+bc+ad-ad}{c(cz+d)} = \frac{azc-ad}{c(cz+d)} + \frac{bc+ad}{c(cz+d)} = \frac{a(cz+d)}{c(cz+d)} + \frac{bc-ad}{c(cz+d)} = \frac{a}{c} + \frac{bc-ad}{c(cz+d)}$$

$$w = \frac{a}{c} + \frac{bc-ad}{c}\cdot\frac{1}{cz+d}, \quad (ad - bc \neq 0). \tag{3}$$

So, once again, the condition $ad - bc \neq 0$ ensures that we do not have a constant function. The transformation $w = 1/z$ is evidently a special case of transformation (1) when $c \neq 0$.

Equation (3) reveals that when $c \neq 0$, a linear fractional transformation is a composition of the mappings.

$$\mathcal{Z} = \frac{1}{cz+d}, \quad \mathcal{W} = \frac{1}{\mathcal{Z}} \Longrightarrow w = \frac{a}{c} + \frac{bc-ad}{c}\mathcal{W}, (ad - bc \neq 0)\ .$$

8.14 Remark:

Solving equation (1) for z, we find that

$$w = \frac{az+b}{cz+d} \Longrightarrow wcz + dw = az + b \Longrightarrow (wc - a)z = -dw + b$$

$$z = \frac{-dw+b}{wc-a}, \ (ad - bc \neq 0) \ . \qquad (4)$$

When a given point w is the image of some point z under transformation (1), the point z is retrieved by means of equation (4). If $c = 0$, so that a and d are both nonzero, each point in the w- plane is evidently the image of one and only one point in the z -plane. The same is true if $c \neq 0$, except when $w = a/c$ since the denominator in equation (4) vanishes if w has that value. We can, however, enlarge the domain of definition of transformation (1) in order to define a linear fractional transformation T on the ***extended z- plane*** such that the point $w = a/c$ is the image of $z = \infty$ when $c \neq 0$. We first write

$$T(z) = \frac{az+b}{cz+d}, \quad (ad - bc \neq 0). \qquad (5)$$

We then write

$$T(\infty) = \infty \qquad \qquad if \ c = 0$$

$$T(\infty) = \frac{a}{c} \text{ and } T\left(-\frac{d}{c}\right) = \infty \quad if \ c \neq 0$$

When its domain of definition is enlarged in this way, the linear fractional transformation (5) is a one to one mapping of the extended z- plane onto the extended w- plane. Hence, associated with the transformation T, there is an inverse transformation T^{-1}, which is defined on the extended w- plane as follows:

$$T^{-1}(w) = z \ \text{ iff } \quad T(z) = w.$$

From equation (4), we see that

$$T^{-1}(w) = \frac{-dw+b}{wc-a}, \ (ad - bc \neq 0) \ . \qquad (6)$$

Evidently, $T^{-1}$is itself a linear fractional transformation, where

$$T^{-1}(\infty) = \infty \qquad \qquad if \ c = 0$$

$$T^{-1}\left(\frac{a}{c}\right) = \infty \text{ and } T^{-1}(\infty) = -\frac{d}{c} \quad if \ c \neq 0 \ .$$

8.15 Example:

Find the special case of transformation $w = \frac{az+b}{cz+d}$, $(ad - bc \neq 0)$ that maps the points $z_1 = 0$, $z_2 = -1$, and $z_3 = 1$ onto the points $w_1 = 1$, $w_2 = -i$, and $w_3 = i$.

Solution:

Since 1 is the image of 0 then $1 = w = \frac{a\cdot 0+b}{c\cdot 0+d} = \frac{b}{d} \Longrightarrow 1 = \frac{b}{d}$ or $b = d$. Thus $w = \frac{az+b}{cz+b}$, $\quad (b(a-c) \neq 0)$.

Since $-i$ is the image of -1 then $-i = w = \frac{a\cdot(-1)+b}{c\cdot(-1)+b} = \frac{b-a}{b-c} \Longrightarrow ic - ib = b - a$.

Since i is the image of 1 then $i = w = \frac{a\cdot 1+b}{c\cdot 1+b} = \frac{b+a}{b+c} \Longrightarrow ib + ic = b + a$.

