محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

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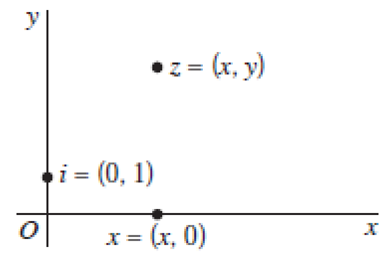
المحاضرة ١

Chapter One

COMPLEX NUMBERS

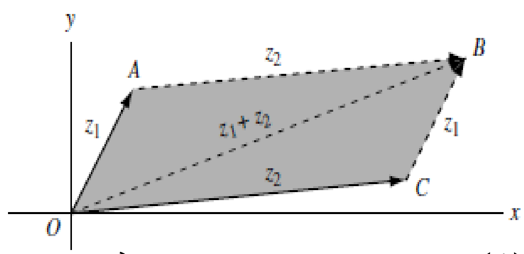
1.1 Definition:

Complex numbers can be defined as ordered pairs (x, y) of real numbers that are to be interpreted as points in the complex plane, with rectangular coordinates x and y. just as real numbers x are thought of as points on the real line.



1.2 Remark:

1. When real numbers x are displayed as points (x, 0) on the real axis, it is clear that the set of complex numbers includes the real numbers as a subset.
2. Complex numbers of the form (0, y) correspond to points on the y axis and are called pure imaginary numbers. when $y \neq 0$ The y axis is then referred to as the imaginary axis.
3. We denote a complex number (x, y) by z, so that **$z = (x, y)$**.
4. The real numbers x and y are, known as the ***real and imaginary parts*** of z, respectively; and we write **$x = \text{Re}\, z$, $y = \text{Im} z$**
5. Two complex numbers z_1 and z_2 are ***equal*** $(z_1 = z_2)$ whenever they have the have the same real parts and the same imaginary parts (i.e. $z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$ were $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$).
6. The ***sum*** and ***product division*** of two complex numbers is also a complex numbers, i.e. if $z_1 = (x_1, y_1), z_2 = (x_2, y_2)$ then



$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \qquad (1)$$

$$z_1 . z_2 = (x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2, x_2y_1 + x_1y_2). \qquad (2)$$

The operations defined by equations (1) and (2) become the usual operations of addition and multiplication when restricted to the real numbers $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$, $(x_1, 0).(x_2, 0) = (x_1x_2, 0)$.

7. Any complex number $z = (x, y)$ can be written $z = (x, 0) + (0, y)$, and it is easy to see that $(0, 1)(y, 0) = (0, y)$. Hence $z = (x, 0) + (0, 1)(y, 0)$ and if we think of a

real number as either x or $(x, 0)$ and let i denote the pure imaginary number (0,1) so $z = x + iy$.

8. $i^2 = (0,1)(0,1) = (-1,0)$, or $i^2 = -1$.

1.3 Remark:

Since $(x, y) = x + iy$ then equations (1) and (2) become

$$(x_1+iy_1)+(x_2+iy_2) = (x_1+x_2)+i(y_1+y_2) \quad (3)$$

$$(x_1+iy_1).(x_2+iy_2) = (x_1x_2-y_1y_2)+i(x_2y_1+x_1y_2). \quad (4)$$

So any complex number times zero is zero, i.e.

$$z \cdot 0 = (x+iy)(0+i0) = 0 + i0 = 0.$$

1.4 Remark:

The ***difference*** and ***division*** of two complex numbers is a complex number, i.e. if $z_1 = (x_1+iy_1), z_2 = (x_2+iy_2), y_1 \neq 0, y_2 \neq 0$ then

$$z_1 - z_2 = (x_1-x_2)+i(y_1-y_2).$$

$$\frac{z_1}{z_2} = \frac{(x_1+iy_1)}{(x_2+iy_2)} = \frac{(x_1+iy_1)}{(x_2+iy_2)}\frac{(x_2-iy_2)}{(x_2-iy_2)} = \frac{x_1x_2-x_1y_2i+x_2y_1i-y_1y_2i^2}{x_2^2-y_2^2i^2}$$

$$= \frac{x_1x_2-y_1y_2+(x_2y_1-x_1y_2)i}{x_2^2+y_2^2} = \frac{x_1x_2-y_1y_2}{x_2^2+y_2^2} + \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}i.$$

So $z^{-1} = \frac{1}{z} = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}i, \quad z \neq 0$.

1.5 Theorem:

Suppose z_1, z_2, z_3 belong to the set $\mathbb{C}$ of complex numbers. Then

(1) $z_1 + z_2$ and $z_1.z_2$ belong to $\mathbb{C}$ (Closure law).

(2) $z_1 + z_2 = z_2 + z_1$ (Commutative law of addition).

(3) $z_1+(z_2+z_3) = (z_1+z_2)+z_3$ (Associative law of addition)

(4) $z_1.z_2 = z_2.z_1$ (Commutative law of multiplication).

(5) $z_1.(z_2.z_3) = (z_1.z_2).z_3$ (Associative law of multiplication).

(6) $z_1.(z_2+z_3) = z_1.z_2 + z_1.z_3$ (Distributive law).

(7) $z_1 + 0 = 0 + z_1 = z_1$, $z_1.1 = 1.z_1 = z_1 1$, 0 is called the identity with respect to addition, 1 is called the identity with respect to multiplication.

(8) For any complex number z_1 there is a unique number z in $\mathbb{C}$ such that $z + z_1 = 0$, [z is called the inverse of z_1 with respect to addition and is denoted by $-z_1$].

(9) For any z_1 there is a unique number z in $\mathbb{C}$ such that $z_1.z_2 = z_2.z_1 = 1$; [z is

called the inverse of z_1 with respect to multiplication and is denoted by z_1^{-1}or $1/z_1$.

1.6 Example:

1. $(3+2i)+(-7-i)=3-7+2i-i=-4+i$
2. $(8-6i)-(2i-7)=8-6i-2i+7=15-8i$
3. $(2-3i)(4+2i)=2(4+2i)-3i(4+2i)=8+4i-12i-6i^2=8+4i-12i+6=14-8i$
4. $(2-i)\{(-3+2i)(5-4i)\}=(2-i)\{-15+12i+10i-8i^2=(2-i)(-7+22i)=-14+44i+7i-22i^2=8+51i$
5. $(-1+2i)\{(7-5i)+(-3+4i)\}=(-1+2i)(4-i)=-4+i+8i-2i^2=-2+9i$

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$$\begin{aligned}(-1+2i)\{(7-5i)+(-3+4i)\}&=(-1+2i)(7-5i)+(-1+2i)(-3+4i)\\&=\{-7+5i+14i-10i^2\}+\{3-4i-6i+8i^2\}\\&=(3+19i)+(-5-10i)=-2+9i\end{aligned}$$

6. $\frac{3-2i}{-1+i}=\frac{3-2i}{-1+i}\cdot\frac{-1-i}{-1-i}=\frac{-3-3i+2i+2i^2}{1-i^2}=\frac{-5-i}{2}=-\frac{5}{2}-\frac{1}{2}i$
7. $\frac{5+5i}{3-4i}+\frac{20}{4+3i}=\frac{5+5i}{3-4i}\cdot\frac{3+4i}{3+4i}+\frac{20}{4+3i}\cdot\frac{4-3i}{4-3i}=\frac{15+20i+15i+20i^2}{9-16i^2}+\frac{80-60i}{16-9i^2}$

$$=\frac{-5+35i}{25}+\frac{80-60i}{25}=3-i$$

8. $\frac{3i^{30}-i^{19}}{2i-1}=\frac{3(i^2)^{15}-(i^2)^9 i}{2i-1}=\frac{3(-1)^{15}-(-1)^9 i}{-1+2i}=\frac{-3+i}{-1+2i}\cdot\frac{-1-2i}{-1-2i}=\frac{3+6i-i-2i^2}{1-4i^2}=\frac{5+5i}{5}=1+i$

1.7 Remark:

The ***binomial formula*** $(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\cdots+\binom{n}{r}a^{n-r}b^r+\cdots+b^n$ involving real numbers remains valid with complex numbers. That is, if z_1 and z_2 are any two nonzero complex numbers, then

$$(z_1+z_2)^n=\sum_{k=0}^{n}\binom{n}{k}z_1^k z_2^{n-k},\ (n=1,2,\cdots,n)$$

Where $\binom{n}{k}=\frac{n!}{k!(n-k)!}$, $(k=0,12,\cdots,n)$.Note that 0!=1.

EXERCISES:

1. Show that
   a) $Re(iz)=-Im\,z$.
   b) $Im(iz)=Re\,z$.

c) $(1 + z)^2 = 1 + 2z + z^2$.

2. Verify that

a) $(\sqrt{2} - i) - i(1 - \sqrt{2}i) = -2i$.

b) $(2, -3)(-2,1) = (-1,8)$.

c) $(3,1)(3,-1)\left(\frac{1}{5}, \frac{1}{10}\right) - (2,1)$.

3. Solve the equation $z^2 + z + 1 = 0$ for $z = (x, y)$ by writing

$$(x, y)(x, y) + (x, y) + (1, 0) = (0, 0).$$

1.8 Remark:

The ***absolute value*** or modulus of a complex number $z = x + iy$ is defined as

$$|x + iy| = \sqrt{x^2 + y^2}$$

1.9 Example:

$$|-4 + 2i| = \sqrt{(-4)^2 + (2)^2} = \sqrt{20} = 2\sqrt{5}.$$

1.10 Remark:

If $z_1, z_2, z_3, \cdots, z_n$ are complex numbers, then the following properties hold

1. $|z_1 . z_2| = |z_1||z_2|$ or $|z_1 . z_2 . z_3 . \cdots . z_n| = |z_1||z_2||z_3| \cdots |z_n|$.

2. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_1|}$ if $z_1 \neq 0$.

3. $|z_1 + z_2| \leq |z_1| + |z_2|$ or $|z_1 + z_2 + z_3 + \cdots + z_n| \leq |z_1| + |z_2| + |z_3| + \cdots + |z_n|$.

4. $|z_1 - z_2| \geq |z_1| - |z_2|$.

1.11 Example:

1. Suoopse $z_1 = 2 + i$ and $z_2 = 3 - 2i$.Evaluate each of the following:

a) $|3z_1 - 4z_2| = |3(2 + i) - 4(3 - 2i)| = |6 + 3i - 12 + 8i|$

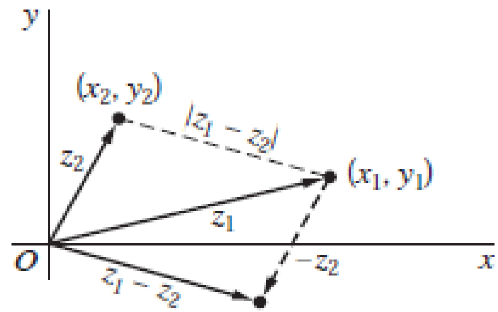
b) $\left|\frac{2z_2 + z_1 - 5 - i}{2z_1 - z_2 + 3 - i}\right|^2 = \left|\frac{2(3 - 2i) + (2 + i) - 5 - i}{2(2 + i) - (3 - 2i) + 3 - i}\right|^2 = \left|\frac{3 - 4i}{4 + 3i}\right|^2 = \frac{|3 - 4i|^2}{|4 + 3i|^2} = \frac{(\sqrt{(3)^2 + (-4)^2})^2}{(\sqrt{(4)^2 + (3)^2})^2} = 1$

1.12 Remark:

1. Since $Re\, z = x$, and $Im\, z = y$ then they are related by the equation

$$|z|^2 = (Re\ z)^2 + (Im\ z)^2.$$

2. $Re\ z \ \le\ |Re\ z|\ \le\ |z|$ and $Imz\ \le\ |Imz|\ \le\ |z|$



$|Z = -2 + 6i| = \sqrt{40} = 2\sqrt{10}$

$Re(2 + 6i) = -2 < |Re\ (2 + 6i)| = 2$

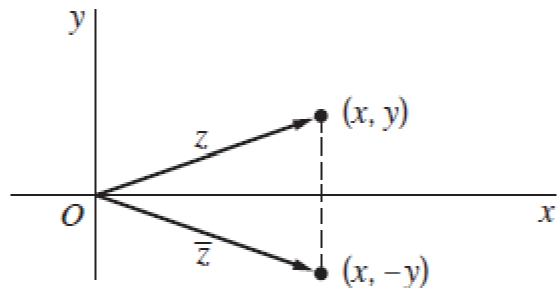
3. Suppose $z_1 = (x_1 + iy_1)$, $z_2 = (x_2 + iy_2)$ then

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

1.13 Definition:

The ***complex conjugate***, or simply the ***conjugate***, of a complex number $z = x + iy$ is defined as the complex number $x - iy$ and is denoted by $\bar{z}$; that is,

$$\bar{z} = x - iy,$$



The number $\bar{z}$ is represented by the point (x, -y), which is the reflection in the real axis of the point (x, y) representing z

1.14 Remark:

1. for all $z \epsilon \mathbb{C}$, $\bar{\bar{z}} = z$ and $|\bar{z}| = |z|$.

Since $\bar{\bar{z}} = \overline{\overline{x + \imath y}} = \overline{x - \imath y} = x + iy$ and $|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2 + y^2} = |z|$.

2. if $z_1 = (x_1 + iy_1)$, $z_2 = (x_2 + iy_2)$ then the conjugate of the sum is the sum of the conjugates

$$\overline{\boldsymbol{z_1 + z_2}} = \overline{(x_1 + \imath y_1) + (x_2 + \imath y_2)} = \overline{(x_1 + x_2) + \imath(y_1 + y_2)} = (x_1 + x_2)\text{-}i(y_1 + y_2)$$
$$= (x_1\text{-}iy_1) + i(x_2\text{-}iy_2) = \overline{\boldsymbol{z_1}} + \overline{\boldsymbol{z_2}}.$$

Also

$$\overline{\boldsymbol{z_1 - z_2}} = \overline{(x_1 + \imath y_1) - (x_2 + \imath y_2)} = \overline{(x_1 - x_2) + \imath(y_1 - y_2)} = (x_1 - x_2)\text{-}i(y_1 - y_2)$$
$$= (x_1 + iy_1)\text{-}i(x_2 + iy_2) = \overline{\boldsymbol{z_1}} - \overline{\boldsymbol{z_2}}\ .$$

$$\overline{\boldsymbol{z_1.z_2}} = \overline{(x_1 + \imath y_1).(x_2 + \imath y_2)} = \overline{(x_1x_2 - y_1y_2) + i(x_2y_1 + x_1y_2)}$$

$= (x_1x_2 - y_1y_2)\text{-}i(x_2y_1 + x_1y_2) = (x_1 - iy_1).(x_2 - iy_2) = \overline{z_1}.\overline{z_2}.$

$\frac{\overline{z_1}}{\overline{z_2}} = \overline{\frac{(x_1+\imath y_1)}{(x_2+\imath y_2)}} = \overline{\frac{x_1x_2-y_1y_2}{x_2^2+y_2^2} + \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}\imath} = \frac{x_1x_2-y_1y_2}{x_2^2+y_2^2} - \frac{x_2y_1-x_1y_2}{x_2^2+y_2^2}i = \frac{(x_1-iy_1)}{(x_2-iy_2)} = \frac{\overline{z_1}}{\overline{z_2}}, z_2 \neq 0.$

3. The sum $z + \bar{z}$ of a complex number $z = x + iy$ and its conjugate $\bar{z} = x - iy$ is the real number $2x$, and the difference $z - \bar{z}$ is the pure imaginary number $2iy$. Hence

$$Re(z) = \frac{z+\bar{z}}{2} \quad \text{and} \quad Im(z) = \frac{z-\bar{z}}{2i}.$$

4. An important identity relating the conjugate of a complex number $z = x + iy$ to its modulus is $\mathbf{z.\bar{z} = |z|^2}$.

Note that we can use remark 1.14 (4) to show that $|z_1.z_2| = |z_1||z_2|$, i.e.

$$|z_1.z_2|^2 = (z_1.z_2)\overline{(z_1.z_2)} = (z_1.z_2)(\overline{z_1}\overline{z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1||z_2|.$$

So $|z|^2 = |z^2|$, in general $|z|^n = |z^n|, n > 1$.

1.15 Example:

If z is a point inside the circle centered at the origin with radius 2, so that $|z| < 2$, then

$$|z^3 + 3z^2 - 2z + 1| \leq |z|^3 + 3|z|^2 + 2|z| + 1 < 25$$

EXERCISES:

1. Show that

a) $\overline{\bar{z} + 3i} = z - 3i$; **b)** $\overline{iz} = -i\bar{z}$; **c)** $\overline{(2+i)^2} = 3 - 4i$; **d)** $|(2\bar{z} + 5)(\sqrt{2} - i)| = \sqrt{3}\,|2z + 5|$.

2. Sketch the set of points determined by the condition

a) $\text{Re}(\bar{z} - i) = 2$; **b)** $|2\bar{z} + i| = 4$.

3. show that

a) $\overline{z_1z_2z_3} = \overline{z_1}\,\overline{z_2}\,\overline{z_3}$; **b)** $\overline{z^4} = \bar{z}^4$. **c)** $|\text{Re}(2 + \bar{z} + z^3)| \leq 4$ when $|z| \leq 1$.

4. Prove that

$$\overline{\bar{z} + 3\imath} = \overline{\overline{x + \imath y} + 3\imath} = \overline{x - \imath y + 3\imath} = \overline{x + \imath(-y + 3)} = x - i(-y + 3)$$
$$= x + iy - 3i = z - 3i$$

a) z is real if and only if $z = \bar{z}$;

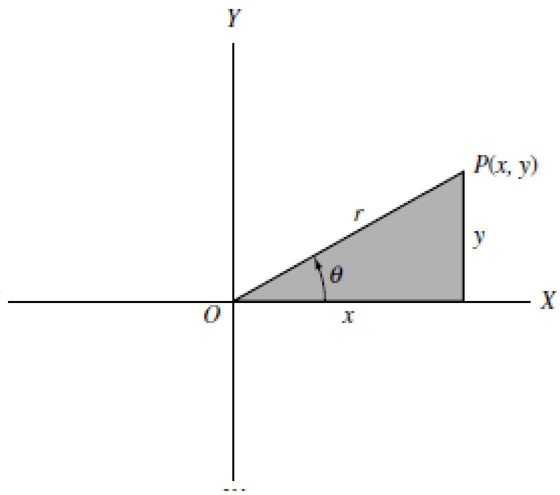
b) z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.

5. Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can be written $|z|^2 - 2Re(z\bar{z_0}) + |z_0|^2 = R^2$.

6. show that the hyperbola $x^2 - y^2 = 1$ can be written $z^2 + \bar{z}^2 = 2$.

1.16 Definition:

Let P be a point in the complex plane corresponding to the non-zero complex number (x, y). Let r and θ be polar coordinates of the point $z = x + iy$. Since $x = r \cos\theta$ and $y = r \sin\theta$, the number z can be written in ***polar form*** as $z = r\,(cos\theta + isin\theta)$.



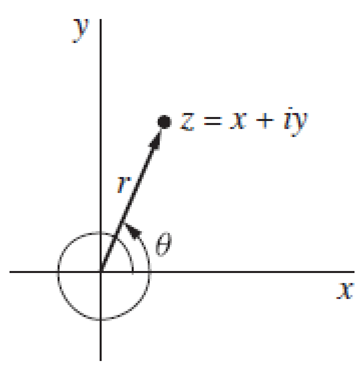
r and u are called ***polar coordinates.***

1.17 Remark:

1. If $z = 0$, the coordinate θ is undefined; and so it is understood that $z \neq 0$ whenever polar coordinates are used.

2. In complex analysis, the real number r is not allowed to be negative and is the length of the radius vector for z ; that is, $r = |z|$.

3. The real number θ represents the angle, measured in radians, that z makes with the positive real axis when z is interpreted as a radius vector. As in calculus, θ has an infinite number of possible values, including negative ones, that differ by integral multiples of 2π . Those values can be determined from the equation $\tan\theta = \frac{y}{x}$, where the quadrant containing the point corresponding to z must be specified. Each value of θ is called an ***argument*** of z, and the set of all such values is denoted by $\boldsymbol{arg\ z}$.The ***principal value*** of $arg\ z$,denoted



denoted by ***Arg z***, is that unique value Θ such that $-\pi < \Theta \leq \pi$. Evidently, then $\boldsymbol{arg\, z = Arg\, z + 2n\pi \quad (n = 0, \pm 1, \pm 2, \cdots)}$. Also, when z is a negative real number, $Arg\ z$ has value π, not $-\pi$.

4. The symbol $e^{i\theta}$, or $exp(i\theta)$, is defined by means of ***Euler's formula*** as

$$\boldsymbol{e^{i\theta} = cos\theta + isin\theta}.$$

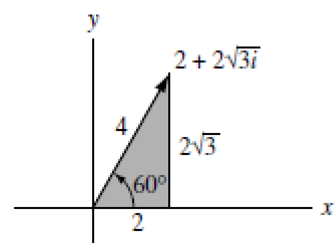
Where θ is to be measured in radians. It enables one to write the polar form $z = r\,(cos\theta + isin\theta)$ more compactly in ***exponential form*** as $z = re^{i\theta}$.

1.18 Example:

Express each of the following complex numbers in polar form.

(a) $2+2\sqrt{3}i$, (b) $-5+5i$, (c) $-\sqrt{6}-\sqrt{2}i$, (d) $-3i$

Solution:

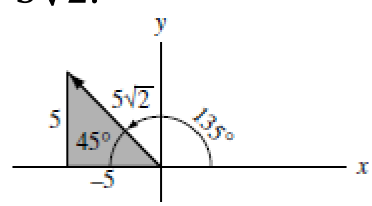


a) Modulus or absolute value, $r = |2+2\sqrt{3i}| = \sqrt{4+12} = 4$.

$Arg\ 2+2\sqrt{3i} = sin^{-1}\frac{\sqrt{3}}{2} = 60^{\circ} = \frac{\pi}{3}$ (radius).

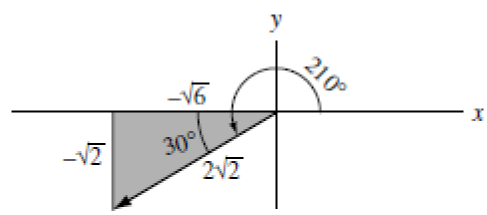
Then $2+2\sqrt{3i} = r\,(cos\theta + isin\theta) = 4(cos60^{\circ} + isin60^{\circ}) = 4(cos\frac{\pi}{3} + isin\frac{\pi}{3})$.

b) Modulus or absolute value, $r = |-5+5i| = \sqrt{25+25} = 5\sqrt{2}$.



$\theta = 180^{\circ} - 45^{\circ} = 135^{\circ} = \frac{3\pi}{4}$ (radius).Then

$-5+5i = 5\sqrt{2}(cos135^{\circ} + isin135^{\circ})$.



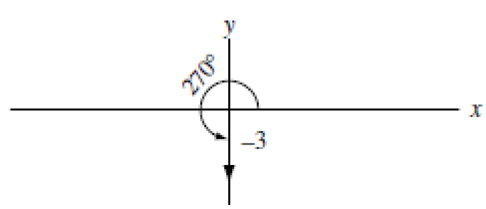
c) Modulus or absolute value, $r = |-\sqrt{6}-\sqrt{2}i|$

$= \sqrt{6+2} = 2\sqrt{2}$.

$\theta = 180^{\circ} + 30^{\circ} = 210^{\circ} = \frac{7\pi}{6}$ (radius).Then

$-\sqrt{6}-\sqrt{2}i = 2\sqrt{2}(cos210^{\circ} + isin210^{\circ})$.

d) Modulus or absolute value,$r = |-3\,i| = \sqrt{0+9} = 3$.



$\theta = 270^\circ = \frac{3\pi}{2}$ (radius).Then

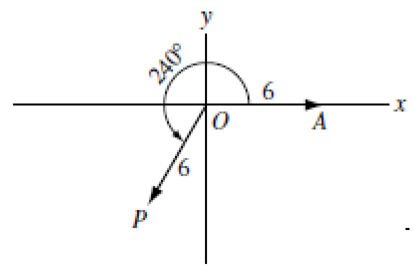
$$-3i = 3(cos270^\circ + isin270^\circ).$$

1.19 Example:

Graph each of the following:

(a) $6(\cos 240^\circ + i \sin 240^\circ)$, (b) $4e^{3\pi i/5}$, (c) $2e^{-\pi i/4}$.

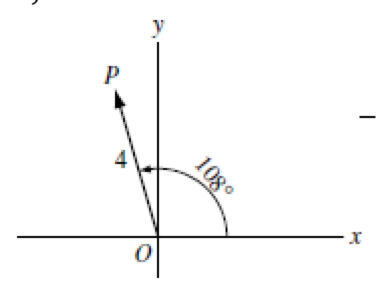
Solution:



a) $6(cos240^\circ + isin240^\circ) = 6\left(cos\frac{4\pi}{3} + isin\frac{4\pi}{3}\right)$

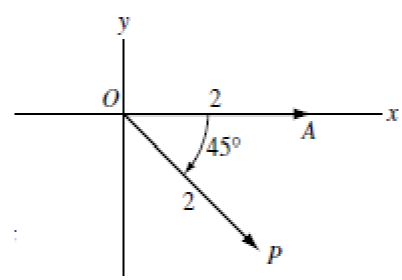
$= \boldsymbol{6e^{\frac{4\pi i}{3}}}$ can be represented graphically by OP.

If we start with vector OA, whose magnitude is 6 and whose direction is that of the positive x axis, we can obtain OP by rotating OA counterclockwise through an angle of 240°. In general, $re^{i\theta}$ is equivalent to a vector obtained by rotating a vector of magnitude r and direction that of the positive x axis, counterclockwise through an angle θ.



b) $4e^{\frac{3\pi i}{5}} = 4\left(cos\frac{3\pi}{5} + isin\frac{3\pi}{5}\right) = 4(cos108^\circ + isin108^\circ)$

is represented by OP.



c) $2e^{\frac{-\pi i}{4}} = 2\left(cos\frac{-\pi}{4} + isin\frac{-\pi}{4}\right)$

$= 2(cos(-45)^\circ + isin(-45)^\circ).$

This complex number can be represented by vector OP.

This vector can be obtained by starting with vector OA,whose magnitude is 2 and whose direction is that of the positive x axis, and rotating it counterclockwise through an angle of $(-45)^\circ$ (which is the same as rotating it clockwise through an angle of 45°).

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ٢

Chapter One

COMPLEX NUMBERS

1.20 Remark:

1. Let $z_1 = r_1\,(cos\theta_1 + isin\theta_1), z_2 = r_2\,(cos\theta_2 + isin\theta_2)$ then

$$\boldsymbol{z_1 . z_2} = r_1 r_2\,(cos\theta_1 + isin\theta_1).(cos\theta_2 + isin\theta_2)$$
$$= r_1 r_2\,(cos\theta_1 cos\theta_2 - sin\theta_1 sin\theta_2) + i(sin\theta_1 cos\theta_2 + cos\theta_1 sin\theta_2)$$
$$= r_1 r_2\,(cos(\theta_1 + \theta_2) + isin(\theta_2 + \theta_2))$$
$$\boldsymbol{= r_1 r_2\, e^{(\theta_1+\theta_2)i}}.$$

Similar $\frac{z_1}{z_2} = \frac{r_1\,(cos\theta_1+isin\theta_1)}{r_2(cos\theta_2+isin\theta_2)} = \frac{r_1}{r_2}\frac{(cos\theta_1+isin\theta_1)}{(cos\theta_2+isin\theta_2)}\frac{(cos\theta_2-isin\theta_2)}{(cos\theta_2-isin\theta_2)}$

$$= \frac{r_1}{r_2}\frac{(cos\theta_1 cos\theta_2+sin\theta_1 sin\theta_2)+ i(sin\theta_1 cos\theta_2-cos\theta_1 sin\theta_2)}{cos\theta_2{}^2+sin\theta_2=1}$$
$$= \frac{r_1}{r_2}(cos(\theta_1 - \theta_2) + isin(\theta_2 - \theta_2))$$
$$\boldsymbol{= \frac{r_1}{r_2} e^{(\theta_1-\theta_2)i}}$$

2. In general $\boldsymbol{z_1 . z_2 \cdots z_n = r_1 r_2 \cdots r_n e^{(\theta_1+\theta_2+\cdots\theta_n)i}, n\epsilon\mathbb{N}}$. If $\boldsymbol{z_1 = z_2} = \cdots = \boldsymbol{z_n} = \boldsymbol{z}$ then

$$z^n = \underbrace{z.z.\cdots.z}_{n\ times} = \underbrace{r.r.\cdots.r}_{n\ times} = z^n = r^n e^{(\theta+\theta+\cdots+\theta)} = r^n e^{n\theta}$$
$$= r^n(cos\,(n\theta) + isin(n\theta)).$$

When $r = 1$ then

$$\boldsymbol{z^n = (cos\theta + isin\theta)^n = cos\,(n\theta) + isin(n\theta)} \qquad (5)$$

which is often called ***De Moivre's theorem***.

1.21 Example:

Prove the identities

a) $cos5\theta = 16cos^5\theta - 20cos^3\theta + 5cos\theta$,

b) $(sin\,5\theta)/(sin\theta) = 16cos^4\theta - 12cos^2\theta + 1, if\ \theta \neq 0, \pm\pi, \pm 2\pi, \cdots$.

Solution:

From equation (5) and by binomial formula we get

$$cos\,(5\theta) + i\sin(5\theta) = (cos\theta + isin\theta)^5$$
$$= \cos^5\theta + \binom{5}{1}(\cos^4\theta)(i\sin\theta) + \binom{5}{2}(\cos^3\theta)(i\sin\theta)^2$$
$$+ \binom{5}{3}(\cos^2\theta)(i\sin\theta)^3 + \binom{5}{4}(\cos\theta)(i\sin\theta)^4 + (i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$
$$- 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$
$$= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$
$$+ i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$$

Hence

a) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$$= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$$
$$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

b) $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$, so
$$\frac{\sin 5\theta}{\sin\theta} = 5\cos^4\theta - 10\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= 5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= 16\cos^4\theta - 12\cos^2\theta + 1$$

Provided $\theta \neq 0, \pm\pi, \pm 2\pi, \cdots$.

1.22 Example:

Show that

a) $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$, b) $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$.

Solution:

Since $e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$ then
$$\frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} = \cos\theta,$$
$$\frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{\cos\theta + i\sin\theta - \cos\theta + i\sin\theta}{2i} = \frac{2i\sin\theta}{2i} = \sin\theta.$$

1.23 Example:

Prove the identities

(a) $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$, (b) $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$.

Solution:

a) $sin^3\theta = \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3 = \frac{\left(e^{i\theta}-e^{-i\theta}\right)^3}{8i^3} = -\frac{1}{8i}\{(e^{i\theta})^3 - 3(e^{i\theta})^2(e^{-i\theta}) + 3(e^{i\theta})(e^{-i\theta})^2 - (e^{-i\theta})^3\}$

$$= -\frac{1}{8i}(e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}) = \frac{3}{4}\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right) - \frac{1}{4}\left(\frac{e^{3i\theta} - e^{-3i\theta}}{2i}\right)$$

$$= \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

b) $\cos^4\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^4 = \frac{(e^{i\theta} + e^{-i\theta})^4}{16}$

$$= \frac{1}{16}\{(e^{i\theta})^4 + 4(e^{i\theta})^3(e^{-i\theta}) + 6(e^{i\theta})^2(e^{-i\theta})^2 + 4(e^{i\theta})(e^{-i\theta})^3 + (e^{-i\theta})^4\}$$

$$= \frac{1}{16}(e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}) = \frac{1}{8}\left(\frac{e^{4i\theta} + e^{-4i\theta}}{2}\right) + \frac{1}{2}\left(\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right) + \frac{3}{8}$$

$$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

1.24 Example:

Prove that $e^{i\theta} = e^{i(\theta+2\pi k)}, k = 0, \pm\pi, \pm 2\pi, \cdots$.

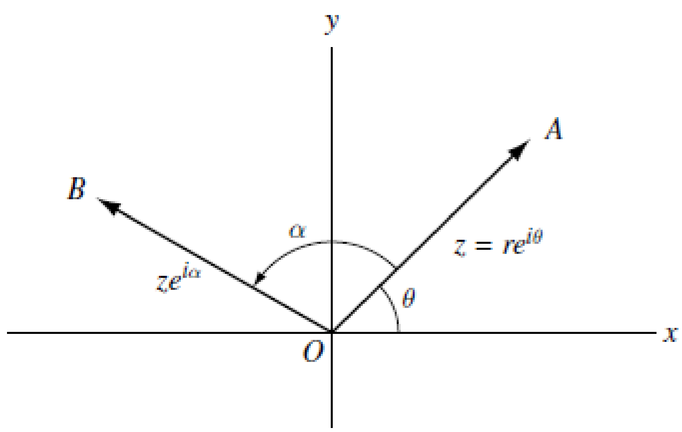
Solution:

$$e^{i(\theta+2\pi k)} = \cos(\theta + 2\pi k) + i\sin(\theta + 2\pi k) = \cos\theta + i\sin\theta = e^{i\theta}$$

1.25 Example:

Given a complex number (vector) z, interpret geometrically $ze^{i\alpha}$ where α is real.

Solution:



Let $z = re^{i\theta}$ be represented graphically by vector OA . Then

$$ze^{i\alpha} = re^{i\theta}.e^{i\alpha} = re^{i(\theta+\alpha)},$$

is the vector represented by OB. Hence multiplication of a vector z by $e^{i\alpha}$amounts to rotating z counterclockwise through angle α.We can consider $e^{i\alpha}$ as an operator that acts on z to produce this rotation.

