Joint Distributions: Two Random Variables

In real life, we are often interested in several random variables that are related to each other. For example, suppose that we choose a random family, and we would like to study the number of people in the family, the household income, the ages of the family members, etc. Each of these is a random variable, and we suspect that they are dependent. In this chapter, we develop tools to study joint distributions of random variables. The concepts are similar to what we have seen so far. The only difference is that instead of one random variable, we consider two or more. In this chapter, we will focus on two random variables, but once you understand the theory for two random variables, the extension to n random variables is straightforward. We will first discuss joint distributions of discrete random variables and then extend the results to continuous random variables.

Joint Probability Mass Function (JPMF)

Remember that for a discrete random variable X, we define the PMF as $P_X(x) = P(X = x)$. Now, if we have two random variables X and Y, and we would like to study them jointly, we define the **joint probability mass function** as follows:

The **joint probability mass function** of two discrete random variables X and Y is defined as

$$P_{XY}(x, y) = P(X = x, Y = y).$$

Joint Probability Mass Function (PMF)

Properties of joint p.m.f.

A joint p.m.f. f(x,y) satisfies two conditions :

1. **F(x,y)** ≥ 0, for all $(x,y) \in R$
2. $\sum_x \sum_y f(x, y) = 1$

Marginal PMFs

The joint PMF contains all the information regarding the distributions of X and Y. This means that, for example, we can obtain PMF of X from its joint PMF with Y. Indeed, we can write

$$
\begin{aligned}
P_X(x) &= P(X = x) \\
&= \sum_{y_j \in R_Y} P(X = x, Y = y_j) \qquad \text{law of total probablity} \\
&= \sum_{y_j \in R_Y} P_{XY}(x, y_j).
\end{aligned}
$$

Here, we call $P_X(x)$ the **marginal PMF** of X. Similarly, we can find the marginal PMF of Y as

$$P_Y(Y) = \sum_{x_i \in R_X} P_{XY}(x_i, y).$$

Marginal PMFs

Then:

Marginal PMFs of X and Y:

$$P_X(x) = \sum_{y_j \in R_Y} P_{XY}(x, y_j), \qquad \text{for any } x \in R_X$$

$$P_Y(y) = \sum_{x_i \in R_X} P_{XY}(x_i, y), \qquad \text{for any } y \in R_Y$$

Example

Consider two random variables X &Y with joint PMF given in the following Table:

	$Y=0$	$Y=1$	$Y=2$
$X=0$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{8}$
$X=1$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$

1. Find $P(X=0, Y \leq 1)$.
2. Find the marginal PMFs of X and Y.
3. Find $P(Y=1 | X=0)$.
4. Are X and Y independent?

Solution

1. To find $P(X=0,Y\leq 1)$, we can write

$$P(X=0,Y\leq 1)=P_{XY}(0,0)+P_{XY}(0,1)$$

$$=1/6+1/4=5/12$$

2. Note that from the table,

$R_X=\{0,1\}$ and $R_Y=\{0,1,2\}$.

Now find the marginal PMFs. For example, to find $P_X(0)$, we can write

$$P_X(0)=P_{XY}(0,0)+P_{XY}(0,1)+P_{XY}(0,2)$$

$$=1/6+1/4+1/8=13/24.$$

Solution

We obtain

$$P_X(x) = \begin{cases} \frac{13}{24} & x = 0 \\ \frac{11}{24} & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$P_Y(y) = \begin{cases} \frac{7}{24} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{7}{24} & y = 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution

3. Find P(Y=1|X=0): Using the formula for conditional probability, we have :

$$P(Y=1|X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = \frac{P_{XY}(0,1)}{P_X(0)} = \frac{\frac{1}{4}}{\frac{13}{24}} = \frac{6}{13}.$$

4. Are X and Y independent? X and Y are not independent, because as we just found out :

P(Y=1|X=0)=6 /13 ≠ P(Y=1) = 5 / 12.

Joint Probability Density Function (JPDF)

Let X and Y be two c.r.v. , a function f defined over XY-plane is a joint p.d.f. of X and Y :

P[(x,y)∈ R] = $\iint_R f(x,y)dxdy = \iint_R f(x,y)dydx$

Properties of joint p.d.f.

A joint p.d.f. f(x,y) satisfies two conditions :

1. **F(x,y)** ≥ 0, for all $(x,y) \in R$
2. $\iint_{-\infty}^{\infty} f(x,y)dxdy = 1$

Marginal (PDFs)

Let X and Y be two c.r.v. , a function f defined over XY-plane is a joint p.d.f. of X and Y :

P[(x,y)∈ R] = $\iint_R f(x,y)dxdy = \iint_R f(x,y)dydx$

also :

$f_1(x)= \int_y f(x,y)dy$, $f_1(x)$ **is called marginal p.d.f. of x.**

$f_2(y)= \int_x f(x,y)dx$, $f_2(y)$ **is called marginal p.d.f. of y.**

Example

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} cx^2y & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

1. **Find the constant c.**
2. **Find marginal PDFs, $f_X(x)$ and $f_Y(y)$.**
3. **Find $P(Y \le X / 2)$.**

Solution

1.

To find the constant c, we can write

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy \\ &= \int_{0}^{1}\int_{0}^{x} cx^2y \;\; dydx \\ &= \int_{0}^{1} \frac{c}{2}x^4 dx \\ &= \frac{c}{10}. \end{aligned}$$

Thus c = 10

Solution

2.

. To find the marginal PDFs, first note that $R_X = R_Y = [0,1]$. For $0 \leq x \leq 1$, we can write

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy \\ &= \int_0^x 10x^2 y dy \\ &= 5x^4. \end{aligned}$$

Thus,

$$f_X(x) = \begin{cases} 5x^4 & 0 \leq x \leq 1 \\ \\ 0 & \text{otherwise} \end{cases}$$

Solution

2.

For $0 \leq y \leq 1$, we can write

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx \\ &= \int_{y}^{1} 10x^2y dx \\ &= \frac{10}{3}y(1-y^3). \end{aligned}$$

Thus,

$$f_Y(y) = \begin{cases} \frac{10}{3}y(1-y^3) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Solution

3. now,

$$
\begin{aligned}
P\left(Y \le \frac{X}{2}\right) &= \int_{-\infty}^{\infty} \int_{0}^{\frac{x}{2}} f_{XY}(x,y)dydx \\
&= \int_{0}^{1} \int_{0}^{\frac{x}{2}} 10x^2y \;\; dydx \\
&= \int_{0}^{1} \frac{5}{4} x^4 dx \\
&= \frac{1}{4}.
\end{aligned}
$$