Mathematical Analysis

Chapter One

The Real Numbers System

Definition (The Field):

Let F be a nonempty set and +, . be two binary operations on F, then $(F, +, .)$ is called field if its satisfy the following conditions:

F1: (Closure Property), $\forall a, b \in F$ we have:

$a + b \in F$ and $a.b \in F$

F2: (Associative Property) ,$\forall a, b, c \in F$ we have:

$a + (b + c) = (a + b) + c \in F$ and $a.(b.c) = (a.b).c \in F$

F3: (Commutative Property), $\forall a, b \in F$ we have:

$a + b = b + a$ and $a.b = b.a$

F4: (Existence of identity element)

There is an element $0 \in F$ such that $a + 0 = 0 + a = a$,$\forall a \in F$, and

There is an element $1 \in F$ such that $a.1 = 1.a = a$,$\forall a \in F$

(Notice that: $1 \neq 0$).

F5: (Existence of inverse element)

$\forall a \in F, \exists -a \in F$ such that $a + (-a) = (-a) + a = 0$

$\forall a \in F, \exists\, a^{-1} \in F$ such that $a.\, a^{-1} = a^{-1}.\, a = 1$

F6: (Distributive Property) , $\forall a, b, c \in F$ we have:

$$a.(b+c) = a.b + a.c \quad and \quad (a+b).c = a.c + b.c$$

Note: The identity element for the binary operations + and . is unique.

Examples: $(\mathbb{R}, +, .)$, $(\mathbb{Q}, +, .)$ are fields.

Note:

$\mathbb{R}$ is the set of real numbers

$\mathbb{Q}$ is the set of rational numbers, where $\mathbb{Q} = \left\{\frac{a}{b} : a, b \;\; integers, b \neq o \; and \; g.c.d(a,b) = 1\right\}$.

Definition (The Relation on A):

Let A be a nonempty set, R is called a relation on A if $R \subset A \times A$, where

$A \times A = \{(a,b): a, b \in A\}$, $(a,b) \in R \;\; i.e. \;\; aRb, \forall a, b \in A$.

Definition (The Order Relation on A) or (Order Set):

Let A be a nonempty set, the relation $R: \leq$ on A is called order relation on A [$(A, \leq)$ order set] if its satisfy the following conditions:

i) $a \leq a, \forall a \in A$ (Reflexive).
ii) If $a \leq b \; and \; b \leq a \implies a = b, \;\; \forall a, b \in A$ (Anti-symmetric).
iii) If $a \leq b \; and \; b \leq c \implies a \leq c, \;\; \forall a, b, c \in A$ (Transitive).

Examples:

The relation $\leq on$ $\mathbb{R}$ ($\mathbb{Q}$) is order relation i.e. $(\mathbb{R}, \leq)$, $(\mathbb{Q}, \leq)$ are order sets.

Definition (The Order Field):

Let $(F,+,.)$ be a field and $\leq$ be a relation on F, we say that $(F,+,.,\leq)$ is an order field if:

i) $a \leq a, \forall a \in F$ (Reflexive)
ii) If $a \leq b$ *and* $b \leq a \Longrightarrow a = b, \ \forall a,b \in F$ (Anti-symmetric)
iii) If $a \leq b$ *and* $b \leq c \Longrightarrow a \leq c, \ \forall a,b,c \in F$ (Transitive)
iv) Either $a \leq b$ or $b \leq a, \forall a,b \in F$
v) If $a \leq b$ *and* $c \leq d \Longrightarrow a + c \leq b + d, \forall a,b,c,d \in F$
vi) If $a \leq b$ *and* $c > 0 \Longrightarrow a.c \leq b.c, \forall a,b,c \in F$

The relation $\leq$ on $(F,+,.)$ is total order relation.

Examples:

$(\mathbb{R},+,.,\leq)$, $(\mathbb{Q},+,.,\leq)$ are order fields.

Bounded Set in Order Field $(F,+,.,\leq)$.

Definitions:

Let $(F,+,.,\leq)$ be an order field and $A \subseteq F$, then:

1) $u \in F$ is called **upper bound** for A $[\mathbf{u.b.}(\boldsymbol{A})]$ if $a \leq u, \forall a \in A$.

2) $\ell \in F$ is called **lower bound** for A $[\boldsymbol{\ell}\mathbf{.b.}(\boldsymbol{A})]$ if $\ell \leq a, \forall a \in A$.

3) A is called **bounded above** if it has upper bound.

4) A is called **bounded below** if it has lower bound.

5) A is called **bounded** if A it has upper bound and lower bound

6) $u^* \in F$ is called **least upper bound** for A $[\boldsymbol{\ell}\mathbf{.u.b.}(\boldsymbol{A})$ ***or*** $\boldsymbol{sup}(\boldsymbol{A})]$ if

i) u^* is an upper bound for A i.e. $\exists\, u^* \in F \ s.t. \ a \leq u^*, \forall a \in A$

ii) For each upper bound u for A we have $u^* \leq u$

7) $\ell^* \in F$ is called **greatest lower bound** for A $[\boldsymbol{g.\ell.b.}(\boldsymbol{A})$ ***or*** $\boldsymbol{inf}(\boldsymbol{A})]$ if

i) ℓ^* is a lower bound for A i.e. $\exists\, \ell^* \in F \ s.t. \ \ell^* \leq a, \forall a \in A$

ii) For each lower bound ℓ for A we have $\ell \leq \ell^*$

Remarks:

1) $\ell - \alpha \le \ell \le a \le u \le u + \beta, \quad \forall a \in A, \quad \alpha, \beta > 0.$

2) If the set A has least upper bound (greatest lower bound) then its unique.

