Chapter 1: Functions

1.1 Functions and their graph

Def: A function f from a set D to a set Y is a rule that assigns a unique (single) element f(x)∈ Y to each element x∈D.

The set D of all possible input values is called the **domain** of the function. The set of all values of f(x) as x varies throughout *D* is called the **range** of the function.

EXAMPLE 1

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

Solution:

1- $y = x^2$ gives a real y-value for any real number x, so the domain is (- ∞,∞).

The range of $y = x^2$ is [0, ∞) because the square of any real number is nonnegative and $x = \sqrt{y}$, x to be real y≥ 0.

2- y = l/x gives a real y-value for every x except x = 0. For the rules of arithmetic, we cannot divide any number by zero. The domain is ℝ\{0}. The range of y = 1/x, can be found by x = 1/y is the input assigned to the output value y. Then range is ℝ\{0}.

3- $y = \sqrt{x}$ gives a real y-value only if x≥ 0 so the domain is [0,∞).

The range of $y = \sqrt{x}$ can be found by y≥0 and $x = y^2$ so range =[0, ∞)

4- $y = \sqrt{4 - x}$: 4 - x ≥0 → 4≥x. The formula gives real y-values for all x≥4.

The range : first y≥0 , second $x = 4 - y^2$→ range =[0, ∞).

5- $y = \sqrt{1 - x^2}$ gives a real y-value if $1 - x^2 \geq 0$ → (1-x)(1+x) ≥0

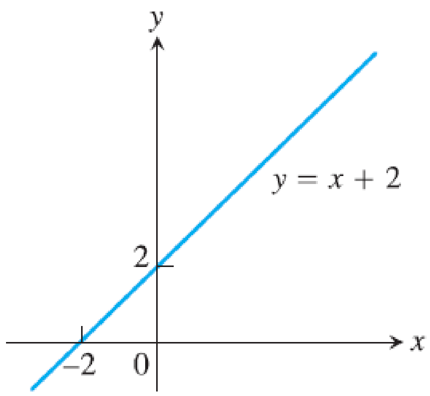
Domain= [-1,1].

Range: . First y≥0 , second $x^2 = 1 - y^2$ → $x = \pm\sqrt{1 - y^2}$ which means that we get the same solution above i.e. y=[-1,1] this implies that the range should be [0.1].

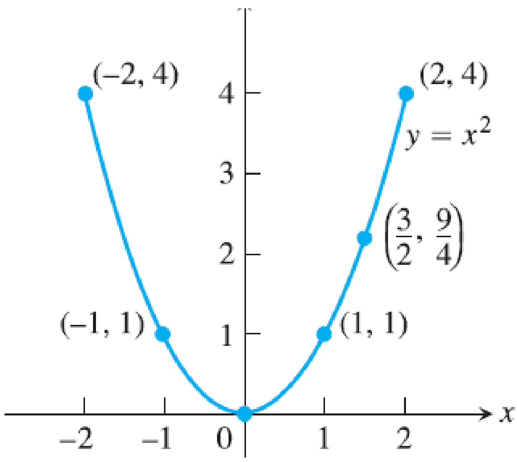
Graphs of Functions

If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for *f*. In set notation, the graph is {(x,f(x)) / *x* ∈*D*}.

***EXAMPLE 1**:* The graph of the function f(x) = x + 2



EXAMPLE 2: Graph the function $y = x^2$ over the interval $[-2, 2]$.

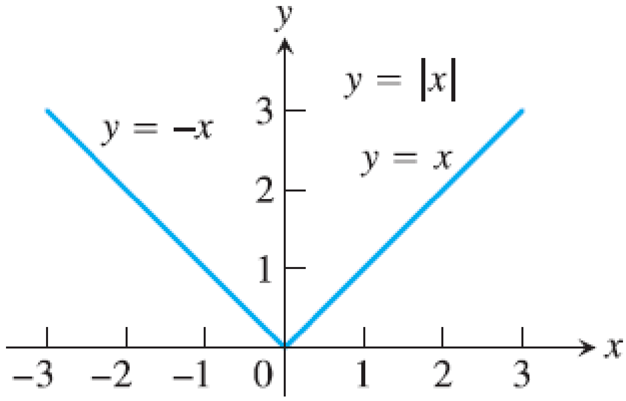


Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function.**

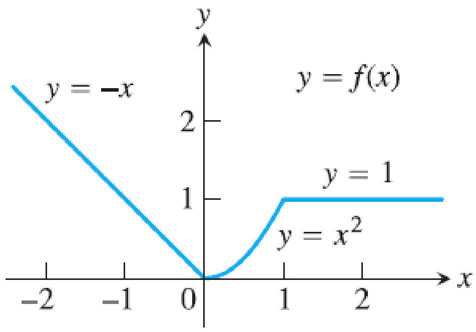
Example 3:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$