Adding corresponding sides of these equations, we find that $2ic = 2b \Longrightarrow ic = b \Longrightarrow c = -ib$; and subtraction reveals that $-2a = -2\,ib \Longrightarrow a = ib$. Consequently $w = \frac{az+b}{cz+b} = \frac{ibz+b}{-ibz+b} = \frac{b(iz+1)}{b(-iz+1)}$. We can cancel out the nonzero number b in this last fraction and write $w = \frac{(iz+1)}{(-iz+1)}$. This is the same as $w = \frac{(iz+1)}{(-iz+1)} \cdot \frac{i}{i} = \frac{i-z}{i+z}$ which is obtained by assigning the value i to the arbitrary number b.

8.16 Example:

Find the images of the points $0, 1 + i, i,$ and ∞ under the linear fractional transformation $T(z) = \frac{2z+1}{z-i}$.

Solution:

For $z = 0$ and $z = 1 + i$ we have:

$$T(0) = \frac{2\cdot 0+1}{0-i} = \frac{1}{-i} \quad \text{and} \quad T(1+i) = \frac{2(1+i)+1}{(1+i)-i} = \frac{3+2i}{1} = 3 + 2i.$$

Identifying $a = 2, b = 1, c = 1,$ and $d = -i$ in (5), we also have:

$$T(i) = \frac{2\cdot i+1}{i-i} = \frac{2\cdot(-\frac{-i}{1})+1}{(-\frac{-i}{1})-i} = T\left(-\frac{d}{c}\right) = \infty \quad \text{and } T(\infty) = \frac{a}{c} = 2\ .$$

8.17 Theorem:

If T is a linear fractional transformation given by

$$\boldsymbol{T(z) = \begin{cases} \frac{az+b}{cz+d}, & z \neq -\frac{d}{c}, z \neq \infty \\ \infty, & z = -\frac{d}{c} \\ \frac{a}{c}, & z = \infty \end{cases}}, \qquad \textbf{(7)}$$

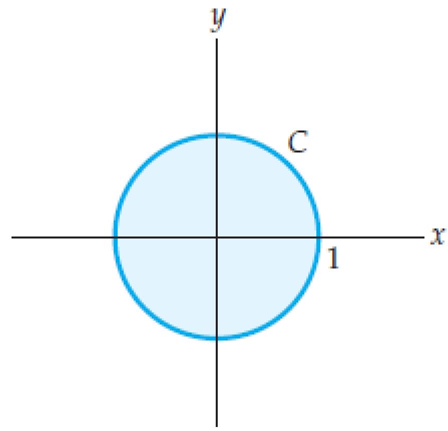
then

1. ***If C is a circle in the z- plane, then the image of C under T is either a circle or a line in the extended w-plane. The image is a line if and only if $c \neq 0$ and the pole $z = -d/c$ is on the circle C.***
2. ***If L is a line in the z- plane, then the image of L under T is either a line or a circle in the extended w-plane. The image is a circle if and only if $c \neq 0$ and the pole $z = -d/c$ is not on the line L.***

8.18 Example:

Find the image of the unit circle $|z| = 1$ under the linear fractional transformation $T(z) = \frac{z+2}{z-1}$. What is the image of the interior $|z| < 1$ of this circle?

Solution:

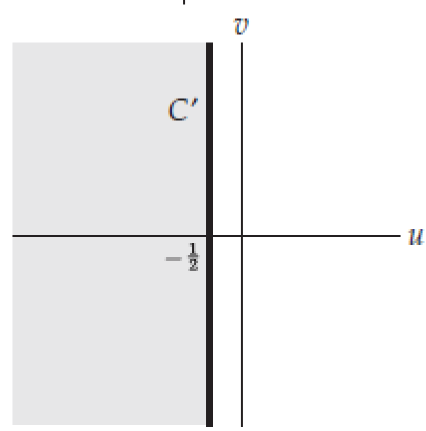


The pole of T is $z = 1$ and this point is on the unit circle $|z| = 1$. Thus, from Theorem 8.17 we conclude that the image of the unit circle is a line.

Since the image is a line (it is determined by any two points on $|z| = 1$, i.e.

$$T(-1) = \frac{-1+2}{-1-1} = -\frac{1}{2},$$

$$T(i) = \frac{i+2}{i-1} \cdot \frac{i+1}{i+1} = \frac{-1+3i+2}{-1-1} = \frac{1+3i}{-2} = -\frac{1}{2} - \frac{3}{2}i\ ).$$



we see that the image is the line $u = -\frac{1}{2}$.