Examples:

1. Let $A = [0,1)$. Find upper bound, lower bound, least upper bound and greatest lower bound.

Answer:

Since $1 \in \mathbb{R}$ s.t. $a < 1, \ \forall a \in [0,1)$

and $1.5 \in \mathbb{R}$ s.t. $a < 1.5, \ \forall a \in [0,1)$

$2 \in \mathbb{R}$ s.t. $a < 2, \ \forall a \in [0,1)$

$\vdots$

$\therefore$ $\text{u. b.}(A) = 1, 1.5, 2, \cdots$ (upper bounds)

$\therefore$ $A = [0,1)$ is bounded above

$\ell.\text{u. b.}(A) = 1$ (least upper bound)

Now, since $0 \in \mathbb{R}$ $s.t.$ $0 \le a, \ \forall a \in [0,1)$, $(0 \in A)$

and $-0.5 \in \mathbb{R}$ $s.t.$ $-0.5 < a, \ \forall a \in [0,1)$

$-1 \in \mathbb{R}$ $s.t.$ $-1 < a, \ \forall a \in [0,1)$

$\vdots$

$\therefore$ $\ell.\text{b.}(A) = 0, -0.5, -1, \cdots$ (lower bounds)

$\therefore$ $A = [0,1)$ is bounded below

$g.\ell.b.(A) = 0$ (greatest lower bound)

$A = [0,1)$ is bounded (since A is bounded above and bounded below).

2. Let $B = \{3,4,5,6\}$. Find upper bound, lower bound, least upper bound and greatest lower bound.

Since $6 \in \mathbb{R}$ s.t. $a \le 6, \forall a \in B = \{3,4,5,6\}$

$\therefore$ $\text{u. b.}(B) = 6, 6.25, 6.5, 7, \cdots$

$\therefore$ $B = \{3,4,5,6\}$ is bounded above

$\ell.\text{u. b.}(B) = 6$

Now, since $3 \in \mathbb{R}$ s.t. $3 \leq a, \ \forall a \in B = \{3,4,5,6\}$

$\therefore \ell.\text{b.}(B) = 3,2.5,2,1,\cdots$

$\therefore B = \{3,4,5,6\}$ is bounded below

$g.\ell.b.(B) = 3$

The set $B = \{3,4,5,6\}$ is bounded (since B is bounded above and bounded below).

3. $\mathbb{N} = \{1,2,3, \dots\}$ is unbounded (since $\mathbb{N}$ is bounded below but unbounded from above)
4. $\mathbb{R}$ is unbounded (since $\mathbb{R}$ unbounded from above and from below).

<u>H.W.</u>

1. Check the $A_1 = \{-n : n \in \mathbb{N}\}$ and $A_2 = (-1,1)$ are bounded.

Theorem:

The equation $x^2 = 2$ has no root in $\mathbb{Q}$.

Proof:

Assume that $x^2 = 2$ has a root in $\mathbb{Q}$, so there is $x = \frac{a}{b} \in \mathbb{Q}$ such that $x^2 = \left(\frac{a}{b}\right)^2 = 2$

$\left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2} = 2 \Longrightarrow a^2 = 2b^2$

$\because b \neq 0 \Longrightarrow a \neq 0$

Suppose a, b are positive numbers such that $g.c.d\ (a, b) = 1$

1. If a, b are odd numbers $\Longrightarrow a^2$ is odd $\Longrightarrow 2b^2$ is odd C! ($2b^2$ is even)

2. If a is odd number and b is even number

   $\Longrightarrow b = 2d \Longrightarrow a^2 = 8d^2 \Longrightarrow a^2$ is even C! (a is odd)

1. If a is even number and b is odd number

   $\Longrightarrow a = 2c \Longrightarrow 4c^2 = 2b^2 \Longrightarrow 2c^2 = b^2 \Longrightarrow b^2$ is even C! (b is odd)

4. If a, b are even numbers impossible since $g.c.d\ (a, b) = 1$

$\therefore$ there is no rational number satisfy $x^2 = 2$. i.e. $\sqrt{2} \notin \mathbb{Q}$.

Theorem:

The equation $x^2 = 2$ has a unique positive real solution.

In general

For each positive integer n and for each positive real number x, the equation $x^n = 2$ has a unique positive real solution.

Definition (Complete Property):

The ordered field $(F, +, ., \leq)$ is said to be complete if every nonempty subset A of F which is bounded above has least upper bound.

Examples:

2. The real numbers system $(\mathbb{R}, +, ., \leq)$ is complete order field.
3. The order field of rational numbers $(\mathbb{Q}, +, ., \leq)$ is not complete. Since

Let $S = \{x \in \mathbb{Q}^+ \text{ such that } x^2 < 2\} \subseteq \mathbb{Q}$ and $1 \in S \neq \emptyset$

S is bounded above but has no least upper bound in $\mathbb{Q}$ because $\sqrt{2} \notin \mathbb{Q}$

i.e. $\exists$ a nonempty subset in $\mathbb{Q}$ which is bounded from above but has no least upper bound.

Theorem: (Archimedean Property):

For all $x, y \in \mathbb{R}$ and $x > 0$, then $\exists\, n \in \mathbb{N}$ such that $nx > y$.