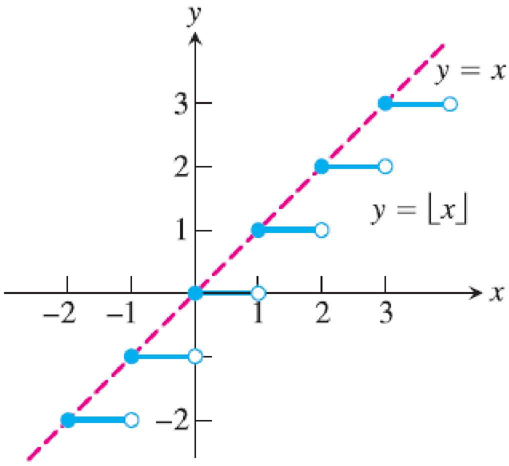
Example 4: the function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

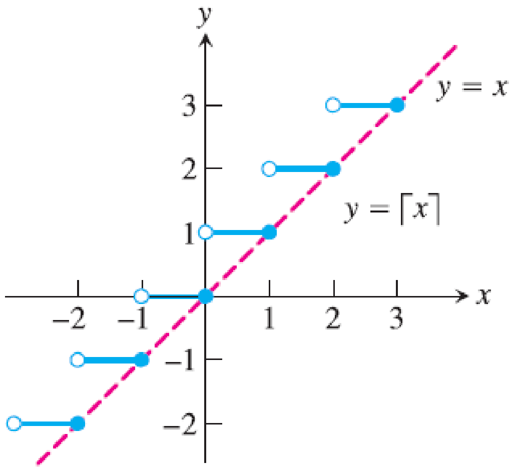


Example 5: greatest integer function or **the integer floor function**: The function whose value at any number x is the *greatest integer less than or equal to x.* It is denoted $\lfloor x \rfloor$. Observe that:

$$\lfloor 2.4 \rfloor = 2, \quad \lfloor 1.9 \rfloor = 1, \quad \lfloor 0 \rfloor = 0, \quad \lfloor -1.2 \rfloor = -2,$$
$$\lfloor 2 \rfloor = 2, \quad \lfloor 0.2 \rfloor = 0, \quad \lfloor -0.3 \rfloor = -1 \quad \lfloor -2 \rfloor = -2.$$



Example 6: least integer function or the integer ceiling function: The function whose value at any number x is the *smallest integer greater than or equal to x*. It is denoted $\lceil x \rceil$.



DEFINITIONS Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_2) > f(x_1)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

EXAMPLE 7: The function graphed in example 4 is decreasing on $(-\infty,0]$ and increasing on $[0, 1]$. The function is neither increasing nor decreasing on the interval $[1,\infty)$.

Even Functions and Odd Functions: Symmetry

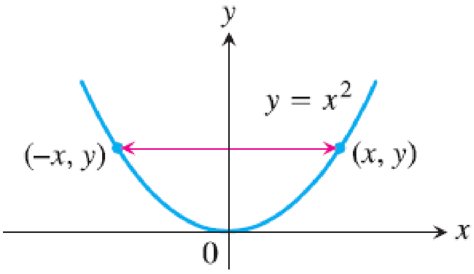
DEFINITIONS A function $y = f(x)$ is an

even function of x if $f(-x) = f(x)$,
odd function of x if $f(-x) = -f(x)$,

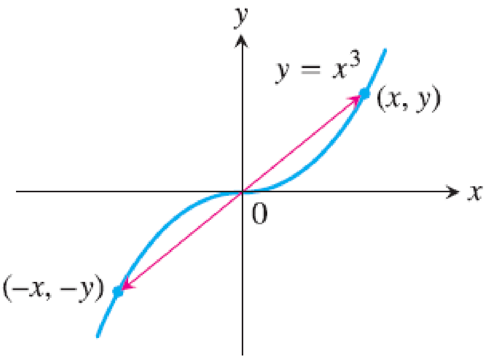
for every x in the function's domain.