1.26 Example:

Evaluate each of the following:

(a) $[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)]$, (b) $\frac{(2(\cos 15^\circ + i\sin 15^\circ))^7}{(4\cos 45^\circ + i\sin 45^\circ)^3}$, (c) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10}$

Solution:

a) $[3(\cos 40^\circ + i\sin 40^\circ)][4(\cos 80^\circ + i\sin 80^\circ)] = 3\cdot 4[\cos(40^\circ + 80^\circ) + i\sin(40^\circ + 80^\circ)]$

$$= 12(\cos 120^\circ + i\sin 120^\circ)$$

$$= 12\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = -6 + 6\sqrt{3}i$$

b) $\frac{\left(2(\cos 15^\circ + i\sin 15^\circ)\right)^7}{(4(\cos 45^\circ + i\sin 45^\circ))^3} = \frac{128(\cos 105^\circ + i\sin 105^\circ)}{64(\cos 135^\circ + i\sin 135^\circ)}$

$$= 2(\cos(105^\circ - 135^\circ) + i\sin(105^\circ - 135^\circ))$$

$$= 2[\cos(-30^\circ) + i\sin(-30^\circ)] = 2[\cos 30^\circ - i\sin 30^\circ] = \sqrt{3} - i$$

c) $\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{2(\cos 60^\circ + i\sin 60^\circ)}{2(\cos -60^\circ + i\sin -60^\circ)}\right)^{10} = (\cos 120^\circ + i\sin 120^\circ)^{10}$

$$= \cos 1200^\circ + i\sin 1200^\circ = \cos 120^\circ + i\sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

1.27 Remark:

We can solve c) in example1.26 by another method

$$\left(\frac{1+\sqrt{3}i}{1-\sqrt{3}i}\right)^{10} = \left(\frac{2e^{\pi i/3}}{2e^{-\pi i/3}}\right)^{10} = (e^{2\pi i/3})^{10} = e^{20\pi i/3}$$

$$= e^{6\pi i}e^{2\pi i/3} = (1)[\cos(2\pi/3) + i\sin(2\pi/3)] = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

EXERCISES:

1. Evaluate each of the following:

a) $(5(\cos 20^\circ + i\sin 20^\circ))(3(\cos 40^\circ + i\sin 40^\circ))$.

b) $(2(\cos 50^\circ + i\sin 50^\circ))^6$.

c) $\frac{\left(8(\cos 40^\circ + i\sin 40^\circ)\right)^3}{\left(2(\cos 60^\circ + i\sin 60^\circ)\right)^4}$

d) $\frac{(3e^{\pi i/6})(2e^{-5\pi i/4})(6e^{5\pi i/3})}{(4e^{2\pi i/3})^2}$

e) $\left(\frac{\sqrt{3}-i}{\sqrt{3}+i}\right)^4\left(\frac{1+i}{1-i}\right)^5$

2. Prove that (a) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$, (b) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

3. Prove that the solutions of $z^4-3z^2+1=0$ are given by

$$z=2\cos 36°, 2\cos 72°, 2\cos 216°, 2\cos 252°.$$

4. Show that (a) $\cos 36° = (\sqrt{5}+1)/4$, (b) $\cos 72° = (\sqrt{5}-1)/4$.

5. Prove that (a) $\sin 4\theta/\sin\theta = 8\cos^3\theta - 4\cos\theta = 2\cos 3\theta + 2\cos\theta$

(b) $\cos 4\theta = 8\sin^4\theta - 8\sin^2\theta + 1$

1.28 Remark:

A number w is called an nth root of a complex number z if $w^n = z$, and we write $w = z^{\frac{1}{n}}$. From De Moivre's theorem we can show that if n is a positive integer,

$$z^{\frac{1}{n}} = (r(\cos\theta + i\sin\theta))^{\frac{1}{n}}$$

$$= r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2\pi k}{n}\right) + i\sin\left(\frac{\theta+2\pi k}{n}\right)\right), \quad k = 0,1,2,\cdots,n-1. \tag{6}$$

from which it follows that there are n different values for $z^{\frac{1}{n}}$, i.e., n different nth roots of z, provided $z \neq 0$.

1.29 Example:

Determine the nth roots of unity.

Solution:

In order to determine the nth roots of unity, we write

$$1 = 1(\cos(0+2\pi k) + i\sin(0+2\pi k)), k = 0, \pm 1, \pm 2, \cdots$$

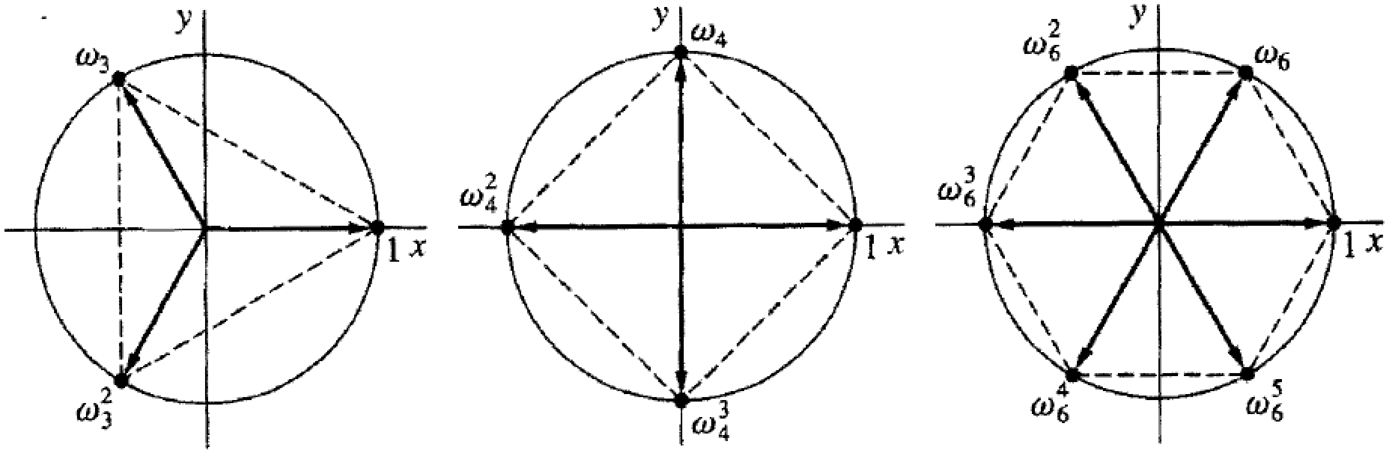
And write that

$$1^{\frac{1}{n}} = \sqrt[n]{1}\left(\cos\left(\frac{0}{n}+\frac{2\pi k}{n}\right) + i\sin\left(\frac{0}{n}+\frac{2\pi k}{n}\right)\right)$$

$$= \sqrt[n]{1}\left(\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)\right), (k = 0,1,2,\cdots,n\text{-}1).$$

When $n = 2$ these roots are ± 1.When $n \geq 3$ the regular polygon at whose vertices are roots lie is incribed in the unit circle $|z| = 1$with one vertex corresponding to the principal root $z = 1(k = 0)$.If we write $\omega_n = e^{\left(\frac{2\pi}{n}i\right)}$then $c_k = \omega_n^k =$

$e^{\left(\frac{2\pi k}{n}i\right)}$,($k = 0,1,2,\cdots,n$-1).Hence the distinct nth roots of z of unity are $1,\omega_n,\omega_n^2,\cdots,\omega_n^{n-1}$.



1.30 Example:

Find all values of $(-8i)^{1/3}$ or find three cube roots of $-8i$.

Solution:

$$-8i = 8e^{\left(-\frac{\pi}{2}+2\pi k\right)i} = 8\left(\cos\left(-\frac{\pi}{2}+2\pi k\right) + i\sin\left(-\frac{\pi}{2}+2\pi k\right)\right), k = 0, \pm 1, \pm 2, \cdots.$$

So the desired roots are

$$c_k = (-8i)^{1/3} = 8^{\frac{1}{3}}\left(\cos\left(\frac{-\frac{\pi}{2}+2\pi k}{3}\right) + i\sin\left(\frac{-\frac{\pi}{2}+2\pi k}{3}\right)\right), \quad k = 0,1,2.$$
$$= 2\left(\cos\left(-\frac{\pi}{6}+\frac{2\pi k}{3}\right) + i\sin\left(-\frac{\pi}{6}+\frac{2\pi k}{3}\right)\right), \quad k = 0,1,2.$$

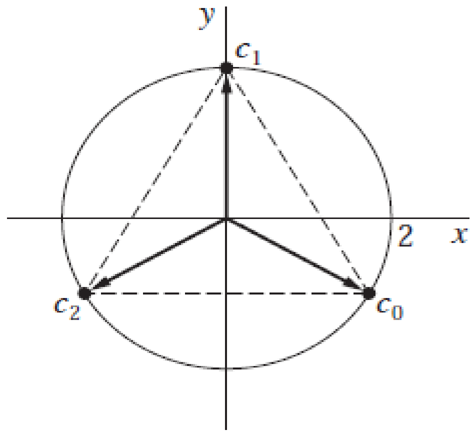
If $k = 0$ then $c_0 = 2(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})) = \sqrt{3} - i$.

If $k = 1$ then $c_1 = 2(\cos(-\frac{\pi}{6}+\frac{2\pi}{3}) + i\sin(-\frac{\pi}{6}+\frac{2\pi}{3})) = 2i$.

If $k = 2$ then $c_2 = 2(\cos(-\frac{\pi}{6}+\frac{4\pi}{3}) + i\sin(-\frac{\pi}{6}+\frac{4\pi}{3})) = -\sqrt{3} - i$.

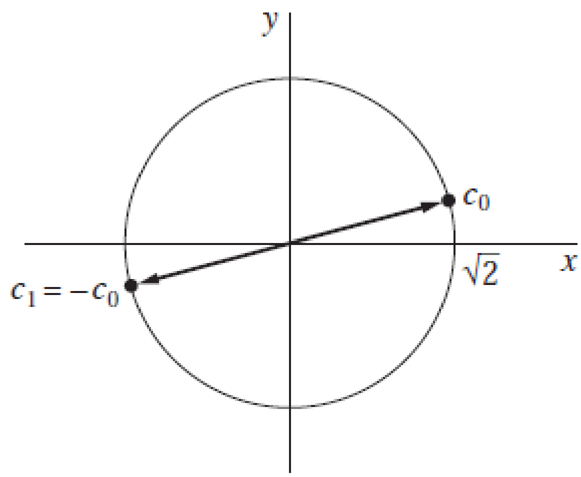
c_0,c_1 and c_2 lie at the vertices of an equilateral triangle, inscribed in the circle $|z| = 2$, and are equally spaced around that circle every $2\pi/3$ radians, starting with the principal root c_0.

Without any further calculations, it is then evident that $c_1 = 2i$; and, since c_2 is symmetric to c_0 with respect to the imaginary axis, we know that $c_2 = -\sqrt{3} - i$.



1.31 Example:

Find all values of $(\sqrt{3} + i)^{1/2}$



Solution:

$\sqrt{3} + i = 2e^{\left(\left(\frac{\pi}{6}+2\pi k\right)i\right)} = 2\left(\cos\left(\frac{\pi}{6} + 2\pi k\right) + i\sin\left(\frac{\pi}{6} + 2\pi k\right)\right),$

$$k = 0, \pm 1, \pm 2, \cdots.$$

$$c_k = (\sqrt{3} + i)^{1/2} = 2^{\frac{1}{2}}\left(\cos\left(\frac{\frac{\pi}{6}+2\pi k}{2}\right) + i\sin\left(\frac{\frac{\pi}{6}+2\pi k}{2}\right)\right), \quad k = 0,1.$$

If $k = 0$ then $c_0 = \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$

Since $cos^2\left(\frac{\alpha}{2}\right) = \frac{1+\cos\alpha}{2}$, $sin^2\left(\frac{\alpha}{2}\right) = \frac{1-\cos\alpha}{2}$ then $cos^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1 + \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{2}\right) = \frac{2+\sqrt{3}}{4}$, $sin^2\left(\frac{\pi}{12}\right) = \frac{1}{2}\left(1 - \cos\frac{\pi}{6}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right) = \frac{2-\sqrt{3}}{4}$. Therefore

$$c_0 = \sqrt{2}\left(\sqrt{\frac{2+\sqrt{3}}{4}} + i\sqrt{\frac{2-\sqrt{3}}{4}}\right) = \frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

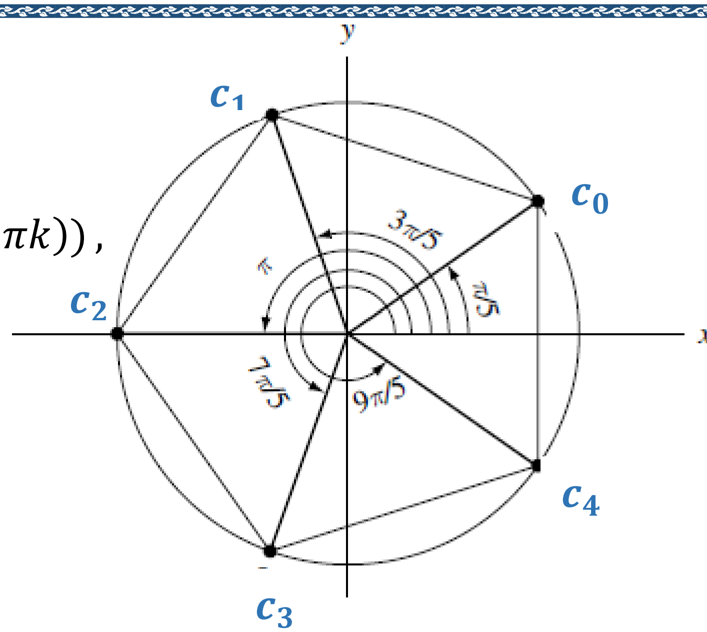
Since $c_1 = -c_0$ then the two square roots of $\sqrt{3} + i$ are

$$\pm\frac{1}{\sqrt{2}}\left(\sqrt{2+\sqrt{3}} + i\sqrt{2-\sqrt{3}}\right).$$

1.32 Example:

(a) Find all values of z for which $z^5 = -32$, and (b) locate these values in the complex plane.

Solution:



$-32 = 32e^{(\pi+2\pi k)i)}$

$= 32\,(\cos(\pi + 2\pi k) + isin(\pi + 2\pi k))\,,$

$k = 0, \pm 1, \pm 2, \cdots.$

$c_k = (-32)^{1/5}$

$= 32^{\frac{1}{5}}\,(\cos(\frac{\pi+2\pi k}{5}) + isin(\frac{\pi+2\pi k}{5}))$

$= 2\,(\cos(\frac{\pi+2\pi k}{5}) + isin(\frac{\pi+2\pi k}{5}))\,, k = 0,1,2,3,4.$

If $k = 0$ then $c_0 = 2(\cos(\frac{\pi}{5}) + isin(\frac{\pi}{5}))$.

If $k = 1$ then $c_1 = 2(\cos(\frac{3\pi}{5}) + isin(\frac{3\pi}{5}))$.

If $k = 2$ then $c_2 = 2(\cos(\frac{5\pi}{5}) + isin(\frac{5\pi}{5})) = -2$.

If $k = 3$ then $c_3 = 2(\cos(\frac{7\pi}{5}) + isin(\frac{7\pi}{5}))$.

If $k = 4$ then $c_4 = 2(\cos(\frac{9\pi}{5}) + isin(\frac{9\pi}{5}))$.

1.33 Example:

a) Find the square roots of $-15 - 8i$.

b) Let $p + iq$, where p and q are real, represent the required square roots. Then $(p + iq)^2 = p^2 - q^2 + 2pqi = -15 - 8i$ or $p^2 - q^2 = -15$, $pq = -4$.

Solution:

a) Since $|-15 - 8i| = 17$ then

$-15 - 8i = 17e^{(\theta+2\pi k)i)} = 8\,(\cos(\theta + 2\pi k) + isin(\theta + 2\pi k))\,, k = 0, \pm 1, \pm 2, \cdots.$

Where $\cos\theta = \frac{-15}{17}$, $\sin\theta = \frac{-8}{17}$ then the sequare roots of $-15 - 8i$ are

$c_0 = \sqrt{17}\,(\cos(\frac{\theta}{2}) + isin(\frac{\theta}{2})),$

$c_1 = \sqrt{17}\,(\cos(\frac{\theta}{2} + \pi) + isin(\frac{\theta}{2} + \pi)) = -\sqrt{17}\,(\cos(\frac{\theta}{2}) + isin(\frac{\theta}{2})).$

Now

$cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{1-\frac{15}{17}}{2}} = \pm\frac{1}{\sqrt{17}}$. $sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos\theta}{2}} = \pm\sqrt{\frac{1+\frac{15}{17}}{2}} = \pm\frac{4}{\sqrt{17}}$.

Since θ is an angle in the third quadrant, $\frac{\theta}{2}$ is an angle in the second quadrant.

Hence, $cos\left(\frac{\theta}{2}\right) = -\frac{1}{\sqrt{17}}$, $sin\left(\frac{\theta}{2}\right) = \frac{4}{\sqrt{17}}$, i.e. the required square roots are $-1+4i$ and $1-4i$ (As a check $(-1+4i)^2 = (1-4i)^2 = -15-8i$.

b) Substituting $q = \frac{-4}{p}$ from $pq = -4$ into $p^2 - q^2 = -15$ it becomes $p^2 - \frac{16}{p^2} = -15$ or $p^4 + 15p^2 - 16 = 0$, i.e. $(p^2+16)(p^2-1) = 0$ or $p^2 = -16, p^2 = 1$ .Since p is real , $p = \pm 1$.If $p = 1$ then $q = -4$ or if $p = -1$ then $q = 4$. Thus the roots are $-1+4i$ and $1-4i$.

EXERCISES:

1. In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain regular polygons, and identify the principal root:

   a) $(-1)^{\frac{1}{3}}$, b) $8^{\frac{1}{6}}$.

2. Let a denote any fixed real number and show that the two square roots of $a+i$ are $\pm\sqrt{A}e^{i\frac{\alpha}{2}}$, where $A = \sqrt{a^2+1}$ and $\alpha = Arg(a+i)$.

3. Find the four zeros of the polynomial z^4+4, one of them $c_0 = \sqrt{2}e^{i\frac{\pi}{4}}$. Then use those zeros to factor z^4+4 into quadratic factors with real coefficients.

4. Show that if c is any nth root of unity other than unity itself, then

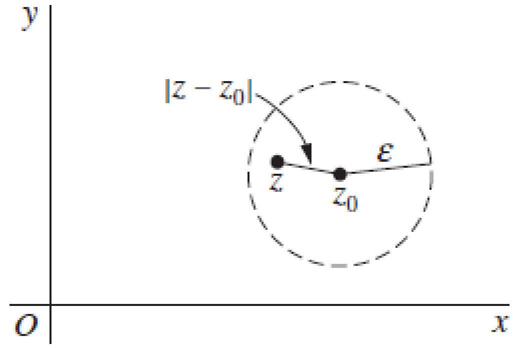
$$1 + c + c^2 + \cdots + c^{n-1} = 0.$$

5. Find each of the indicated roots and locate them graphically.

   (a) $(2\sqrt{3}-2i)^{1/2}$, (b) $(-4+4i)^{1/5}$, (c) $(2+2\sqrt{3}i)^{1/3}$, (d) $(-16i)^{1/4}$, (e) $(64)^{1/6}$, (f) $(i)^{2/3}$.

6. Find all the indicated roots and locate them in the complex plane.
   a) Cube roots of 8,**b)** square roots of $4\sqrt{2}+4\sqrt{2}i$, **c)** fifth roots of $-16+16\sqrt{3}i$,
7. Solve the equations **a)** $z^4+81=0$, **b)** $z^6+1=\sqrt{3}i$.
8. Find the square roots of **a)** $5-12i$, **b)** $8-4\sqrt{5}i$.
9. Find the cube roots of $-11-2i$.

1.34 Remark:



1. An ***ε neighborhood*** of a point z_0 is the set of all points z lying inside but not on a circle centered at z_0 and with a specified positive radius ε, i.e.

$$|z - z_0| < \varepsilon.$$

2. A ***deleted neighborhood*** of a point z_0 is the set of all points z in an ε neighborhood of z_0 except for the point z_0 itself, i.e. $0 < |z - z_0| < \varepsilon$.

3. A point z_0 is said to be an ***interior point*** of a set $S \subseteq \mathbb{C}$ whenever there is some neighborhood of z_0 that contains only points of S.

4. A point z_0 is said to be an ***exterior point*** of $S \subseteq \mathbb{C}$ when there exists a neighborhood of it containing no points of S.

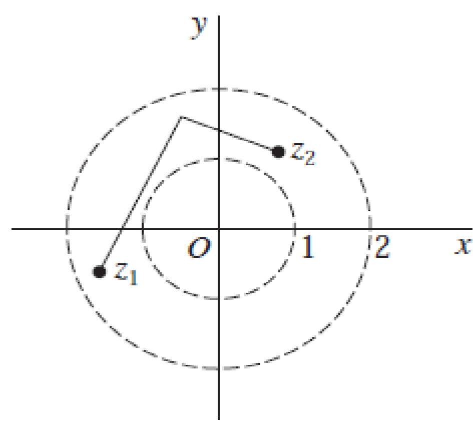
5. If z_0 is neither interior point nor exterior point then it is a ***boundary point*** of S, i.e. a boundary point is, therefore, a point all of whose neighborhoods contain at least one point in S and at least one point not in S. The circle $|z| = 1$,for instance, is the boundary of each of the sets $|z| < 1$ and $|z| \leq 1$.

6. An ***open set*** is a set which consists only of interior points. For example, the set Of points z such that $|z| < 1$ is an open set.

7. A set is ***closed*** if it contains all of its boundary points, and the ***closure*** of a set S is the closed set consisting of all points in S together with the boundary of S.

8. Some sets are neither open nor closed, for example the punctured disk $0 < |z| \leq 1$ is neither open nor closed.

9. An open set S is ***connected*** if each pair of points z_1 and z_2 in it can be joined by a polygonal line, consisting of a finite number of line segments joined end



to end, that lies entirely in S. The open set $|z| < 1$ is connected. The annulus $1 < |z| < 2$ is open and it is also connected.

10. A nonempty open set that is connected is called a ***domain***. Note that any neighborhood is a domain.

11. A domain together with some or all of its boundary points is referred to as a ***region*** .

12. A set S is ***bounded*** if every point of S lies inside some circle $|z| = R$;otherwise, it is ***unbounded***. Both of the sets $|z| < 1$ and $|z| \leq 1$ are bounded regions,and the half plane $Re\, z \geq 0$ is unbounded.

13. A point z_0 is said to be an ***accumulation point*** of a set S if each deleted neighborhood of z_0 contains at least one point of S. It follows that if a set S is closed, then it contains each of its accumulation points.

14. A point z_0 is ***not an accumulation*** point of a set S whenever there exists some deleted neighborhood of z_0 that does not contain at least one point of S. For example the origin is the only accumulation point of the set $z_n = \frac{i}{n}, n = 1,2, \cdots$.

EXERCISES:

1. Sketch the following sets and determine which are domains, neither open nor closed and bounded

(a) $|z - 2 + i| \leq 1$; (b) $|2z + 3| > 4$;
(c) $\mathrm{Im}\, z > 1$; (d) $\mathrm{Im}\, z = 1$;
(e) $0 \leq \arg z \leq \pi/4 \ (z \neq 0)$; (f) $|z - 4| \geq |z|$.

2. Let S be the open set consisting of all points z such that $|z| < 1$ or $|z - 2| < 1$. State why S is not connected.

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Chapter Two

Analytic Functions

2.1 Definition:

Let S be a set of complex numbers. A ***function f*** defined on S is a rule that assigns to each z in S a complex number w. The number w is called the ***value*** of f at z and is denoted by $f(z)$; that is, $w = f(z)$. The set S is called the ***domain of definition* of *f*** .

2.2 Remark:

1. Suppose that $w = u + iv$ is the value of a function f at $z = x + iy$, so that

$$u + iv = f(x + iy).$$

   Each of the real numbers u and v depends on the real variables x and y, and it follows that $f(z)$ can be expressed in terms of a pair of real-valued functions of the real variables x and y:

$$f(z) = u(x, y) + iv(x, y). \qquad (1)$$

2. If the polar coordinates r and θ, instead of x and y, are used, then

$$u + iv = f(re^{i\theta}).$$

   where $w = u + iv$ and $z = re^{i\theta}$. In that case, we may write

$$f(z) = u(r, \theta) + iv(r, \theta). \qquad (2)$$

2.3 Remark:

If in either of equations (1) and (2) the function v always has value zero then the value of f is always real. That is, f is a ***real-valued function*** of a complex variable.

2.4 Example:

If $f(z) = z^2$, then $f(x + iy) = (x + iy)^2 = x^2 - y^2 + i2xy$. Hence $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

2.5 Remark:

In example 2.4 if we take the polar coordinates then

$$f(re^{i\theta}) = (re^{i\theta})^2 = r^2 e^{i2\theta} = r^2 \cos 2\theta + ir^2 \sin 2\theta.$$

Consequently, $u(r, \theta) = r^2 \cos 2\theta$ and $v(r, \theta) = r^2 \sin 2\theta$.

2.6 Remark:

If only one value of w corresponds to each value of z, we say that w is a ***single-valued function*** of z or that $f(z)$ is **single-valued**. If more than one value of w

corresponds to each value of z, we say that w is a ***multiplevalued*** or ***many-valued function*** of z.

2.7 Example:

1. If $w = z^2$, then to each value of z there is only one value of w. Hence, is a single-valued function of z.
2. If $w = z^{\frac{1}{2}}$, then to each value of z there are two values of w. Hence, $w = z^{\frac{1}{2}}$ defines a multiple-valued (in this case two-valued) function of z.

2.8 Example:

For each of the functions below, describe the domain of definition that is understood:

a) $f(z) = \frac{1}{z^2+1}$; b) $f(z) = Arg(\frac{1}{z})$;

c) $f(z) = \frac{z}{\bar{z}+z}$; d) $f(z) = \frac{1}{1-|z|^2}$.

Solution:

a) The function $f(z) = \frac{1}{z^2+1}$ is defined everywhere in the finite plane except the points $z = \pm i$ where $z^2 + 1 = 0$.

b) The function $f(z) = Arg(\frac{1}{z})$ is defined throughout the entire finite plane except the points $z = 0$.

c) The function $f(z) = \frac{z}{\bar{z}+z}$ is defined everywhere in the finite plane except for the imaginary axis. This is because the equation $\bar{z} + z = 0$ is the same as $x = 0$.

d) The function $f(z) = \frac{1}{1-|z|^2}$ is defined everywhere in the finite plane except on the circle $|z| = 1$, where $1 - |z|^2 = 0$

EXERCISES:

1. Write the function $f(z) = z^3 + z + 1$ in the form $f(z) = u(x, y) + iv(x, y)$.
2. Suppose that $f(z) = x^2 - y^2 - 2y + i(2x - 2xy)$, where $z = x + iy$. Use the expressions $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$ to write *f (z)* in terms of z, and simplify the result.
3. Write the function $f(z) = z + \frac{1}{z}, z \neq 0$ in the form $f(z) = u(r, \theta) + iv(r, \theta)$.

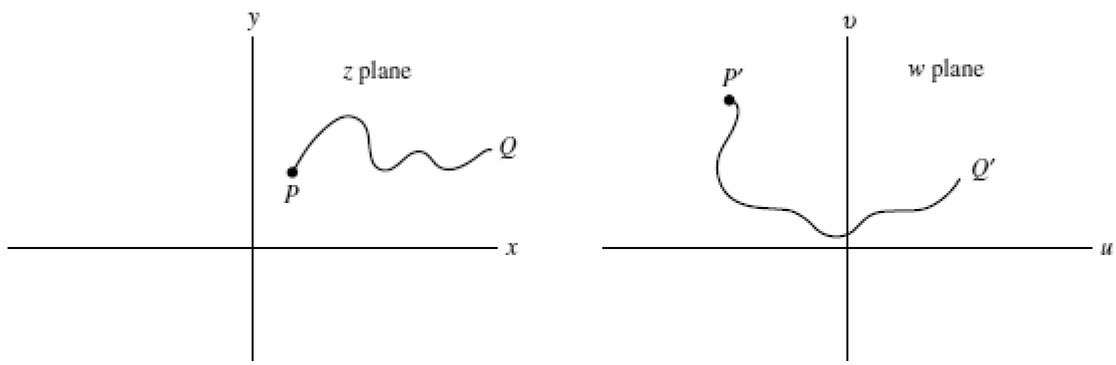
2.9 Definition:

If $w = f(z)$ then we can also consider z as a function, possibly multiple-valued, of w, written $z = g(w) = f^{-1}(w)$. The function f^{-1} is often called the ***inverse***

function corresponding to f. Thus, $w = f(z)$ and $z = f^{-1}(w)$. are inverse functions of each other.

2.10 Remark:

Given a point $P = (x, y)$ in the z - plane there corresponds a point $P'(u, v)$ in the w- plane. The set of equations $u = u(x, y)$and $v = v(x, y)$ [or the equivalent, $w = f(z)$] is called a ***transformation***. We say that point P is ***mapped*** or ***transformed*** into point P' by means of the ***transformation*** and call P' the image of P.



In general, under a transformation, a set of points such as those on curve PQ of is mapped into a corresponding set of points, called the image, such as those on curve $P'Q'$. The particular characteristics of the image depend of course on the type of function f (z), which is sometimes called a mapping function. If f (z) is multiple-valued, a point (or curve) in the z - plane is mapped in general into more than one point (or curve) in the w - plane.

2.11 Example:

The mapping $w = z + 1 = (x + 1) + iy$ where $z = x + iy$, can be thought of as a translation of each point z one unit to the right.

2.12 Example:

the mapping $w = \bar{z} = x - iy$transforms each point $z = x + iy$ into its reflection in the real axis.

2.13 Remark:

In the following three examples, we illustrate the transformation $w = z^2$. We begin by finding the images of some curves in the z – plane:

2.14 Example:

According to Example 2.4 the mapping $w = z^2$ can be thought of as the transformation $u = x^2 - y^2$ and $v = 2xy$ from the xy - plane into the uv- plane. This form of the mapping is especially useful in finding the images of certain hyperbolas. For instance, that each branch of a hyperbola $x^2 - y^2 = c_1$,$(c_1 > 0)$ is mapped in a one to one manner onto the vertical line $u = c_1$.

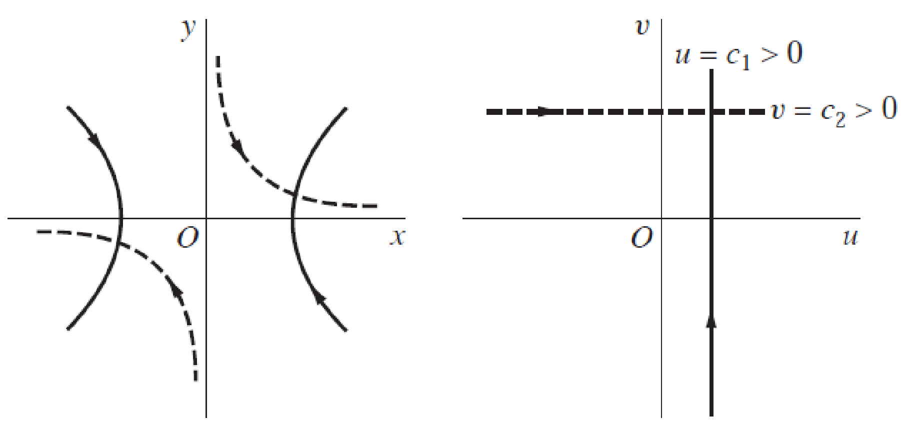
We start by noting from the equation $u = x^2 - y^2$ that $u = c_1$ when (x, y) is a point lying on either branch. When, in particular, it lies on the right-hand branch. The second equation $v(x, y) = 2xy$ tells us that $v = 2y\sqrt{y^2 + c_1}$. Thus the image of the right-hand branch can be expressed parametrically as

$$u = c_1 , v = 2y\sqrt{y^2 + c_1}. \quad (-\infty < y < \infty) ,$$

and it is evident that the image of a point (x, y) on that branch moves upward along the entire line as (x, y) traces out the branch in the upward direction. Likewise, since the pair of equations

$$u = c_1 , v = -2y\sqrt{y^2 + c_1}. \quad (-\infty < y < \infty) ,$$

furnishes a parametric representation for the image of the left-hand branch of the hyperbola, the image of a point going downward along the entire left-hand branch is seen to move up the entire line $u = c_1$.



On the other hand each branch of a hyperbola $2xy = c_2$,($c_2 > 0$) is transformed into the line $v = c_2$. To verify this, we note from the equation $v = 2xy$ that $v = c_2$ when (*x*, *y*) is a point on either branch. Suppose that (*x*, *y*) is on the branch lying in the first quadrant. Then, since $y = \frac{c_2}{2x}$, the equation $u = x^2 - y^2$ reveals that the branch's image has parametric representation

$$u = x^2 - \frac{c_2^2}{4x^2}, \qquad v = c_2 \qquad (0 < x < \infty)$$

Observe that $\lim_{\substack{x \to 0 \\ x > 0}} u = -\infty$ and $\lim_{x \to \infty} u = \infty$. Since u depends continuously on *x*, then, it is clear that as (*x*, *y*) travels down the entire upper branch of hyperbola $2xy = c_2$,($c_2 > 0$), its image moves to the right along the entire horizontal line $v = c_2$. Inasmuch as the image of the lower branch has parametric representation

$$u = \frac{c_2^2}{4x^2} - y^2, \qquad v = c_2 \qquad (-\infty < x < 0),$$

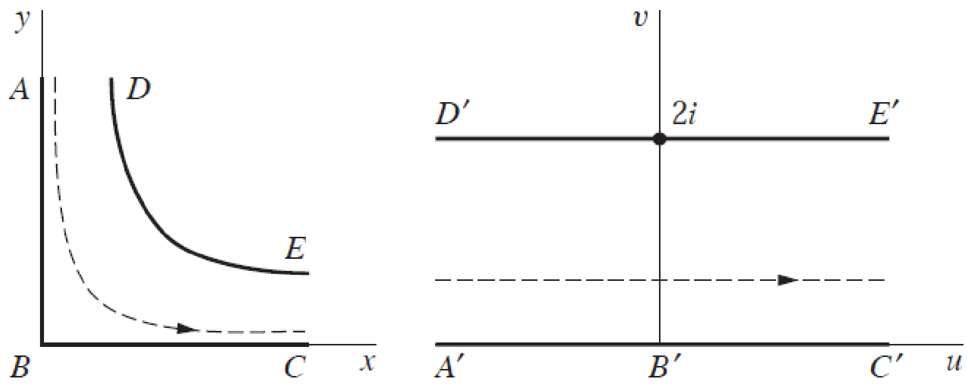
and since $\lim_{y \to -\infty} u = -\infty$ and $\lim_{\substack{y \to 0 \\ y < 0}} u = \infty$. It follows that the image of a point moving upward along the entire lower branch also travels to the right along the entire line $v = c_2$.

2.15 Remark:

We shall now use Example 2.14 to find the image of a certain region.

2.16 Example:

The domain *x* > 0, *y* > 0, *xy* < 1 consists of all points lying on the upper branches of hyperbolas from the family 2*xy* = *c*, where 0 < *c* < 2. We know from Example 2.14 that as a point travels downward along the entirety of such a branch, its image under the transformation $w = z^2$ moves to the right along the entire line *v* = *c*. Since, for all values of *c* between 0 and 2, these upper branches fill out the domain $x > 0, y > 0, xy < 1$ that domain is mapped onto the horizontal strip $0 < v < 2$.



In view of equations $u = x^2 - y^2$ and $v = 2xy$, the image of a point $(0, y)$ in the z - plane is $(-y^2, 0)$. Hence as $(0, y)$ travels downward to the origin along the y - axis, its image moves to the right along the negative u axis and reaches the origin in the w- plane. Then, since the image of a point $(x, 0)$ is $(x^2, 0)$, that image moves to the right from the origin along the u - axis as $(x, 0)$ moves to the right from the origin along the x axis. The image of the upper branch of the hyperbola $xy = 1$ is, of course, the horizontal line $v = 2$. Evidently, then, the closed region $x \geq 0, y \geq 0, xy \leq 1$ is mapped onto the closed strip $0 \leq v \leq 2$.

2.17 Remark:

The next example illustrates how polar coordinates can be useful in analyzing certain mappings.