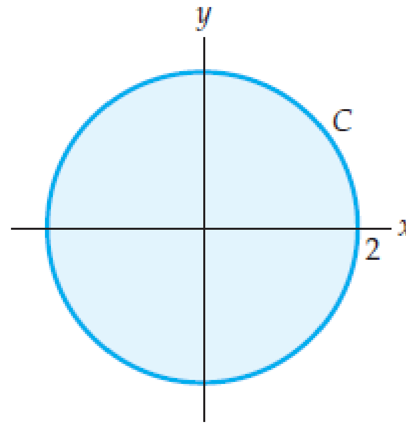
To answer the second question, we first note that a linear fractional transformation is a rational function, and so it is continuous on its domain. As a consequence the image of the interior $|z| < 1$ of the unit circle is either the half-plane $u < \frac{-1}{2}$ or $u > \frac{-1}{2}$.Using $z = 0$ as a test point, we find $T(0) = \frac{0+2}{0-1} = -2$ which is to the left of the line $u = \frac{-1}{2}$ and so the image is the half-plane $u < \frac{-1}{2}$.

8.19 Example:

Find the image of the unit circle $|z| = 2$ under the linear fractional transformation $T(z) = \frac{z+2}{z-1}$. What is the image of the disk $|z| \leq 2$ under T?

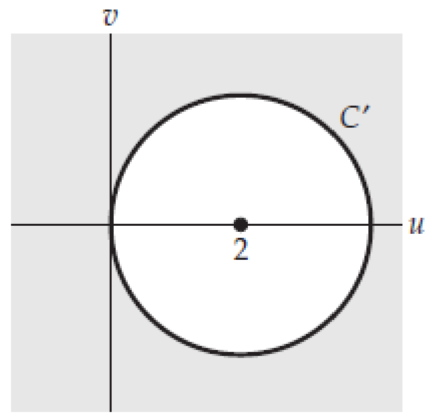
Solution:

The pole $z = 1$ does not lie on the circle $|z| = 2$, and so Theorem 8.17 indicates that the image of $|z| = 2$ is a circle C'. To find an algebraic description of C', we first note that the circle $|z| = 2$ is symmetric with respect to the x-axis. That is, if z is on the circle $|z| = 2$, then so is $\bar{z}$. Furthermore, we observe that for all z,



$$T(\bar{z}) = \frac{\bar{z}+2}{\bar{z}-1} = \frac{\overline{z+2}}{\overline{z-1}} = \overline{\left(\frac{z+2}{z-1}\right)} = \overline{T(z)}.$$

Hence, if z and $\bar{z}$ are on the circle $|z| = 2$, then we must have that both $w = T(z)$ and $\bar{w} = \overline{T(z)} = T(\bar{z})$ are on the circle C'. It follows that $C'$is symmetric with respect to the u-axis. Since $z = 2$ and -2 are on the circle $|z| = 2$, the two points $T(2) = 4$ and $T(-2) = 0$ are on C'. The symmetry of C' implies that 0 and 4 are endpoints of a diameter, and so C' is the circle $|w - 2| = 2$. Using $z = 0$ as a test point, we find that $w = T(0) = -2$, which is outside the circle $|w - 2| = 2$. Therefore, the image of the interior of the circle $|z| = 2$ is the exterior of the circle $|w - 2| = 2$. In summary, the disk $|z| \leq 2$ is mapped onto the region $|w - 2| \geq 2$ by the linear fractional transformation $T(z) = (z + 2)/(z - 1)$.



8.20 Remark:

To determine a general method to construct a linear fractional transformation $w = T(z)$, which maps three given distinct points z_1, z_2, and z_3 on the boundaryof D to three given distinct points w_1, w_2, and w_3 on the boundaryof D'. This is accomplished using the ***cross-ratio***, which is defined as follows.

8.21 Definition:

The ***cross-ratio*** of the complex numbers z, z_1, z_2, and z_3 is the complex number

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \qquad (8)$$

8.22 Remark:

When computing a cross-ratio, we must be careful with the order of the complex numbers. For example, you should verify that the cross-ratio of 0, 1, i, and 2 is

$$\frac{(0-1)(i-2)}{(0-2)(i-1)} = \frac{i-2}{2i-2} \cdot \frac{2i+2}{2i+2} = \frac{-2-4i+2i-4}{-4-4} = \frac{-6-2i}{-8} = \frac{3+i}{4} = \frac{3}{4} + \frac{1}{4}i,$$

whereas the cross-ratio of 0, i, 1, and 2 is

$$\frac{(0-i)(1-2)}{(0-2)(1-i)} = \frac{i}{-2+2i} \cdot \frac{-2-2i}{-2-2i} = \frac{-2i+2}{4+4} = \frac{2(1-i)}{8} = \frac{1-i}{4} = \frac{1}{4} - \frac{1}{4}i.$$