The graph of an even function is symmetric about they-axis. Since $f(-x) = f(x)$, a point (x,y) lies on the graph if and only if the point (-x, y) lies on the graph . A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is symmetric about the origin. Since $f(-x) = -f(x)$, a point (x,y) lies on the graph if and on1y if the point $(-x, -y)$ lies on the graph.



(a)

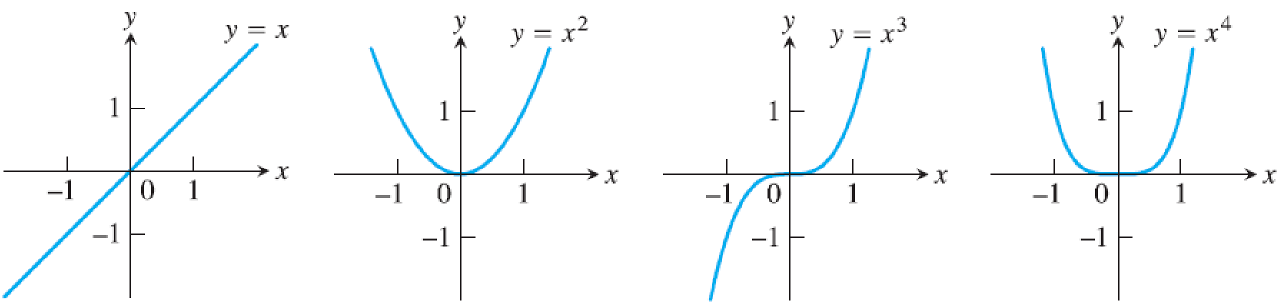


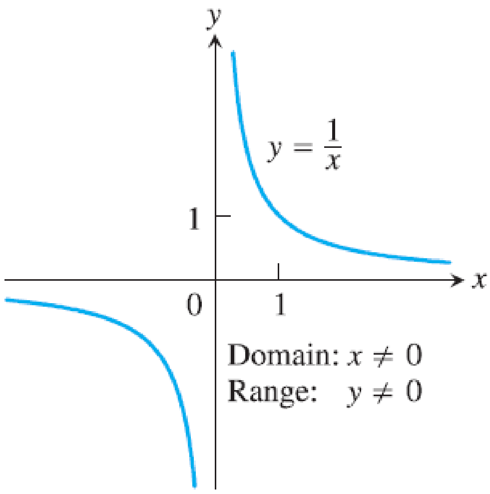
(b)

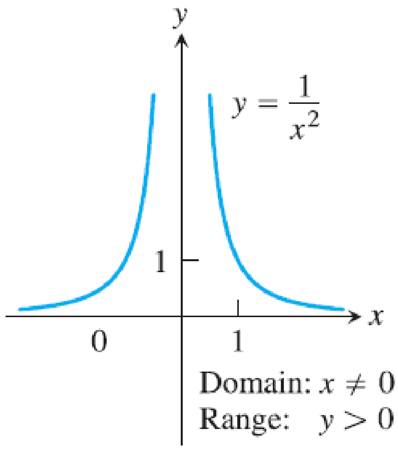
EXAMPLE 8:

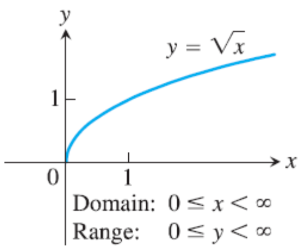
$f(x) = x^2$	Even function: $(-x)^2 = x^2$ for all x; symmetry about y-axis.
$f(x) = x^2 + 1$	Even function: $(-x)^2 + 1 = x^2 + 1$ for all x; symmetry about y-axis
$f(x) = x$	Odd function: $(-x) = -x$ for all x; symmetry about the origin.
$f(x) = x + 1$	Not odd: $f(-x) = -x + 1$, but $-f(x) = -x - 1$. The two are not equal. Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$