Proof:

Assume that $\forall n \in \mathbb{N},\ \exists\, x, y \in \mathbb{R}\ (x > 0)\ s.t.\ nx \leq y$

Let $S = \{nx : n \in \mathbb{N}\} \subseteq \mathbb{R}$ and $x \in S \neq \emptyset$

y is an upper bound of S

Since $\mathbb{R}$ is complete $\Longrightarrow S$ has least upper bound say α

$\alpha = \ell.\text{u.b.}(S)$

$\because x > 0 \Longrightarrow -x < 0 \Longrightarrow \alpha - x < \alpha$

i.e. $\alpha - x$ can not be upper bound of S

$\therefore \exists\, mx \in S\ \ s.t.\ \alpha - x < mx \Longrightarrow \alpha < x(m+1)$

But $x(m+1) \in S$ and this is contradiction that $\alpha = \ell.\text{u.b}(S)$

$\therefore \exists\, n \in \mathbb{N}$ s.t $nx > y$.

Corollary:

$\forall \varepsilon > 0, \exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$.

Proof:

Given $\varepsilon > 0$, by A.P. (Archimedean Property), $\forall\, x, y \in \mathbb{R}$ and $x > 0$, $\exists\, n \in \mathbb{N}$ s.t. $nx > y$

Let $x = \varepsilon > 0$ and $y = 1 \Longrightarrow n\varepsilon > 1 \Longrightarrow 0 < \frac{1}{n} < \varepsilon$.

Theorem: (Density of Rational Numbers in $\mathbb{R}$):

If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\, r \in \mathbb{Q}$ such that $x < r < y$.

Proof:

Let $x, y \in \mathbb{R}$ and $x < y$

If $x < 0 < y \Longrightarrow 0 \in \mathbb{Q}$ result holds.

If $x > 0$ $(y > 0)$ we have $y - x > 0$ $\quad (x < y)$

By Archimedean property $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < y - x$.

$\Longrightarrow 1 < n(y - x) = ny - nx$

$1 < ny - nx \Longrightarrow 1 + nx < ny \cdots (1)$

$nx > 0 \Longrightarrow \exists m \in \mathbb{N}$ such that $m - 1 \leq nx < m \cdots (2)$

From (1) and (2) we have $nx < m \leq nx + 1 < ny$

$\Longrightarrow nx < m < ny$

$\therefore x < \frac{m}{n} < y$ $\quad (n \neq 0$ since $n \in \mathbb{N})$.

Theorem: (Density of Irrational Numbers in $\mathbb{R}$):

If $x, y \in \mathbb{R}$ and $x < y$, then $\exists\, s \in \mathbb{Q}'$ (irrational number) such that $x < s < y$.

Proof:

Let $x, y \in \mathbb{R}$ and $x < y, \sqrt{2} \in \mathbb{Q}' \subseteq \mathbb{R} \Longrightarrow \sqrt{2} \in \mathbb{R}$

$\sqrt{2}\, x < \sqrt{2}\, y \in \mathbb{R}$

By (D. $\mathbb{Q}$ in $\mathbb{R}$), $\exists\, r \in \mathbb{Q}$ such that $\sqrt{2}x < r < \sqrt{2}y \Longrightarrow x < \frac{r}{\sqrt{2}} < y$.

H.W.

Prove that if $x, y \in \mathbb{Q}'$, then $\exists\, r \in \mathbb{Q}$ such that $x < r < y$.

Chapter Two

Sequence of Real Numbers

Definition (Sequence of Real Numbers):

The sequence of real numbers S_n is a function from $\mathbb{N}$ into $\mathbb{R}$

i.e. $S: \mathbb{N} \to \mathbb{R}$ defined as $S(n) = S_n \in \mathbb{R}, \forall\, n \in \mathbb{N}$, denoted as S_n, $(S_n), < S_n >, \{S_n\}$.

$\{S_n : n \in \mathbb{N}\}$ the range of the sequence.

Examples:

1) $S_n = n$ 2) $S_n = 1$ 3) $S_n = (-1)^n$ 4) $S_n = \frac{1}{n}$

Definition (Convergent Sequence of Real Numbers):

Let S_n be a sequence of real numbers, $S \in \mathbb{R}$ we say S_n converges to S if:

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \forall n > n_0(\varepsilon)$.

S is called convergent point of S_n, write $S_n \to S$ as $n \to \infty$ or $\lim_{n\to\infty} S_n = S$.

Geometric Meaning of Convergent Sequence of Real Numbers.

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \forall n > n_0(\varepsilon)$.

$$\Downarrow$$

$$-\varepsilon < S_n - S < \varepsilon$$

$$\Downarrow$$

$$S - \varepsilon < S_n < S + \varepsilon$$

i.e $S_n \in (S-\varepsilon, S+\varepsilon)$ the open interval $(S-\varepsilon, S+\varepsilon)$ contain all terms of sequence S_n except finite numbers of terms.

Examples:

1) The sequence of real numbers $S_n = C$ is convergent.

Answer:

We have to prove that $S_n = C \to C$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \ \forall n > n_0(\varepsilon)$.

$|S_n - S| = |\,C - C\,| = 0 < \varepsilon, \forall n > n_0(\varepsilon)$.

$\therefore \ S_n = C \to C$

2) The sequence of real numbers $S_n = \frac{1}{n}$ is convergent.