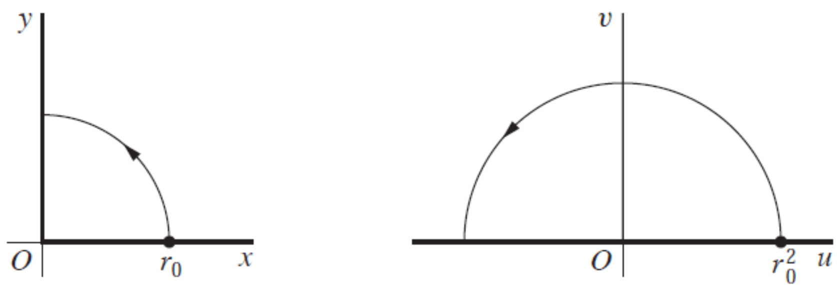
2.18 Example:

The mapping $w = z^2$ becomes $w = r^2 e^{2i\theta}$ when $z = re^{i\theta}$. Evidently, then, the image $w = \rho e^{i\phi}$ of any nonzero point z is found by squaring the modulus $r = |z|$ and doubling the value θ of $arg\ z$ that is used:

$$\rho = r^2 \text{ and } \phi = 2\theta.$$

Observe that points $z = r_0 e^{i\theta}$ on a circle $r = r_0$ are transformed into points

$w = r_0^2 e^{2i\theta}$ on the circle $\rho = r_0^2$. As a point on the first circle moves counterclockwise from the positive real axis to the positive imaginary axis, its image on the second circle moves counterclockwise from the positive real axis to the negative real axis. So, as all possible positive values of r_0 are chosen, the corresponding arcs in the z and w planes fill out the first quadrant and the upper half plane, respectively. The transformation $w = z^2$ is, then, a one to one mapping of the first quadrant $r \geq 0$, $0 \leq \theta \leq \pi/2$ in the z - plane onto the upper half $\rho \geq 0$, $0 \leq \varphi \leq \pi$ of the w- plane. The point $z = 0$ is mapped onto the point $w = 0$.



The transformation $w = z^2$ also maps the upper half plane $r \geq 0$, $0 \leq \theta \leq \pi$ onto the entire w plane. However, in this case, the transformation is not one to one since both the positive and negative real axes in the z - plane are mapped onto the positive real axis in the w- plane.

2.19 Example:

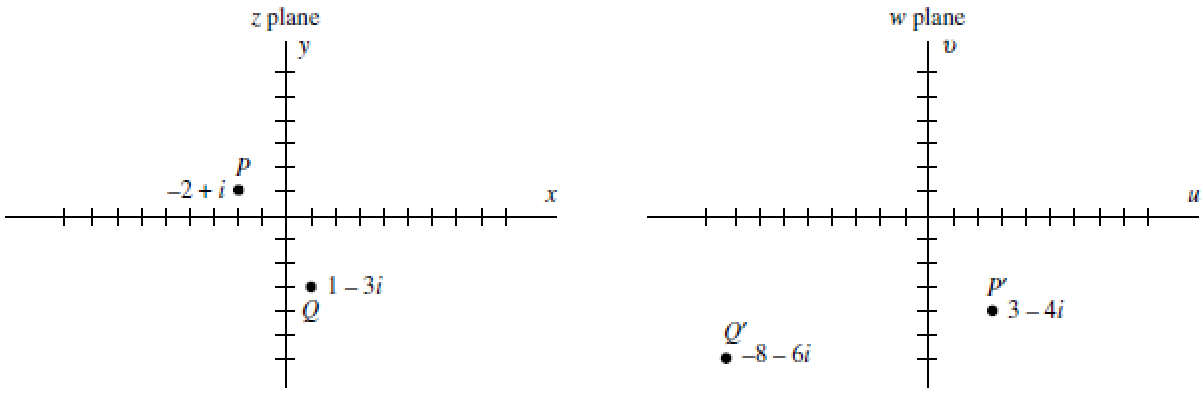
Let $w = f(z) = z^2$. Find the values of w that correspond to (a) $z = -2 + i$ and (b) $z = 1 - 3i$, and show how the correspondence can be represented graphically.

Solution:

(a) $w = f(-2 + i) = (-2 + i)^2 = 4 - 4i + i^2 = 3 - 4i.$

(b) $w = f(1 - 3i) = (1 - 3i)^2 = 1 - 6i + 9i^2 = -8 - 6i.$

The point $z = -2 + i$, represented by point P in the z- plane has the image point $w = 3 - 4i$ represented by P' in the w- plane. We say that P is mapped into P' by means of the mapping function or transformation $w = z^2$. Similarly, $z = 1 - 3i$ [point Q] is mapped into $w =$ [point Q']. To each point in the z- plane, there corresponds one and only one point (image) in the w plane, so that w is a single-valued function of z.



2.20 Example:

Show that the line joining the points P and Q in the z- plane of example 2.19 is mapped by $w = z^2$ into curve joining points $P'Q'$ and determine the equation of this curve.

Solution:

Points P and Q have coordinates (-2,1) and (1, -3). Then, the parametric equations of the line joining these points are given by

$$\frac{y-1}{x-(-2)} = \frac{-3-1}{1-(-2)} = t \quad \text{or} \quad \frac{x-(-2)}{1-(-2)} = \frac{y-1}{-3-1} = t \quad \text{or } x = 3t - 2\,, y = 1 - 4t$$

The equation of the line PQ can be represented by $z = 3t - 2 + (1 - 4t)i$. The curve in the w- plane into which this line is mapped has the equation

$$w = z^2 = (3t - 2 + (1 - 4t)i)^2 = (3t - 2)^2 - (1 - 4t)^2 + 2(3t - 2)(1 - 4t)i$$

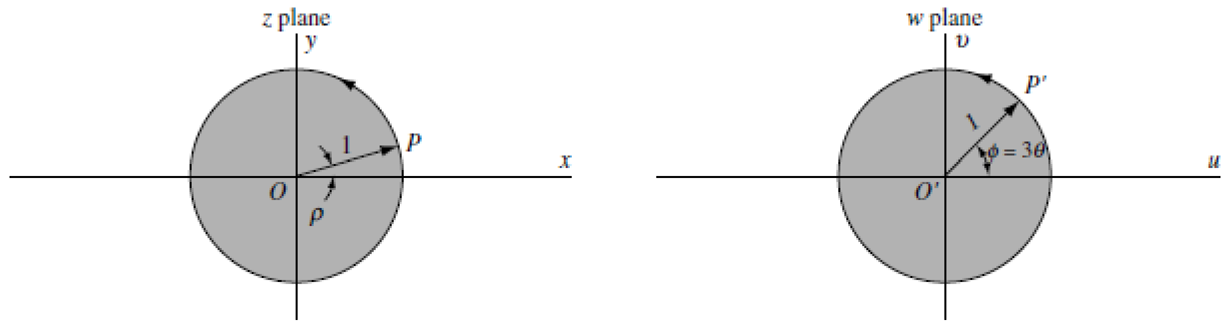
$$= 3 - 4t - 7t^2 + (-4 + 22t - 24t^2)i.$$

Then, since $w = u + iv$, the parametric equations of the image curve are given by $u = 3 - 4t - 7t^2$, $v = -4 + 22t - 24t^2$. By assigning various values to the parameter t, this curve may be graphed.

2.21 Example:

A point P moves in a counterclockwise direction around a circle in the z - plane having center at the origin and radius 1. If the mapping function is $w = z^3$, show that when P makes one complete revolution, the image P' of P in the w- plane makes three complete revolutions in a counterclockwise direction on a circle having center at the origin and radius 1.

Solution:



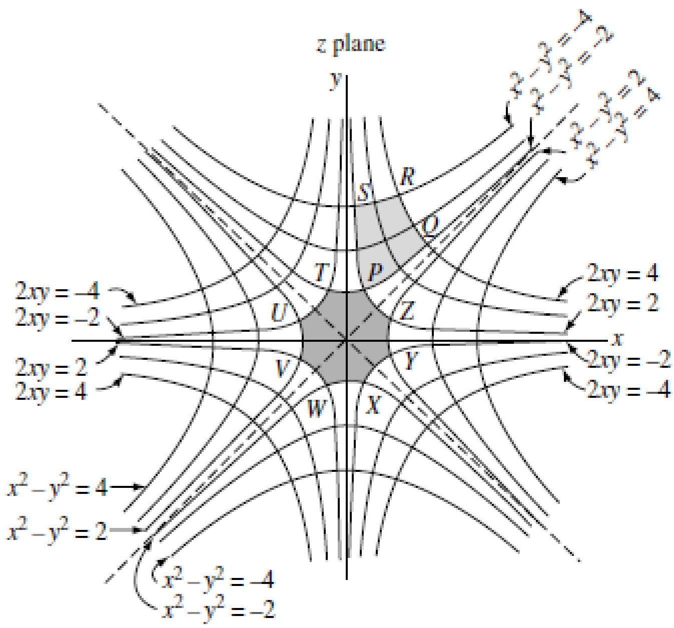
Let $z = re^{i\theta}$. Then, on the circle |z|=1, $r = 1$ and $z = re^{i\theta}$. Hence, $w = z^3 = \left(e^{i\theta}\right)^3 = e^{3i\theta}$. Letting$(\rho,\phi)$ denote polar coordinates in the w- plane, we have $w = \rho e^{i\phi} = e^{3i\theta}$. so that $\rho = 1$, $\phi = 3\theta$.

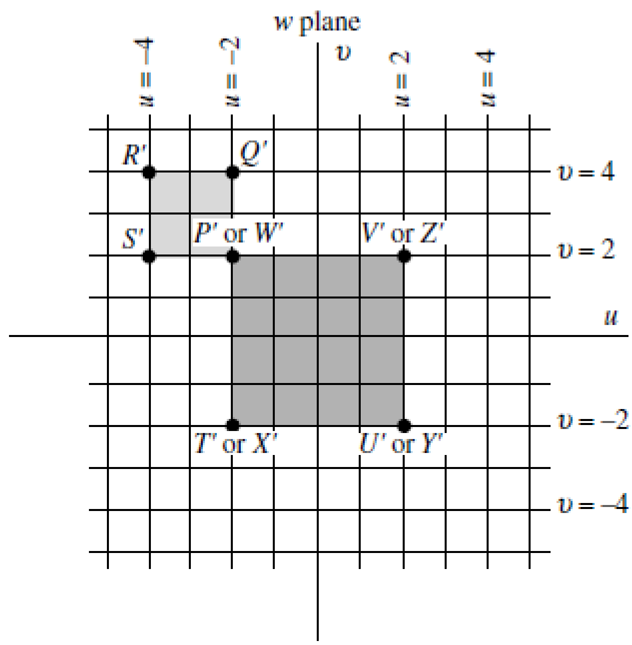
Since $\rho = 1$, it follows that the image point P' moves on a circle in the w- plane of radius 1 and center at the origin. Also, when P moves counterclockwise through an angle θ, P' moves counterclockwise through an angle 3θ. Thus, when P makes one complete revolution, P' makes three complete revolutions. In terms of vectors, it means that vector $O'P'$ is rotating three times as fast as vector OP.

2.22 Example:

In example 2.14 suppose that $c_1 = 2,4,-2,-4$ and $c_2 = 2,4,-2,-4$. Determine the set of all points in the z- plane that map into the lines (a) $u = c_1$, (b) $v = c_2$ in the w plane by means of the mapping function $w = z^2$.

Since $u = x^2 - y^2$ and $v = 2xy$ then lines $u = c_1$ and $v = c_2$ in the w - plane correspond respectively to hyperbolas $x^2 - y^2 = c_1$ and $2xy = c_2$ in the z –plane.





2.23 Example:

Referring to example 2.22, determine:

(a) The image of the region in the first quadrant bounded by $x^2 - y^2 = -2$, $xy = 1$, $x^2 - y^2 = -4$ and $xy = 2$ the image of the region in the z plane,

(b) The image of the region in the z- plane bounded by all the branches of $x^2 - y^2 = 2$, $xy = 1$, $x^2 - y^2 = -2$ and $xy = -1$,

(c) The curvilinear coordinates of that point in the xy - plane whose rectangular coordinates are (2,-1).

Solution:

(a) The region in the z- plane is indicated by the shaded portion $PQRS$. This region maps into the required image region $P'Q'R'S'$. It should be noted that curve PQRSP is traversed in a counterclockwise direction and the image curve $P'Q'R'S'$ is also traversed in a counterclockwise direction.

(b) The region in the z- plane is indicated by the shaded portion *PTUVWXYZ*. This region maps into the required image region *P'T'U'V'*.It is of interest to note that when the boundary of the region *PTUVWXYZ* is traversed only once, the boundary of the image region *P'T'U'V'* is traversed twice. This is due to the fact that the eight points P and W, T and X, U and Y, V and Z of the z- plane map into the four points *P'* or *W'*, *T'* or *X'*, *U'* or *Y'*, *V'* or *Z'*, respectively. However, when the boundary of region PQRS is traversed only once, the boundary of the image region is also traversed only once. The difference is due to the fact that in traversing the curve *PTUVWXYZP*, we are encircling the origin $z = 0$, whereas when we are traversing the curve PQRSP, we are not encircling the origin.

(c) $u = x^2 - y^2 = 2^2 - (-1)^2 = 3$, $v = 2xy = 2(2)(-1) = -4$.Then the curvilinear coordinates are $u = 3, v = -4$.

2.23 Example:

Let $w^5 = z$ and suppose that corresponding to the particular value $z = z_1$, we have $w = w_1$.

(a) If we start at the point z_1 in the z -plane and make one complete circuit counterclockwise around the origin, show that the value of w on returning to $z_1$is $w_1 e^{\frac{2\pi i}{5}}$.

(b) What are the values of w on returning to z_1, after 2, 3, . . . complete circuits around the origin?

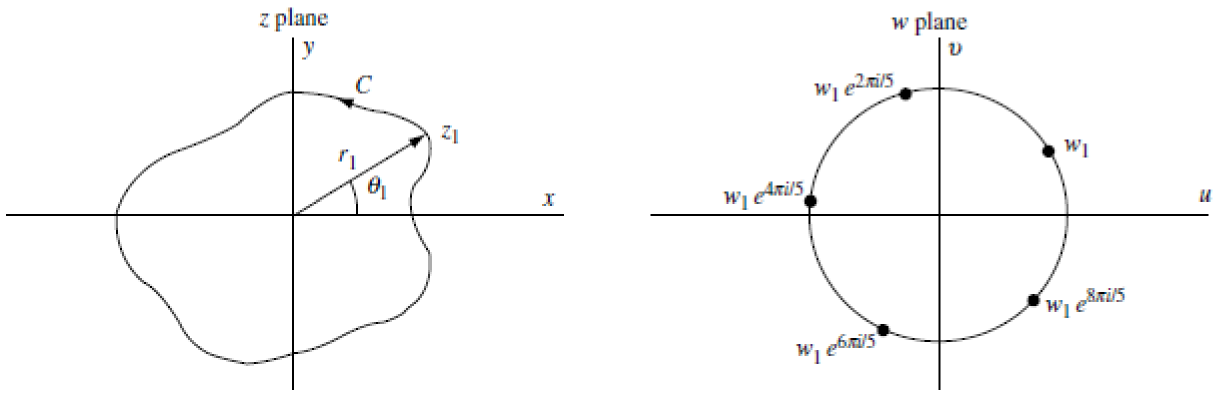
(c) Discuss parts (a) and (b) if the paths do not enclose the origin.

Solution:

(a) We have $z = re^{i\theta}$, so that $w = z^{\frac{1}{5}} = r^{\frac{1}{5}} e^{\frac{i\theta}{5}}$. If $r = r_1$ and $\theta = \theta_1$, then $w = r_1^{\frac{1}{5}} e^{\frac{i\theta_1}{5}}$.

As u increases from θ_1 to $\theta_1 + 2\pi$, which is what happens when one complete circuit counterclockwise around the origin is made, we find

$$w = r_1^{\frac{1}{5}} e^{\frac{i(\theta_1+2\pi)}{5}} = r_1^{\frac{1}{5}} e^{\frac{i\theta_1}{5}} e^{\frac{2\pi i}{5}} = w_1 e^{\frac{2\pi i}{5}}.$$



(b) After two complete circuits around the origin, we find

$$w = r_1^{\frac{1}{5}} e^{\frac{i(\theta_1+4\pi)}{5}} = r_1^{\frac{1}{5}} e^{\frac{i\theta_1}{5}} e^{\frac{4\pi i}{5}} = w_1 e^{\frac{4\pi i}{5}}.$$

Similarly, after three and four complete circuits around the origin, we find $w = w_1 e^{\frac{6\pi i}{5}}$ and $w = w_1 e^{\frac{8\pi i}{5}}$. After five complete circuits, the value of w is $w_1 e^{\frac{10\pi i}{5}} = w_1$, so that the original value of w is obtained after five revolutions about the origin. Thereafter, the cycle is repeated.

(c) If the path does not enclose the origin, then the increase in arg z is zero and so the increase in arg w is also zero. In this case, the value of w is w_1, regardless of the number of circuits made.

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Chapter Two

Analytic Functions

2.24 Remark:

We shall introduce and develop properties of ***exponential function***

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y), (z = x + iy),$$

where e is the natural base of logarithms and the two factors e^x and e^{iy} being well defined at this time.

2.25 Example:

Prove that

a) $e^{z_1}e^{z_2} = e^{z_1+z_2}$, b) $|e^z| = e^x$, c) $e^{z+2\pi ki} = e^z$, $k = 0, \pm 1, \pm 2, \cdots$.

Solution:

a) By definition $e^z = e^x(\cos y + i\sin)$ where $z = x + iy$ then if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$e^{z_1}e^{z_2} = e^{x_1}(\cos y_1 + i\sin y_1)e^{x_2}(\cos y_2 + i\sin y_2)$$
$$= e^{x_1}e^{x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)$$
$$= e^{x_1}e^{x_2}(\cos(y_1 + y_2) + i\sin(y_1 + y_2)) = e^{z_1+z_2}$$

b) $|e^z| = |e^x(\cos y + i\sin)| = |e^x||\cos y + i\sin| = e^x.1 = e^x$.

c) By part a) $e^{z+2\pi ki} = e^z\, e^{2\pi ki} = e^z(\cos 2\pi k + i\sin 2\pi k) = e^z$

This shows that the function e^z has period $2\pi ki$. In particular, it has period $2\pi i$.

2.26 Example:

Find numbers $z = x + iy$ such that $e^z = 1 + i$.

Solution:

Since $e^z = e^x e^{iy}$ and $\sqrt{2}e^{i\frac{\pi}{4}}$ then $e^z = 1 + i$ imply $e^x e^{iy} = \sqrt{2}e^{i\frac{\pi}{4}}$, i.e.

$$e^x = \sqrt{2} \quad \text{and } y = \frac{\pi}{4} + 2\pi k,\ k = 0, \pm 1, \pm 2, \cdots.$$

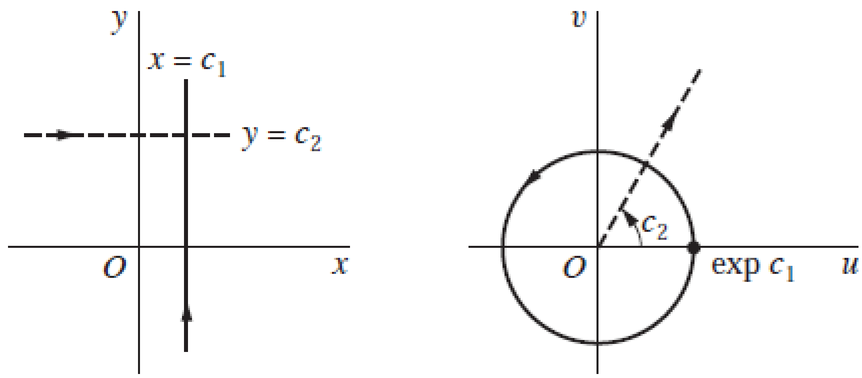
Since $\ln e^x = x$ then $x = \ln\sqrt{2} = \frac{1}{2}\ln 2$ and $y = (\frac{1}{4} + 2k)\pi$, $k = 0, \pm 1, \pm 2, \cdots$. So $z = \frac{1}{2}\ln 2 + (\frac{1}{4} + 2k)\pi$, $k = 0, \pm 1, \pm 2, \cdots$.

2.27 Example:

The transformation $w = e^z$ can be written $w = e^x e^{iy}$, where $z = x + iy$. Thus, if $w = \rho e^{i\emptyset}$, transformation $w = e^z$ can be expressed in the form $\rho = e^x$, $\emptyset = y$.

The image of a typical point $z = (c_1, y)$ on a vertical line $x = c_1$ has polar coordinates $\rho = e^{c_1}$ and $\emptyset = y$ in the w - plane. That image moves counterclockwise around the circle as z moves up the line. The image of the line is evidently the entire circle; and each point on the circle is the image of an infinite number of points, spaced 2π units apart, along the line

A horizontal line $y = c_2$ is mapped in a one to one manner onto the ray $\emptyset = c_2$. Since we note that the image of a point $z = (x, c_2)$ has polar coordinates $\rho = e^x$ and $\emptyset = c_2$. Consequently, as that point z moves along the entire line from left to right, its image moves outward along the entire ray $\emptyset = c_2$.



2.28 Remark:

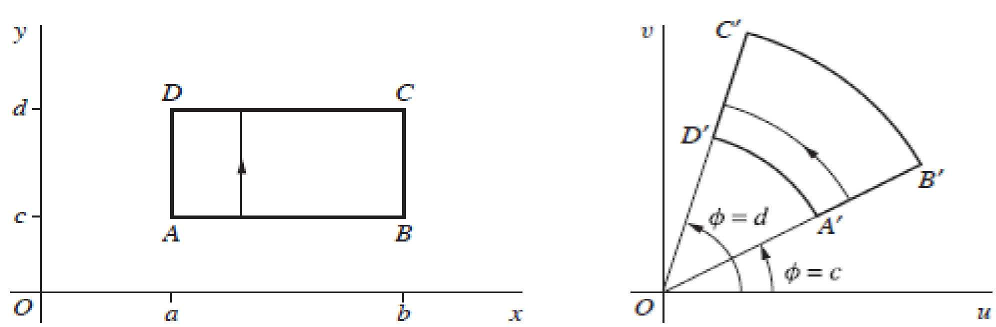
Vertical and horizontal line segments are mapped onto portions of circles and rays, respectively, and images of various regions are readily obtained from observations made in example 2.27. This is illustrated in the following example.

2.29 Example:

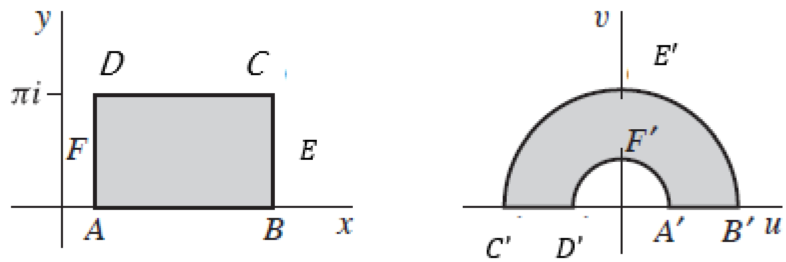
Given $w = e^z$ show that it transform the rectangular region $a \leq x \leq b, c \leq y \leq d$ onto the region $e^a \leq \rho \leq e^b, c \leq \emptyset \leq d$.

Solution:

The vertical line segment AD is mapped onto the arc $\rho = e^a, c \leq \emptyset \leq d$, which is labeled $A'D'$. The images of vertical line segments to the right of AD and joining the horizontal parts of the boundary are larger arcs; eventually, the image of the line segment BC is the arc $\rho = e^b, c \leq \emptyset \leq d$, labeled $B'C'$.



The mapping is one to one if $d - c < 2\pi$. In particular, if $c = 0$ and $d = \pi$, then $0 \leq \emptyset \leq \pi$; and the rectangular region is mapped onto half of a circular ring,



2.30 Remark:

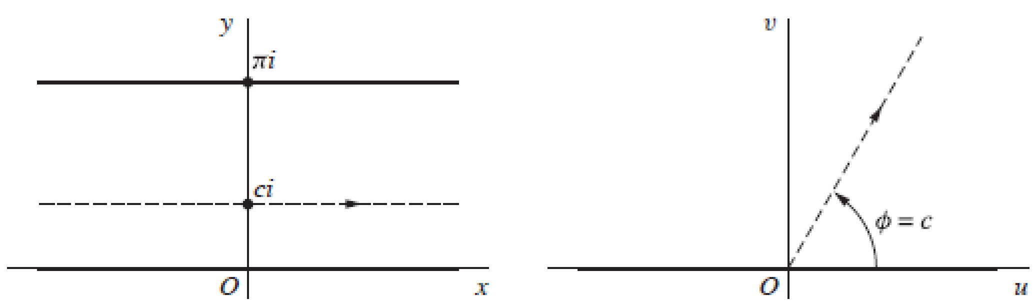
The next example uses the images of horizontal lines to find the image of a horizontal strip.

2.31 Example:

Given $w = e^z$ show that the image of the infinite strip $0 \leq y \leq \pi$ is the upper half $v \geq 0$ of the w- plane.

Solution:

From example 2.27 we saw how a horizontal line $y = c$ is transformed into a ray $\emptyset = c$ from the origin. As the real number c increases from $c = 0$ to $c = \pi$, the y intercepts of the lines increase from 0 to π and the angles of inclination of the rays increase from $\emptyset = 0$ to $\emptyset = \pi$.



EXERCISES:

1. Sketch the region onto which the sector $r \leq 1$, $0 \leq \theta \leq \pi/4$ is mapped by the transformation
   a) $w = z^2$, b) $w = z^3$, c) $w = z^4$.
2. Show that the lines $ay = x$ $(a \neq 0)$ are mapped onto the spirals $\rho = e^{a\emptyset}$ under the transformation $w = e^z$, where $w = \rho\, e^{i\emptyset}$.

3. By considering the images of horizontal line segments, verify that the image of the rectangular region $a \le x \le b,\ c \le y \le d$ under the transformation $w = e^z$ is the region $e^a \le \rho \le e^b, c \le \emptyset \le d$.
4. Find the image of the semi-infinite strip $x \ge 0, 0 \le y \le \pi$ under the transformation $w = e^z$, and label corresponding portions of the boundaries.
5. Show that

   a) $e^{(2\pm3\pi i)} = -e^2$, b) $e^{\left(\frac{2+\pi i}{4}\right)} = \sqrt{\frac{e}{2}}(1+i)$, c) $e^{(z+\pi i)} = -e^z$.

6. Write $\left|e^{2z+i}\right|$ and $\left|e^{iz^2}\right|$ in terms of x and y. Then show that

$$\left|e^{2z+i} + e^{iz^2}\right| \le e^{2x} + e^{-2xy}$$

7. Show that $\left|e^{z^2}\right| \le e^{|z|^2}$.
8. Prove that $|e^{-2z}| < 1$ if and only if $Re(z) > 0$.
9. Find all values of z such that

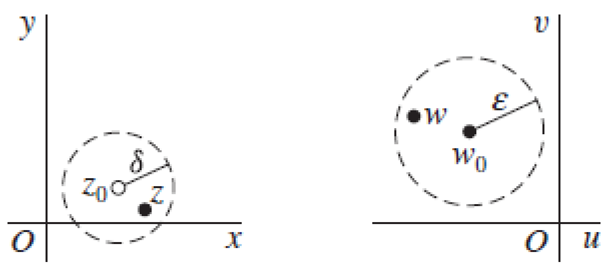
   a) $e^z = -2$, b) $e^z = 1 + \sqrt{3}i$, c) $e^{2z-1} = 1$.

10. Sow that $\overline{e^{iz}} = e^{i\bar{z}}$ if and only if $z = k\pi, k = 0, \pm1, \pm2, \cdots$.
11. a) Show that if e^z is real then $Im\ z = k\pi, k = 0, \pm1, \pm2, \cdots$.

    b) If e^z is pure imaginary, what restriction is placed on z?
12. Prove that there cannot be any finite values of z such that $e^z = 0$.

2.32 Definition:

Let f (z) be defined and single-valued in a neighborhood of $z = z_0$ with the possible exception of $z = z_0$ itself (i.e., in a deleted d neighborhood of z_0). We say that the number ***w_0 is the limit of f (z) as z approaches z_0*** and write $\lim_{z \to z_0} f(z) = w_0$ if for any positive number ϵ (however small), we can find some positive number δ (usually depending on ϵ) such that $|f(z) - w_0| < \epsilon$ whenever $|z - z_0| < \delta$.



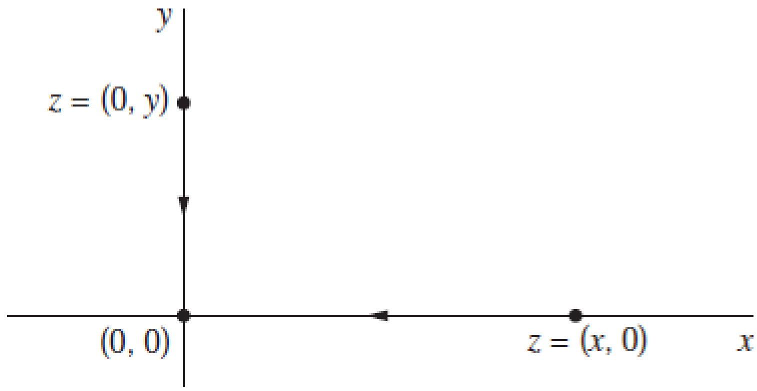
2.33 Remark:

When a limit of a function f (z) exists at a point z_0, it is unique.

2.34 Example:

If $f(z) = \frac{z}{\bar{z}}$ the limit $\lim_{z\to 0} f(z)$ does not exist.

Solution:



Assume it did exist, it could be found by letting the point $z = (x, y)$ approach the origin in any manner.

But when $z = (x, 0)$ is a nonzero point on the real axis

$$f(z) = \frac{x+0i}{x-0i} = 1;$$

and when $z = (0, y)$ is a nonzero point on the imaginary axis,

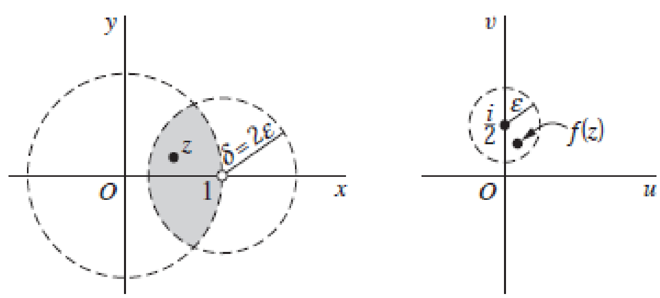
$$f(z) = \frac{0+iy}{x-iy} = -1.$$

Thus, by letting z approach the origin along the real axis, we would find that the desired limit is 1. An approach along the imaginary axis would, on the other hand, yield the limit -1. Since a limit is unique, we must conclude that limit $\lim_{z\to 0} f(z)$ does not exist.

2.35 Example:

Show that if $f(z) = \frac{i\bar{z}}{2}$ in the open disk $|z| < 1$, then $\lim_{z\to 1} f(z) = \frac{i}{2}$.

Solution:



The point 1 being on the boundary of the domain of definition of f. Observe that when z is in the disk $|z| < 1$,

$$\left|f(z) - \frac{i}{2}\right| = \left|\frac{i\bar{z}}{2} - \frac{i}{2}\right| = \left|\frac{i(\bar{z}-1)}{2}\right| = \frac{|i||\bar{z}-1|}{2} = \frac{|z-1|}{2}$$

Hence, for any such z and each positive number ϵ, $\left|f(z) - \frac{i}{2}\right| < \epsilon$ whenever $0 < |z-1| < 2\epsilon$. Thus it satisfied by points in the region $|z| < 1$ when δ is equal to 2ϵ or any smaller positive number.

2.36 Example:

a) Suppose $f(z) = z^2$. Prove that $\lim_{z\to z_0} f(z) = z_0^2$.

b) Find $\lim_{z\to z_0} f(z)$ if $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$.

Solution:

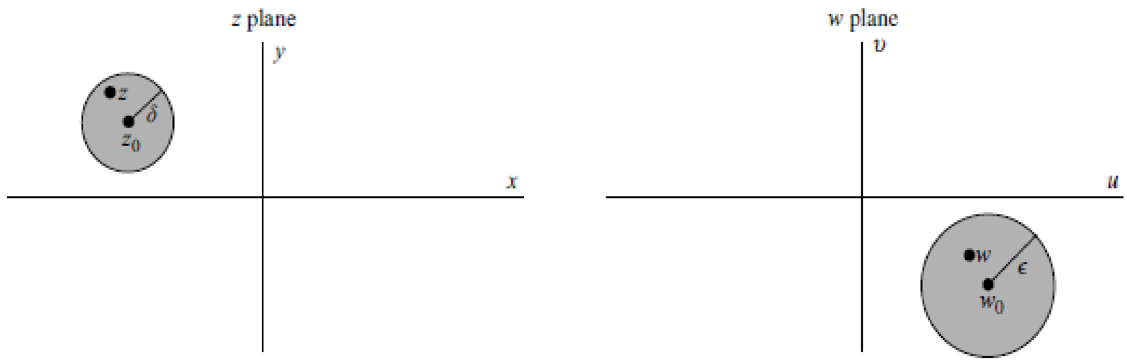
a) We must show that, given any $\epsilon > 0$, we can find δ (depending in general on ϵ) such that $|z^2 - z_0^2| < \epsilon$ whenever $0 < |z - z_0| < \delta$.

If $\delta \leq 1$ then $0 < |z - z_0| < \delta$ implies that

$|z^2 - z_0^2| = |z - z_0||z + z_0| < \delta|z - z_0 + 2z_0| < \delta(|z - z_0| + |2z_0|) < \delta(1 + 2|z_0|)$.

Take $\delta = \min\{1, \frac{\epsilon}{1+2|z_0|}$. Then, we have $|z^2 - z_0^2| < \epsilon$ whenever $|z - z_0| < \delta$, and the required result is proved.

b) There is no difference between this problem and that in part (a), since in both cases we exclude $z = z_0$ from consideration. Hence, $\lim\limits_{z \to z_0} f(z) = f(z_0)$. Note that the limit of $f(z)$ as $z \to z_0$ has nothing whatsoever to do with the value of $f(z)$ at z_0.



2.37 Example:

Prove that $\lim\limits_{z \to i} \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = 4 + 4i$.

Solution:

We must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left|\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} - (4 + 4i)\right| < \epsilon, \text{ when } 0 < |z - z_0| < \delta.$$

Since $z \neq i$, we can write

$$\frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} = \frac{3z^4 - 2z^3 + 5z^2 + 3z^2 - 2z + 5}{z - i}$$

$$= \frac{3z^4 - 2z^3 + 3iz^3 + -3iz^3 + 5z^2 - 2iz^2 + 2iz^2 - 3i^2z^2 + 5iz - 5iz + 2i^2z - 5i^2}{z - i}$$

$$= \frac{3z^4 - (2 - 3i)z^3 + (5 - 2i)z^2 + 5iz - 3iz^3 + (2 - 3i)iz^2 - (5 - 2i)iz - 5i^2}{z - i}$$

$$= \frac{(3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i)z - (3z^3 - (2 - 3i)z^2 + (5 - 2i)z - 5i)i}{z - i}$$

$$= \frac{(3z^3-(2-3i)z^2+(5-2i)z+5i)(z-i)}{z-i} = (3z^3-(2-3i)z^2+(5-2i)z+5i)$$

on cancelling the common factor $z - i \neq 0$. Then, we must show that for any $\epsilon > 0$, we can find $\delta > 0$ such that

$$\begin{aligned}
|(3z^3-(2-3i)z^2+(5-2i)z+5i)-(4+4i)| &= |3z^3-(2-3i)z^2+(5-2i)z-4+i| \\
&= |3z^3+6iz^2-3iz^2-2z^2+6z-z-4iz+2iz+i+4i^2| \\
&= |3z^3+6iz^2-2z^2-z-4iz-3iz^2+6z+2iz+i+4i^2| \\
&= \left|\left(3z^2+(6i-2)z+(-1-4i)\right)z+(3z^2+(-6i-2)z+(-1-4i))(-i)\right| \\
&= \left|\left(3z^2+(6i-2)z+(-1-4i)\right)(z-i)\right| \\
&= |z-i||3z^2+(6i-2)z+(-1-4i)| \\
&= |z-i|\,|3(z-i+i)^2+(6i-2)(z-i+i)+(-1-4i)| \\
&= |z-i|\,|3((z-i)^2+2i(z-i)-1)+(6i-2)(z-i)+i(6i-2)+(-1-4i)| \\
&= |z-i|\,|3(z-i)^2+6i(z-i)-3+(6i-2)(z-i)-6-2i-1-4i| \\
&= |z-i|\,|3(z-i)^2+(12i-2)(z-i)-10-6i| \\
&< \delta\,(3|z-i|^2+|12i-2||z-i|+|10-6i|) \\
&< \delta\,(3|z-i|^2+13|z-i|+12) \quad \begin{cases} |12i-2| = \sqrt{148} < \sqrt{159} = 13 \\ |10-6i| = \sqrt{136} < \sqrt{144} = 12 \end{cases} \\
&< \delta\,(3+13+12) = 28\delta.
\end{aligned}$$

Take $\delta = \min\{1, \frac{\epsilon}{28}$. the required result follows.