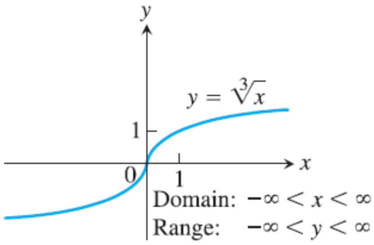
Common Function

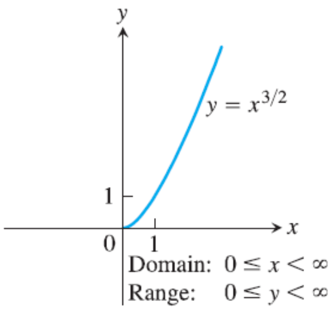












1.2 Combining Functions , Shifting and Scaling Graphs

Sums, Differences, Products, and Quotients

$$(f + g)(x) = f(x) + g(x).$$

$$(f - g)(x) = f(x) - g(x).$$

$$(fg)(x) = f(x)g(x).$$

At each of these functions the domain = domain (f) $\cap$ $domain(g)$

At any point of domain (f) $\cap$ $domain(g)$ at which g(x) $\neq$ 0, we can also define the function f/g by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0)$$

Functions can also be multiplied by constants: If c is a real number, then the function cf is defined for all x in the domain of f by (cf)(x) = cf(x).

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

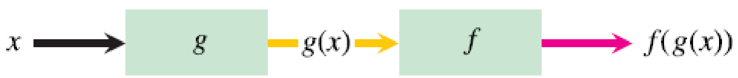
Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Composite Functions

Definition: If f and g are functions, the **composite** function fog is defined by

(f 0g)(x) = f(g(x)).

The domain of *fog* consists of the numbers x in the domain of g for which g(x) lies in the domain of f.



EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

To see why the domain of $f \circ g$ is $[-1, \infty)$, notice that $g(x) = x + 1$ is defined for all real x but belongs to the domain of f only if $x + 1 \geq 0$, that is to say, when $x \geq -1$. ■

Notice that if $f(x) = x^2$ and $g(x) = \sqrt{x}$, then $(f \circ g)(x) = (\sqrt{x})^2 = x$. However, the domain of $f \circ g$ is $[0, \infty)$, not $(-\infty, \infty)$, since $\sqrt{x}$ requires $x \geq 0$.

Shifting a Graph of a Function

Vertical Shifts

$y = f(x) + k$	Shifts the graph of f *up* k units if $k > 0$ Shifts it *down* $\lvert k \rvert$ units if $k < 0$

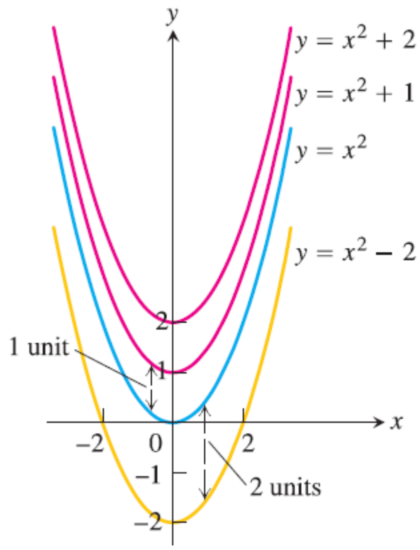
Horizontal Shifts

$y = f(x + h)$	Shifts the graph of f *left* h units if $h > 0$ Shifts it *right* $\lvert h \rvert$ units if $h < 0$

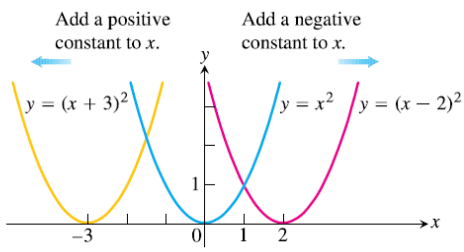
EXAMPLE 3

(a) Adding 1 to the right-hand side of the formula $y = x^2$ to get $y = x^2 + 1$ shifts the graph up 1 unit (Figure 1.29).