Answer:

We have to prove that $S_n = \frac{1}{n} \to 0$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \forall n > n_0(\varepsilon)$.

$$|S_n - S| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right|$$

By Archimedean property $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $0 < \frac{1}{n_0(\varepsilon)} < \varepsilon$.

$$\forall\, n > n_0(\varepsilon) \Longrightarrow \frac{1}{n} < \frac{1}{n_0(\varepsilon)} < \varepsilon \Longrightarrow \therefore \frac{1}{n} < \varepsilon\,, \forall n > n_0(\varepsilon)$$

i.e $|S_n - 0| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon, \quad \forall n > n_0(\varepsilon)$

$$\therefore \ S_n = \frac{1}{n} \to 0.$$

3) Discuss the convergent of the sequence of real numbers $S_n = \frac{1}{n+1}$.

<u>Answer:</u>

We have to prove that $S_n = \frac{1}{n+1} \to 0$

$\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|S_n - S| < \varepsilon, \forall n > n_0(\varepsilon)$.

$$|S_n - S| = \left|\frac{1}{n+1} - 0\right| = \left|\frac{1}{n+1}\right|$$

By Archimedean Property $\forall \varepsilon > 0$, $\exists\, n_0(\varepsilon) > 0$ such that $0 < \frac{1}{n_0(\varepsilon)} < \varepsilon$.

$$\forall n > n_0(\varepsilon) \Longrightarrow n + 1 > n_0(\varepsilon) + 1 > n_0(\varepsilon)$$

$$\Longrightarrow \frac{1}{n+1} < \frac{1}{n_0(\varepsilon)+1} < \frac{1}{n_0(\varepsilon)} < \varepsilon$$

$$\Longrightarrow \therefore \frac{1}{n+1} < \varepsilon$$

$i.e.$ $|S_n - S| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} < \varepsilon, \forall n > n_0(\varepsilon)$ i.e. $S_n = \frac{1}{n+1} \to 0$.

4) Discuss the convergent of the sequence of real numbers $S_n = (-1)^n$.

<u>Answer:</u> We have to prove that $S_n = (-1)^n$ does not convergent (divergent $S_n = (-1)^n \nrightarrow$)

<u>Case 1:</u> If $S \in \mathbb{R}, S \neq 1, S \neq -1$,

We can find $\varepsilon > 0$ such that $(S - \varepsilon, S + \varepsilon)$ does not contain any terms of $S_n = (-1)^n$

$\therefore S_n = (-1)^n$ does not convergent sequence.

<u>Case 2:</u> If $S = 1$ we can find $\varepsilon > 0$ such that $(1 - \varepsilon, 1 + \varepsilon)$ contains all even terms but not contain odd terms

i.e. $S_n = (-1)^n$ divergent.

<u>Case 3:</u> If $S = -1$ by same way we can prove that S_n diverges.

$\therefore S_n = (-1)^n$ divergent (not convergent).

Theorem (Uniqueness of Convergent Point):

If the sequence of real numbers a_n convergent then it has unique limit point.

Proof:

Assume that $a_n \to a,\ a_n \to b$ such that $a \neq b \Longrightarrow |b-a| > 0$

$a_n \to a \Longrightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$a_n \to b \Longrightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|a_n - b| < \varepsilon, \forall n > n_1(\varepsilon)$.

Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$|b-a| = |b - a_n + a_n - a| \leq |a_n - a| + |a_n - b| < \varepsilon + \varepsilon = 2\varepsilon$

Let $\varepsilon = \frac{|b-a|}{2} > 0 \Longrightarrow |b-a| < 2\frac{|b-a|}{2} = |b-a|$ C!

$\therefore a = b$.

Definition (Bounded Sequence of Real Numbers):

Let a_n be a sequence of real numbers, we say that a_n is bounded iff $\exists\ M > 0$, $(M \in \mathbb{R})$, such that $|a_n| < M, \forall\ n \in \mathbb{N}$.

Theorem:

Every convergent sequence of real numbers a_n is bounded.

Proof:

Since a_n is a convergent sequence of real numbers, so $\exists\ a \in \mathbb{R}$ such that $a_n \to a$

$\Longrightarrow \forall\ \varepsilon > 0, \exists\ n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon,\ \forall\ n > n_0(\varepsilon)$

i.e. $a_n \in (a-\varepsilon, a+\varepsilon), \forall\ n > n_0(\varepsilon)$

Let $M = \max\ \{|a_1|, |a_2|, \cdots, |a_{n_0}|, a-\varepsilon, a+\varepsilon\}$

$\therefore |a_n| < M, \forall\ n \in \mathbb{N}$

$\therefore a_n$ bounded.

Remark:

The converse may not be true, for example $a_n = (-1)^n$ is bounsed sequence but not convergent.

(Algebra of Convergent Sequence of Real Numbers)

Theorem: Let $a_n \to a, b_n \to b$ be two convergent sequence in $\mathbb{R}$, then:

i) $a_n + b_n \to a + b$
ii) $a_n - b_n \to a - b$
iii) $a_n . b_n \to a. b$
iv) $Ca_n \to Ca, \quad \forall C \in \mathbb{R}$
v) $\frac{a_n}{b_n} \to \frac{a}{b}, \quad b_n \neq 0$ and $b \neq 0$.