We extend the concept of the cross-ratio to include points in the extended complex plane by using the limit formula ($\lim_{Z\to\infty} f(Z) = L \ iff \ \lim_{z\to z_0} \left(\frac{1}{z}\right) = L$. For example, the cross-ratio of, say, ∞, z_1, z_2 and z_3 is given by the limit

$$\lim_{Z\to\infty} \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} \tag{9}$$

8.23 Remark:

The following theorem illustrates the importance of cross-ratios in the study of linear fractional transformations. In particular, the cross-ratio is invariant under a linear fractional transformation.

8.24 Theorem (Cross-Ratios andLinear Fractional Transformations):

If $w = T(z)$ is a linear fractional transformation that maps the distinct points z_1, z_2, and z_3 onto the distinct points w_1, w_2, and w_3, respectively, then

$$\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \tag{10}$$

for all z.

8.25 Example:

Construct a linear fractional transformation that maps the points 1, i, and -1 on the unit circle $|z| = 1$ onto the points -1, 0, 1 on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.

Solution:

Identifying $z_1 = 1, z_2 = i, z_3 = -1, w_1 = -1, w_2 = 0$, and $w_3 = 1$ in (10) we see from Theorem 8.24 that the desired mapping $w = T(z)$ must satisfy

$$\frac{(z-1)(i-(-1))}{(z-(-1))(i-1)} = \frac{(w-(-1))(0-1)}{(w-1)(0-(-1))} \Longrightarrow \frac{(z-1)(i+1)}{(z+1)(i-1)} = \frac{-(w+1)}{(w-1)} \Longrightarrow \frac{iz+z-i-1}{iz-z+i-1} = \frac{-w-1}{w-1} \Longrightarrow$$

$$(iz+z-i-1)w - (iz+z-i-1) = -(iz-z+i-1)w - (iz-z+i-1) \Longrightarrow$$

$$(iz+z-i-1)w + (iz-z+i-1)w = (iz+z-i-1) - (iz-z+i-1) \Longrightarrow$$

$$(2iz-2)w = 2z-2i \Longrightarrow w = \frac{2(z-i)}{2(iz-1)} \Longrightarrow w = \frac{(z-i)}{(iz-1)} = T(z).$$

Using the test point $z = 0$, we obtain $T(0) = i$. Therefore, the image of the interior $|z| < 1$ is the upper half-plane $v > 0$.

8.26 Example:

Construct a linear fractional transformation that maps the points $-i$, 1, and ∞ on the line $y = x - 1$ onto the points 1, i, and -1 on the unit circle $|w| = 1$.

Solution:

We proceed as in Example 8.25. Using remaek 8.22 we find that the cross-ratio of z, $z_1 = -i$, $z_2 = 1$, $z_3 = \infty$ is

$$\lim_{z_3\to\infty}\frac{(z+i)(1-z_3)}{(z-z_3)(1+i)} = \lim_{z_3\to 0}\frac{(z+i)\left(1-\frac{1}{z_3}\right)}{\left(z-\frac{1}{z_3}\right)(1+i)} = \lim_{z_3\to 0}\frac{\frac{1}{z_3}(z+i)(z_3-1)}{\frac{1}{z_3}(zz_3-1)(1+i)} = \lim_{z_3\to 0}\frac{(z+i)(z_3-1)}{(zz_3-1)(1+i)} = \frac{-(z+i)}{-(1+i)} = \frac{z+i}{1+i}$$

Now from (10) of Theorem 8.24 with $w_1 = 1$, $w_2 = i$, and $w_3 = -1$, the desired mapping $w = T(z)$ must satisfy

$$\frac{z+i}{1+i} = \frac{(w-1)(i+1)}{(w+1)(i-1)} \Longrightarrow \frac{(z+i)(i-1)}{(1+i)(i+1)} = \frac{(w-1)}{(w+1)} \Longrightarrow \frac{iz-z-1-i}{1+2i-1} = \frac{(w-1)}{(w+1)} \Longrightarrow$$

$$(iz - z - 1 - i)(w + 1) = 2i(w - 1) \Longrightarrow (iz - z - 1 - i)w - 2iw = -iz + z + 1 + i - 2i \Longrightarrow$$

$$(iz - z - 1 - 3i)w = (-iz + z + 1 - i) \Longrightarrow w = T(z) = \frac{-iz+z+1-i}{iz-z-1-3i} = \frac{(1-i)z+1-i}{(i-1)z-1-3i}.$$

8.27 Example:

a) Show that the linear fractional transformation $\varphi(z) = i\,\frac{1-z}{1+z}$ maps the unit disk onto the upper half-plane.