2.38 Theorem:

Suppose that $f(z) = u(x,y) + iv(x,y)$, $(z = x+iy)$ ***and*** $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$. ***Then*** $\lim_{z\to z_0} f(z) = w_0$ ***iff*** $\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0$ ***and*** $\lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0$.

2.39 Theorem:

Suppose that $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$ **. Then**

1. $\lim_{z\to z_0}[f(z)+g(z)] = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z) = A + B.$
2. $\lim_{z\to z_0}[f(z).g(z)] = \lim_{z\to z_0} f(z).\lim_{z\to z_0} g(z) = A.\ B.$
3. $\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{A}{B}.\quad g(z) \neq 0, B \neq 0.$

2.40 Theorem:

If z_0 ***and*** w_0 ***are points in the*** z ***and*** w ***- planes, respectively, then***

1. $\lim_{z\to z_0} f(z) = \infty$ **if and only if** $\lim_{z\to z_0}\frac{1}{f(z)} = 0$ **and**

2. $\lim_{z\to\infty} f(z) = w_0$ **if and only if** $\lim_{z\to 0} f(\frac{1}{z}) = w_0$ **if** $w_0 \neq 0$ **,Moreover;**

3. $\lim_{z\to\infty} f(z) = \infty$ **if and only if** $\lim_{z\to 0} \frac{1}{f\left(\frac{1}{z}\right)} = 0.$

2.41 Remark:

1. In theorem 2.40(1) the first limit means that for each $\epsilon > 0$ there is a positive number $\delta > 0$ such that $|f(z)| > \frac{1}{\epsilon}$ whenever $0 < |z - z_0| < \delta$, i.e. the point $w = f(z)$ lies in the ε neighborhood $|w| > 1/\varepsilon$ of ∞ whenever z lies in the deleted neighborhood $0 < |z - z0| < \delta$ of z_0.Since $|f(z)| > \frac{1}{\epsilon}$ whenever $0 < |z - z_0| < \delta$ then it can be written $\left|\frac{1}{f(z)} - 0\right| < \varepsilon$ whenever $0 < |z - z0| < \delta$ then the second of limits of theorem 2.40(1) follows.
2. In theorem 2.40(2) the first limit means that for each $\epsilon > 0$ there is a positive number $\delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $|z| > \frac{1}{\delta}$. Replacing z by $1/z$ we get $\left|f\left(\frac{1}{z}\right) - w_0\right| < \epsilon$ whenever $0 < |z - 0| < \delta$ then the second of limits of theorem 2.40(2) follows.
3. In theorem 2.40(3) the first limit means that for each $\epsilon > 0$ there is a positive number $\delta > 0$ such that $|f(z)| > \frac{1}{\epsilon}$ whenever $|z| > \frac{1}{\delta}$. Replacing z by $1/z$ we get $\left|\frac{1}{f(z)} - 0\right| < \varepsilon$ whenever $0 < |z - 0| < \delta$ then the second of limits of theorem 2.40(3) follows.

2.42 Example:

Observe that

1. $\lim_{z\to -1} \frac{iz+3}{z+1} = \infty$ since $\lim_{z\to -1} \frac{z+1}{iz+3} = 0.$
2. $\lim_{z\to\infty} \frac{2z+i}{z+1} = 2$ since $\lim_{z\to 0} \frac{2\frac{1}{z}+i}{\frac{1}{z}+1} = \lim_{z\to 0} \frac{2+iz}{1+z} = 2$.
3. $\lim_{z\to\infty} \frac{2z^3-1}{z^2+1} = \infty$ since $\lim_{z\to 0} \frac{\frac{1}{z^2}+1}{2\frac{1}{z^3}-1} = \lim_{z\to 0} \frac{\frac{1+z^2}{z^2}}{\frac{2-z^3}{z^3}} = \lim_{z\to 0} \frac{z+z^3}{2-z^3} = 0.$

2.43 Definition:

A function f is ***continuous*** at a point z_0 if all three of the following conditions are satisfied:

1. $\lim_{z\to z_0} f(z)$ exists,

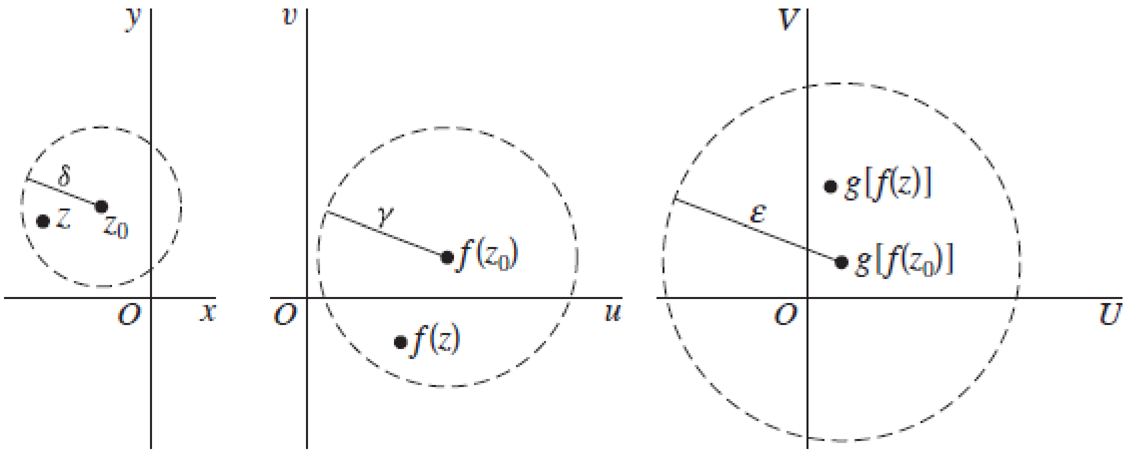
2. $f(z_0)$ exists,
3. $\lim_{z \to z_0} f(z) = f(z_0)$.

2.44 Definition:

An alternative definition of continuity is we say that f (z) as continuous at $z = z_0$ if for any $\epsilon > 0$ there is a positive number $\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < \delta$.

2.45 Theorem:

A composition of continuous functions is itself continuous.



2.46 Theorem:

If a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

2.47 Theorem:

If a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that $|f(z)| \leq M$ for all points z in R.

2.48 Theorem:

If $f(z)$ and $g(z)$ are continuous at $z = z_0$. Then so are the functions $f(z) + g(z)$, $f(z) - g(z)$, $f(z)g(z)$ and $f(z)/g(z)$, the last if $g(z_0) \neq 0$. Similar results hold for continuity in a region.

2.49 Example:

a) Suppose that $f(z) = \begin{cases} z^2 & z \neq i \\ 0 & z = i \end{cases}$ then $\lim_{z \to i} f(z) = i^2 = -1$ but $f(i) = 0$ hence $\lim_{z \to i} f(z) \neq f(i)$, so f is not continuous at $z = i$.

b) Suppose that $f(z) = z^2$for all z then $\lim_{z \to i} f(z) = f(i) = i^2 = -1$ is continuous at $z = i$.

2.50 Example:

Is the function $f\,(z) = \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$ continuous at $z = i$?

Solution:

$f\,(i)$ does not exist, i.e., $f\,(x)$ is not defined at $z = i$. Thus $f(z)$ is not continuous at $z = i$.

2.51 Example:

For what values of z are each of the following function continuous?

Solution:

$f\,(z) = \frac{z}{z^2+1} = \frac{z}{(z-i)(z+i)}$. Since the denominator is zero when $z = \pm i$, the function is continuous everywhere except $z = \pm i$.

EXERCISES:

1. Use definition 2.32 of limit to prove that

   a) $\lim_{z \to z_0} Re(z) = Re(z_0)$, b) $\lim_{z \to z_0} \bar{z} = \overline{z_0}$, c) $\lim_{z \to 0} \frac{\bar{z}^2}{z} = 0$.

2. Show that the limit of the function $f(z) = \left(\frac{z}{\bar{z}}\right)^2$ as z tends to 0 does not exist.

3. Show that

   a) $\lim_{z \to \infty} \frac{4z^2}{(z-1)^2} = 4$; b) $\lim_{z \to 1} \frac{1}{(z-1)^3} = \infty$; c) $\lim_{z \to \infty} \frac{z^2+1}{z-1} = \infty$.

4. Show that when $T(z) = \frac{az+b}{cz+d}$, $(ad - bc) \neq 0$,

   a) $\lim_{z \to \infty} T(z) = \infty$ if $c = 0$.

   b) $\lim_{z \to \infty} T(z) = \frac{a}{c}$ and $\lim_{z \to d/c} T(z) = \infty$ if $c \neq 0$.

5. Find all points of discontinuity for the following functions.

   (a) $f(z) = \frac{2z - 3}{z^2 + 2z + 2}$, (b) $f(z) = \frac{3z^2 + 4}{z^4 - 16}$

6. Prove that $f\,(z) = \frac{z^2+1}{z^3+9}$ is (a) continuous and (b) bounded in the region $|z| \leq 2$.

7. Show that $f\,(z) = \frac{z^2+1}{z^2-3z+2}$ is continuous for all z outside $|z| = 2$.

8. Prove that $f\,(z) = \frac{1}{z}$ is continuous for all z such that $|z| > 1$, but that it is not bounded.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ٥

Chapter Two

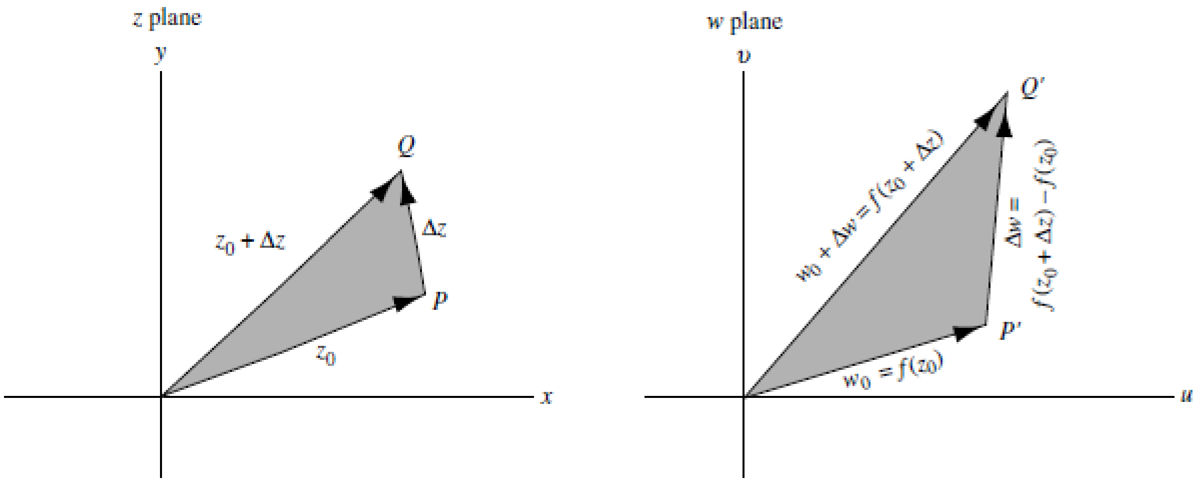
Analytic Functions

2.52 Definition:

Let $f\ (z)$ be a single-valued in some region R of the z - plane, the ***derivative*** of $f\ (z)$ at z_0 is defined as

$$\frac{d}{dz} f(z_0) = \left.\frac{df}{dz}\right|_{z_0} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$

and the function f is said to be ***differentiable at z*** when $f'(z)$ exists.



2.53 Remark:

1. If the derivative $f'(z_0)$ exists at all points z_0 of a region R, then $f\ (z)$ is said to Be ***analytic*** in R and is referred to as an ***analytic function*** in R or a function analytic in R.
2. A function $f\ (z)$ is said to be analytic at a point z_0 if there exists a neighborhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

2.54 Example:

Using the definition of derivative and find the derivative of $f(z) = z^3 - 2z$ at the point where

a) $z = z_0$, **b)** $z = -1$.

Solution:

a) By definition, the derivative at $z = z_0$ is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^3 - 2(z_0+\Delta z) - ({z_0}^3 - 2z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - {z_0}^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{3z_0^2 \Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2)\Delta z}{\Delta z}$$

$$= \lim_{\Delta z \to 0} 3z_0^2 + 3z_0 \Delta z + (\Delta z)^2 - 2 = 3z_0^2 - 2.$$

In general, $f'(z_0) = 3z_0^2 - 2$ for all z.

b) From a) if $z_0 = -1$ then $f'(-1) = 3(-1)^2 - 2 = 1$.

2.55 Example:

Using the definition of derivative and find the derivative of $f(z) = z^2$.

Solution:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0 \Delta z + (\Delta z)^2 - {z_0}^2}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(2z_0 + \Delta z)\Delta z}{\Delta z} = \lim_{\Delta z \to 0} 2z_0 + \Delta z = 2z_0.$$

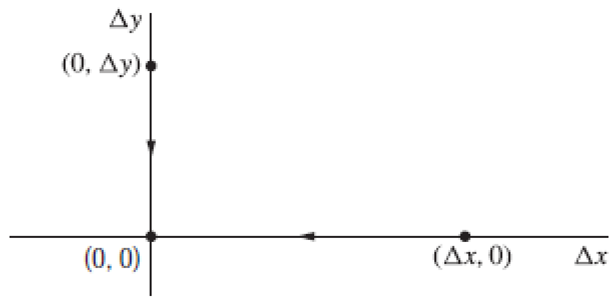
2.56 Example:

Show that $\frac{d}{dz}\bar{z}$ does not exist anywhere, i.e., $f(z) = \bar{z}$ is non-analytic anywhere.

Solution

By definition

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$



if this limit exists independent of the manner in which $\Delta z = \Delta x + i\Delta y$ approaches zero. Then

$$f'(z) = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$. If $\Delta x = 0$, the required limit is $\lim_{\Delta y \to 0} \frac{-\Delta y}{\Delta y} = -1$. Then, since the limit depends on the manner in which $\Delta z \to 0$, the derivative does not exist, i.e., $f(z) = \bar{z}$ is non-analytic anywhere.

2.57 Example:

Given $w = f(z) = \frac{(1+z)}{(1-z)}$, find (a) $\frac{dw}{dz}$ and (b) determine where $f(z)$ is non-analytic.

Solution:

(a) By definition

$$\frac{dw}{dx} = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{\frac{1+(z+\Delta z)}{1-(z+\Delta z)}-\frac{(1+z)}{(1-z)}}{\Delta z} = \lim_{\Delta z\to 0}\frac{\frac{(1-z)(1+z+\Delta z)-(1+z)(1-z-\Delta z)}{(1-(z+\Delta z))(1-z)}}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{\frac{(1+z+\Delta z-z-z^2-z\Delta z-(1-z-\Delta z+z-z^2-z\Delta z)}{(1-(z+\Delta z))(1-z)}}{\Delta z} = \lim_{\Delta z\to 0}\frac{\frac{1+z+\Delta z-z-z^2-z\Delta z-1+z+\Delta z-z+z^2+z\Delta z}{(1-(z+\Delta z))(1-z)}}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{\frac{2\Delta z}{(1-(z+\Delta z))(1-z)}}{\Delta z} = \lim_{\Delta z\to 0}\frac{2}{(1-(z+\Delta z))(1-z)} = \frac{2}{(1-z)(1-z)} = \frac{2}{(1-z)^2}.$$

independent of the manner in which $\Delta z \to 0$, provided $z \neq 1$.

(b) The function $f\,(z)$ is analytic for all finite values of z except $z = 1$ where the derivative does not exist and the function is non-analytic. The point $z = 1$ is a singular point of $f\,(z)$.

2.58 Example:

(a) If f (z) is analytic at z_0, prove that it must be continuous at z_0.

(b) Give an example to show that the converse of (a) is not necessarily true.

Solution:

(a) Since $f(z_0+h)-f(z_0) = \frac{f(z_0+h)-f(z_0)}{h}.h$ where $h = \Delta z \neq 0$, we have

$$\lim_{h\to 0} f(z_0+h)-f(z_0) = \lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}.\lim_{h\to 0} h = f'(z_0).0 = 0.$$

because $f'(z_0)$ exists by hypothesis. Thus

$$\lim_{h\to 0} f(z_0+h)-f(z_0) = 0 \text{ or } \lim_{h\to 0} f(z_0+h) = f(z_0).$$

Showing that $f\,(z)$ is continuous at z_0.

(b) The function $f(z) = \bar{z}$ is continuous at z_0. However, by example 2.56, $f\,(z)$ is not analytic anywhere. This shows that a function, which is continuous, need not have a derivative, i.e., need not be analytic.

2.59 Example:

Determine whether $f(z) = |z|^2$ has a derivative anywhere.

Solution:

Since

$$f'(z) = \lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = \lim_{\Delta z\to 0}\frac{|z+\Delta z|^2-|z|^2}{\Delta z} = \lim_{\Delta z\to 0}\frac{(z+\Delta z)(\overline{z+\Delta z})-z\bar{z}}{\Delta z} = \lim_{\Delta z\to 0}\frac{(z+\Delta z)(\bar{z}+\overline{\Delta z})-z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z\to 0}\frac{z\bar{z}+z\overline{\Delta z}+\bar{z}\Delta z+\Delta z\overline{\Delta z}-z\bar{z}}{\Delta z} = \lim_{\Delta z\to 0}\frac{z\overline{\Delta z}+\bar{z}\Delta z+\Delta z\overline{\Delta z}}{\Delta z} = \lim_{\Delta z\to 0}\bar{z}+\overline{\Delta z}+z\frac{\overline{\Delta z}}{\Delta z}$$

$$= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\overline{x+\imath y}+\overline{\Delta x+\imath\Delta y}+(x+iy)\frac{\overline{\Delta x+\imath\Delta y}}{\Delta x+i\Delta y} = \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} x-iy+\Delta x-i\Delta y+(x+iy)\frac{\Delta x-i\Delta y}{\Delta x+i\Delta y}.$$

If $\Delta y = 0$, the required limit is $\lim_{\Delta x\to 0} x + \Delta x + x\frac{\Delta x}{\Delta x} = \lim_{\Delta x\to 0} x + \Delta x + x = 2x$. If $\Delta x = 0$, the required limit is $\lim_{\Delta y\to 0} -iy - i\Delta y + iy\frac{-i\Delta y}{+i\Delta y} = \lim_{\Delta y\to 0} -iy - i\Delta y - iy = -2iy$. Then, $2x = -2iy$ only when $x = y = 0$. So $f'(z)$ exist when $z = 0$ its value there being 0 and does not exist when $z \neq 0$.

2.60 Rules for Differentiation:

Suppose $f(z)$, $g(z)$, and $h(z)$ are analytic functions of z. Then the following differentiation rules are valid:

1. $\frac{d}{dz}c = 0$, c is any constant.
2. $\frac{d}{dz}cf(z) = c\frac{d}{dz}f(z) = cf'(z)$, c is any constant .
3. $\frac{d}{dz}z^n = nz^{n-1}$, n is positive integer.
4. $\frac{d}{dz}\big(f(z) \pm g(z)\big) = \frac{d}{dz}f(z) \pm \frac{d}{dz}g(z) = f'(z) \pm g'(z)$.
5. $\frac{d}{dz}\big(f(z)\cdot d(z)\big) = f(z)\frac{d}{dz}g(z) + g(z)\frac{d}{dz}f(z) = f(z)g'(z) + g(z)f'(z)$.
6. $\frac{d}{dz}\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)\frac{d}{dz}f(z) - f(z)\frac{d}{dz}g(z)}{\big(g(z)\big)^2} = \frac{g(z)f'(z) - f(z)g'(z)}{\big(g(z)\big)^2}$ if $g(z) \neq 0$.
7. If $w = f(w_0)$ where $w_0 = g(z)$ then

$$\frac{dw}{dz} = \frac{dw}{dw_0}\cdot\frac{dw_0}{dz} = f'(w_0)\cdot g'(z) = f'(g(z))\cdot g'(z) \qquad (1)$$

Similarly, if $w = f(w_0)$ where $w_0 = g(w_1)$ and $w_1 = h(z)$, then

$$\frac{dw}{dz} = \frac{dw}{dw_0}\cdot\frac{dw_0}{dw_1}\cdot\frac{dw_1}{dz} = f'(w_0)\cdot g'(w_1)h'(z) = f'(g(h(z)))\cdot g'(h(z))\cdot h'(z) \qquad (2)$$

The results (1) and (2) are often called ***chain rules for differentiation of composite functions***.

8. If $w = f(z)$ has a single-valued inverse f^{-1}, then $z = f^{-1}(w)$, and $\frac{dw}{dz}$ and $\frac{dz}{dw}$ are related by $\frac{dw}{dz} = \frac{1}{dz/dw}$.
9. If $z = f(t)$ and $w = g(t)$ where t is a parameter, then $\frac{dw}{dz} = \frac{dw/dt}{dz/dt} = \frac{g'(t)}{f'(t)}$.

2.61 Example:

Find the derivative of $\frac{d}{dz}(2z^2 + i)^5$.

Solution:

$$\frac{d}{dz}(2z^2 + i)^5 = 5(2z^2 + i)^4(4z) = 20z(2z^2 + i)^4.$$

2.62 Example:

Given $w = f(z) = z^3 - 2z^2$. Find: **(a)** Δw, **(b)** dw, **(c)** $\Delta w - dw$.

Solution:

(a) $\Delta w = f(z+\Delta z) - f(z) = ((z+\Delta z)^3 - 2(z+\Delta z)^2) - (z^3 - 2z^2)$

$= z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2(z^2 + 2z\Delta z + (\Delta z)^2) - z^3 + 2z^2$

$= 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3 - 2z^2 - 4z\Delta z - 2(\Delta z)^2 + 2z^2$

$= (3z^2 - 4z)\Delta z + (3z-2)(\Delta z)^2 + (\Delta z)^3.$

(b) dw =principal part of $\Delta w = (3z^2-4z)\Delta z = (3z^2-4z)dz$, since by definition $\Delta z = dz$. Note that $f'(z) = 3z^2 - 4z$ and $dw = (3z^2-4z)dz$,i.e. $\frac{dw}{dz} = 3z^2 - 4z$.

(b) From (a) and(b), $\Delta w - dw = (3z-2)(\Delta z)^2 + (\Delta z)^3 = \epsilon \Delta z$ where $\epsilon = (3z-2)\Delta z + (\Delta z)^2$. Note that $\epsilon \to 0$ as $\Delta z \to 0$, i.e. $\frac{\Delta w - dw}{\Delta z} \to 0$ as $\Delta z \to 0$. It follows that $\Delta w - dw$ is an infinitesimal of higher order than Δz.

EXERCISES:

1. Using the definition, find the derivative of each function at the indicated points.

a) $f(z) = 3z^2 + 4iz - 5 + i\,; z = 2,$

b) $f(z) = \frac{2z-i}{z+2i}\,; z = -i,$

c) $f(z) = 3z^{-2}\,; z = 1+i.$

2. Apply the definition of derivative to give a direct proof that $\frac{dw}{dz} = -\frac{1}{z^2}$ when $w = \frac{1}{z}$, $(z \neq 0)$.

3. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Use the definition of derivative to show that $\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$.

4. Prove that $\frac{d}{dz}(z^2\bar{z})$ does not exist anywhere.

5. For each of the following functions determine the singular points, i.e., points at which the function is not analytic. Determine the derivatives at all other points.

a) $\frac{z}{z+i}$, **b)** $\frac{3z-2}{z^2+2z+5}$.

6. show that $f'(z)$ does not exist at any point when

a) $f(z) = Re(z)$, **b)** $f(z) = Im(z)$.

7. Let f denote the function whose values are $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$ then $f'(0)$ does not exist.

8. Using differentiation rules, find the derivatives of each of the following functions:

a) $f(z) = (1 + 4i)z^2 - 3z - 2$, **b)** $f(z) = (2z + 3i)(z - i)$,

c) $f(z) = \frac{2z-i}{z+2i}$ $(z \neq -2i)$, **d)** $f(z) = (2iz + 1)^2$,

e) $f(z) = (iz - 1)^{-3}$ **f)** $f(z) = \frac{z-1}{2z+1}$ $(z \neq -\frac{1}{2})$,

g) $f(z) = \frac{(1-z^2)^4}{z^2}$ $(z \neq 0)$.

9. Find the derivatives of each of the following at the indicated points:

a) $f(z) = (z + 2i)(i - z)(2z - 1), z = i$,

b) $f(z) = (z + (z^2 + 1)^2)^2, z = 1 + i$.

2.63 Cauchy–Riemann equations:

A necessary and sufficient condition that $w = f(z) = u(x, y) + iv(x, y)$ be analytic in a region R is that the ***Cauchy–Riemann equations***

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{2}$$

are satisfied in R where it is supposed that these partial derivatives are continuous in R.

A necessary condition to show that $f(z)$ is analytic in a region R the limit

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - (u(x, y) + iv(x, y))}{\Delta x + i\Delta y} \tag{3}$$

must exist independent of the manner in which Δz (or Δx and Δy) approaches zero. We consider two possible approaches.

case (1): $\Delta y = 0, \Delta x \to 0$ .In this case, (1) becomes

$$\lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - (u(x, y) + iv(x, y))}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y) + i(v(x+\Delta x, y) - v(x, y))}{\Delta x}$$

$$= \lim_{\Delta x \to 0} [\frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i\frac{(v(x+\Delta x, y) - v(x, y)}{\Delta x}]$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{(v(x+\Delta x, y) - v(x, y)}{\Delta x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

provided the partial derivatives exist.

case (2): $\Delta x = 0, \Delta y \to 0$ .In this case, (1) becomes

$$\lim_{\Delta x \to 0} \frac{u(x, y+\Delta y) + iv(x, y+\Delta y) - (u(x, y) + iv(x, y))}{i\Delta y} = \lim_{\Delta x \to 0} \frac{u(x, y+\Delta y) - u(x, y) + i(v(x,y+\Delta y) - v(x, y))}{i\Delta y}$$

$$= \lim_{\Delta x \to 0} [\frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + \frac{(v(x, y+\Delta y) - v(x, y)}{\Delta y}]$$

$$= \lim_{\Delta x \to 0} \frac{u(x,y+\Delta y) - u(x, y)}{i\Delta y} + \lim_{\Delta x \to 0} \frac{(v(x,y+\Delta y) - v(x, y)}{\Delta y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

Now $f(z)$ cannot possibly be analytic unless these two limits are identical. Thus, a necessary condition that $f(z)$ be analytic is

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{or} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

A sufficient condition. Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are supposed to be continuous, we have

$$\Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$

$$= u(x + \Delta x, y + \Delta y) - u(x, y + \Delta y) + u(x, y + \Delta y) - u(x, y)$$

$$= \frac{u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)}{\Delta x}\Delta x + \frac{u(x, y+\Delta y) - u(x, y)}{\Delta y}\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + \epsilon_1\right)\Delta x + \left(\frac{\partial u}{\partial y} + \eta_1\right)\Delta y = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1\Delta x + \eta_1\Delta y$$

where $\epsilon_1 \to 0$ and $\eta_1 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.

Similarly, Since $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are supposed to be continuous, we have

$$\Delta v = \left(\frac{\partial v}{\partial x} + \epsilon_2\right)\Delta x + \left(\frac{\partial v}{\partial y} + \eta_2\right)\Delta y = \frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \epsilon_2\Delta x + \eta_1\Delta y.$$

where $\epsilon_2 \to 0$ and $\eta_2 \to 0$ as $\Delta x \to 0$ and $\Delta y \to 0$.Then

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

where $\epsilon = \epsilon_1 + \epsilon_2$ and $\eta = \eta_1 + \eta_2$ as $\Delta x \to 0$ and $\Delta y \to 0$.

By the Cauchy–Riemann equations we get

$$\Delta w = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(-\frac{\partial v}{\partial y} + i\frac{\partial u}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\Delta x + i\Delta y) + \epsilon\Delta x + \eta\Delta y .$$

Then, on dividing by $\Delta z = \Delta x + i\Delta y$ and taking the limit as $\Delta z \to 0$, we see that

$$\frac{dw}{dz} = f'(z) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

so that the derivative exists and is unique, i.e., $f(z)$ is analytic in R.

2.64 Example:

Prove that **(a)** $\frac{d}{dz}e^z = e^z$, **(b)** $\frac{d}{dz}e^{az} = ae^{az}$ where a is any constant.

Solution:

(a) By definition $w = e^z = e^x(\cos y + isin) = u + iv$ or

$$u(x,y) = e^x \cos y, \qquad v(x,y) = e^x \sin y$$

$$\frac{\partial u}{\partial x} = e^x \cos y \qquad \frac{\partial v}{\partial x} = e^x \sin y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \qquad \frac{\partial v}{\partial y} = e^x \cos y$$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ the Cauchy–Riemann equations are satisfied. The required derivative exists and is equal to

$$\frac{d}{dz}e^z = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = e^x \cos y + ie^x \sin y = e^z$$

(b) Let $w = e^{w_0}$ where $w_0 = az$. Then by part (a)

$$\frac{d}{dz}e^{az} = \frac{d}{dz}e^{w_0} = \frac{d}{dw_0}e^{w_0} \cdot \frac{dw_0}{dz} = e^{w_0} \cdot a = ae^{az}$$

2.65 Example:

Showed that the function $f(z) = z^2 = x^2 - y^2 + 2xyi$ is differentiable everywhere and that $f'(z) = 2z$.

Solution:

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ then $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 2yi = 2(x + iy) = 2z.$

2.66 Example:

Showed that the function $f(z) = |z|^2 = x^2 + y^2$ is differentiable does not exist at any nonzero point.

Solution:

$$u(x,y) = x^2 + y^2, \qquad v(x,y) = 0$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0$$

If the Cauchy–Riemann equations are to hold at a point *(x, y)*, it follows that $\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 0$,i.e, $2x = 0 \Longrightarrow x = 0$ and $\frac{\partial u}{\partial y} = 2y = -\frac{\partial v}{\partial x} = 0$,i.e, $2y = 0 \Longrightarrow y = 0$. Consequently $f'(z)$ does not exist at any nonzero point.

2.67 Example(Cauchy–Riemann equations in polar form):

Prove that the polar form of the Cauchy–Riemann equations can be written

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution:

We have $z=x+iy=re^{i\theta}=rcos\theta+irsin\theta$, $x=rcos\theta$, $y=rsin\theta$ or $r=\sqrt{x^2+y^2}$ and $\theta=tan^{-1}(\frac{y}{x})$.Then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right)\qquad \{\frac{d}{d\alpha}tan^{-1}\alpha=\frac{1}{1+\alpha^2}\}$$

$$=\frac{\partial u}{\partial r}\frac{rcos\theta}{r}+\frac{\partial u}{\partial \theta}\frac{-rsin\theta}{r^2}=\frac{\partial u}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta. \tag{4}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial u}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right)+\frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right)=\frac{\partial u}{\partial r}\frac{rsin\theta}{r}+\frac{\partial u}{\partial \theta}\frac{rcos\theta}{r^2}$$

$$=\frac{\partial u}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta. \tag{5}$$

Similarly,

$$\frac{\partial v}{\partial x}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial x}=\frac{\partial v}{\partial r}\cos\theta-\frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta. \tag{6}$$

$$\frac{\partial v}{\partial y}=\frac{\partial v}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial v}{\partial \theta}\cdot\frac{\partial \theta}{\partial y}=\frac{\partial v}{\partial r}\sin\theta+\frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta. \tag{7}$$

From the Cauchy–Riemann equation $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ we have, using (4) and (7),

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta-\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta=0 \tag{8}$$

From the Cauchy–Riemann equation $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$ we have, using (5) and (6),

$$\left(\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta+\left(\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta=0 \tag{9}$$

Multiplying (8) by $\cos\theta$, (9) by $\sin\theta$ and adding yields

$$\frac{\partial u}{\partial r}-\frac{1}{r}\frac{\partial v}{\partial \theta}=0\quad \text{or}\ \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}.$$

Multiplying (8) by $-\sin\theta$, (9) by $\cos\theta$ and adding yields

$$\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial \theta}=0\quad \text{or}\ \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

2.68 Example:

Consider the function $f(z)=\frac{1}{z}$, use the polar form of the Cauchy–Riemann equations to find $f'(z)$.

Solution:

$$f(z)=\frac{1}{z}=\frac{1}{re^{i\theta}}=\frac{1}{r}e^{-i\theta}=\frac{1}{r}(\cos\theta-isin\,\theta),\ (z\neq 0).$$

Since $u(r,\theta) = \frac{\cos\theta}{r}$ and $v(r,\theta) = \frac{-\sin\theta}{r}$ then

$$\frac{\partial u}{\partial r} = -\frac{\cos\theta}{r^2}, \qquad \frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2}$$

$$\frac{\partial u}{\partial \theta} = \frac{-\sin\theta}{r}, \qquad \frac{\partial v}{\partial \theta} = \frac{-\cos\theta}{r}$$

Now $\frac{\partial u}{\partial r} = -\frac{\cos\theta}{r^2} = \frac{1}{r}\frac{-\cos\theta}{r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2} = -\frac{1}{r}\frac{-\sin\theta}{r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Hence the derivative of f exists when $z \neq 0$; and

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -e^{-i\theta}\frac{e^{-i\theta}}{r^2} = -\frac{1}{\left(re^{i\theta}\right)^2} = -\frac{1}{z^2}.$$

2.69 Example:

Consider the function $f(z) = z^{\frac{1}{3}}$, use the polar form of the Cauchy–Riemann equations to show that $f'(z) = \frac{1}{3\left(\sqrt[3]{z}\right)^2}$.