Proof: (i) To prove $a_n + b_n \to a + b$

Since $a_n \to a \Rightarrow \forall\, \varepsilon > 0, \exists\, n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \ \forall\, n > n_0(\varepsilon)$

Since $b_n \to b \Rightarrow \forall\, \varepsilon > 0, \exists\, n_1(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \ \forall\, n > n_1(\varepsilon)$

Let $\varepsilon = \frac{\varepsilon}{2} > 0$

We have to find $n_2(\varepsilon) > 0$ such that $|(a_n + b_n) - (a + b)| < \varepsilon\,, \ \forall\, n > n_2(\varepsilon)$

We choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall\, n > n_2(\varepsilon)$$

$\therefore a_n + b_n \to a + b.$

Proof: (iii) To prove $a_n . b_n \to a. b$

1) Since a_n converges to a, so a_n is bounded $\Rightarrow \exists M_1 > 0$ such that $|a_n| < M_1, \ \forall n \in \mathbb{N}$

2) $a_n \to a \Rightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \ \forall n > n_0(\varepsilon)$

Let $\varepsilon = \frac{\varepsilon}{2|b|} > 0 \Longrightarrow |a_n - a| < \frac{\varepsilon}{2|b|}, \ \forall n > n_0(\varepsilon)$

$b_n \to b \Longrightarrow \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \ \forall n > n_1(\varepsilon)$.

Let $\varepsilon = \frac{\varepsilon}{2M_1} > 0 \Longrightarrow |b_n - b| < \frac{\varepsilon}{2M_1}, \ \forall n > n_1(\varepsilon)$.

3) Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$$\begin{aligned} |a_n . b_n - a. b| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |(a_n)(b_n - b) + (a_n - a)(b)| \\ &\leq |a_n||b_n - b| + |b||a_n - a| \\ &< M_1 \frac{\varepsilon}{2M_1} + |b| \frac{\varepsilon}{2|b|} = \varepsilon, \forall n > n_2(\varepsilon) \end{aligned}$$

i.e. $a_n . b_n \to a. b$.

Proof: **(iv)** To prove $Ca_n \to Ca, \quad \forall C \in \mathbb{R}$

Case 1: If $c = 0 \Longrightarrow 0 \to 0.$

Case 2: If $c \neq 0 \Longrightarrow |c| > 0$

$a_n \to a, \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

Let $\varepsilon = \frac{\varepsilon}{|c|} > 0$

$|ca_n - ca| = |c||a_n - a| < |c| \frac{\varepsilon}{|c|} = \varepsilon, \forall n > n_0(\varepsilon)$ i.e. $Ca_n \to Ca, \ \forall C \in \mathbb{R}$

Proof: **(v)** To prove $\frac{a_n}{b_n} \to \frac{a}{b}, \ b_n \neq 0, b \neq 0$

1) To prove $\frac{1}{b_n} \to \frac{1}{b}, b_n \neq 0, b \neq 0$

$\because b_n \to b \Longrightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|b_n - b| < \varepsilon, \ \forall n > n_0(\varepsilon)$

$\because b \neq 0 \Longrightarrow b > 0$ $(-b > 0$).

Let $\varepsilon = \dfrac{b}{2} > 0$

$|b_n - b| < \varepsilon$ means

$-\varepsilon < b_n - b < \varepsilon$

$b - \varepsilon < b_n < b + \varepsilon$

$$b - \frac{b}{2} < b_n < b + \frac{b}{2} \Longrightarrow 0 < \frac{b}{2} < b_n < \frac{3b}{2} \Longrightarrow 0 < \frac{2}{3b} < \frac{1}{b_n} < \frac{2}{b}$$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b - b_n}{b_n b}\right| = \frac{1}{|b_n||b|}|b_n - b| < \frac{2}{b^2}.\varepsilon = \frac{2}{b^2}.\frac{b^2\varepsilon}{2} = \varepsilon, \quad \text{(we choose } \varepsilon = \frac{b^2\varepsilon}{2})$$

$$\therefore \frac{1}{b_n} \to \frac{1}{b}$$

2) By using part (iii) $\Longrightarrow \therefore \frac{a_n}{b_n} \to \frac{a}{b}.$

Note:

$(1+a)^n \geq 1+na,\ a>0.$

Theorem: Let a_n be a sequence of real numbers, if $a_n \to a$, then:

i) If $a>0$, then $\frac{1}{1+na} \to 0$

ii) If $0<a<1$, then $a^n \to 0$

iii) If $a_n \geq 0 \Longrightarrow a \geq 0$

iv) $|a_n| \to |a|$

v) If $a_n \geq 0, a \geq 0$, then $\sqrt{a_n} \to \sqrt{a}$.

Proof: For (i)

By Archimedean Property $\frac{1}{n} \to 0$, and $c>0$, $\frac{1}{n}.c \to 0$

$$1+na>na \Longrightarrow \frac{1}{1+na} < \frac{1}{na} = (c)\frac{1}{n}, \qquad \text{where } c=\frac{1}{a}>0$$

$$\therefore \frac{1}{1+na} \to 0.$$

Proof: For (ii)

$$\because 0<a<1 \Longrightarrow a=\frac{1}{1+b}, \qquad b>0$$

$$a^n = \left(\frac{1}{1+b}\right)^n \leq \frac{1}{1+nb} \text{ , (by note)}$$

$$< \frac{1}{nb} \quad = c.\frac{1}{n} \to 0, \quad \text{(By A. P.} \frac{1}{n} \to 0 \text{ and } c=\frac{1}{b}>0)$$

$$\therefore a^n \to 0.$$

Proof: **For (iii)**

Let $a_n \geq 0$. Assume that $a < 0 \Longrightarrow -a > 0$

$a_n \to a \Longrightarrow \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$|a_n - a| < \varepsilon$ means

$a - \varepsilon < a_n < a + \varepsilon, \qquad \forall n > n_0(\varepsilon)$

we choose $\varepsilon = -a > 0$

$\Longrightarrow a_n < a + \varepsilon = a + (-a) = 0 \Longrightarrow a_n < 0$ C! which is impossible $\Longrightarrow \therefore a \geq 0$.