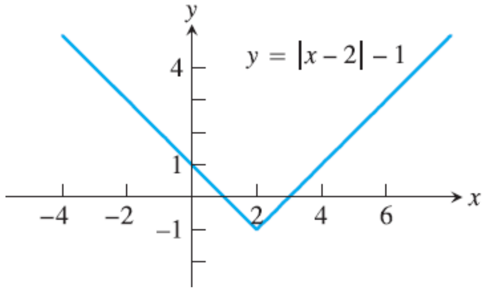
(b) Adding -2 to the right-hand side of the formula $y = x^2$ to get $y = x^2 - 2$ shifts the graph down 2 units (Figure 1.29).



(c) Adding 3 to x in $y = x^2$ to get $y = (x + 3)^2$ shifts the graph 3 units to the left



(d) Adding -2 to x in $y = |x|$, and then adding -1 to the result, gives $y = |x - 2| - 1$ and shifts the graph 2 units to the right and 1 unit down (Figure 1.31). ■



Vertical and Horizontal Scaling and Reflecting Formulas

For $c > 1$, the graph is scaled:

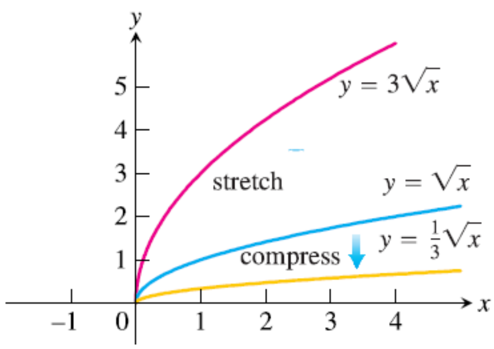
$y = cf(x)$	Stretches the graph of f vertically by a factor of c.
$y = \frac{1}{c} f(x)$	Compresses the graph of f vertically by a factor of c.
$y = f(cx)$	Compresses the graph of f horizontally by a factor of c.
$y = f(x/c)$	Stretches the graph of f horizontally by a factor of c.

For $c = -1$, the graph is reflected:

$y = -f(x)$	Reflects the graph of f across the x-axis.
$y = f(-x)$	Reflects the graph of f across the y-axis.

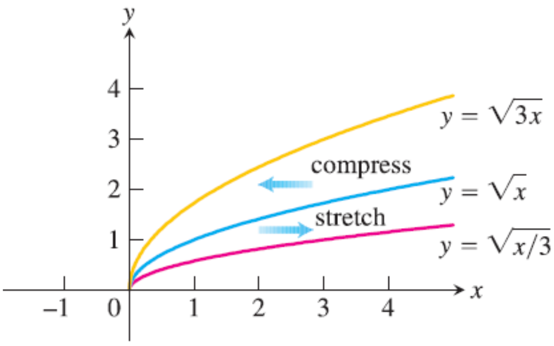
EXAMPLE 4 Here we scale and reflect the graph of $y = \sqrt{x}$.

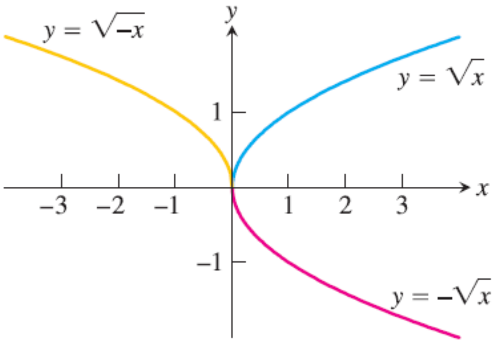
(a) **Vertical:** Multiplying the right-hand side of $y = \sqrt{x}$ by 3 to get $y = 3\sqrt{x}$ stretches the graph vertically by a factor of 3, whereas multiplying by $1/3$ compresses the graph by a factor of 3 (Figure 1.32).



(b) **Horizontal:** The graph of $y = \sqrt{3x}$ is a horizontal compression of the graph of $y = \sqrt{x}$ by a factor of 3, and $y = \sqrt{x/3}$ is a horizontal stretching by a factor of 3 (Figure 1.33). Note that $y = \sqrt{3x} = \sqrt{3}\sqrt{x}$ so a horizontal compression *may* correspond to a vertical stretching by a different scaling factor. Likewise, a horizontal stretching may correspond to a vertical compression by a different scaling factor.