Solution:

$z^{\frac{1}{3}} = \sqrt[3]{r}e^{i\frac{\theta}{3}}$, $(r > 0, \alpha < \theta < \alpha + 2\pi)$. Hence $u(r,\theta) = \sqrt[3]{r}\cos\frac{\theta}{3}$ and $v(r,\theta) = \sqrt[3]{r}\sin\frac{\theta}{3}$.

$$\frac{\partial u}{\partial r} = \frac{1}{3\sqrt[3]{r^2}}\cos\frac{\theta}{3}, \qquad \frac{\partial v}{\partial r} = \frac{1}{3\sqrt[3]{r^2}}\sin\frac{\theta}{3}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sqrt[3]{r}}{3}\sin\frac{\theta}{3}, \qquad \frac{\partial v}{\partial \theta} = \frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3}$$

Now

$$\frac{\partial u}{\partial r} = \frac{1}{3\sqrt[3]{r^2}}\cos\frac{\theta}{3} = \frac{1}{3\sqrt[3]{r^2}}\frac{\sqrt[3]{r}}{\sqrt[3]{r}}\cos\frac{\theta}{3} = \frac{\sqrt[3]{r}}{3r}\cos\frac{\theta}{3} = \frac{1}{r}\frac{\sqrt[3]{r}}{3}\cos\frac{\theta}{3} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

And $\frac{\partial v}{\partial r} = \frac{1}{3\sqrt[3]{r^2}}\sin\frac{\theta}{3} = \frac{1}{3\sqrt[3]{r^2}}\frac{\sqrt[3]{r}}{\sqrt[3]{r}}\sin\frac{\theta}{3} = \frac{\sqrt[3]{r}}{3r}\sin\frac{\theta}{3} = -\frac{1}{r}\frac{-\sqrt[3]{r}}{3}\sin\frac{\theta}{3} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Hence the derivative of f exists exists at each point where $f\ (z)$ is defined.

$$f'(z) = e^{-i\theta}\left[\frac{1}{3\left(\sqrt[3]{r}\right)^2}\cos\frac{\theta}{3} + i\frac{1}{3\left(\sqrt[3]{r}\right)^2}\sin\frac{\theta}{3}\right] = \frac{1}{3\left(\sqrt[3]{r}\right)^2}e^{-i\theta}e^{i\frac{\theta}{3}} = \frac{1}{3\left(\sqrt[3]{r}\right)^2\cdot\left(e^{i\frac{\theta}{3}}\right)^2}$$

$$= \frac{1}{3\left(\sqrt[3]{re^{i\theta}}\right)^2} = \frac{1}{3\left(\sqrt[3]{z}\right)^2}.$$

EXERCISES:

1. Show that $f'(z)$ does not exist at any point if

a) $f(z) = \bar{z}$, **b)** $f(z) = z - \bar{z}$, **c)** $f(z) = 2x + ixy^2$, **d)** $f(z) = e^x e^{-iy}$.

2. Show that $f'(z)$ and its derivative $f''(z)$ exist every whereand find $f''(z)$ when

a) $f(z) = iz + 2,$ **b)** $f(z) = e^{-x}e^{-iy},$ **c)** $f(z) = z^3.$

3. Determine where $f'(z)$ exists and find its value when

a) $f(z) = x^2 + iy^2,$ **b)** $f(z) = z \cdot Im(z),$

4. Show that each of these functions is differentiable in the indicated domain of definition, and also to find $f'(z)$

a) $f(z) = \frac{1}{z^4}$ $(z \neq 0),$ **b)** $f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$ $(r > 0, \alpha < \theta < \alpha + 2\pi).$

5. Verify that the real and imaginary parts of the following functions satisfy the Cauchy–Riemann equations and thus deduce the analyticity of each function:

a) $f(z) = z^2 + 5iz + 3 - i,$ **b)** $f(z) = ze^{-z}.$

6. Show that the function $f(z) = x^2 + iy^3$ is not analytic anywhere. Reconcile this with the fact that the Cauchy–Riemann equations are satisfied at $x = 0, y = 0.$

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Chapter Two

Analytic Functions

2.70 Definition:

A real-valued function H of two real variables x and y is said to be ***harmonic*** in a given domain of the xy- plane if, throughout that domain, it has continuous partial derivatives of the first and second order and satisfies the partial differential equation

$$\boldsymbol{H_{xx}(x,y) + H_{yy}(x,y) = 0,}$$

known as ***Laplace's equation***.

2.71 Example:

Show that the function $T\,(x, y) = e^{-y}\ sin\, x$ is harmonic in any domain of the xy- plane.

Solution:

$$T_{x}(x, y) = e^{-y}\ cos\, x \qquad T_y\,(x, y) = -e^{-y}\ sin\, x$$

$$T_{xx}\,(x, y) = -e^{-y}\ sin\, x \qquad T_{yy}\,(x, y) = e^{-y}\ sin\, x$$

$$T_{xx}\,(x, y) + T_{yy}\,(x, y) = -e^{-y}\ sin\, x + e^{-y}\ sin\, x = 0.$$

So T is harmonic in any domain of the xy- plane.

2.72 Theorem:

If a function $f\,(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then its component functions u and v are harmonic in D.

Proof:

Assuming that f is analytic in D, then the first order partial derivatives of its component functions must satisfy the Cauchy–Riemann equations throughout D:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}\ ,\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\ . \qquad (1)$$

Differentiating both sides of these equations in (1) with respect to x, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x}\ ,\ \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x^2}.$$

Likewise, differentiation (1) with respect to y yields

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}\ ,\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}\ .$$

Now, by a theorem in advanced calculus, the continuity of the partial derivatives of u and v ensures that $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y}$. It then follows that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2}, \quad \text{i.e.} \quad \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

$$-\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 v}{\partial y^2}, \quad \text{i.e.} \quad \frac{\partial^2 v}{\partial y^2} - \frac{\partial^2 v}{\partial x^2} = 0.$$

That is, u and v are harmonic in D.■

2.73 Example:

Since the function $f(z) = z^2 = x^2 - y^2 - 2xyi$ is analytic in xy- plane (example 2.55) then its component functions $u = (x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ are harmonic in xy- plane.

2.74 Example:

a) Prove that $u = (x,y) = e^{-x}\ (x\sin y - y\cos y)$ is harmonic.

b) Find v such that $f(z) = u(x,y) + iv(x,y)$ is analytic.

c) Find $f(z)$.

Solution:

a) $\frac{\partial u}{\partial x} = e^{-x}(\sin y) + (-e^{-x})(x\sin y - y\cos y) = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y$

$\frac{\partial^2 u}{\partial x^2} = -e^{-x}\sin y + xe^{-x}\sin y - e^{-x}\sin y - ye^{-x}\cos y = -2e^{-x}\sin y + xe^{-x}\sin y - ye^{-x}\cos y.$

$\frac{\partial u}{\partial y} = e^{-x}(x\cos y + y\sin y - \cos y) = xe^{-x}\cos y + ye^{-x}\sin y - e^{-x}\cos y .$

$\frac{\partial^2 u}{\partial y^2} = -xe^{-x}\sin y + ye^{-x}\cos y + e^{-x}\sin y + e^{-x}\sin y = -xe^{-x}\sin y + ye^{-x}\cos y + 2e^{-x}\sin y$

Then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and u is harmonic.

b) From the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^{-x}\sin y - xe^{-x}\sin y + ye^{-x}\cos y. \tag{2}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y . \tag{3}$$

Integrate (2) with respect to y, keeping x constant. Then

$$v = -e^{-x}\cos y + xe^{-x}\cos y + e^{-x}(y\sin y + \cos y) + F(x)$$

$$= xe^{-x}\cos y + ye^{-x}\sin y + F(x). \tag{4}$$

$$\frac{\partial v}{\partial x} = -xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + F'(x) \tag{5}$$

where $F(x)$ is an arbitrary real function of x. Substitute (5) into (3) and obtain

$-xe^{-x}\cos y + e^{-x}\cos y - ye^{-x}\sin y + F'(x) = -xe^{-x}\cos y - ye^{-x}\sin y + e^{-x}\cos y$

or $F'(x) = 0$ and $F(x) = c$, a constant. Then, from (4)

$$v = xe^{-x}\cos y + ye^{-x}\sin y + c = e^{-x}(x\cos y + y\sin y) + c.$$

c) We have $f(z) = u(x,y) + iv(x,y)$.Putting $y = 0$ then $f(x) = u(x,0) +$

$iv(x,0)$. Replacing x by z, we get $f(z) = u(z,0) + iv(z,0)$.From a) $u(z,0) = 0$ and from b) $v(z,0) = ze^{-z}$and so $f(z) = zie^{-z}$ apart from an arbitrary additive constant.

2.75 Remark:

We can solve example 2.74 (c) in another method apart from an arbitrary additive constant

$$\begin{aligned} f(z) &= u + iv = e^{-x}(x\sin y - y\cos y) + ie^{-x}(y\sin y + x\cos y) \\ &= e^{-x}\left\{x\left(\frac{e^{iy}-e^{-iy}}{2i}\right) - y\left(\frac{e^{iy}+e^{-iy}}{2}\right)\right\} + ie^{-x}\left\{y\left(\frac{e^{iy}-e^{-iy}}{2i}\right) + x\left(\frac{e^{iy}+e^{-iy}}{2}\right)\right\} \\ &= i(x+iy)e^{-(x+iy)} = ize^{-z} \end{aligned}$$

2.76 Remark:

If two given functions u and v are harmonic in a domain D and their first-order partial derivatives satisfy the Cauchy–Riemann equations throughout D, then v is said to be a ***harmonic conjugate*** of u (example 2.74).

2.77 Theorem:

A function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D if and only if v is a harmonic conjugate of u.

2.78 Remark:

If v is a harmonic conjugate of u in some domain, it is not, in general, true that u is a harmonic conjugate of v there.

EXERCISES:

1. Show that $u(x,y)$ is harmonic in some domain and find a harmonic conjugate $v(x,y)$ when

a) $u(x,y) = 2x(1-y)$, b) $u(x,y) = 2x - x^3 + 3xy^2$,

c) $u(x,y) = \frac{y}{(x^2+y^2)}$.

2. a) Prove that $u(x,y) = y^3 - 3x^2y$ is harmonic and show that v such that $f(z) = u(x,y) + iv(x,y)$ is analytic. Find $f(z)$.

b) Prove that $u(x,y) = 2x(1-y)$ is harmonic and show that v such that $f(z) = f(z) = u(x,y) + iv(x,y)$ is analytic. Find $f(z)$.

c) Prove that $u(x,y) = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic and show that v such that $f(z) = u(x,y) + iv(x,y)$ is analytic. Find $f(z)$.

3. Determine which of the following functions u are harmonic

(a) $3x^2y + 2x^2 - y^3 - 2y^2$, (b) $2xy + 3xy^2 - 2y^3$, (c) $xe^z\cos y - ye^z\sin y$, (d) $e^{-2xy}\sin(x^2 - y^2)$.

Chapter Three

ELEMENTARY FUNCTIONS

3.1 Remark:

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$. In remark 2.24 we introduce and develop properties of exponential function $e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$, $(z = x + iy)$. In this chapter we will use it to develop the others.

3.2 THE LOGARITHMIC FUNCTION:

The definition of the logarithmic function is based on solving the equation $e^w = z$ for w, where z is any nonzero complex number. To do this, we note that when z and w are written $z = re^{i\Theta}$ $(-\pi < \Theta \leq \pi)$ and $w = u + iv$, then $e^w = z$ becomes $e^w = e^u e^{iv} = re^{i\Theta}$, i.e. $e^u = r$ and $v = \Theta + 2n\pi$ where n is any integer. Since the equation $e^u = r$ is the same as $u = \ln r$ it follows that equation $e^w = z$ is satisfied if and only if w has one of the values

$$w = u + iv = \ln r + i(\Theta + 2n\pi), \quad (n = 0, \pm 1, \pm 2, \cdots).$$

Thus, if we write

$$\mathbf{\log z = \ln r + i(\Theta + 2n\pi), \quad (n = 0, \pm 1, \pm 2, \cdots).} \tag{1}$$

Equation $e^w = z$ tells us that $e^{\log z} = z, (z \neq 0)$ which serves to motivate expression (1) as the definition of the (multiple-valued) **logarithmic function** of a nonzero complex variable $z = re^{i\Theta}$.

3.3 Example:

Find $log(-1 - \sqrt{3}i)$.

Solution:

$z = -1 - \sqrt{3}i$ then $r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$ and $\Theta = sin^{-1}\frac{-\sqrt{3}}{2} = -\frac{2\pi}{3}$, so

$$log(-1 - \sqrt{3}i) = ln2 + i\left(-\frac{2\pi}{3} + 2n\pi\right) = ln2 + 2\left(n - \frac{1}{3}\right)\pi i, (n = 0, \pm 1, \pm 2, \cdots).$$

3.4 Remark:

It should be emphasized that it is not true that the left-hand side of equation

$e^{\log z} = z$ with the order of the exponential and logarithmic functions reversed reduces to just z. More precisely, since expression (1) can be written $\log z = \ln|r| + iarg\, z$ and since $|e^z| = e^x$ and $arg\,(e^z) = y + 2n\pi$,$(n = 0, \pm 1, \pm 2, \cdots)$. When $z = x + iy$, we know that

$$\log(e^z) = \ln|e^z| + iarg\,(e^z) = \ln(e^x) + i(y + 2n\pi),\quad (n = 0, \pm 1, \pm 2, \cdots).$$

That is $\log(e^z) = z + 2\,n\pi i$,$(n = 0, \pm 1, \pm 2, \cdots)$.

The principal value of $log\, z$ is the value obtained from equation (1) when $n = 0$ there and is denoted by $Log\, z$. Thus $Log\, z = ln\, r + i\Theta$.

Note that Log z is well defined and single-valued when $z \neq 0$ and that

$$log\, z = Log\, z + 2n\pi i \;,(n = 0, \pm 1, \pm 2, \cdots).$$

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$.To see this, one need only write $z = re^{i0}$, in which case equation $Log\, z = ln\, r + i\Theta$ becomes $Log\, z = ln\, r$. That is, $Log\, r = ln\, r$.

3.5 Example:

$log\, 1 = ln\, 1 + i(0 + 2n\pi) = 2n\pi i$, $(n = 0, \pm 1, \pm 2, \cdots)$. As anticipated, $Log\, 1 = 0$.

3.6 Remark:

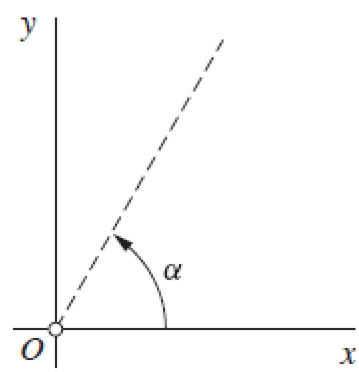
The following example reminds us that although we were unable to find logarithms of negative real numbers in calculus, we can now do so.

3.7 Example:

$log(-1) = ln\, 1 + i(\pi + 2n\pi) = (2n + 1)\pi i$, $(n = 0, \pm 1, \pm 2, \cdots)$ and that $Log\,(-1) = \pi i$.

3.8 Remark:

If $z = re^{i\theta}$ is a nonzero complex number, the argument θ has any one of the values $\theta = \Theta + 2n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$, where $\Theta = Arg\, z$. Hence the definition $log\, z = ln\, r + i(\Theta + 2n\pi)$ $(n = 0, \pm 1, \pm 2, \cdots)$ of the multiple-valued logarithmic function which can be written $\log z = \ln r + i\theta$.



If we let α denote any real number and restrict the value of θ in expression $\log z = \ln r + i\theta$ so that $\alpha < \theta < \alpha + 2\pi$, the function $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$, with components $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$, is single-valued and continuous in the stated domain. Note that if the function were to be defined on the $ray\; \theta = \alpha$, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near α and also points such that the values of v are near $\alpha + 2\pi$.

The function $\log z = \ln r + i\theta\ (r > 0, \alpha < \theta < \alpha + 2\pi)$ is not only continuous but also analytic throughout the domain $r > 0, \alpha < \theta < \alpha + 2\pi$ since the first-order partial derivatives of u and v are continuous there and satisfy the polar form $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$, $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$ of the Cauchy–Riemann equations. Furthermore

$$\frac{d}{dz}\log z = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\left(\frac{1}{r} + 0i\right) = \frac{1}{re^{i\theta}}.$$

that is,

$$\frac{d}{dz}\log z = \frac{1}{z}. \qquad (|z| > 0, \alpha < arg\ z < \alpha + 2\pi).$$

In particular,

$$\frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z}. \qquad (|z| > 0, -\pi < Arg\ z < \pi).$$

3.9 Definition:

Branch Points of multiple-valued functions are non-isolated singular points since a multiple-valued function is not continuous and, therefore, not analytic in a deleted neighborhood of a branch point.

3.10 Remark:

Observe that for each fixed α , the single-valued function $\log z = \ln r + i\theta$ $(r > 0, \alpha < \theta < \alpha + 2\pi)$ is a branch of the multiple-valued function $\log z = \ln r + i\theta$. The function $\mathrm{Log}\, z = \ln r + i\Theta$, $(r > 0, -\pi < \Theta < \pi)$. is called the ***principal branch***.

Special care must be taken in using branches of the logarithmic function, especially since expected identities involving logarithms do not always carry over from calculus.

3.11 Example:

$\mathrm{Log}\,(i^3) = \mathrm{Log}\,(-i) = \ln 1 - i\frac{\pi}{2} = -\frac{\pi}{2}i$, and $3\mathrm{Log}\, i = 3(\ln 1 + i\frac{\pi}{2}) = \frac{3\pi}{2}i$.

Hence $\mathrm{Log}\,(i^3) \neq 3\mathrm{Log}\, i$.

EXERCISES:

1. Show that **a)** $Log(-ei) = 1 - \frac{\pi}{2}i$, **b)** $Log(1-i) = \frac{1}{2}ln2 - \frac{\pi}{4}i$.

2. Show that

a) $loge = 1 + 2n\pi i \quad (n = 0, \pm 1, \pm 2, \cdots)$.

b) $logi = \left(2n + \frac{1}{2}\right)\pi i \ (n = 0, \pm 1, \pm 2, \cdots)$.

c) $\log\left(-1 + \sqrt{3}i\right) = ln2 + 2(n + \frac{1}{3})\pi i \ (n = 0, \pm 1, \pm 2, \cdots)$.

3. Sho that

a) $\text{Log}\ (1 + i)^2 = 2Log(1 + i)$, **b)** $Log(-1 + i)^2 = 2Log(-1 + i)$.

4. Show that

a) $\log(i^2) = 2 \log i$ when $\log z = \ln r + i\theta \ (r > 0, \frac{\pi}{4} < \theta < \frac{9\pi}{4})$,

b) $\log(i^2) \neq 2 \log i$ when $\log z = \ln r + i\theta \ (r > 0, \frac{3\pi}{4} < \theta < \frac{11\pi}{4})$,

5. Show that

a) The set of values of $\log\left(i^{1/2}\right) = \left(n + \frac{1}{4}\right)\pi i$, $(n = 0, \pm 1, \pm 2, \cdots)$ and that the same is true of $(1/2)\ log\ i$;

b) The set of values of $\log(i^2)$ is not the same as the set of values of $2\ log\ i$.

6. Find all roots of the equation $log\ z = i\frac{\pi}{2}$.

7. Show in two ways that the function $ln(x^2 + y^2)$ is harmonic in every domain that does not contain the origin.

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Chapter Three

ELEMENTARY FUNCTIONS

3.12 Remark:

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$$\log(z_1 z_2) = \log z_1 + \log z_2. \tag{1}$$

This statement involving a multiple valued function, is to be interpreted in the same way that the statement $\arg(z_1 z_2) = \arg z_1 + \arg z_2$. Since $|z_1 z_2| = |z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that $ln\,|z_1 z_2| = ln\,|z_1| + ln\,|z_2|$.So it follows from this and equation (1) that

$$ln\,|z_1 z_2| + i\,arg(z_1 z_2) = (ln\,|z_1| + i\,arg\,z_1) + (ln\,|z_2| + i\,arg\,z_2) \tag{2}$$

Finally, because of the way in which equations (1) and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ are to be interpreted, equation (2) is the same as equation (1).Also the same way as statement (1),we can interpreted

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

3.13 Example:

Let $z_1 = z_2 = -1$ .Since $log\,1 = 2n\pi i$ and $log(-1) = (2n + 1)\pi i, (n = 0, \pm 1, \pm 2, \cdots)$. Noting that $z_1 z_2 = 1$ and using the values $\log(z_1 z_2) = 0$ and $log\,z_1 = \pi i$ we find that equations (1) is satisfied when the value $log\,z_2 = -\pi i$ is chosen.

If, on the other hand, the principal values $Log\,1 = 0$ and $Log(-1) = \pi i$, are used $Log(z_1 z_2) = 0$ and $Log\,z_1 + log\,z_2 = 2\pi i$, for the same numbers z_1 and z_2. Thus statement (1), which is sometimes true when log is replaced by Log, is not always true when principal values are used in all three of its terms.

3.14 Remark:

If z is a nonzero complex number, then

$$z^n = e^{nlog\,z}, \ (n = 0, \pm 1, \pm 2, \cdots). \tag{3}$$

For any value of $log\,z$ that is taken. It is also true that when $z \neq 0$,

$$z^{\frac{1}{n}} = \exp(\frac{1}{n}\log z), \quad (n = 1,2, \cdots).$$

That is, the term on the right here has n distinct values, and those values are the n th roots of z. To prove this, we write $z = re^{i\Theta}$, where Θ is the principal value

value of $arg\ z$. Then, from the definition of $log\ z$,

$$\exp(\frac{1}{n}\log z) = \exp(\frac{1}{n}\ln r + \frac{i(\Theta+2\pi k)}{n}), \quad (k = 0, \pm 1, \pm 2, \cdots). \quad (4)$$

Thus

$$\exp(\frac{1}{n}\log z) = \sqrt[n]{r}\cdot \exp(i\left(\frac{\Theta}{n} + \frac{2\pi k}{n}\right)), \quad (k = 0, \pm 1, \pm 2, \cdots). \quad (5)$$

Because $\exp(i\frac{2\pi k}{n}$ has distinct values only when $k = 0, 1, \ldots, n - 1$, the right hand side of equation (5) has only *n* values. That right-hand side is, in fact, an expression for the *n*th roots of *z*, and so it can be written $z^{\frac{1}{n}}$. This establishes property (4), which is actually valid when *n* is a negative integer too.

EXERCISES:

1. Show that if $Re\ z_1 > 0$ and $Re\ z_2 > 0$, then $Log(z_1 z_2) = Log\ z_1 + Log\ z_2$.

2. Show that for any two nonzero complex numbers z_1 and z_2,

$$Log(z_1 z_2) = Log\ z_1 + Log\ z_2 + 2N\pi i\ ,$$

where *N* has one of the values $0, \pm 1$.

3.15 (TRIGONOMETRIC FUNCTIONS):

Since from Euler's formula we have $e^{ix} = cosx + isinx$ and $e^{-ix} = cosx - isinx$ for every real number *x*. Hence $e^{ix} - e^{-ix} = 2isin\ x$ and $e^{ix} + e^{-ix} = 2cos\ x$. That is $sin\ x = \frac{e^{ix}-e^{-ix}}{2i}$ and $cos\ x = \frac{e^{ix}+e^{-ix}}{2}$. Therefore, to define the ***sine and cosine functions of a complex variable*** *z* as follows:

$$\boldsymbol{sin\ z = \frac{e^{iz}-e^{-iz}}{2i}} \qquad \text{and} \qquad \boldsymbol{cos\ z = \frac{e^{iz}+e^{-iz}}{2}}.$$

3.16 Example:

Prove that: **(a)** $\frac{d}{dz}\sin z = \cos z$, **(b)** $\frac{d}{dz}\cos z = -\sin z$

Solution:

(a) $\frac{d}{dz}\sin z = \frac{d}{dz}\frac{e^{iz}-e^{-iz}}{2i} = \frac{1}{2i}\left(ie^{iz} + ie^{-iz}\right) = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$

(b) $\frac{d}{dz}\cos z = \frac{d}{dz}\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2}\left(ie^{iz} - ie^{-iz}\right) = -\frac{e^{iz}-e^{-iz}}{2i} = -\sin z.$

3.17 Example:

Prove that: **(a)** $sin -z = -sin\ z$, **(b)** $cos -z = cos\ z$

Solution:

(a) $sin -z = \frac{e^{i(-z)}-e^{-i(-z)}}{2i} = \frac{e^{-iz}-e^{iz}}{2i} = \frac{-(e^{iz}-e^{-iz})}{2i} = -\left(\frac{e^{iz}-e^{-iz}}{2i}\right) = -\sin z.$

(b) $cos - z = \frac{e^{i(-z)}+e^{-i(-z)}}{2} = \frac{e^{-iz}+e^{iz}}{2} = \frac{e^{iz}+e^{-iz}}{2} = \cos z.$

3.18 Remark:

Functions of z having the property that $f(-z) = -f(z)$ are called ***odd functions***, while those for which $f(-z) = f(z)$ are called ***even functions***. Thus $sin\ z$ is an odd functions, while cos z is an even function.

3.19 Example:

Prove that: **(a)** $sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2,$

(b) $cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$

(c) $cos^2 z + sin^2 = 1$

Solution:

a) $sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i} = \frac{e^{iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{2i} = \frac{\frac{2(e^{iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2})}{2}}{2i}$

$$= \frac{e^{iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}+e^{iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{4i}$$

$$= \frac{e^{iz_1}e^{iz_2}+e^{iz_1}e^{-iz_2}-e^{-iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}+e^{iz_1}e^{iz_2}-e^{iz_1}e^{-iz_2}+e^{-iz_1}e^{iz_2}-e^{-iz_1}e^{-iz_2}}{4i}$$

$$= \frac{e^{iz_1}(e^{iz_2}+e^{-iz_2})-e^{-iz_1}(e^{iz_2}+e^{-iz_2})+e^{iz_1}(e^{iz_2}-e^{-iz_2})+e^{-iz_1}(e^{iz_2}-e^{-iz_2})}{4i}$$

$$= \frac{(e^{iz_1}-e^{-iz_1})(e^{iz_2}+e^{-iz_2})+(e^{iz_1}+e^{-iz_1})(e^{iz_2}-e^{-iz_2})}{4i}$$

$$= \frac{(e^{iz_1}-e^{-iz_1})(e^{iz_2}+e^{-iz_2})}{4i} + \frac{(e^{iz_1}+e^{-iz_1})(e^{iz_2}-e^{-iz_2})}{4i}$$

$$= \frac{(e^{iz_1}-e^{-iz_1})}{2i} \cdot \frac{(e^{iz_2}+e^{-iz_2})}{2} + \frac{(e^{iz_1}+e^{-iz_1})}{2} \cdot \frac{(e^{iz_2}-e^{-iz_2})}{2i}$$

$$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

(b) $cos(z_1 + z_2) = \frac{e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}}{2} = \frac{e^{iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}}{2} = \frac{\frac{2(e^{iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2})}{2}}{2}$

$$= \frac{e^{iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}+e^{iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}}{4}$$

$$= \frac{e^{iz_1}e^{iz_2}+e^{iz_1}e^{-iz_2}+e^{-iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}+e^{iz_1}e^{iz_2}-e^{iz_1}e^{-iz_2}-e^{-iz_1}e^{iz_2}+e^{-iz_1}e^{-iz_2}}{4}$$

$$= \frac{e^{iz_1}(e^{iz_2}+e^{-iz_2})+e^{-iz_1}(e^{iz_2}+e^{-iz_2})+e^{iz_1}(e^{iz_2}-e^{-iz_2})-e^{-iz_1}(e^{iz_2}-e^{-iz_2})}{4}$$

$$= \frac{(e^{iz_1}+e^{-iz_1})(e^{iz_2}+e^{-iz_2})+(e^{iz_1}-e^{-iz_1})(e^{iz_2}-e^{-iz_2})}{4}$$

$$= \frac{(e^{iz_1}+e^{-iz_1})(e^{iz_2}+e^{-iz_2})}{4} + \frac{(e^{iz_1}-e^{-iz_1})(e^{iz_2}-e^{-iz_2})}{4}$$

$$= \frac{(e^{iz_1}+e^{-iz_1})}{2} \cdot \frac{(e^{iz_2}+e^{-iz_2})}{2} - \frac{(e^{iz_1}-e^{-iz_1})}{2i} \cdot \frac{(e^{iz_2}-e^{-iz_2})}{2i}$$
$$= \cos z_1 \cos z_2 - \sin z_1 \sin z_2,$$

(c) $cos^2 z + sin^2 z = \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2 + \left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 = \frac{e^{2iz}+2e^{iz}e^{-iz}+e^{-2iz}}{4} - \frac{e^{2iz}-2e^{iz}e^{-iz}+e^{-2iz}}{4}$

$$= \frac{e^{2iz}+2+e^{-2iz}-e^{2iz}+2-e^{-2iz}}{4} = \frac{4}{4} = 1$$

3.20 Remark:

1. When y is any real number we can use the definitions of hyperbolic functions are $sinh\ y = \frac{e^y-e^{-y}}{2}$ and $cosh\ y = \frac{e^y+e^{-y}}{2}$ to write

$sin\ iy = \frac{e^{-y}-e^y}{2i} = \frac{-(e^y-e^{-y})}{2i} = i\frac{e^y-e^{-y}}{2} = isinh\ y$ and $cos\ iy = \frac{e^{-y}+e^y}{2} = \frac{e^y-e^{-y}}{2} = cosh\ y$

2. Also, the real and imaginary components of $sin\ z$ and $cos\ z$ can be displayed in terms of those hyperbolic functions:

$sin\ z = sin\ (x+iy) = \sin x \cos iy + \cos x \sin iy = sin\ xcosh\ y + i \cos x\ sinh\ y$

$cos\ z = cos\ (x+iy) = \cos x \cos iy - \sin x \sin iy = cos\ xcosh\ y - i \sin x\ sinh\ y.$

3. $|sin\ z|^2 = (sin\ x\ cosh\ y)^2 + (\cos x\ sinh\ y)^2 = (sin\ x\ )^2(\ cosh\ y)^2 + (\cos x)^2(sinh\ y)^2$
$= (sin\ x\ )^2(\ cosh\ y)^2 - (sin)^2(sinh\ y)^2 + (sinh\ y)^2$
$= (sin\ x\ )^2((\ cosh\ y)^2 - (sinh\ y)^2) + (sinh\ y)^2 \quad \{(\ cosh\ y)^2 - (sinh\ y)^2 = 1\}$
$= (sin\ x\ )^2 + (sinh\ y)^2.$

$|cos\ z|^2 = (cos\ x\ cosh\ y)^2 + (\sin x\ sinh\ y)^2 = (cos\ x\ )^2(\ cosh\ y)^2 + (\sin x)^2(sinh\ y)^2$
$= (cos\ x\ )^2(\ cosh\ y)^2 - (cos)^2(sinh\ y)^2 + (sinh\ y)^2$
$= (cos\ x\ )^2((\ cosh\ y)^2 - (sinh\ y)^2) + (sinh\ y)^2$
$= (cos\ x\ )^2 + (sinh\ y)^2.$

Inasmuch as sinh y tends to infinity as y tends to infinity, it is clear from these two equations that sin z and cos z are *not bounded* on the complex plane, whereas the absolute values of sin x and cos x are less than or equal to unity for all values of x.

3.21 Definition:

A ***zero*** of a given function $f\ (z)$ is a number z_0 such that $f\ (z_0) = 0$.

3.22 Remark:

Since $sin\ z$ becomes the usual sine function in calculus when z is real, we know that the real numbers $z = n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$ are all zeros of $sin\ z$. To show that there are no other zeros, we assume that $sin\ z = 0$ and note how it follows from equation $|sin\ z|^2 = (sin\ x\ )^2 + (sinh\ y)^2$ that $(sin\ x\ )^2 + (sinh\ y)^2 = 0$. This sum of two squares reveals that $sin\ x = 0$ and $sinh\ y = 0$

$sinh\, y = 0 \Rightarrow \frac{e^y - e^{-y}}{2} = 0 \Rightarrow e^y - e^{-y} = 0 \Rightarrow e^y = e^{-y} \Rightarrow y = -y \Rightarrow y = 0.$

$sin\, x = 0$ iff $x = n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$.

3.23 Remark:

As we saw for the case with $sin\, z$, the zeros of $cos\, z$ are all real, $z = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$. We assume that $cos\, z = 0$, it follows from equation $|cos\, z|^2 = (cos\, x)^2 + (sinh\, y)^2$ that $(cos\, x)^2 + (sinh\, y)^2 = 0$. This sum of two squares reveals that $cos\, x = 0$ and $sinh\, y = 0$

$sinh\, y = 0 \Rightarrow \frac{e^y - e^{-y}}{2} = 0 \Rightarrow e^y - e^{-y} = 0 \Rightarrow e^y = e^{-y} \Rightarrow y = -y \Rightarrow y = 0.$

$cos\, x = 0$ iff $x = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$.

3.24 Remark:

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}, \qquad \cot z = \frac{\cos z}{\sin z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}},$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}, \qquad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz} - e^{-iz}}.$$

Observe that the quotients

a) $tan\, z$ and $sec\, z$ are analytic everywhere except at the singularities $z = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$ which are the zeros of $cos\, z$.

b) $cot\, z$ and $csc\, z$ are analytic everywhere except at the singularities $z = n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$ which are the zeros of $sin\, z$.

Also the differentiation formulas are

3.25 Example:

Prove that:

(a) $\frac{d}{dz} \tan z = sec^2 z$, **(b)** $\frac{d}{dz} \cot z = -csc^2 z$

(c) $\frac{d}{dz} \sec z = \sec z\, tan\, z$, **(d)** $\frac{d}{dz} \csc z = -\csc z\, cot\, z$.

Solution:

(a) $\frac{d}{dz} \tan z = \frac{d}{dz} \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = \frac{-(e^{iz} + e^{-iz})(e^{iz} + e^{-iz}) + (e^{iz} - e^{-iz})(e^{iz} - e^{-iz})}{-(e^{iz} + e^{-iz})^2}$

$= \frac{-e^{2iz} - 2 - e^{-2iz} + e^{2iz} - 2 + e^{-2iz}}{-(e^{iz} + e^{-iz})^2} = \frac{-4}{-(e^{iz} + e^{-iz})^2} = \left(\frac{2}{e^{iz} + e^{-iz}}\right)^2 = sec^2 z.$

(b) $\frac{d}{dz} \cot z = \frac{d}{dz} \frac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})} = \frac{-(e^{iz} - e^{-iz})(e^{iz} - e^{-iz}) + (e^{iz} + e^{-iz})(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})^2}$

$$= \frac{-e^{2iz}+2-e^{-2iz}+e^{2iz}+2+e^{-2iz}}{(e^{iz}+e^{-iz})^2} = \frac{4}{(e^{iz}+e^{-iz})^2}\frac{i^2}{i^2} = -\left(\frac{2}{i(e^{iz}+e^{-iz})}\right)^2 = -csc^2 z.$$

(c) $\frac{d}{dz}\sec z = \frac{d}{dz}\frac{2}{(e^{iz}+e^{-iz})} = \frac{-2i(e^{iz}-e^{-iz})}{(e^{iz}+e^{-iz})^2} = \frac{2}{e^{iz}+e^{-iz}} \cdot \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})} = \sec z \, tan\, z.$

(d) $\frac{d}{dz}\csc z = \frac{d}{dz}\frac{2i}{(e^{iz}-e^{-iz})} = \frac{-2i^2(e^{iz}+e^{-iz})}{(e^{iz}-e^{-iz})^2} = -\frac{2i}{e^{iz}-e^{-iz}} \cdot \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}} = -\sec z \, tan\, z.$

3.26 Example:

Show that **(a)** $\tan(z+\pi) = \tan z$, **(b)** $tan - z = -tan\, z$**.**

Solution:

(a) Since $e^{i\pi} = \cos\pi + i\sin\pi = 1$ and $e^{-i\pi} = \cos\pi - i\sin\pi = 1$

$$\tan(z+\pi) = \frac{e^{i(z+\pi)}-e^{-i(z+\pi)}}{i(e^{i(z+\pi)}+e^{-i(z+\pi)})} = \frac{e^{iz}e^{i\pi}-e^{-iz}e^{-i\pi}}{i(e^{iz}e^{i\pi}+e^{iz}e^{-i\pi})} = \tan z.$$

(b) $tan - z = \frac{\sin -z}{\cos -z} = \frac{-\sin z}{\cos z} = -\tan z.$

EXERCISES:

1. Prove that:

(1) $sin(z_1 - z_2) = \sin z_1 \cos z_2 - \cos z_1 \sin z_2$,

(2) $cos(z_1 - z_2) = \cos z_1 \cos z_2 + \sin z_1 \sin z_2$,

(3) $sin(2z) = 2\sin z \cos z$

(4) $cos(2z) = cos^2 z - sin^2 z$,

(5) $sin(z + \frac{\pi}{2}) = \cos z$,

(6) $sin(z - \frac{\pi}{2}) = -\cos z$,

(7) $sin(z + 2\pi) = \sin z$,

(8) $sin(z + \pi) = -\sin z$,

(9) $cos(z + 2\pi) = \cos z$,

(10) $cos(z + \pi) = -\cos z$,

(11) $1 + tan^2 z = sec^2 z$,

(12) $1 + cot^2 z = csc^2 z$,

(13) $|sin\, z| \geq |sin\, x|$,

(14) $|cos\, z| \geq |cos\, x|$,

(15) $|sinh\, y| \leq |sin\, z| \leq cosh\, y$,

(16) $|sinh\, y| \leq |cos\, z| \leq cosh\, y$,

(17) $2sin(z_1 + z_2)\sin(z_1 - z_2) = cos(2z_2) - cos(2z_1)$,

(18) if $\cos z_1 = \cos z_2$, then at least one of the numbers $z_1 + z_2$ and $z_1 - z_2$ is an integral multiple of 2π.