Proof: **For (iv)** If $a_n \to a$, then $|a_n| \to |a|$

$a_n \to a, \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

$\big||a_n| - |a|\big| \leq |a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

$\therefore |a_n| \to |a|$.

Remark: The converse may not be true.

For example:

$a_n = (-1)^n, \ \ |a_n| = |(-1)^n| = 1 \to 1.$

But a_n does not converge.

Proof: **For (v)** If $a_n \to a$, then $\sqrt{a_n} \to \sqrt{a}$

$a_n \to a$, i.e. $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

Let $\varepsilon = \sqrt{a}\,\varepsilon > 0$

$$\left|\sqrt{a_n} - \sqrt{a}\right| = \left|\sqrt{a_n} - \sqrt{a} \times \frac{\sqrt{a_n} + \sqrt{a}}{\sqrt{a_n} + \sqrt{a}}\right| = \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \leq \frac{|a_n - a|}{\sqrt{a}} < \frac{\varepsilon}{\sqrt{a}} < \frac{\sqrt{a}\,\varepsilon}{\sqrt{a}} = \varepsilon, \forall n > n_0(\varepsilon)$$

$\therefore \sqrt{a_n} \to \sqrt{a}$.

Theorem (Sandwich Theorem):

If $a_n \to a, b_n \to a,$ (c_n) be a sequence of real numbers such that $a_n \leq c_n \leq b_n$, then $c_n \to a$.

Proof:

$a_n \to a, \Longrightarrow \ \forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$.

$b_n \to a \Longrightarrow \ \forall \varepsilon > 0, \exists n_1(\varepsilon) > 0$ such that $|b_n - a| < \varepsilon, \forall n > n_1(\varepsilon)$.

Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$-\varepsilon < a_n - a \leq c_n - a \leq b_n - a < \varepsilon$

$\Longrightarrow -\varepsilon < c_n - a < \varepsilon$

i.e. $|c_n - a| < \varepsilon, \forall n > n_2(\varepsilon)$

$\therefore c_n \to a$.

Example:

Discuss the convergent of $a_n = \dfrac{\sin(n)}{n}$.

Answer:

$-1 \leq \sin(n) \leq 1$

$-\dfrac{1}{n} \leq \dfrac{\sin(n)}{n} \leq \dfrac{1}{n}$

By Archimedean property $\dfrac{1}{n} \to 0$ *and* $-\dfrac{1}{n} \to 0$

By Sandwich theorem $a_n = \dfrac{sin(n)}{n} \to 0$.

Definition (Monotone Sequence of Real Numbers):

Let (a_n) be a sequence of real numbers, then:

(a_n) is called increasing sequence (↑) if $a_n \leq a_{n+1}, \forall n \in \mathbb{N}$.

(a_n) is called decreasing sequence (↓) if $a_n \geq a_{n+1}, \forall n \in \mathbb{N}$.

(a_n) is called monotone equence (↕) if a_n increasing (↑) or a_n decreasing (↓).

For example:

$a_n = n\ (\uparrow), \qquad a_n = \frac{1}{n}(\downarrow), \qquad a_n = k(\leftrightarrow).$

Theorem (Monotone Theorem of Sequence):

Let (a_n) be a monotone sequence of real numbers. (a_n) convergent iff (a_n) is bounded.

Proof:

$\Longrightarrow$) It has been proved.

$\Longleftarrow$)

Let $S = \{a_n : n \in \mathbb{N}\}, \emptyset \neq S \subseteq \mathbb{R}$, S is bounded (since range is bounded set)

By completeness of $\mathbb{R} \Longrightarrow S$ has least upper bound say a

We claim $a_n \to a$

$\forall \varepsilon > 0$, $a - \varepsilon < a$

$a - \varepsilon$ is not upper bound for $S \Longrightarrow \exists\ a_{n0}(\varepsilon) > 0$ such that $a - \varepsilon < a_{n0}(\varepsilon)$

Since (a_n) monotone (increasing) $\Longrightarrow a_{n0}(\varepsilon) \leq a_n, \forall n > n_0(\varepsilon)$

$\Longrightarrow a - \varepsilon < a_n \Longrightarrow |a_n - a| < \varepsilon, \forall n > n_0(\varepsilon)$

$\therefore a_n \to a$.

Example:

Discuss the convergent of following sequence

1) $a_1 = 1$, $a_{n+1} = \frac{1}{4}(2a_n + 3)$, $\forall n \geq 1$.

Answer: To prove a_n convergent

1) monotone (increasing)

$$a_1 = 1, a_2 = \frac{1}{4}(2.1 + 3) = \frac{5}{4}$$

$$a_n = \left(1, \frac{5}{4}, \dots\right) \text{ is increasing}$$

We have to prove that $a_n \leq a_{n+1}$

by using mathematical induction

for $n = 1 \Longrightarrow a_1 \leq a_2$

Assume that it is true for $n = k \Longrightarrow a_k \leq a_{k+1}$

$$\underbrace{\frac{1}{4}(2a_k + 3)}_{a_{k+1}} \leq \underbrace{\frac{1}{4}(2a_{k+1} + 3)}_{a_{k+2}}$$

$\therefore a_n$ is increasing.