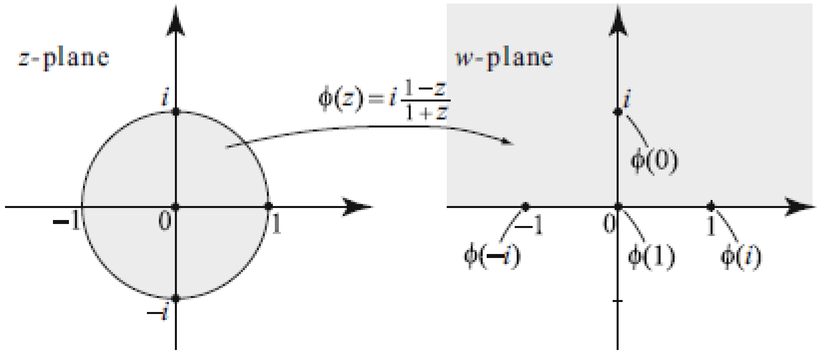
b) Show that the linear fractional transformation $\varphi(z) = \frac{i-z}{i+z}$ maps the upper half-plane onto the unit disk.

Solution:

a) Since the image of the circle C is either a line or a circle in the w-plane. Since three points will determine either a line or circle, it suffices to check the images of three points on C. Let $1, i, -i$ be three points on C then we have

$$\varphi(1) = i\,\frac{1-1}{1+1} = 0; \quad \varphi(i) = i\,\frac{1-i}{1+i} = \frac{i+1}{1+i} = 1; \quad \varphi(-i) = i\,\frac{1+i}{1-i} = \frac{i-1}{1-i} = \frac{-(1-i)}{(1-i)} = -1$$

Thus $\varphi(1)$, $\varphi(i)$, and $\varphi(-i)$ lie on the u-axis (the real axis in the w-plane), and so the image of C is the u-axis. As φ is one-to-one, it maps the boundary C onto the boundary of the image of the unit disk. Thus the image of the unit disk



is either the upper half-plane or the lower half-plane. Since $\varphi\,(0) = i\frac{1-0}{1+0} = i$ (a point in the upper half-plane), we conclude that φ maps the unit disk one-to-one onto the upper half-plane.

Note also that since φ maps the closed unit disk to an unbounded region (the upper half-plane), it has to be discontinuous somewhere in the closed unit disk. Indeed it is singular at $z = -1$.

b) Let $0, i, -1$ be three points on C in the upper half-plane then

$\varphi(0) = \frac{i-0}{i+0} = 1;\quad \varphi(1) = \frac{i-1}{i+1}\cdot\frac{i-1}{i-1} = \frac{-2i}{-2} = i;\quad \varphi(-i) = \frac{i+1}{i-1}\cdot\frac{i-1}{i-1} = \frac{-2}{-2i} = \frac{1}{i}\cdot\frac{i}{i} = -i.$

Since the images of the three points are not collinear, we conclude that the real axis is mapped onto the circle that goes through the points 1, i, and $-i$, which is clearly the unit circle. (Here again, we are using the fact that three points determine a circle.) Also, $\varphi(i) = \frac{i-i}{i+i} = 0$; hence φ maps the upper half-plane onto the unit disk.

EXERCISES:

1. Find the linear fractional transformation that maps the points $z_1 = 2$, $z_2 = i$, $z_3 = -2$ onto the points $w_1 = 1, w_2 = i, w_3 = -1$.

2. Find the linear fractional transformation that maps the points $z_1 = -i, z_2 = 0, z_3 = i$ onto the points $w_1 = -1, w_2 = i, w_3 = 1$. Into what curve is the imaginary axis $x = 0$ transformed?

3. Find the bilinear transformation that maps the points $z_1 = \infty, z_2 = i, z_3 = 0$ onto the points $w_1 = 0, w_2 = i, w_3 = \infty$.

4. Find the bilinear transformation that maps distinct points z_1, z_2, z_3 onto the points $w_1 = 0, w_2 = 1, w_3 = \infty$.