2. Show that neither $sin\ \bar{z}$ nor $cos\ \bar{z}$ is an analytic function of z anywhere.

3. Show that for all z,

(1) $\overline{sin\ z} = sin\bar{z}$, for all z,

(2) $\overline{cos\ z} = cos\bar{z}$, for all z,

(3) $\overline{cos\ iz} = \cos i\bar{z}$, for all z,

(4) $\overline{sin\ iz} = \sin i\bar{z}$ iff $z = n\pi i$, $(n = 0, \pm 1, \pm 2, \cdots)$.

4. Find all roots of the equation $sin\ z = cosh\ 4$.

5. show that the roots of the equaion $cos\ z = 2$ are $z = 2n\pi + i\ cosh^{-1}2$ **,** $(n = 0, \pm 1, \pm 2, \cdots)$.Then express them in the form $z = 2n\pi \pm i\ ln(2 + \sqrt{3})$, $(n = 0, \pm 1, \pm 2, \cdots)$.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ٨

Chapter Three

ELEMENTARY FUNCTIONS

3.27 (HYPERBOLIC FUNCTIONS):

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is,

$$\sinh z = \frac{e^z - e^{-z}}{2} \quad \text{and} \quad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{1}$$

3.28 Remark:

1. Since e^z and e^{-z} are entire, it follows from definitions (1) that $\sinh z$ and $\cosh z$ are entire. Furthermore

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$$
$$\frac{d}{dz}\cosh z = \frac{d}{dz}\frac{e^z + e^{-z}}{2} = \frac{e^z - e^{-z}}{2} = \sinh z.$$

2. The hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$$-i\sinh iz = -i\frac{e^{iz} - e^{-iz}}{2} = -i\frac{i}{i}\frac{e^{iz} - e^{-iz}}{2} = -i^2\frac{e^{iz} - e^{-iz}}{2i} = \frac{e^{iz} - e^{-iz}}{2i} = \sin z,$$
$$\cosh iz = \frac{e^{iz} + e^{-iz}}{2} = \cos z,$$
$$-i\sin iz = -i\frac{e^{i^2 z} - e^{-i^2 z}}{2i} = -\frac{e^{-z} - e^{z}}{2} = \frac{e^z - e^{-z}}{2} = \sinh z,$$
$$\cos iz = \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$$

3. Some of the most frequently used identities involving hyperbolic sine and cosine functions are

a) $\sinh(-z) = \frac{e^{-z} - e^{z}}{2} = \frac{-(e^z - e^{-z})}{2} = -\frac{e^z - e^{-z}}{2} = -\sinh z.$

b) $\cosh(-z) = \frac{e^{-z} + e^{z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z.$

c) $\cosh^2 z - \sinh^2 z = \left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2 = \frac{e^{2z} + 2 + e^{-2z}}{4} - \frac{e^{2z} - 2 + e^{-2z}}{4} = \frac{4}{4} = 1.$

d) $\sinh(z_1 + z_2) = \frac{e^{z_1 + z_2} - e^{-(z_1 + z_2)}}{2} = \frac{e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2}}{2} = \frac{2(e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2})}{4}$

$$= \frac{e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2} + e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}e^{z_2}+e^{z_1}e^{-z_2}-e^{-z_1}e^{z_2}-e^{-z_1}e^{-z_2}+e^{z_1}e^{z_2}-e^{z_1}e^{-z_2}+e^{-z_1}e^{z_2}-e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}e^{z_2}+e^{z_1}e^{-z_2}-e^{-z_1}e^{z_2}-e^{-z_1}e^{-z_2}}{4} + \frac{e^{z_1}e^{z_2}-e^{z_1}e^{-z_2}+e^{-z_1}e^{z_2}-e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}(e^{z_2}+e^{-z_2})-e^{-z_1}(e^{z_2}+e^{-z_2})}{4} + \frac{e^{z_1}(e^{z_2}+e^{-z_2})+e^{-z_1}(e^{z_2}-e^{-z_2})}{4}$$

$$= \frac{e^{z_1}-e^{-z_1}}{2}\cdot\frac{e^{z_2}+e^{-z_2}}{2} + \frac{e^{z_1}+e^{-z_1}}{2}\cdot\frac{e^{z_2}-e^{-z_2}}{2}$$

$$= \sinh z_1 \cdot \cosh z_2 + \cosh z_1 \cdot \sinh z_2 .$$

e) $\cosh(z_1+z_2) = \frac{e^{z_1+z_2}+e^{-(z_1+z_2)}}{2} = \frac{e^{z_1}e^{z_2}+e^{-z_1}e^{-z_2}}{2} = \frac{2(e^{z_1}e^{z_2}+e^{-z_1}e^{-z_2})}{4}$

$$= \frac{e^{z_1}e^{z_2}+e^{-z_1}e^{-z_2}+e^{z_1}e^{z_2}+e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}e^{z_2}+e^{z_1}e^{-z_2}+e^{-z_1}e^{z_2}+e^{-z_1}e^{-z_2}+e^{z_1}e^{z_2}-e^{z_1}e^{-z_2}-e^{-z_1}e^{z_2}+e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}e^{z_2}+e^{z_1}e^{-z_2}+e^{-z_1}e^{z_2}+e^{-z_1}e^{-z_2}}{4} + \frac{e^{z_1}e^{z_2}-e^{z_1}e^{-z_2}-e^{-z_1}e^{z_2}+e^{-z_1}e^{-z_2}}{4}$$

$$= \frac{e^{z_1}(e^{z_2}+e^{-z_2})+e^{-z_1}(e^{z_2}+e^{-z_2})}{4} + \frac{e^{z_1}(e^{z_2}-e^{-z_2})-e^{-z_1}(e^{z_2}-e^{-z_2})}{4}$$

$$= \frac{e^{z_1}+e^{-z_1}}{2}\cdot\frac{e^{z_2}+e^{-z_2}}{2} + \frac{e^{z_1}-e^{-z_1}}{2}\cdot\frac{e^{z_2}-e^{-z_2}}{2}$$

$$= \cos z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$$

3.29 Example:

Prove that $|sinh\, z|^2 = sinh^2 x + sin^2 y.$

Solution:

$$|sinh\, z|^2 = |-i\, sin\, iz|^2 = |-i|^2 \cdot |sin\, iz|^2 = |\sin i(x+iy)|^2 = |\sin(-y+ix)|^2$$
$$= (sin - y\,)^2 + (sinh\, x)^2 = sinh^2 x + sin^2 y.$$

3.30 Remark:

The hyperbolic tangent and secant of z is defined by means of the equations

$$\boldsymbol{tanh\, z = \frac{\sinh z}{\cosh z} = \frac{e^z-e^{-z}}{e^z+e^{-z}}}\ , \qquad \boldsymbol{sech\, z = \frac{1}{cosh\, z} = \frac{2}{e^z+e^{-z}}}.$$

and are analytic in every domain in which $cosh\, z \neq 0$. The hyperbolic $coth$ z and csch z is defined by means of the equations

$$\boldsymbol{coth\, z = \frac{\cosh z}{\sinh z} = \frac{e^z+e^{-z}}{e^z-e^{-z}}}\ , \qquad \boldsymbol{csch\, z = \frac{1}{sinh\, z} = \frac{2}{e^z-e^{-z}}}\ .$$

and are analytic in every domain in which $sinh\, z \neq 0$.

EXERCISES:

1. Prove that :

a) $sinh\, 2z = 2sinh\, z \cdot cosh\, z.$

b) $\sinh(z_1 - z_2) = sinh\, z_1 \cdot cosh\, z_2 - cosh\, z_1 \cdot sinh\, z_2$

c) $\cosh(z_1 - z_2) = cosh\, z_1 \cdot cosh\, z_2 - sinh\, z_1 \cdot sinh\, z_2.$

d) $\sinh z = sinhx \cdot cos\, y + i\, cosh\, x \cdot sin\, y.$

e) $\cosh z = coshx \cdot cos\, y + i\, sinh\, x \cdot sin\, y.$

f) $|cosh\, z|^2 = sinh^2\, x + cos^2\, y.$

g) $\frac{d}{dz} tanh\, z = sech^2\, z.$

h) $\frac{d}{dz} coth\, z = -csch^2\, z$

i) $\frac{d}{dz} sech\, z = -sech\, z \cdot \tanh z.$

j) $\frac{d}{dz} csch\, z = -csch\, z \cdot \coth z.$

2. Show that:

a) $|sinh\, x| \le |cosh\, z| \le cosh\, x.$

b) $|sinh\, y| \le |cos\, z| \le cosh\, y.$

c) $sinh\,(z + \pi i) = -\sinh z.$

d) $cosh\,(z + \pi i) = \cosh z.$

e) $tanh\,(z + \pi i) = \tanh z.$

f) $\overline{\sinh z} = \sinh \bar{z}$, for all z.

g) $\overline{\cosh z} = \cosh \bar{z}$, for all z.

h) $\overline{\tanh z} = \tanh \bar{z}$ at points where $\cosh z \neq 0$.

i) $cosh^2\, z - sinh^2\, z = 1.$

j) $\sinh z + \cosh z = e^z.$

k) $\sinh z = 0 \quad iff \quad z = n\pi i\,,\ (n = 0, \pm 1, \pm 2, \cdots).$

l) $\cosh z = 0 \quad iff \quad z = (\frac{\pi}{2} + n\pi)i\,,\ (n = 0, \pm 1, \pm 2, \cdots).$

m) locate all zeros and singularities of the hyperbolic tangent function

n) Why is the function $sinh(e^z)$ entire? Write its real component as a function of *x* and *y*, and state why that function must be harmonic everywhere.

3. Find all roots of the equations:

a) $\sinh z = i.$

b) $\cosh z = \frac{1}{2}.$

c) $\cosh z = -2.$

3.31 (INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS):

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

3.32 Remark:

In order to define the inverse sine function $sin^{-1}z$, we write $w = sin^{-1}z$ when $z = sin\, w = \frac{e^{iw}-e^{-iw}}{2i}$. If we put this equation in the form

$$2zi = e^{iw} - e^{-iw} \Longrightarrow e^{iw} - e^{-iw} - 2zi = 0 \Longrightarrow e^{iw}(e^{iw} - e^{-iw} - 2zi = 0)$$

$$\Longrightarrow \left(e^{iw}\right)^2 - e^{-iw}e^{iw} - 2ize^{iw} = 0 \Longrightarrow \left(e^{iw}\right)^2 - e^{-iw+iw} - 2ize^{iw} = 0$$

$$\Longrightarrow \left(e^{iw}\right)^2 - 2ize^{iw} - 1 = 0.$$

which is quadratic in e^{iw}, and solve for e^{iw}

$$e^{iw} = \frac{2iz+\sqrt{(-2iz)^2+4}}{2} = \frac{2iz+\sqrt{-4z^2+4}}{2} = \frac{2iz+\sqrt{4(-z^2+1)}}{2} = \frac{2iz+2\sqrt{(1-z^2)}}{2} = \frac{2(iz+\sqrt{(1-z^2)}}{2}$$

$$= iz + \sqrt{(1-z^2)}$$

we find that $e^{iw} = iz + \sqrt{(1-z^2)}$ where $\sqrt{(1-z^2)}$ is a double-valued function of z. Taking logarithms of each side of equation

$$iw = \log\left(iz + \sqrt{(1-z^2)}\right) \Longrightarrow w = \frac{1}{i}\log\left(iz + \sqrt{(1-z^2)}\right) \Longrightarrow w = -i\log(iz + \sqrt{(1-z^2)}).$$

Recalling that $w = sin^{-1}z$, we arrive at the expression

$$sin^{-1}z = -i\log(iz + \sqrt{(1-z^2)}). \quad (1)$$

3.33 Remark:

The following example emphasizes the fact that $sin^{-1}z$ is a multiple-valued function with infinitely many values at each point z.

3.34 Example:

Find $sin^{-1}(-i)$.

Solution:

$$sin^{-1}(-i) = -i\log\left(i(-i) + \sqrt{(1-(-i)^2)}\right) = -ilog(1+\sqrt{2}).$$

But $2^{\frac{1}{2}} = \sqrt{2}\,(\cos(\frac{0+2\pi k}{2}) + isin(\frac{0+2\pi k}{2})), k = 0,1 \Longrightarrow 2^{\frac{1}{2}} = \pm\sqrt{2}$, so

$$sin^{-1}(-i) = -ilog(1 \pm \sqrt{2})$$

$log\left(1+\sqrt{2}\right) = ln\left(1+\sqrt{2}\right) + 2n\pi i,\ (n = 0, \pm 1, \pm 2, \cdots)$, and

$$log\left(1-\sqrt{2}\right) = ln\left(\sqrt{2}-1\right) + (2n+1)\pi i,\ (n = 0, \pm 1, \pm 2, \cdots).$$

Since $ln(\sqrt{2}-1) = ln(\sqrt{2}-1)\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1} = ln\frac{1}{1+\sqrt{2}} = ln(1+\sqrt{2})^{-1} = -ln(1+\sqrt{2})$. Then the numbers $(-1)^n ln(1+\sqrt{2}) + n\pi i, (n = 0, \pm 1, \pm 2, \cdots)$,constitute the set of values of $log(1 \pm \sqrt{2})$. Thus

$$sin^{-1}(-i) = n\pi + i(-1)^{n+1} ln(1+\sqrt{2}), (n = 0, \pm 1, \pm 2, \cdots).$$

3.35 Example:

Find $cos^{-1}z$.

Solution:

Let $w = cos^{-1}z$ then $z = cos\, w = \frac{e^{iw}+e^{-iw}}{2}$. If we put this equation in the form

$$2z = e^{iw} + e^{-iw} \Longrightarrow e^{iw} + e^{-iw} - 2z = 0 \Longrightarrow e^{iw}(e^{iw} + e^{-iw} - 2z = 0)$$

$$\Longrightarrow (e^{iw})^2 + e^{-iw}e^{iw} - 2ze^{iw} = 0 \Longrightarrow (e^{iw})^2 + e^{-iw+iw} - 2ze^{iw} = 0$$

$$\Longrightarrow (e^{iw})^2 - 2ze^{iw} + 1 = 0.$$

$$e^{iw} = \frac{2z+\sqrt{(-2z)^2-4}}{2} = \frac{2z+\sqrt{4z^2-4}}{2} = \frac{2z+\sqrt{4i^2(1-z^2)}}{2} = \frac{2z+2i\sqrt{1-z^2}}{2} = \frac{2(z+i\sqrt{1-z^2}}{2} = z + i\sqrt{1-z^2}.$$

$$iw = \log(z + i\sqrt{1-z^2}) \Longrightarrow w = -i\log(z + i\sqrt{1-z^2}), \text{ so}$$

$$cos^{-1}z = -i\log(z + i\sqrt{1-z^2}).$$

3.36 Example:

Find $tan^{-1}z$.

Solution:

$$w = tan^{-1}z \Longrightarrow z = \tan w \Longrightarrow z = \frac{e^{iw}-e^{-iw}}{i(e^{iw}+e^{-iw})}$$

$$iz(e^{iw} + e^{-iw}) = e^{iw} - e^{-iw} \Longrightarrow e^{iw} - e^{-iw} - zie^{iw} - ize^{-iw}$$

$$\Longrightarrow (1-zi)e^{iw} - (1+zi)e^{-iw} = 0$$

$$\Longrightarrow (1-zi)e^{2iw} - (1+zi) = 0$$

$$\Longrightarrow (1-zi)e^{2iw} = (1+zi)$$

$$\Longrightarrow e^{2iw} = \frac{(1+zi)}{(1-zi)} = \frac{i-z}{i+z}$$

$$\Longrightarrow 2iw = \log(\frac{i-z}{i+z})$$

$$\Longrightarrow w = \frac{1}{2i}\log\left(\frac{i-z}{i+z}\right) = \frac{-i}{2}\log\left(\frac{i-z}{i+z}\right) = \frac{i}{2}\log\left(\frac{i-z}{i+z}\right)^{-1}$$

$$\Longrightarrow w = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right),$$

So $tan^{-1}z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$.

3.37 Remark:

The functions $cos^{-1}z$ and $tan^{-1}z$ are also multiple-valued. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions. The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

$$\frac{d}{dz}sin^{-1}z = \frac{d}{dz}\left(-i\log\left(iz+\sqrt{(1-z^2)}\right)\right) = \frac{-i(i+\frac{-2z}{2\cdot\sqrt{(1-z^2)}})}{(iz+\sqrt{(1-z^2)})} = \frac{\frac{\sqrt{(1-z^2)}+iz}{\sqrt{(1-z^2)}}}{(iz+\sqrt{(1-z^2)})} = \frac{1}{\sqrt{(1-z^2)}}.$$

$$\frac{d}{dz}cos^{-1}z = \frac{d}{dz}\left(-i\log(z+i\sqrt{1-z^2})\right) = \frac{-i(1+\frac{-2iz}{2i\cdot\sqrt{(1-z^2)}})}{(z+i\sqrt{(1-z^2)})} = \frac{\frac{-(\sqrt{(1-z^2)}+iz)}{\sqrt{(1-z^2)}}}{(iz+\sqrt{(1-z^2)})} = \frac{-1}{\sqrt{(1-z^2)}}.$$

$$\frac{d}{dz}tan^{-1}z = \frac{d}{dz}\frac{i}{2}\log\left(\frac{i+z}{i-z}\right) = \frac{i}{2}\frac{1}{\left(\frac{i+z}{i-z}\right)}\frac{(i-z)+(i+z)}{(i-z)^2} = \frac{i}{2}\cdot\frac{i-z}{i+z}\cdot\frac{2i}{(i-z)^2} = \frac{-1}{(i+z)(i-z)}$$

$$= \frac{-1}{-1-z^2} = \frac{1}{1+z^2}.$$

The derivative of the $tan^{-1}z$ does not depend on the manner in which the function is made single valued.

3.38 Remark:

Inverse hyperbolic functions can be treated in a corresponding manner.

3.39 Example:

Find $sinh^{-1}z$.

Solution:

$$w = sinh^{-1}z \Longrightarrow z = \sinh w \Longrightarrow z = \frac{\boldsymbol{e^w - e^{-w}}}{\boldsymbol{2}} \Longrightarrow 2z = \boldsymbol{e^w} - \boldsymbol{e^{-w}}$$

$$\Longrightarrow (e^w)^2 - 2ze^w - 1 = 0.$$

$$e^w = \frac{2z+\sqrt{4z^2+4}}{2} = \frac{2z+2\sqrt{z^2+1}}{2} = z+\sqrt{z^2+1} \Longrightarrow w = \log(z+\sqrt{z^2+1}).$$

So $sinh^{-1}z = \log(z+\sqrt{z^2+1})$.

3.40 Example:

Find $cosh^{-1}z$.

Solution:

$$w = cosh^{-1}z \Rightarrow z = \cosh w \Rightarrow z = \frac{e^w + e^{-w}}{2} \Rightarrow 2z = e^w + e^{-w}$$

$$\Rightarrow (e^w)^2 - 2ze^w + 1 = 0.$$

$$e^w = \frac{2z + \sqrt{4z^2 - 4}}{2} = \frac{2z + 2\sqrt{z^2 - 1}}{2} = z + \sqrt{z^2 - 1} \Rightarrow w = \log(z + \sqrt{z^2 - 1}).$$

So $cosh^{-1}z = \log(z + \sqrt{z^2 - 1})$.

3.41 Example:

Find $tanh^{-1}z$.

Solution:

$$w = tanh^{-1}z \Rightarrow z = \tanh w \Rightarrow z = \frac{e^w - e^{-w}}{e^w + e^{-w}} \Rightarrow z(e^w + e^{-w}) = e^w - e^{-w}$$

$$\Rightarrow ze^w + ze^{-w} = e^w - e^{-w} \Rightarrow ze^w + ze^{-w} - e^w + e^{-w} = 0.$$

$$\Rightarrow (z-1)e^w + (z+1)e^{-w} = 0 \Rightarrow (z-1)e^{2w} + (z+1) = 0$$

$$\Rightarrow e^{2w} = \frac{-(z+1)}{(z-1)} = \frac{(z+1)}{(1-z)} \Rightarrow 2w = log\frac{(1+z)}{(1-z)} \Rightarrow w = \frac{1}{2}log\frac{(1+z)}{(1-z)}.$$

So $tanh^{-1}z = \frac{1}{2}log\frac{(1+z)}{(1-z)}$.

EXERCISES:

1. Find

a) $cot^{-1}z$.

b) $sec^{-1}z$.

c) $csc^{-1}z$.

d) $coth^{-1}z$.

e) $sech^{-1}z$.

f) $csch^{-1}z$.

g) $\frac{d}{dz}cot^{-1}z$.

h) $\frac{d}{dz}sec^{-1}z$.

i) $\frac{d}{dz}csc^{-1}z$.

j) $\frac{d}{dz}coth^{-1}z$.

k) $\frac{d}{dz} sech^{-1} z.$

l) $\frac{d}{dz} csch^{-1} z$

2. Find all the values of

a) $tan^{-1}(2i)$, **b)** $tan^{-1}(1 + i)$, **c)** $cosh^{-1}(-1)$, **d)** $tanh^{-1}0$.

3. Solve the equation $sin\, z = 2$ for z by

a) equating real parts and then imaginary parts in that equation;

b) using expression for $sin^{-1} z$.

4. Solve the equation $cos\, z = \sqrt{2}$ for z.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ٩

Chapter Four

INTEGRALS

Integrals are extremely important in the study of functions of a complex variable. The theory of integration, to be developed in this chapter, is noted for its mathematical elegance. The theorems are generally concise and powerful, and many of the proofs are short.

4.1 Daefinition (Integrals Of Complex–Valued Function Of a Real Valued):

Let $w(t) = u(t) + iv(t)$ be a complex – valued function of a real valued t where the functions u and v are real – valued functions of t. The definite integral of $w(t)$ over an interval $a \le t \le b$ is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt, \tag{1}$$

provided the individual integrals on the right exist. Thus

$$Re\int_a^b w(t)dt = \int_a^b Re[w(t)]dt \quad \text{and} \quad Im\int_a^b w(t)dt = \int_a^b Im[w(t)]dt \tag{2}$$

4.2 Example:

$$\int_0^1 (1+it)^2\,dt = \int_0^1 (1-t^2)dt + i\int_0^1 2tdt = t - \frac{t^3}{3}\Big]_0^1 + it^2]_0^1 = 1 - \frac{1}{3} + i = \frac{2}{3} + i\ .$$

Where $(1+it)^2 = 1 + 2it - t^2 = 1 - t^2 + 2it.$

4.3 Remark:

1. Improper integrals of $w(t)$ over unbounded intervals are defined in a similar way.
2. The existence of the integrals of u and v in definition (2) is ensured if those functions are ***piecewise continuous*** on the interval $a \le t \le b$. Such a function is continuous everywhere in the stated interval except possibly for a finite number of points where, although discontinuous, it has one-sided limits. Of course, only the right-hand limit is required at a; and only the left-hand limit is required at b. When both u and v are piecewise continuous, the function w is said to have that property. Anticipated rules for integrating a complex constant times a function $w(t)$, for integrating sums of such functions, and for interchanging limits of integration are all valid. Those rules, as well as the property

$$\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt,$$

are easy to verify by recalling corresponding results in calculus.

4.4 Theorem(Fundamental Theorem Of Calculus):

suppose that the functions $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ are continuous on the interval $a \le t \le b$. If $W'(t) = w(t)$ when $a \le t \le b$, then $U'(t) = u(t)$ and $V'(t) = v(t)$.Hence

$$\int_a^b w(t)dt = U(t)]_a^b + iV(t)]_a^b = [U(b) + iV(b)] - [U(a) + iV(a)].$$

That is,

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)]_a^b$$

4.5 Example:

$$\int_0^{\pi/4} e^{it}\,dt = \frac{e^{it}}{i}\Big]_0^{\pi/4} = \frac{e^{i\pi/4}}{i} - \frac{1}{i} = \frac{1}{i}\left(e^{i\pi/4} - 1\right) = \frac{1}{i}\frac{i}{i}(\cos\pi/4 + i\sin\pi/4 - 1)$$

$$= -i\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} - 1\right) = \frac{-i}{\sqrt{2}} + \frac{1}{\sqrt{2}} + i = \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right).$$

4.6 Theorem(mean value theorem for derivatives in calculus):

Suppose that $w(t)$ is continuous on an interval $a \le t \le b$; that is, its component functions $u(t)$ and $v(t)$ are continuous there. Even if $w'(t)$ exists when $a < t < b$ then there is a number c in the interval $a < t < b$ such that

$$w'(c) = \frac{W(b) - W(a)}{b - a}.$$

4.7 Remark:

1. The mean value theorem for derivatives in calculus does not carry over to complex - valued functions $w(t)$, i.e. it is not necessarily true that there is a number c in the interval $a < t < b$ such that $w'(c) = \frac{W(b)-W(a)}{b-a}$,for example the function $w(t) = e^{it}$ on the interval $0 \le t \le 2\pi$ then $|w'(t)| = |ie^{it}| = 1$and this means that the derivative $w'(t)$ is never zero, while $w(2\pi) - w(0) = e^{2\pi i} - e^0 = \cos 2\pi + i\sin 2\pi - 1 = 1 - 1 = 0$.

2. The following example shows that the mean value theorem for integrals does not carry over either. Thus special care must continue to be used in applying rules from calculus

4.8 Example:

Let $w(t)$ be a complex – valued function of a real valued t defined on an interval $a \le t \le b$. In order to show that it is not necessarily true that there is a number c

in the interval $a < t < b$ such that

$$\int_a^b w(t)dt = w(c)(b-a) ,$$

we write $a = 0$, $b = 2\pi$ and use the same function $w(t) = e^{it}$ $(0 \le t \le 2\pi)$ as in the remark 4.7. It is easy to see that

$$\int_a^b w(t)dt = \int_0^{2\pi} e^{it}dt = \frac{e^{it}}{i}\bigg]_0^{2\pi} = \frac{e^{2\pi i}}{i} - \frac{e^0}{i} = \frac{1}{i} - \frac{1}{i} = 0.$$

But, for any number c such that $0 < c < 2\pi$

$$|w(c)(b-a)| = \left|e^{ic}\right|2\pi = 2\pi,$$

and this means that $w(c)(b - a)$ is not zero.

EXERCISES:

1. Evaluate the following integrals:

a) $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt$; **b)** $\int_0^{\pi/6} e^{2it}dt$; **c)** $\int_0^{\infty} e^{-zt}dt$, $(Re(z) > 0)$.

2. Show that if m and n are integers,

$$\int_0^{2\pi} e^{im\theta}e^{-in\theta}d\theta = \begin{cases} 0 & \text{when } n \ne m \\ 2\pi & \text{when } n = m \end{cases}.$$

3. According to definition (1) of definite integrals of complex-valued functions of Real variable

$$\int_0^{\pi} e^{(1+i)x}dx = \int_0^{\pi} e^x \cos x\, dx + i\int_0^{\pi} e^x \sin x\, dx.$$

Evaluate the two integrals on the right here by evaluating the single integral on the left and then using the real and imaginary parts of the value found.

4. Let $w(t) = u(t) + iv(t)$ denote a continuous complex – valued function defined on an interval $-a \le t \le b$.

a) Suppose that $w(t)$ is *even*; that is, $w(-t) = w(t)$ for each point t in the given interval. Show that

$$\int_{-a}^{a} w(t)dt = 2\int_0^a w(t)dt.$$

b) Show that if $w(t)$ is an *odd* function, one where $w(-t) = -w(t)$ for each point t in the given interval, then

$$\int_{-a}^{a} w(t)dt = 0.$$

4.9 (CONTOURS):

Integrals of complex-valued functions of a complex variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are

1. A set of points $z = (x, y)$ in the complex plane is said to be an ***arc*** if

$$x = x(t),\ y = y(t).\ \ (a \le t \le b). \tag{3}$$

Where $x(t)$ and $y(t)$ are continuous functions of the real parameter t. This definition establishes a continuous mapping of the interval $a \le t \le b$ into the xy, or z, plane; and the image points are ordered according to increasing values of t. It is convenient to describe the points of ***C*** by means of the equation

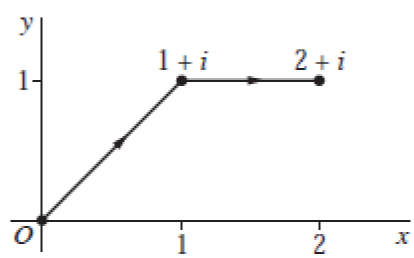
$$z(t) = x(t) + iy(t).$$

2. The **arc *C*** is a ***simple arc***, or a ***Jordan arc***, if it does not cross itself ; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$.

3. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a ***simple closed curve***, or a Jordan curve.

4. Such a curve is ***positively oriented*** when it is in the counterclockwise direction.

4.10 Example:

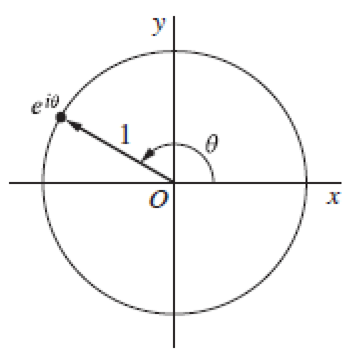


The ***polygonal line*** defined by means of the equations

$$z = \begin{cases} x + ix & when\ 0 \le x \le 1 \\ x + i & when\ 1 \le x \le 2 \end{cases},$$

and consisting of a line segment from 0 to 1 + *i* followed by one from 1 + *i* to 2 + *i* is a simple arc.

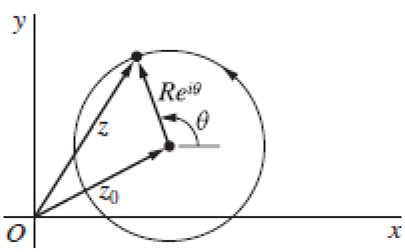
4.11 Example:



The unit circle

$$z = e^{i\theta}, 0 \le \theta \le 2\pi,$$

about the origin is a simple closed curve, oriented in the *counterclockwise* direction.



So is the circle

$$z = z_0 + Re^{i\theta}, 0 \le \theta \le 2\pi,$$

centered at the point z_0 and with radius R. The same set of points can make up different arcs.

4.12 Example:

The arc $z = e^{-i\theta}, 0 \leq \theta \leq 2\pi$ is not the same as the arc described by equation $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$. The set of points is the same, but now the circle is traversed in the *clockwise* direction.

4.13 Remark:

1. A ***contour*** or ***piecewise smooth arc*** is an arc consisting of a finite number of Smooth arcs joined end to end. Hence if equation $z = z(t) = x(t) + iy(t)$, $a \leq t \leq b$ represents a contour, $z(t)$ is continuous, whereas its derivative $z'(t)$ is piecewise continuous. The polygonal line in example 4.10 is a contour.
2. When only the initial and final values of $z(t)$ are the same, a contour ***C*** is called a ***simple closed contour***. The circles in examples 4.11 and 4.12 are simple closed contour as well as the boundary of a triangle or a rectangle taken in a specific direction.
3. The length of a contour or a simple closed contour is the sum of the lengths of the smooth arcs that make up the contour.
4. The points on any simple closed curve or simple closed contour ***C*** are **boundary points** of two distinct domains, one of which is the ***interior*** of ***C*** and is bounded. The other, which is the ***exterior*** of ***C***, is unbounded. It will be convenient to accept this statement, known as the ***Jordan curve theorem***.

4.14 Remark:

We turn now to integrals of complex-valued functions f of the complex variable z. Such an integral is defined in terms of the values $f(z)$ along a given contour C, extending from a point $z = z_1$ to a point $z = z_2$ in the complex plane. It is, therefore, a line integral ; and its value depends in general on the contour C as well as on the function f . It is written

$$\int_C f(z)dz \quad \text{or} \quad \int_{z_1}^{z_2} f(z)dz$$

the latter notation often being used when the value of the integral is independent of the choice of the contour taken between two fixed end points.

4.15 Remark:

Suppose that the equation $z = z(t)$, $a \leq t \leq b$ represents a contour C, extending from a point $z_1 = z(a)$ to a point $z_2 = z(b)$. We assume that $f[z(t)]$ is piecewise continuouson the interval $a \leq t \leq b$ and refer to the function $f(z)$ as being piecewise continuous on C. We then define the line integral, or ***contour integral, of f along C*** in terms of the parameter t :

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt.$$

Note that since C is a contour $z'(t)$ is also piecewise continuous on $a \le t \le b$; and so the existence of integral is ensured.

4.16 Remark:

The properties of integrals of complex-valued functions $w(t)$ are:

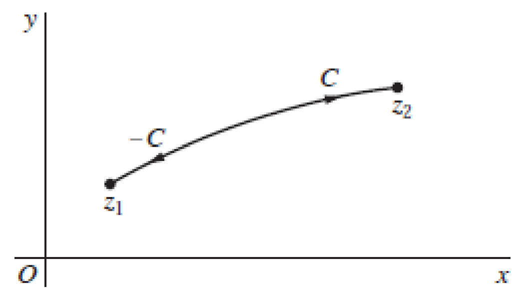
1. For any complex constant z_0 we have

$$\int_C z_0 \cdot f(z)dz = z_0 \cdot \int_C f(z)dz.$$

2. $$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz.$$

3. Associated with the contour C used in integral

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$



is the contour $-C$ consisting of the same set of points but with the order reversed so that the new contour extends from the point z_2 to the point z_1. The contour $-C$ has parametric representation.

$$z = z(-t), \ -b \le t \le -a.$$

Hence, $\int_{-C} f(z)dz = \int_{-b}^{-a} f[z(-t)] \frac{d}{dt} z(-t)dt = -\int_{-b}^{-a} f[z(-t)]z'(-t)dt.$

where $z'(-t)$ denotes the derivative of $z(t)$ with respect to t , evaluated at $-t$.