2) To prove a_n is bounded

$$a_1 = 1, a_2 = \frac{5}{4} < 2$$

To prove $a_n \leq 2$

by mathematical induction

for $a_1 = 1 < 2$

Assume that it is true when $n = k \Longrightarrow a_k < 2$

we have to prove that $a_{k+1} < 2$

$$\frac{1}{4}(2a_k + 3) < \frac{1}{4}(2.2 + 3)$$

$$\begin{array}{ccc} \| & & \| \\ a_{k+1} & & \frac{7}{4} < 2 \end{array}$$

$$\therefore a_{k+1} < \frac{7}{4} < 2 \Longrightarrow a_{k+1} < 2$$

$\therefore a_{k+1}$ is bounded above.

By (Monotone Theorem) a_n convergent , $(a_n \to a)$

Now, to calculate the convergent point (a)

we have $\quad a_{n+1} = \frac{1}{4}(2a_n + 3)$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ a & & \frac{1}{4}(2a + 3) \end{array}$$

$$\Longrightarrow a = \frac{1}{4}(2a + 3)$$

$$4a = 2a + 3 \Longrightarrow a = \frac{3}{2}$$

$$\therefore a_n \to \frac{3}{2}.$$

Definition (Cauchy Sequence):

Let (a_n) be a sequence of real numbers. (a_n) is called Cauchy sequence if $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a_m| < \varepsilon,\ \forall n, m > n_0(\varepsilon)$.

Remark:

i) If (a_n) convergent to a, then (a_n) is Cauchy.
ii) The converse of (i) is not true.

Proof: **(i)** If $(a_n) \to a$, then (a_n) is Cauchy.

$a_n \to a$ means $\forall \varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a| < \frac{\varepsilon}{2}, \forall n > n_0(\varepsilon)$

$$\begin{aligned} |a_n - a_m| &= |a_n - a + a - a_m| \\ &\leq |a_n - a| + |a_m - a| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall n, m > n_0(\varepsilon) \end{aligned}$$

(ii) The converse of (i) is not true.

For example: Let $X = \mathbb{R}\backslash\{0\}, (a_n) = \frac{1}{\mathrm{n}}$

$(a_n) = \frac{1}{\mathrm{n}} \to 0$ in $\mathbb{R}$ (By Archimedean Property) $\Longrightarrow \therefore (a_n) = \frac{1}{\mathrm{n}}$ is Cauchy

But does not convergent in $\mathbb{R}\backslash\{0\}$.

Note: If (a_n) Cauchy sequence of real numbers, then (a_n) is bounded.

Definition (Subsequence):

Let (a_n) be a sequence of real numbers. The sequence (a_{nk}) is called subsequence.

Example:

$a_n = (-1)^n$

$a_{nk} = -1$ subsequence of a_n

$a_{nk} = 1$ subsequence of a_n

Theorem: Let a_{nk} be any subsequence of the sequence of real numbers a_n , then:

i) If a_n convergent, then a_{nk} is convergent
ii) If a_n bounded, then a_{nk} is bounded
iii) If a_n monotone, then a_{nk} is monotone.

Theorem: (Bolezano-Weierstrass)

Every bounded sequence of real numbers has convergent subsequence.

Example:

$a_n = (-1)^n$ bounded sequence

$a_{nk} = -1$ convergent subsequence $(a_{nk} = -1 \to -1)$

$a_{nk} = 1$ convergent subsequence $(a_{nk} = 1 \to 1)$

Theorem: If (a_n) is a Cauchy sequence in $\mathbb{R}$ then it is convergent.

Proof:

1. (a_n) is a Cauchy sequence $\Longrightarrow (a_n)$ bounded.
2. (a_n) has convergent subsequence a_{nk} $(a_{nk} \to a)$ (by Bolezano-Weierstrass theorem).
3. Now, to prove that $a_n \to a$.

a_n Cauchy sequence $\Longrightarrow \forall\varepsilon > 0, \exists n_0(\varepsilon) > 0$ such that $|a_n - a_m| < \frac{\varepsilon}{2}, \ \forall n, m > n_0(\varepsilon)$

$a_{nk} \to a \Longrightarrow \exists n_1(\varepsilon) > 0$ s.t. $|a_{nk} - a| < \frac{\varepsilon}{2}, \ \forall n_k > n_1(\varepsilon)$

Choose $n_2(\varepsilon) = \max\{n_0(\varepsilon), n_1(\varepsilon)\}$

$|a_n - a| = |a_n - a_{nk} + a_{nk} - a| \leq |a_n - a_{nk}| + |a_{nk} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \forall n > n_2(\varepsilon)$

$\therefore a_n \to a.$

<u>Theorem:</u> In $\mathbb{R}$, (a_n) is a Cauchy sequence $\Leftrightarrow$ (a_n) is convergent.

Chapter Three
Metric Space

Definition (Metric Space):

Let X be any nonempty set, the function $d: X \times X \to \mathbb{R}$ is called metric on X if d satisfies:

M_1: $d(x,y) \geq 0$

M_2: $d(x,y) = 0 \Leftrightarrow x = y$

M_3: $d(x,y) = d(y,x)$

M_4: $d(x,y) \leq d(x,z) + d(z,y)$

$\forall\, x, y, z \in X$

The pair (X,d) is called metric space.

Example (1):

Let $X = \mathbb{R}$, $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined as follows $d(x,y) = |x-y|$, $\forall x, y \in \mathbb{R}$.

Show that $(\mathbb{R}, d)$ is a metric space.