Making the substitution $\tau = -t$ in this last integral we obtain the expression

$$\int_{-C} f(z)dz = -\int_a^b f[z(\tau)]z'(\tau)dt.$$

which is the same as

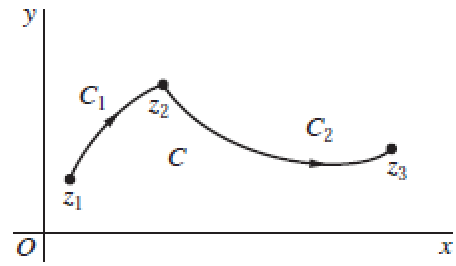
$$\int_{-C} f(z)dz = -\int_C f(z)dz.$$

4. Consider now a path C, with representation $z = z(t)$, $a \le t \le b$, that consists of a contour C_1 from z_1 to z_2 followed by a contour C_2 from z_2 to z_3, the initial point of C_2 being the final point of C_1. There is a value c of t, where $a < c < b$, such that $z(c) = z_2$. Consequently, C_1 is represented by $z = z(t)$,$(a \le t \le c)$ and C_2 is represented by $z = z(t)$,$(c \le t \le b)$.So

$$\int_a^b f[z(t)]z'(t)dt = \int_a^c f[z(t)]z'(t)dt + \int_c^b f[z(t)]z'(t)dt.$$

Evidently $\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_1} f(z)dz$.

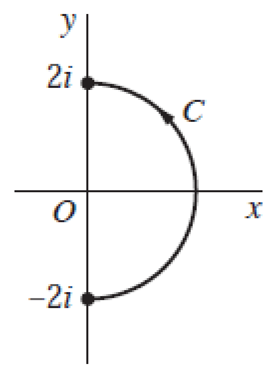
Sometimes the contour C is called the ***sum*** of its Legs C_1 and C_2 and is denoted by $C_1 + C_2$. The sum of two contours C_1 and $-C_2$ is well defined when C_1 and C_2 have the same final points and it is written $C_1 - C_2$.



4.17 Example:

Find the value of the integral $I = \int_C \bar{z}dz$. When C is the right - hand half $z = 2e^{i\theta}, (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$ of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$.

Solution:



$$I = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} \left(2e^{i\theta}\right)' d\theta = 4 \int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}} \left(e^{i\theta}\right)' d\theta,$$

and since $\overline{e^{i\theta}} = e^{-i\theta}$ and $\left(e^{i\theta}\right)' = ie^{i\theta}$ this mean that

$$I = 4 \int_{-\pi/2}^{\pi/2} e^{-i\theta}\, ie^{i\theta} d\theta = 4i \int_{-\pi/2}^{\pi/2} e^{i\theta - i\theta}\, d\theta = 4i \int_{-\pi/2}^{\pi/2} d\theta$$

$$= 4i\theta\Big]_{-\pi/2}^{\pi/2} = 4i[\pi/2 - (-\pi/2)] = 4i\left[\frac{\pi}{2} + \frac{\pi}{2}\right] 4\pi i.$$

4.18 Remark:

Note that $z\bar{z} = |z|^2 = 4$ when z is a point on the semicircle C. Hence the result $\int_C \bar{z}dz = 4\pi i$ can also be written $\int_C \frac{1}{z} dz = \pi i$.

4.19 Example:

Let C_1 denote the polygonal line OAB shown in Fig.1 , and evaluate the integral

$$I_1 = \int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz.$$

Where $f(z) = y - x - i3x^2, (z = x + iy)$.

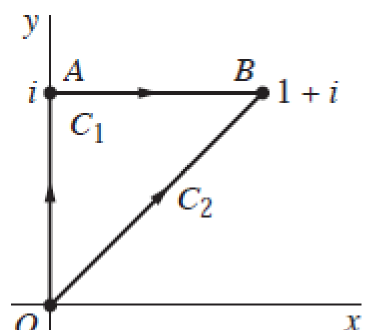


Fig. 1

Solution:

The leg OA may be represented parametrically as $z = 0 + iy \ (0 \le y \le 1)$; and, since $x = 0$ at points on that line segment, the values of f there vary with the parameter y according to the equation $f(z) = y \ (0 \le y \le 1)$. Consequently,

$$\int_{OA} f(z)dz = \int_0^1 yi\, dy = i\int_0^1 y\, dy = i\frac{y^2}{2}\Big]_0^1 = \frac{i}{2}.$$

On the leg AB, the points are $z = x + i\ (0 \le x \le 1)$; and, since $y = 1$ on this segment,

$$\int_{AB} f(z)dz = \int_0^1 (1 - x - i3x^2) \cdot 1dx = \left(x - \frac{x^2}{2} - i3\frac{x^3}{3}\right)\Big]_0^1 = \left(1 - \frac{1}{2} - i\right) = \frac{1}{2} - i.$$

We now see that

$$I_1 = \int_{C_1} f(z)dz = \int_{OA} f(z)dz + \int_{AB} f(z)dz = \frac{i}{2} + \frac{1}{2} - i = \frac{1}{2} - \frac{i}{2} = \frac{1-i}{2}.$$

If C_2 denotes the segment OB of the line $y = x$ in Fig. 1, with parametric representation $z = x + ix\ (0 \le x \le 1)$, the fact that $y = x$ on OB enables us to write

$$I_2 = \int_{C_2} f(z)dz = \int_0^1 -i3x^2(1+i)dx = -3i(1+i)\int_0^1 x^2 dx = 3(1-i)\frac{x^3}{3}\Big]_0^1$$

$$= 3(1-i)\frac{1}{3} = (1-i) = (1-i).$$

Evidently, then, the integrals of $f\ (z)$ along the two paths C_1 and C_2 have different values even though those paths have the same initial and the same final points. Observe how it follows that the integral of $f\ (z)$ over the simple closed contour $OABO$, or $C_1 - C_1$, has the nonzero value

$$I_1 - I_2 = \frac{1-i}{2} - (1-i) = \frac{1-i-2+2i}{2} = \frac{-1+i}{2}.$$

4.20 Example:

Let C denote an arbitrary smooth arc $z = z(t),\ (a \le t \le b)$ from a fixed Point z_1 to a fixed point z_2 (Fig. 2).Evaluate the integral

$$\int_C z \cdot dz = \int_a^b z(t) \cdot z'(t)dt.$$

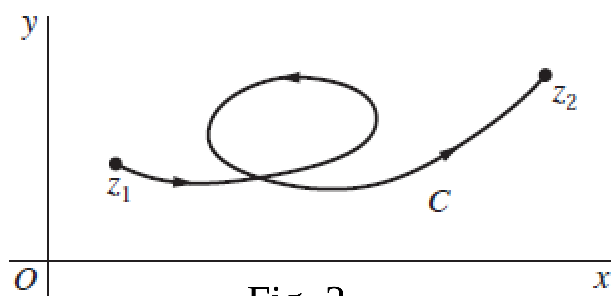


Fig. 2

Solution:

Since $\frac{d}{dz} \cdot \frac{[z(t)]^2}{2} = z(t) \cdot z'(t)$ and since $z(a) = z_1$ and $z(b) = z_2$, we have

$$\int_C z \cdot dz = \int_a^b z(t) \cdot z'(t)dt = \frac{[z(t)]^2}{2}\Big]_a^b = \frac{[z(b)]^2 - [z(a)]^2}{2} = \frac{z_2^2 - z_1^2}{2}.$$

Inasmuch as the value of this integral depends only on the end points of C and is otherwise independent of the arc that is taken, we may write

$$\int_{z_1}^{z_2} zdz = \frac{z_2^2 - z_1^2}{2}$$

4.21 Remark:

Expression $\int_{z_1}^{z_2} z dz = \frac{z_2^2 - z_1^2}{2}$ is also valid when C is a contour that is not necessarily smooth since a contour consists of a finite number of smooth arcs C_k $(k = 1, 2, \cdots, n)$ joined end to end. More precisely, suppose that each C_k extends from z_k to z_{k+1}.Then

$$\int_C z \cdot dz = \sum_{k=1}^{n} \int_{C_k} z \cdot dz = \sum_{k=1}^{n} \int_{z_k}^{z_{k+1}} z dz = \sum_{k=1}^{n} \frac{z_{k+1}^2 - z_k^2}{2}$$

$$= \frac{z_{n+1}^2 - z_n^2}{2} + \frac{z_n^2 - z_{n-1}^2}{2} + \cdots + \frac{z_3^2 - z_2^2}{2} + \frac{z_2^2 - z_1^2}{2} = \frac{z_{n+1}^2 - z_1^2}{2}. \quad (4)$$

Where this last summation has telescoped and z_1 is the initial point of C and z_{n+1} is its final point. It follows from expression (4) that the integral of the function $f(z) = z$ around each closed contour in the plane has value zero.

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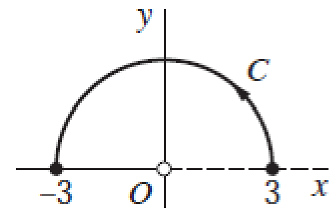
Chapter Four

INTEGRALS

4.22 Remark:

The path in a contour integral can contain a point on a branch cut of the integrand involved. The next two examples illustrate this.

4.23 Example:



Let C denote the semicircular path $z = 3e^{i\theta}, 0 \le \theta \le \pi$ from the point $z = 3$ to the point $z = -3$. Although the branch

$$f(z) = z^{\frac{1}{2}} = \exp(\tfrac{1}{2}\log z), \quad (|z| > 0, 0 < \arg(z) < 2\pi),$$

of the multiple-valued function $z^{\frac{1}{2}}$ is not defined at the initial point $z = 3$ of the contour C, the integral $I = \int_C z^{\frac{1}{2}} dz$ nevertheless exists. For the integrand is piecewise continuous on C. To see that this is so, we first observe that when $z(\theta) = 3e^{i\theta}$,

$$f[z(\theta)] = \exp\left[\frac{1}{2}(ln3 + i\theta)\right] = \exp\left[(ln3^{\frac{1}{2}} + \frac{i\theta}{2})\right] = \exp ln\sqrt{3} \cdot \exp\frac{i\theta}{2} = \sqrt{3}e^{i\frac{\theta}{2}}.$$

Hence the right-hand limits of the real and imaginary components of the function

$$f[z(\theta)] \cdot z'(\theta) = \sqrt{3}e^{i\frac{\theta}{2}} \cdot 3ie^{i\theta} = 3\sqrt{3}ie^{i\frac{3\theta}{2}} = -3\sqrt{3}\sin\frac{3\theta}{2} + i3\sqrt{3}\cos\frac{3\theta}{2}, \quad (0 < \theta \le \pi)$$

at $\theta = 0$ exist, those limits being 0 and $i3\sqrt{3}$ respectively. This means that $f[z(\theta)] \cdot z'(\theta)$ is continuous on the closed interval $0 \le \theta \le \pi$ when its value at $\theta = 0$ is defined as $i3\sqrt{3}$. Consequently $I = i3\sqrt{3}\int_0^\pi e^{i\frac{3\theta}{2}} d\theta$.

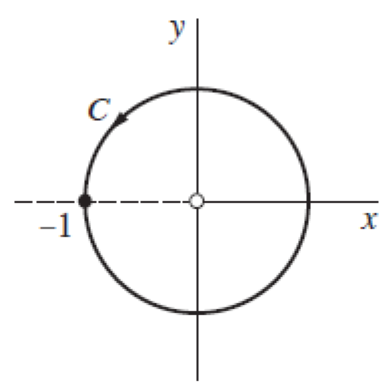
Since

$$\int_0^\pi e^{i\frac{3\theta}{2}} d\theta = \frac{2}{3i}\int_0^\pi \frac{3i}{2} e^{i\frac{3\theta}{2}} d\theta = \frac{2}{3i} e^{i\frac{3\theta}{2}}\Big]_0^\pi = \frac{2}{3i}\left(e^{i\frac{3\pi}{2}} - e^0\right) = \frac{2}{3i}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} - 1\right) = -\frac{2}{3i}(1+i)$$

we now have the value

$$I = i3\sqrt{3}\int_0^\pi e^{i\frac{3\theta}{2}} d\theta = i3\sqrt{3} \cdot -\frac{2}{3i}(1+i) = -2\sqrt{3}(1+i).$$

4.24 Example:



Suppose that C is the positively oriented circle

$$z = Re^{i\theta}, (-\pi \le \theta \le \pi),$$

about the origin, and left a denote any nonzero real number. Using the principal branch

$f(z) = z^{a-1} = e^{Log z^{a-1}} = e^{(a-1)Log z} = , (|z| > 0, -\pi < Arg\, z < \pi),$
of the power function z^{a-1}, let us evaluate the integral $I = \int_C z^{a-1} dz$. When $z(\theta) = Re^{i\theta}$, it is easy to see that

$$f[z(\theta)] \cdot z'(\theta) = iR^a e^{ia\theta} = iR^a(\cos a\theta + i \sin a\theta) = -R^a \sin a\theta + iR^a \cos a\theta,$$

where the positive value of R^a is to be taken. Inasmuch as this function is piecewise continuous on $-\pi < \theta < \pi$, integral $I = \int_C z^{a-1} dz$ exists. In fact,

$$I = iR^a \int_{-\pi}^{\pi} e^{ia\theta}\, d\theta = iR^a \frac{1}{ia} \int_{-\pi}^{\pi} ia \cdot e^{ia\theta}\, d\theta = \frac{R^a}{a} e^{ia\theta}\Big]_{-\pi}^{\pi} = \frac{R^a}{a}\left[e^{ia\pi} - e^{-ia\pi}\right] =$$

$$= i\frac{2R^a}{a}\left[\frac{e^{ia\pi} - e^{-ia\pi}}{2i}\right] = i\frac{2R^a}{a} \sin a\pi.$$

Note that if a is a nonzero integer n, this result tells us that

$$I = \int_C z^{n-1} dz = 0, \quad (n = \pm 1, \pm 2, \cdots).$$

If a is allowed to be zero, we have

$$I = \int_C z^{-1} dz = \int_C \frac{1}{z} dz = \int_{-\pi}^{\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta}\, d\theta = i\int_{-\pi}^{\pi} d\theta = i\theta\Big]_{-\pi}^{\pi} = 2\pi i.$$

EXERCISES:

For the functions f and contours C in Exercises 1 through 7, use parametric representations for C, or legs of C, to evaluate $\int_C f(z) dz$:

1. $f(z) = \frac{z+2}{z}$ and C is

a) the semicircle $z = 2e^{i\theta}$ $(0 \le \theta \le \pi)$;

b) the semicircle $z = 2e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

c) the circle $z = 2e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

2. $f(z) = z - 1$ and C is the arc from $z = 0$ to $z = 2$ consisting of

a) the semicircle $z = 1 + e^{i\theta}$ $(\pi \le \theta \le 2\pi)$;

b) the segment $z = x$ $(0 \le x \le 2)$ of the real axis.

3. $f(z) = \pi e^{\pi \bar{z}}$ and C is the boundary of the square with vertices at the points 0, 1,1+i and i, the orientation of C being in the counterclockwise direction.

4. $f(z)$ is defined by means of the equations

$$f(z) = \begin{cases} 1 & \text{when } y < 0 \\ 4y & \text{when } y > 0\text{'} \end{cases}$$

and C is the arc from $z = -1 - i$ to $z = 1 + i$ along the curve $y = x^3$.

5. $f(z) = 1$ and C is an arbitrary contour from any fixed point z_1 to any fixed point z_2 in the z- plane.

6. $f(z)$ is the branch $z^{-1+i} = e^{(-1+i)log z}$, $(|z| > 0, 0 < Arg\, z < 2\pi)$, of the indicated power function, and C is the unit circle $z = e^{i\theta}$ $(0 \le \theta \le 2\pi)$.

7. $f(z)$ is the principal branch $z^i = e^{iLogz}$, $(|z| > 0, -\pi < Arg\, z < \pi)$ of this power function, and C is the semicircle $z = e^{i\theta}$ $(0 \le \theta \le \pi)$.

8. Evaluate the integral $\int_C z^m z^{-n} dz$, where m and n are integers and C is the unit circle $|z| = 1$, taken counterclockwise.

9. Evaluate the integral $I = \int_C \bar{z} dz$ for C is $z = \sqrt{4 - y^2} + iy$, $(-2 \le \theta \le 2\pi)$.

10. Let C_0 and C denote the circles $z = z_0 + Re^{i\theta}$ $(-\pi \le \theta \le \pi)$ and $z = Re^{i\theta}$ $(-\pi \le \theta \le \pi)$, respectively.

a) Use these parametric representations to show that $\int_{C_0} f(z - z_0) dz = \int_C f(z) dz$ when f is piecewise continuous on C.

b) Apply the result in part ***a)*** to integrals $\int_C z^{n-1} dz$ and $\int_C \frac{dz}{z}$ to show that $\int_{C_0} (z - z_0)^{n-1} dz = 0$, $(n = \pm 1, \pm 2, \cdots)$ and $\int_{C_0} \frac{dz}{z} = 2\pi i$.

4.25 (UPPER BOUNDS FOR MODULI OF CONTOUR INTEGRALS):

We turn now to an inequality involving contour integrals that is extremely important in various applications.

4.26 Lemma:

If $w(t)$ is a piecewise continuous complex-valued function defined on an interval a ≤ t ≤ b, then

$$\left|\int_a^b w(t)dt\right| \le \int_a^b |w(t)|\, dt.$$

4.27 Theorem:

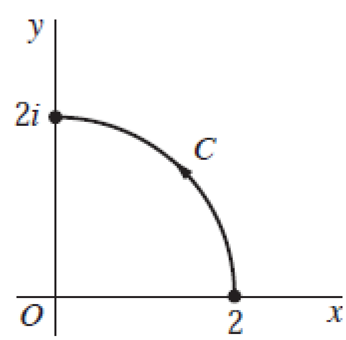
Let C denote a contour of length L, and suppose that a function $f(z)$ is piecewise continuous on C. If M is a nonnegative constant such that $|f(z)| \le M$ for all points z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)dz\right| \le ML.$$

4.28 Example:

Let C be the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Show that

$$\left|\int_C \frac{z+4}{z^3-1} dz\right| \le \frac{6\pi}{7}$$



Solution:

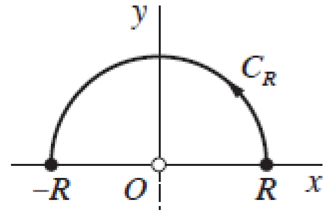
If z is a point on C, so that $|z| = 2$, then

$$|z + 4| \le |z| + 4 = 6 \text{ and } |z^3 - 1| \ge ||z|^3 - 1| = 7.$$

Thus, when z lies on C, $\left|\frac{z+4}{z^3-1}\right| = \frac{|z+4|}{|z^3-1|} \le \frac{6}{7}$. Writing $M = 6/7$ and observing that $L = \pi$ is the length of C. by theorem 4.27 we obtain

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \le ML = \frac{6\pi}{7}.$$

4.29 Example:



Let C_R is the semicircular path $z = Re^{i\theta}, (0 \le \theta \le \pi)$, and $z^{\frac{1}{2}}$ denotes the branch

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} = \sqrt{r}e^{i\frac{\theta}{2}},\ (r > 0,\ -\frac{\pi}{2} < \theta < \frac{3\pi}{2})$$

of the square root function. Show that $\lim\limits_{R\to\infty}\int_{C_R}\frac{z^{\frac{1}{2}}}{z^2+1}dz = 0.$

Solution:

For, when $|z| = R > 1$, $\left|z^{\frac{1}{2}}\right| = \left|\sqrt{R}e^{i\frac{\theta}{2}}\right| = \sqrt{R}$ and $|z^2+1| \ge ||z|^2-1| = R^2-1$. Consequently, at points on C_R, $\left|\frac{z^{\frac{1}{2}}}{z^2+1}\right| \le M_R$ where $M_R = \frac{\sqrt{R}}{R^2-1}$. Since the length of C_R is the number $L = \pi R$, it follows from theorem 4.27 that $\left|\int_{C_R}\frac{z^{\frac{1}{2}}}{z^2+1}dz\right| \le M_R L$. But $M_R L = \frac{\pi R\sqrt{R}}{R^2-1}\cdot\frac{\frac{1}{R^2}}{\frac{1}{R^2}} = \frac{\frac{\pi}{\sqrt{R}}}{1-\frac{1}{R^2}}$, and it is clear that the term on the far right here tends to zero as R tends to infinity, i.e. $\lim\limits_{R\to\infty}\frac{\frac{\pi}{\sqrt{R}}}{1-\frac{1}{R^2}} = 0$.Then $\lim\limits_{R\to\infty}\int_{C_R}\frac{z^{\frac{1}{2}}}{z^2+1}dz = 0.$

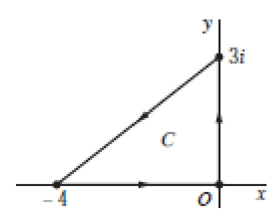
inequality

EXERCISES:

1. Show that $\left|\int_C\frac{dz}{z^2-1}\right| \le \frac{\pi}{3}$ when C is the arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant.

2. Let C denote the line segment from $z = i$ to $z = 1$. By observing that of all the points on that line segment, the midpoint is the closest to the origin, show that $\left|\int_C\frac{dz}{z^4}\right| \le 4\sqrt{2}$ without evaluating the integral.

3. Show that if C is the boundary of the triangle with vertices at the points $0, 3i,$ and -4, oriented in the counterclockwise direction, then $\left|\int_C(e^z-\bar{z})dz\right| \le 60.$



4. Let C_R denote the upper half of the circle |z| = R (R > 2), taken in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{2z^2-1}{z^4+5z^2+4} dz\right| \le \frac{\pi R(2R^2+1)}{(R^2-1)(R^2-4)}.$$

Then, by dividing the numerator and denominator on the right here by R^4, show that the value of the integral tends to zero as R tends to infinity.

5. Let C_R be the circle $|z| = R\ (R > 1)$, described in the counterclockwise direction. Show that

$$\left|\int_{C_R} \frac{Log\ z}{z^2} dz\right| \le 2\pi(\frac{\pi+lnR}{R},$$

and show that the value of this integral tends to zero as R tends to infinity.

6. Let C_ρ denote a circle $|z| = \rho\ (0 < \rho < 1)$, oriented in the counterclockwise direction, and suppose that $f\ (z)$ is analytic in the disk $|z| \le 1$. Show that if $z^{-\frac{1}{2}}$ represents any particular branch of that power of z, then there is a nonnegative constant M, independent of ρ, such that

$$\left|\int_{C_\rho} z^{-\frac{1}{2}} f(z) dz\right| \le 2\pi M\sqrt{\rho}\ .$$

Thus show that the value of the integral here approaches 0 as ρ tends to 0.

4.30 ANTIDERIVATIVES:

Although the value of a contour integral of a function $f\ (z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path that is taken, there are certain functions whose integrals from z_1 to z_2 have values that are independent of path. (Recall Examples 4.19 and 4.20) The examples just cited also illustrate the fact that the values of integrals around closed paths are sometimes, but not always, zero. Our next theorem is useful in determining when integration is independent of path and, moreover, when an integral around a closed path has value zero.

4.31 Definition:

Let $f\ (\ z\ )$ be a single-valued analytic function on a domain D . Then a function $\Phi(\ z\ )$ is said to be ***an indefinite integral*** (or ***antiderivative***) of $f\ (\ z\ )$ on D if $\Phi(\ z\ )$ is single-valued and analytic on D , and $\Phi'(\ z\ ) = f(z), \forall z \epsilon D.$

4.32 Theorem:

Suppose that a function f (z) is continuous on a domain D. If any one of the following statements is true, then so are the others:

(a) ***f (z) has an antiderivative F(z) throughout D;***

(b) ***The integrals of*** $f\ (z)$ ***along contours lying entirely in D and extending from any fixed point*** z_1 ***to any fixed point*** z_2 ***all have the same value, namely***

$$\int_{z_1}^{z_2} f(z)dz = F(x)\Big]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

***where F(z) is the antiderivative in statement* (a);**

(c) ***The integrals of*** $f\ (z)$ ***around closed contours lying entirely in D all have Value zero.***

4.33 Example:

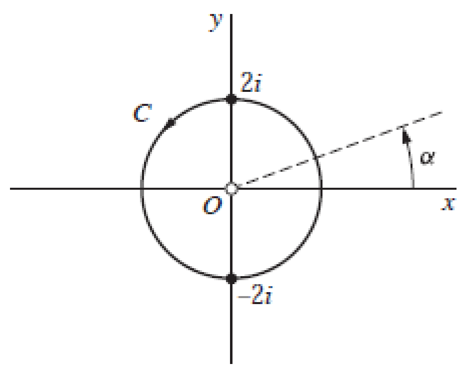
The continuous function $f\ (z) = z^2$ has an antiderivative $F(z) = \frac{z^3}{3}$ throughout the plane. Hence

$$\int_0^{1+i} z^2 dz = \frac{z^3}{3}\Big]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{1}{3}(1 + 3i + 3i^2 + i^3)$$

$$= \frac{1}{3}(1 + 3i - 3 - i) = \frac{2}{3}(-1 + i),$$

for every contour from $z = 0$ to $z = 1 + i$.

4.34 Example:

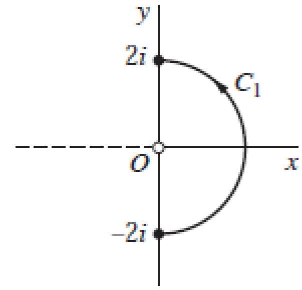


The function $f\ (z) = \frac{1}{z^2}$, which is continuous Everywhere except at the origin, has an antiderivative $F(z) = -\frac{1}{z}$ in the domain $|z| > 0$, consisting of the entire plane with the origin deleted. Consequently $\int_C \frac{dz}{z^2} = 0$ when C is the positively oriented circle $z = 2\,e^{i\theta}\ (-\pi \le \theta \le \pi)$ about the origin.

4.35 Remark:

Note that the integral of the function $f\ (z) = \frac{1}{z}$ around the same circle cannot be evaluated in a similar way. For, although the derivative of any branch $F(z)$ of $log\ z$ is $1/z$, $F(z)$ is not differentiable, or even defined, along its branch cut. In particular, if a $ray\ \theta = \alpha$ from the origin is used to form the branch cut, $F'(\,z\,)$ fails to exist at the point where that ray intersects the circle C. So C does not lie in any domain throughout which $F'(\,z\,) = \frac{1}{z}$, and one cannot make direct use of an antiderivative.The next example illustrates how a combination of two different antiderivatives can be used to evaluate $f\ (z) = \frac{1}{z}$ around C.

4.36 Example:



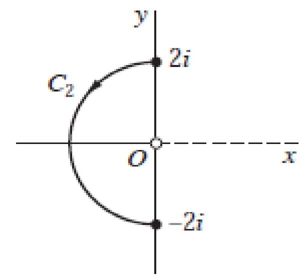
Let C_1 denote the right half $z = 2e^{i\theta} (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$ of the circle C in Example 4.34. The principal branch

$$Logz = lnr + i\Theta, \quad (r > 0, -\pi < \Theta < \pi),$$

of the logarithmic function serves as an antiderivative of the function 1/*z* in the evaluation of the integral of 1/*z* along C_1.

$$\int_{C_1} \frac{dz}{z} = \int_{-2i}^{2i} \frac{dz}{z} = Logz]_{-2i}^{2i} = Log(2i) - Log(-2i) = \left(ln2 + i\frac{\pi}{2}\right) - \left(ln2 - i\frac{\pi}{2}\right) = \pi i.$$

This integral was evaluated in another way in Example 4.17, where representation $z = 2e^{i\theta} (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$ for the semicircle was used.



Next, let C_2 denote the left half $z = 2e^{i\theta} (\frac{\pi}{2} \le \theta \le \frac{3\pi}{2})$ of the same circle C and consider the branch

$$logz = lnr + i\theta, \quad (r > 0, 0 < \theta < 2\pi),$$

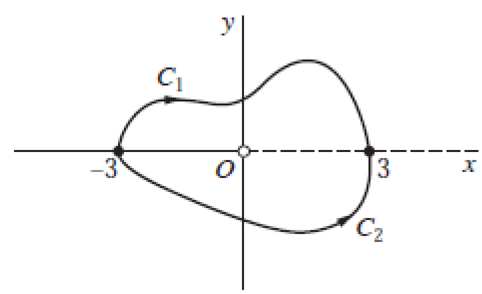
of the logarithmic function. One can write

$$\int_{C_2} \frac{dz}{z} = \int_{2i}^{-2i} \frac{dz}{z} = logz]_{2i}^{-2i} = log(-2i) - log(2i) = \left(ln2 + i\frac{3\pi}{2}\right) - \left(ln2 + i\frac{\pi}{2}\right) = \pi i.$$

The value of the integral of 1/*z* around the entire circle $C = C_1 + C_2$ is thus obtained:

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = \pi i + \pi i = 2\pi i.$$

4.37 Example:



Let us use an antiderivative to evaluate the integral $\int_{C_1} z^{\frac{1}{2}} dz$ where the integrand is the branch

$$f(z) = z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} = \sqrt{r} e^{i\frac{\theta}{2}}, \ (r > 0, 0 < \theta < 2\pi) \qquad \ldots(1)$$

of the square root function and where C_1 is any contour from $z = -3$ to $z = 3$ that, except for its end points, lies above the *x* axis. Although the integrand is piecewise continuous on C_1, and the integral therefore exists, the branch (1) of $z^{\frac{1}{2}}$ is not defined on the $ray\ \theta = 0$, in particular at the point $z = 3$. But another branch,

$$f_1(z) = \sqrt{r} e^{i\frac{\theta}{2}}, \ (r > 0, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}),$$

is defined and continuous everywhere on C_1. The values of $f_1(z)$ at all points on C_1 except $z = 3$ coincide with those of our integrand (1); so the integrand can be replaced by $f_1(z)$. Since an antiderivative of $f_1(z)$ is the function

$$F_1(z) = \frac{2}{3} z^{\frac{3}{2}} = \frac{2}{3} r\sqrt{r} e^{i\frac{3\theta}{2}},\ (r > 0,\ -\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}).$$

we can now write

$$\int_{C_1} z^{\frac{1}{2}} dz = \int_{-3}^{3} f_1(z)dz = F_1(z)]_{-3}^{3} = 2\sqrt{3}(e^{i0} - e^{i\frac{3\pi}{2}}) = 2\sqrt{3}(1+i).$$

(Compare with Example 4.23), The integral $\int_{C_2} z^{\frac{1}{2}} dz$ of the function (1) over any contour C_2 that extends from $z = -3$ to $z = 3$ below the real axis can be evaluated in a similar way. In this case, we can replace the integrand by the branch

$$f_2(z) = \sqrt{r} e^{i\frac{\theta}{2}},\ (r > 0,\ \frac{\pi}{2} \le \theta \le \frac{5\pi}{2}),$$

whose values coincide with those of the integrand at $z = -3$ and at all points on C_2 below the real axis. This enables us to use an antiderivative of $f_2(z)$ to evaluate integral $\int_{C_2} z^{\frac{1}{2}} dz$. (Details are left to the exercises).

EXERCISES:

1. Use an antiderivative to show that for every contour C extending from a point z_1 to a point z_2.

$$\int_C z^n dz = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1}),\ (n = 0,1,2,\cdots).$$

2. By finding an antiderivative, evaluate each of these integrals, where the path is any contour between the indicated limits of integration:

a) $\int_i^{i/2} e^{\pi z} dz$; **b)** $\int_0^{\pi+2i} \cos\frac{z}{3} dz$; **c)** $\int_1^3 (z-2)^3 dz$.

3. Show that $\int_{C_0} (z - z_0)^{n-1} dz = 0,\ (n = 0, \pm 1, \pm 2, \cdots)$. when C_0 is any closed contour which does not pass through the point z_0.

4. Show that $\int_{-1}^{1} z^i dz = \frac{1+e^{-\pi}}{2}(1-i)$, where the integrand denotes the principal branch $z^i = e^{i\,\mathrm{Log}\,z},\ (|Z| > 0,\ -\pi < Arg\ z < \pi)$ of z^i and where the path of integration is any contour from $z = -1$ to $z = 1$ that, except for its end points, lies above the real axis.

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ١١

Chapter Four

INTEGRALS

4.38 CAUCHY–GOURSAT THEOREM:

We saw that when a continuous function f has an antiderivative in a domain D, the integral of $f\,(z)$ around any given closed contour C lying entirely in D has value zero. Now we shall present a theorem giving other conditions on a function f which ensure that the value of the integral of $f\,(z)$ around a simple closed contour is zero. The theorem is central to the theory of functions of a complex variable; and some modifications of it, involving certain special types of domains.

We let C denote a simple closed contour $z = z(t)$ $(a \le t \le b)$, described in the positive sense (counterclockwise), and we assume that f is analytic at each point interior to and on C.

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt. \qquad (1)$$

and if $f(z) = u(x,y) + iv(x,y)$, $z(t) = x(t) + iy(t)$ the integral $f[z(t)]z'(t)$ in expression (1) is the product of the functions

$$u[x(t),y(t)] + iv[x(t),y(t)]\,,\ x'(t)+iy'(t)\,.$$

of the real variable t ,i.e.

$$[u(x,y) + iv(x,y)\,]\cdot[x'(t)+iy'(t)] = [u(x,y)x'(t) - v(x,y)y'(t)] + i[v(x,y)x'(t) + u(x,y)y'(t)].$$

Thus

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt = \int_a^b (ux' - vy')dt + i\int_a^b (vx' - uy')dt \qquad (2)$$

In terms of line integrals of real-valued functions of two real variables, then,

$$\int_C f(z)dz = \int_C (udx - vdy) + i\int_C (vdx + udy) \qquad (3)$$

Observe that expression (3) can be obtained formally by replacing $f\,(z)$ and dz on the left with the binomials $u + iv$ and $dx + idy$ respectively, and expanding their product. Expression (3) is, of course, also valid when C is any contour, not necessarily a simple closed one, and when $f\,[z(t)]$ is only piecewise continuous on it.

4.39 (*Green's theorem*):

Suppose that two real-valued functions $P(x,y)$ and $Q(x,y)$, together with their first-order partial derivatives, are continuous throughout the closed region R consisting of all points interior to and on the simple closed contour C.

$$\int_C f(z)dz = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

4.40 Remark:

The Green's theorem enables us to express the line integrals on the right in equation (3) as double integrals $\int_C Pdx + Qdy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy.$

Now f is continuous in *R*, since it is analytic there. Hence the functions *u* and *v* are also continuous in *R*. Likewise, if the derivative $f'$of f is continuous in *R*, so are the first-order partial derivatives of *u* and *v*. Green's theorem then enables us to rewrite equation (3) as

$$\int_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\, dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\, dy.$$

But, in view of the Cauchy–Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ the integrands of these two double integrals are zero throughout *R*. So when *f* is analytic in *R* and f' *is* continuous there,

$$\int_C f(z)dz = \int_{-C} f(z)dz = 0.$$

4.41 Example:

If *C* is any simple closed contour, in either direction, then $\int_C e^{z^3} dz = 0.$

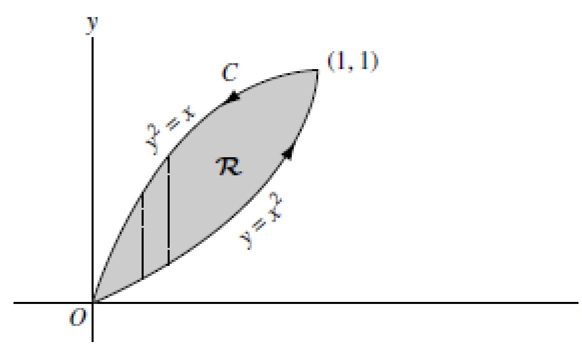
Solution:

The function $f(z) = e^{z^3}$ is analytic everywhere and its derivative $f'(z) = 3z^2e^{z^3}$ is continuous everywhere.

4.42 Example:

Verify Green's theorem in the plane for $\int_C [(2xy - x^2)dx + (x + y^2)dy]$, where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

Solution:



The plane curves $y = x^2$ and $y^2 = x$ intersect at (0 , 0) and (1 , 1). The positive direction in traversing C.Along $y = x^2$the line integral equals

$$\int_{x=0}^{x=1}[\big((2x)x^2 - x^2\big)dx + (x + (x^2)^2)d(x^2)]$$

$$= \int_0^1 (2x^3 + x^2 + 2x^5)dx = \left(\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right)\Big]_0^1$$

$$= \left(\frac{x^4}{2} + \frac{x^3}{3} + \frac{x^6}{3}\right)\Big]_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3} = \frac{7}{6}.$$

Along $y^2 = x$, the line integral equals

$$\int_{y=0}^{y=1}[(2y^2y - (y^2)^2)d(y^2) + (y^2 + y^2)dy] = \int_0^1 (4y^4 - 2y^5 + 2y^2)dy$$

$$= \left(\frac{4y^5}{5} - \frac{2y^6}{6} + \frac{2y^3}{3}\right)\Big]_0^1 = \frac{4}{5} - \frac{1}{3} + \frac{2}{3} = \frac{17}{15}.$$

Then the required integral $\frac{7}{6}-\frac{17}{15}=\frac{1}{30}$. On the other hand,

$$\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy=\iint_R\left(\frac{\partial}{\partial x}(x+y^2)-\frac{\partial}{\partial y}(2xy-x^2)\right)dx\,dy=\iint_R(1-2y)dx\,dy$$
$$=\int_{x=0}^{x=1}\int_{y=x^2}^{y=\sqrt{x}}(1-2x)dy\,dx=\int_0^1 y-2xy\Big]_{y=x^2}^{y=\sqrt{x}}dx=\int_0^1\left(\sqrt{x}-2x^{3/2}-x^2+2x^3\right)dx$$
$$=\left(\frac{2x^{3/2}}{3}-\frac{4x^{5/2}}{5}-\frac{x^3}{3}+\frac{x^4}{2}\right)\Big]_0^1=\frac{2}{3}-\frac{4}{5}-\frac{1}{3}+\frac{1}{2}=\frac{1}{30}.$$

Hence, Green's theorem is verified.

4.43 Remark:

We now state the revised form of Cauchy's result, known as the *Cauchy–Goursat theorem*.

4.44 (*Cauchy–Goursat theorem*):

If a function f is analytic at all points interior to and on a simple closed contour C, then $\int_C f(z)dz=0$.

4.45 Definition:

A ***simply connected domain*** *D* is a domain such that every simple closed contour within it encloses only points of *D*.

4.46 Remark:

The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is not simply connected.

4.47 Remark:

The closed contour in the Cauchy–Goursat theorem need not be simple when the theorem is adapted to simply connected domains. More precisely, the contour can actually cross itself. The following theorem allows for this possibility.

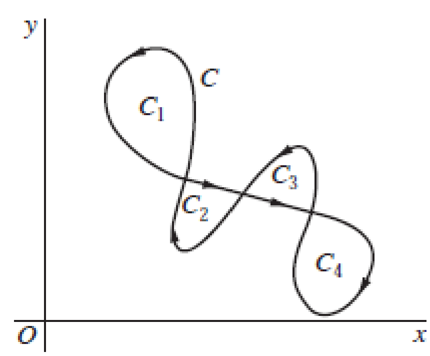
4.48 Theorem:

If a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)dz=0\,,$$

for every closed contour C lying in D.

4.49 Remark:



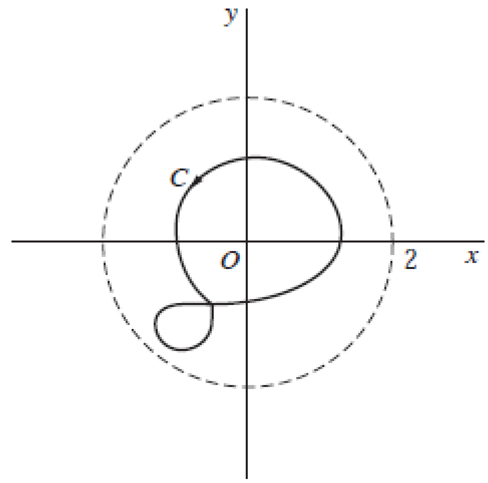
If C is closed but intersects itself a finite number of times, it consists of a finite number of simple closed contours. Where the simple closed contours C_k $(k=1,2,3,4)$ make up C. Since the value of the integral around each C_k is zero, according to the Cauchy–Goursat theorem, it follows that

$$\int_C f(z)dz=\sum_{k=1}^{4}\int_{C_k}f(z)dz=0$$

4.50 Example:

If C denotes any closed contour lying in the open disk $|z| < 2$ then

$$\int_C \frac{ze^z}{(z^2+9)^5} dz = 0.$$



Solution:

The disk $z| < 2$ is a simply connected domain and the two singularities $z = \pm 3i$ of the integrand are exterior to the disk then

$$\int_C \frac{ze^z}{(z^2+9)^5} dz = 0.$$

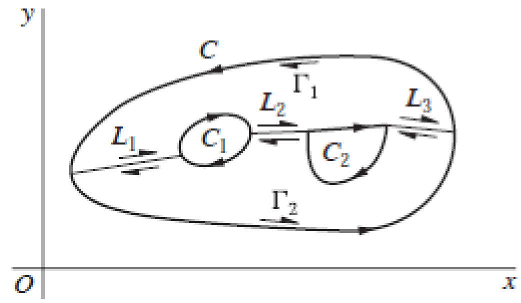
4.51, Corollary:

A function f that is analytic throughout a simply connected domain D must have an antiderivative everywhere in D.

4.52 Remark:

A domain that is not simply connected is said to be ***multiply connected***. The following theorem is an adaptation of the Cauchy–Goursat theorem to multiply connected domains.

4.53 Theorem:



Suppose that

(a) C is a simple closed contour, described in the counterclockwise direction;

(b) C_k $(k = 1, 2, \cdots, n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common.

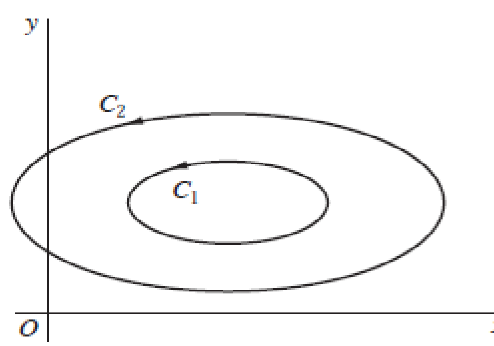
If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each Ck, then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz = \mathbf{0}. \quad (4)$$

4.54 Remark:

Note that in equation (4), the direction of each path of integration is such that the multiply connected domain lies to the left of that pa

4.55 Corollary:



Let C_1 and C_2 denote positively oriented simple Closed contours, where C_1 is interior to C_2. If a function f is analytic in the closed region consisting of those contours and all points between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz. \quad (5)$$

4.56 Remark:

The corollary 4.54 is known as the ***principle of deformation of paths*** since it tells us that if C_1 is continuously deformed into C_2, always passing through points at which *f* is analytic, then the value of the integral of f over C_1 never changes. To verify the corollary, we need only write equation (5) as

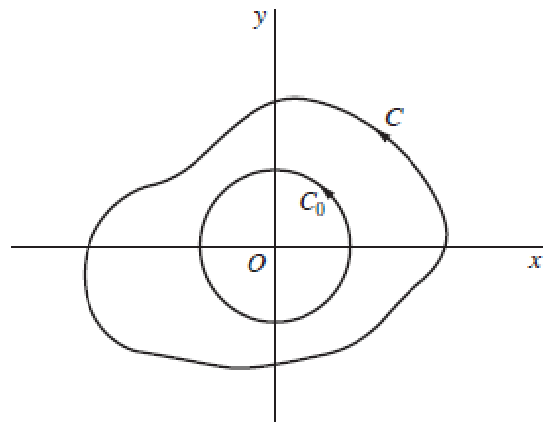
$$\int_{C_2} f(z)dz + \int_{-C_1} f(z)dz = 0$$

and apply the theorem.

4.57 Example:

When C is any positively oriented simple closed contour surrounding the origin, show that

$$\int_C \frac{dz}{z} = 2\pi i.$$



Solution:

Constructing a positively oriented circle C_0 with center at the origin and radius so small that C_0 lies entirely inside C. Since $\int_{C_0} \frac{dz}{z} = 2\pi i$ (Example 4.24) and since $1/z$ is analytic everywhere except at $z = 0$, the desired result follows.

Note that the radius of C_0 could equally well have been so large that C lies entirely inside C_0.

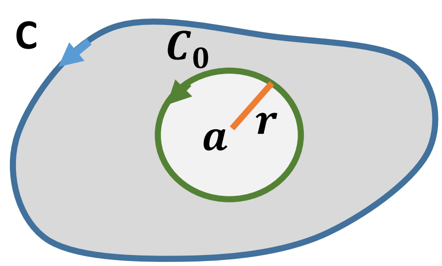
4.58 Example:

Evaluate $\int_C \frac{dz}{z-a}$ where C is any simple closed curve C and $z = a$ is **(a)** outside C, **(b)** inside C.

Solution:

(a) If a is outside C, then $f(z) = \frac{dz}{z-a}$ is analytic everywhere inside and on C.

Hence, by Cauchy's theorem, $\int_C \frac{dz}{z-a} = 0.$



(b) Suppose a is inside C and let C_0 be a circle of radius r with center at $z = a$ so that C_0 is inside C (this can Be done since $z = a$ is an interior point).Then

$$\int_C \frac{dz}{z-a} = \int_{C_0} \frac{dz}{z-a}$$

Now on C_0, $|z-a| = r$ or $z - a = re^{i\theta}$ i.e., $z = a + re^{i\theta}$, $0 \le \theta < 2\pi$. Thus, since $dz = ire^{i\theta}d\theta$ the right side of becomes

$$\int_{C_0} \frac{dz}{z-a} = \int_{\theta=0}^{\theta=2\pi} \frac{ire^{i\theta}d\theta}{re^{i\theta}} = i\int_0^{2\pi} d\theta = i\theta]_0^{2\pi} = 2\pi i = \int_C \frac{dz}{z-a}.$$

4.59 Example:

Evaluate $\int_C \frac{dz}{(z-a)^n}, n = 2,3,4,\cdots$ where $z = a$ is inside the simple closed curve C.

Solution:

As in example 4.58 we have

$$\int_C \frac{dz}{(z-a)^n} = \int_{C_0} \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{i\theta}d\theta}{r^n e^{in\theta}} = \frac{i}{r^{n-1}} \int_0^{2\pi} e^{(1-n)i\theta} d\theta$$

$$= \frac{1}{r^{n-1}.(1-n)} \int_0^{2\pi} i(1-n)e^{(1-n)i\theta} d\theta$$

$$= \frac{1}{r^{n-1}.(1-n)} e^{(1-n)i\theta}\Big]_0^{2\pi} = \frac{1}{r^{n-1}.(1-n)} (e^{2(1-n)\pi i} - e^0)$$

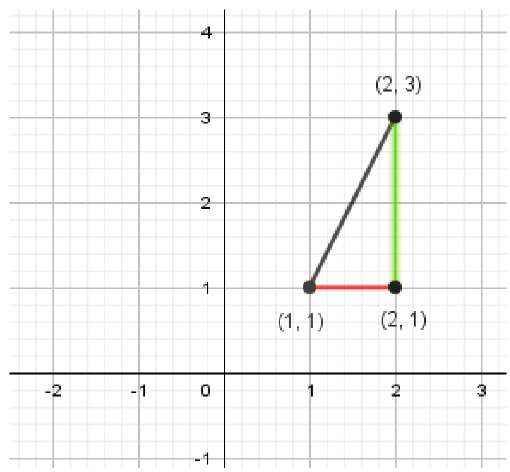
$$= \frac{1}{r^{n-1}.(1-n)} (1-1) = 0, \text{ where } n \neq 1.$$

4.60 Example:

Let C be the curve $y = x^3 - 3x^2 + 4x - 1$ joining points (1, 1) and (2, 3). Find the value of $\int_C (12z^2 - 4iz)dz$.

Solution:

There are two methods to solve this examples:



Method 1:

The integral is independent of the path joining (1,1) And (2 , 3) . Hence, any path can be chosen. In particular, let us choose the straight line paths from (1, 1) to (2, 1) and then from (2, 1) to (2, 3).

Along the path from (1,1) to (2,1), $y = 1$, $dy = 0$ so that $z = x + iy = x + i$, $dz = dx$. Then , the integral equals

$$\int_{x=1}^{x=2} (12(x+i)^2 - 4i(x+i))dx = 4(x+i)^3 - 2i(x+i)^2]_1^2$$

$$= 4(2+i)^3 - 2i(2+i)^2 - 4(1+i)^3 + 2i(1+i)^2 = 20 + 30i.$$

Along the path from (2,1) to (2,3), $x = 2$, $dx = 0$ so that $z = x + iy = 2 + iy$, $dz = idy$. Then, the integral equals

$$\int_{y=1}^{y=3} (12(2+iy)^2 - 4i(2+iy))dy = 4(2+iy)^3 - 2i(2+iy)^2]_1^3$$

$$= 4(2+3i)^3 - 2i(2+3i)^2 - 4(2+i)^3 + 2i(2+i)^2 = -176 + 8i.$$

Then adding the required value $= (20 + 30i) + (-176 + 8i) = -156 + 38i.$

Method 2:

The given integral equals

$$\int_C (12z^2 - 4iz)dz = \int_{1+i}^{2+3i} (12z^2 - 4iz)dz = (4z^3 - 2iz^2)]_{1+i}^{2+3i}$$

$$= 4(2+3i)^3 - 2i(2+3i)^2 - 4(1+i)^3 + 2i(1+i)^2 = -156 + 38i.$$

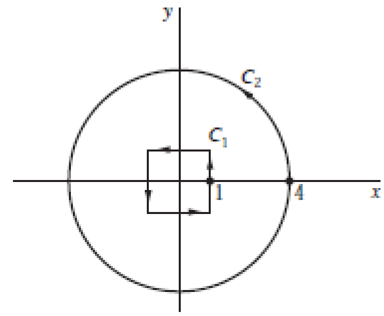
EXERCISES:

1. Apply the Cauchy–Goursat theorem to show that $\int_C f(z)dz = 0$ when the contour C is the unit circle $|z| = 1$, in either direction, and when

a) $f(z) = \frac{z^2}{z-3}$; **b)** $f(z) = ze^{-z}$; **c)** $f(z) = \frac{1}{z^2+2z+2}$;

d) $f(z) = senh\, z$; **e)** $f(z) = \tan z$; **f)** $f(z) = Log(z+2)$.

2. Let C_1 denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 1$, $y = \pm 1$ and let C_2 be the positively oriented circle $|z| = 4$. Show that $\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$ when



a) $f(z) = \frac{1}{3z^2+1}$; **b)** $f(z) = \frac{z+2}{\sin\frac{z}{2}}$; **c)** $f(z) = \frac{z}{1-e^z}$.

3. Let C denote the positively oriented boundary of the half disk $0 \le r \le 1$, $0 \le \theta \le \pi$, and let $f(z)$ be a continuous function defined on that half disk by writing $f(0) = 0$ and using the branch

$$f(z) = \sqrt{r}e^{i\frac{\theta}{2}}, \quad (r > 0, -\frac{\pi}{2} < \theta < \frac{3\pi}{2}).$$

of the multiple- valued function $z^{\frac{1}{2}}$. Show that $\int_C f(z)dz = 0$ by evaluating separately the integrals of $f(z)$ over the semicircle and the two radii which make up C. Why does the Cauchy–Goursat theorem not apply here?

محاضرات مادة التحليل العقدي

المرحلة الرابعة/ الكورس الاول

ا. م. د. بان جعفر الطائي

المحاضرة ١٢

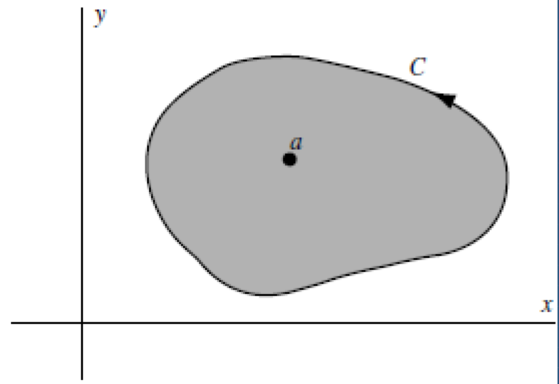
Chapter Four

INTEGRALS

4.61 (CAUCHY INTEGRAL FORMULA):

Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If a is any point interior to C, then

$$\boldsymbol{f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-a}}\,. \qquad (1)$$



Formula (1) is called the ***Cauchy integral formula***.

4.62 Example:

Let C be the positively oriented circle $|z| = 2$. Evaluate $\int_C \frac{zdz}{(9-z^2)(z+i)}$.

Solution:

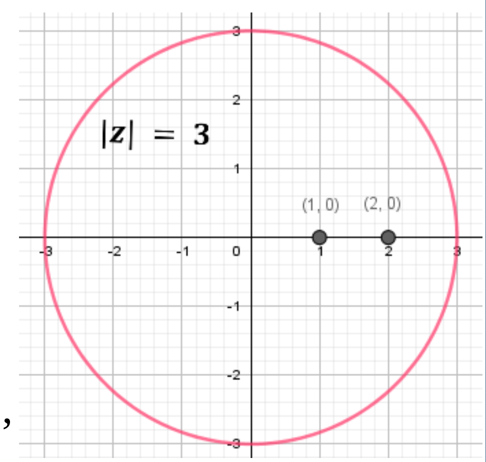
Since the function $f(z) = \frac{z}{(9-z^2)}$ is analytic within and on C and since the point $a = -i$ is interior to C, formula (1) tells us that

$$\int_C \frac{zdz}{(9-z^2)(z+i)} = \int_C \frac{z/(9-z^2)}{z-(-i)}dz = 2\pi i \cdot f(-i) = 2\pi i \cdot \frac{-i}{(9-(-i)^2)} = 2\pi i \cdot \frac{-i}{10} = \frac{\pi}{5}\,.$$

4.63 Example:

Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}dz$, where C is the circle $|z| = 3$.

Solution:



$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-2)} + \frac{B}{(z-1)} \implies 1 = A(z-1) + B(z-2) \implies$$

$$1 = (A+B)z - (A+2B) \implies \begin{matrix} A + B = 0 \\ -A - 2B = 1 \end{matrix} \implies \begin{matrix} B = -1 \\ A = 1 \end{matrix}$$

then $\frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$, so

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}.$$

By Cauchy's integral formula with $a = 2$ and $a = 1$, respectively, we have

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)} = 2\pi i(\sin \pi(2)^2 + \cos \pi(2)^2) = 2\pi i,$$

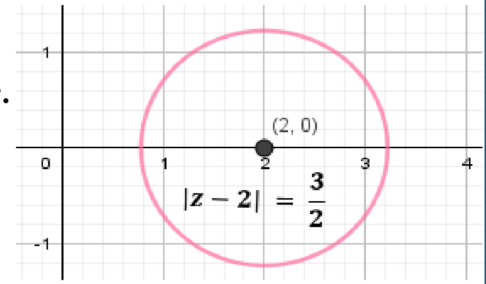
$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)} = 2\pi i(\sin \pi(1)^2 + \cos \pi(1)^2 = -2\pi i.$$

Since $z = 1$ and $z = 2$ are inside C and $\sin \pi z^2 + \cos \pi z^2$ is analytic inside C. Then, the required integral has the value

$$\int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)(z-2)} = \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-2)} - \int_C \frac{\sin \pi z^2+\cos \pi z^2}{(z-1)} = 2\pi i - (-2\pi i) = 4\pi i.$$

4.64 Example:

Evaluate $\int_C \frac{z}{(z-2)} dz$, where C is the circle $|z-2| = \frac{3}{2}$.



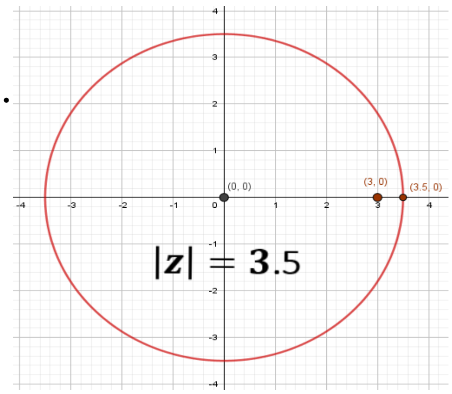
Solution:

By Cauchy's integral formula with $a = 2$ we have

$$\int_C \frac{f(z)}{(z-2)} dz = \int_C \frac{z}{(z-2)} dz = 2\pi i f(2) = 2\pi i(2) = 4\pi i.$$

4.65 Example:

Evaluate $\int_C \frac{dz}{(z^2-7z+12)} dz$, where C is the circle $|z| = 3.5$.



Solution

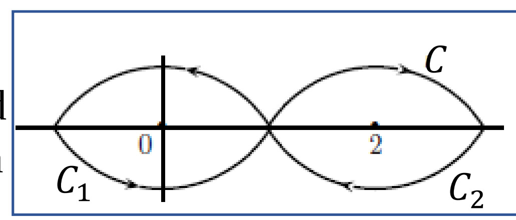
$$\int_C \frac{1}{(z^2-7z+12)} dz = \int_C \frac{1}{(z-4)(z-3)} dz = \int_C \frac{1/(z-4)}{(z-3)} dz.$$

Since C is a circle with center (0,0) and radius 3.5 then $z = 2$ lies inside C, so let $f(z) = \frac{1}{z-4}$ which is analytic everywhere inside and on C . By Cauchy's integral formula with $a = 3$, we have

$$\int_C \frac{1/(z-4)}{(z-3)} dz = 2\pi i f(3) = 2\pi i \frac{1}{3-4} = -2\pi i.$$

4.66 Example:

Evaluate $\int_C \frac{e^z dz}{z(z-2)} dz$, where C is the figure-eight contour.



Solution

Let C_1 and C_2 be the positively oriented left lobe and the negatively oriented right lobe, respectively. Then we have

$$\int_C \frac{e^z}{z(z-2)} dz = \int_{C_1} \frac{e^z/(z-2)}{z} dz + \int_{C_2} \frac{e^z/z}{(z-2)} dz = 2\pi i \left.\frac{e^z}{(z-2)}\right|_{z=0} + 2\pi i \left.\frac{e^z}{z}\right|_{z=2} = -\pi i + \pi i e^2.$$

4.67 Remark:

The Cauchy integral formula can be extended to the *nth* derivative of $f(z)$ at $z = a$ is given by

$$\boldsymbol{f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-a)^{n+1}}, \quad n = 1,2,3,4,\cdots} \qquad (2)$$

The result (1) can be considered a special case of (2) with $n = 0$ if we define $0! = 1$ and $f^{(0)}(a) = f(a)$.

4.68 Example:

If C is the positively oriented unit circle $|z| = 1$. Evaluate $\int_C \frac{e^{2z} dz}{z^4}$.

Solution:

Since the function $f(z) = e^{2z}$ is analytic within and on C and since the point $a = 0$ is interior to C, formula (2) tells us that

$$\int_C \frac{e^{2z}dz}{z^4} = \int_C \frac{f(z)dz}{(z-0)^{3+1}} = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{6} \cdot 8e^0 = \frac{8\pi i}{3}.$$

Where $f(z) = e^{2z}, \quad f'(z) = 2e^{2z}, \quad f''(z) = 4e^{2z}, \quad f'''(z) = 8e^{2z}$.

4.69 Example:

Let a be any point interior to a positively oriented simple closed contour C. When $f(z) = 1, f^{(n)}(z) = 0, n = 1,2,3,\cdots$ then expression (2) shows that

$$\int_C \frac{dz}{(z-a)} = 2\pi i, \quad \int_C \frac{dz}{(z-a)^{n+1}} = 0, \quad n = 1,2,3,4,\cdots.$$

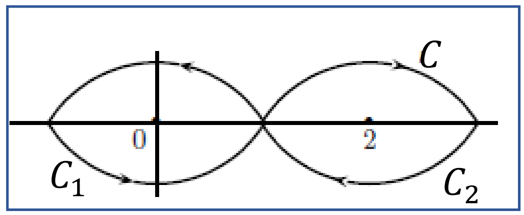
4.70 Example:

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 3$.

Solution

Let $f(z) = e^{2z}$ and $a = -1$ then $f'(z) = 2e^{2z}; \quad f''(z) = 4e^{2z}; \quad f'''(z) = 8e^{2z}$.
in the Cauchy integral formula $f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$. If $n = 3$ then $f'''(-1) = 8e^{-2}$.
Hence $\int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{n!} f'''(-1) = \frac{2\pi i}{6} \cdot 8e^{-2} = \frac{8\pi i}{3} e^{-2}$.

4.71 Example:



Evaluate $\int_C \frac{3z+1}{z(z-2)^2} dz$, where C is the figure-eight Contour (example 4.66) .

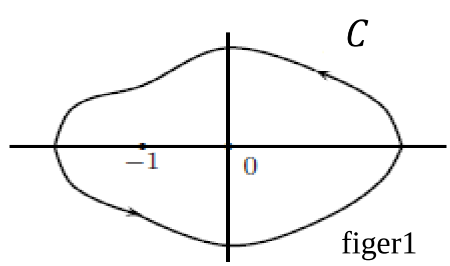
Solution

Let C_1 and C_2 be the positively oriented left lobe and the negatively oriented right lobe, respectively. Then we have

$$\int_C \frac{3z+1}{z(z-2)^2} dz = \int_{C_1} \frac{(3z+1)/(z-2)^2}{z} dz + \int_{C_2} \frac{(3z+1)/z}{(z-2)^2} dz = 2\pi i \left.\frac{3z+1}{(z-2)^2}\right|_{z=0} - \frac{2\pi i}{1!} \cdot \frac{d}{dz}\left(\frac{3z+1}{z}\right)\Big|_{z=2} = \frac{\pi i}{2} + \frac{\pi i}{2} = \pi i.$$

4.72 Example:

Evaluate $\int_C \frac{\cosh z}{z(z+1)^2} dz$, where C is the Contour shown in figer1.



figer1

Solution

$$\frac{1}{z(z+1)^2} = \frac{A}{z} + \frac{B}{z+1} + \frac{D}{(z+1)^2} = \frac{1}{z} - \frac{1}{z+1} - \frac{1}{(z+1)^2}, \ A = 1, B =$$

$$\int_C \frac{\cosh z}{z(z+1)^2} dz = \int_C \frac{\cosh z}{z} dz - \int_C \frac{\cosh z}{z+1} dz - \int_C \frac{\cosh z}{(z+1)^2} dz = 2\pi i \cosh z|_{z=0}$$

$$-2\pi i \cosh z|_{z=-1} - 2\pi i \cosh z|_{z=-1} = 2\pi i - 2\pi i \cosh 1 - 2\pi i \cosh 2 .$$

4.73 Remark:

The following is a list of some important theorems that are consequences of Cauchy's integral formulas.

4.74 (Morera's theorem (converse of Cauchy's theorem):

If $f\ (z)$ is continuous in a simply-connected region R and if $\int_C f(z)dz = 0$ around every simple closed curve C in R, then $f\ (z)$ is analytic in R (f is called ***holomorphic*** function).

4.75 Example:

Consider the function $f(z)\ = \int_0^1 e^{-z^2 t}dt$. Let C be any simple closed contour in the complex plane. Changing the order of integration, we have

$$\int_C f(z)dz = \int_0^1 (\int_C e^{-z^2 t}dz) = 0\ ,$$

Hence, in view of Theorem 4.74, the function $f(z)\ = \frac{(1-e^{-z^2})}{z^2}$ is analytic.

4.76 (Cauchy's inequality):

Suppose $f\ (z)$ is analytic inside and on a circle C of radius r and center at $z =\ a$. Then $\left|f^{(n)}(a)\right| \le \frac{M\cdot n!}{r^n}$, $n = 0,1,2,3,\cdots$, where M is a constant such that $|f\ (z)| < M$ on C, i.e., M is an upper bound of $|f\ (z)|$ on C.

4.77 (Liouville's theorem):

Suppose that for all z in the entire complex plane,

(i) $f\ (z)$ is analytic and

(ii) $f\ (z)$ is bounded, i.e., that $|f\ (z)| < M$.

Then $f\ (z)$ must be a constant.

4.78 Example:

Suppose that $f(z)$ is entire and that the harmonic function $u(x,y)\ =\ Re[f(z)]$ has an upper bound u_0. Show $u(x,y)$ must be a constant.

Solution:

The function $g(z) = e^{f(z)}$is entire ,and

$$|g(z)| = \left|e^{f(z)}\right| = \left|e^{u+iv}\right| = \left|e^u e^{iv}\right| = |e^u|\left|e^{iv}\right| = |e^u| \le e^{u_0}.$$

By Liouville's theorem $g(z)$ is constant so $g'(z) = 0$. Now, $g'(z) = e^{f(z)}f'(z)$ so $f'(z) = 0$ since the exponential cannot be 0, implying that $f(z)$ is constant.

4.79 Example:

Suppose $f(z)$ and $g(z)$ are entire functions,$g(z) \ne 0$ and $|f(z) \le |g(z)|, z\epsilon\boldsymbol{C}$. Show that there is a constant c such that $f(z)\ =\ cg(z)$.

Solution:

Observe that $f(z)/g(z)$ is entire and $|f(z)/g(z)| \leq 1$. Now use Theorem 4.77 then $f(z)/g(z) \leq c \Leftrightarrow f(z) = cg(z)$.

4.80 (Fundamental theorem of algebra):

Every polynomial equation $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n = 0$ with degree $n \geq 1$ and $a_n \neq 0$ has at least one root.

From this it follows that $P(z) = 0$ has exactly n roots, due attention being paid to multiplicities of roots.

4.81 Example:

Show that if the coefficients of the polynomial equation $P(z) = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n = 0$ are positive and non decreasing; i.e., $0 < a_n \leq a_{n-1} \leq \cdots \leq a_0$, then $P(z)$ has no root in the circle $|z| \leq 1$, except perhaps at $z = -1$.

Solution:

Obviously, $z = 1$ is not a solution. Consider $|z| \leq 1$ except at $z = \pm 1$. It suffices to show that

$$|(1-z)(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0)| > 0.$$

Since $(1-z)(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0) = a_0 - [a_nz^{n+1} + (a_{n-1} - a_n)z^n + \cdots + (a_0 - a_1)z]$, it follows that

$$|(1-z)(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0)| \geq |a_0| - |a_nz^{n+1} + (a_{n-1} - a_n)z^n + \cdots + (a_0 - a_1)z|$$

Now

$$|a_nz^{n+1} + (a_{n-1} - a_n)z^n + \cdots + (a_0 - a_1)z| \leq a_n|z^{n+1}| + (a_{n-1} - a_n)|z^n| + \cdots + (a_0 - a_1)|z| \quad (1)$$

with equality if and only if $z \epsilon \mathbb{R}$ and $z \geq 0$. However for such z, $P(z) > 0$ $(a_0 > 0)$. Thus, in we need to consider only strict inequality. Then, it follows that

$$|a_nz^{n+1} + (a_{n-1} - a_n)z^n + \cdots + (a_0 - a_1)z| < a_n + (a_{n-1} - a_n) + \cdots + (a_0 - a_1) = a_0. \quad (2)$$

Using (2) in (1), we get the required inequality

$$|(1-z)(a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0)| > 0.$$

4.82 (Gauss' mean value theorem):

Suppose $f(z)$ is analytic inside and on a circle C with center at a and radius r. Then $f(a)$ is the mean of the values of $f(z)$ on C, i.e.,

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f\left(a + re^{i}\right)d\theta.$$

4.83 (Maximum modulus theorem):

Suppose $f(z)$ is analytic inside and on a simple closed curve C and is not identically equal to a constant. Then the maximum value of $|f(z)|$ occurs on C.

4.84 (Minimum modulus theorem):

Suppose $f(z)$ is analytic inside and on a simple closed curve C and $f(z) = 0$ inside C. Then $|f(z)|$ assumes its minimum value on C.

4.85 Example:

Find the maximum modulus of $f(z) = 2z + 5i$ on the closed circular region defined by $|z| \leq 2$.

Solution:

Since $|z|^2 = z\bar{z}$. By replacing the symbol z by $2z + 5i$ we have

$$|2z + 5i|^2 = (2z + 5i)\overline{(2z + 5i)} = (2z + 5i)(2\bar{z} - 5i) = 4z\bar{z} - 10i(z - \bar{z}) + 25$$

Since $z - \bar{z} = 2iIm(z)$ and so $|2z + 5i|^2 = 4|z|^2 + 20Im(z) + 25$. Because f is a polynomial, it is analytic on the region defined by $|z| \leq 2$. By theorem 4.84, $max_{|z|\leq 2} \leq |2z + 5i|$ occurs on the boundary $|z| = 2$. Therefore on $|z| = 2$, we get

$$|2z + 5i| = \sqrt{41 + 20Im(z)}.$$

The last expression attains its maximum when $Im(z)$ attains its maximum on $|z| = 2$, namely, at the point $z = 2i$. Thus, $max_{|z|\leq 2} \leq |2z + 5i| = \sqrt{81} = 9$.

4.86 Remark:

Note in Example 4.85 that $f(z) = 0$ only at $z = -\frac{5}{2}i$ and that this point is outside the region defined by $|z| \leq 2$. Hence we can conclude that $|2z + 5i| = \sqrt{41 + 20Im(z)}$ attains its minimum when $Im(z)$ attains its minimum on $|z| = 2$ at $z = -2i$. As a result, $max_{|z|\leq 2} \leq |2z + 5i| = \sqrt{1} = 1$.

EXERCISES:

1. Let C denote the positively oriented boundary of the square whose sides lie along the lines $x = \pm 2$ and $y = \pm 2$. Evaluate each of these integrals:

a) $\int_C \frac{e^{-z}}{z-(\pi i/2)} dz$; **b)** $\int_C \frac{\cos z}{z(z^2+8)} dz$; **c)** $\int_C \frac{z}{2z+1} dz$;

d) $\int_C \frac{\cosh z}{z^4} dz$; **e)** $\int_C \frac{\tan(z/2)}{(z-x_0)^2} dz, \ -2 < x_0 < 2)$.

2. Find the value of the integral of $g(z)$ around the circle $|z - i| = 2$ in the positive sense when

a) $g(z) = \frac{1}{z^2+4}$; **b)** $g(z) = \frac{1}{(z^2+4)^2}$.

3. Let C be the circle $|z| = 3$, described in the positive sense. Show that if

$$g(z) = \int_C \frac{2s^2 - s - 2}{s - z} ds, \quad (|z| \neq 3)$$

then $g(2) = 8\pi i$. What is the value of $g(z)$ when $|z| > 3$?

4. Let C be any simple closed contour, described in the positive sense in the z- plane and Write $g(z) = \int_C \frac{s^3+2s}{(s-z)^s} ds$. Show that $g(z) = 6\pi i z$ when z is inside C and that $g(z) = 0$ when z is outside.

5. Show that if f is analytic within and on a simple closed contour C and z_0 is not on C, then $\int_C \frac{f'(x)}{z-z_0} dz = \int_C \frac{f(z)}{(z-x_0)^2} dz.$

6. Let C be the unit circle $z = e^{i\theta}$ $(-\pi \leq \theta \leq \pi)$. First show that for any real constant a, $\int_C \frac{e^{az}}{z} dz = 2\pi i$. Then write this integral in terms of θ to derive the integration formula $\int_0^{\pi} e^{\mathrm{a}\cos\theta} \cos(a\ sin\theta) d\theta = \pi.$