Answer:

Let $x, y, z \in \mathbb{R}$

M_1: $\because |x-y| \geq 0 \implies \therefore d(x,y) = |x-y| \geq 0$

M_2: $d(x,y) = 0 \Leftrightarrow |x-y| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$

M_3: $d(x,y) = |x-y| = |y-x| = d(y,x)$

M_4: $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$.

$\therefore d$ is metric on $\mathbb{R}$

$(\mathbb{R}, d)$ is metric space called absolute metric (usual metric space).

Some Important Inequality:

1. Cauchy-Schwartz Inequality

   Let $a_1,\ a_2, \dots ,\ a_n,\ b_1, b_2, \dots ,\ b_n$ are real numbers then

   $$\sum_{i=1}^{n}|a_i + b_i| \le \sqrt{\sum_{i=1}^{n} a_i^2}\ .\sqrt{\sum_{i=1}^{n} b_i^2}$$

2. Minkowski Inequality

   Let $a_1,\ a_2, \dots ,\ a_n,\ b_1, b_2, \dots ,\ b_n$ are real numbers then

   $$\sqrt{\sum_{i=1}^{n}(a_i + b_i)^2} \le \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}$$

Example (2):

Let $X = \mathbb{R}^2$, $d: \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, defined as follows $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ $\forall x = (x_1, y_1), y = (x_2, y_2) \in \mathbb{R}^2$. Is $(\mathbb{R}^2, d)$ forms metric space ?

Answer:

Let $x = (x_1, y_1), y = (x_2, y_2),\ Z = (x_3, y_3) \in \mathbb{R}^2$

M_1: $\because \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ge 0 \implies \therefore d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \ge 0$

M_2: $d(x, y) = 0 \Leftrightarrow \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = 0$

$\Leftrightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$

$\Leftrightarrow x_1 - x_2 = 0$ and $y_1 - y_2 = 0$

$\Leftrightarrow x_1 = x_2$ and $y_1 = y_2 \Leftrightarrow x = y$.

M_3: $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d(y, x)$.

M_4: $d(x, y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

$= \sqrt{(x_1 - x_3 + x_3 - x_2)^2 + (y_1 - y_3 + y_3 - y_2)^2}$

$\le \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} = d(x, z) + d(z, y)$. (By using Minkowski Inequality)

$\therefore d$ is metric on $\mathbb{R}^2$, $(\mathbb{R}^2, d)$ is a metric space called (Euclidian metric space).

<u>Example (3)</u>: Let X be any nonempty set, $d: X \times X \to \mathbb{R}$ defined as follows

$$d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}, \forall\, x, y \in X$$

Show that (X, d) is a metric space.

<u>Answer</u>:

M_1: $d(x,y) \geq 0, \forall\, x, y \in X$

M_2: $d(x,y) = 0 \Leftrightarrow x = y, \forall\, x, y \in X$

M_3: $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases} = \begin{cases} 1, & y \neq x \\ 0, & y = x \end{cases} = d(y,x), \forall\, x, y \in X$

M_4: $d(x,y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}$

1. If $x = y$ $\;and\;$ $y = z \Longrightarrow x = z$

   $d(x,y) = 0 \leq d(x,z) + d(z,y) = 0$

2. If $x \neq y$ $\;and\;$ $y \neq z \Longrightarrow x \neq z$

   $d(x,y) = 1 \leq d(x,z) + d(z,y) = 2$

3. If $x = y$ $\;and\;$ $y \neq z \Longrightarrow x \neq z$

   $d(x,y) = 0 \leq d(x,z) + d(z,y) = 2$

4. If $x \neq y$ $\;and\;$ $y = z \Longrightarrow x \neq z$

   $d(x,y) = 1 \leq d(x,z) + d(z,y) = 1$

$\therefore d(x,y) \leq d(x,z) + d(z,y), \forall\, x, y, z \in X$

$\therefore (X, d)$ is metric space

Example (4):

Let $X = \mathrm{C}[a,b]$, $d: \mathrm{C}[a,b] \times \mathrm{C}[a,b] \to \mathbb{R}$, defined as follows

$d(f,g) = \max\{|f(x) - g(x)|: x \in [a,b]\}, \forall f,g \in \mathrm{C}[a,b]$

Show that $(\mathrm{C}[a,b], d)$ is a metric space.

Answer:

Let $f,g,h \in \mathrm{C}[a,b]$

M_1: $\because |f(x) - g(x)| \geq 0, \forall x \in [a,b] \Longrightarrow \therefore d(f,g) = \max\{|f(x) - g(x)|: x \in [a,b]\} \geq 0$

M_2:

$d(f,g) = 0 \Leftrightarrow \max\{|f(x) - g(x)|: x \in [a,b]\} = 0$

$\Leftrightarrow |f(x) - g(x)| = 0 \Leftrightarrow f(x) - g(x) = 0 \Leftrightarrow f(x) = g(x), \forall x \in [a,b] \Leftrightarrow f = g$

M_3: $d(f,g) = \max\{|f(x) - g(x)|: x \in [a,b]\} = \max\{|g(x) - f(x)|: x \in [a,b]\} = d(g,f)$

M_4:

$d(f,g) = \max\{|f(x) - g(x)|: x \in [a,b]\}$

$= \max\{|f(x) - h(x) + h(x) - g(x)|: x \in [a,b]\}$

$\leq \max\{|f(x) - h(x)|: x \in [a,b]\} + \max\{|h(x) - g(x)|: x \in [a,b]\} = d(f,h) + d(h,g)$

$\therefore (\mathrm{C}[a,b], d)$ is a metric space.