(c) **Reflection:** The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis, and $y = \sqrt{-x}$ is a reflection across the y-axis (Figure 1.34). ■

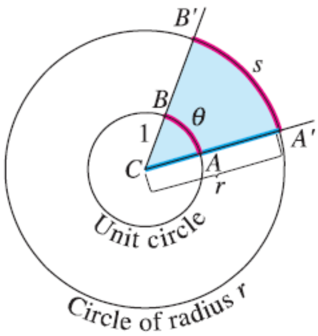




1.3 Trigonometric Functions

Angles

Angles are measured in degrees or radians. One radian is the angle subtended at the centre of a circle by an arc that is equal in length to the radius of the circle, that is, θ = s / r, where θ is the subtended angle in radians, s is arc length, and r is radius.



Let the circle is a unit circle having radius $r = 1$, one complete revolution of the unit circle is 360 degree has arc length 2r*π =2π radians, so we have

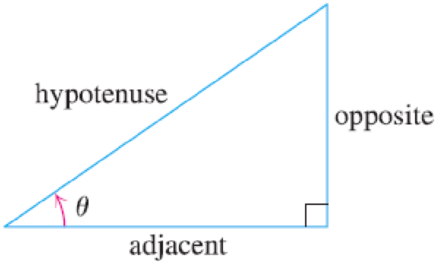
$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \quad \text{or} \quad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
***θ* (radians)**	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

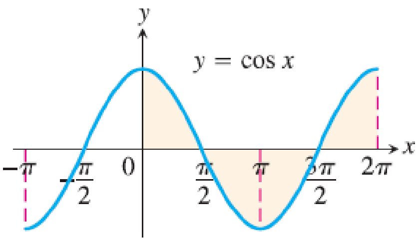
$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

Even

$$\cos(-x) = \cos x$$
$$\sec(-x) = \sec x$$

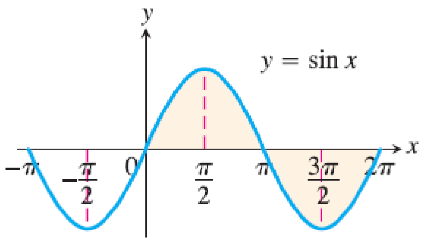
Odd

$$\sin(-x) = -\sin x$$
$$\tan(-x) = -\tan x$$
$$\csc(-x) = -\csc x$$
$$\cot(-x) = -\cot x$$



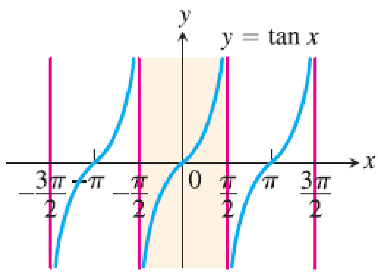
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



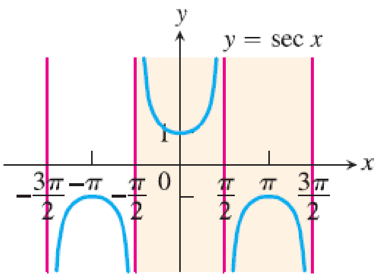
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



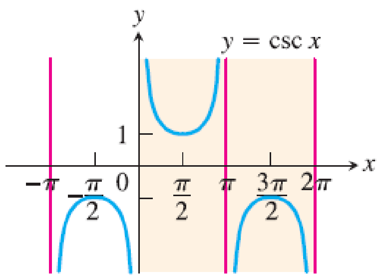
Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



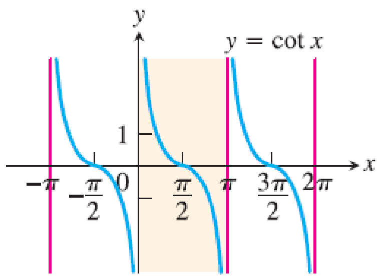
Domain: $x \ne \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(d)



Domain: $x \ne 0, \pm\pi, \pm2\pi, \ldots$
Range: $y \le -1$ or $y \ge 1$
Period: 2π

(e)



Domain: $x \ne 0, \pm\pi, \pm2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

$$\cos^2\theta + \sin^2\theta = 1.$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Chapter 2: LIMITS AND CONTINUITY

1.2 Limit of a Function and Limit Laws

The limit of a function is the behaviour of that function near a particular input.

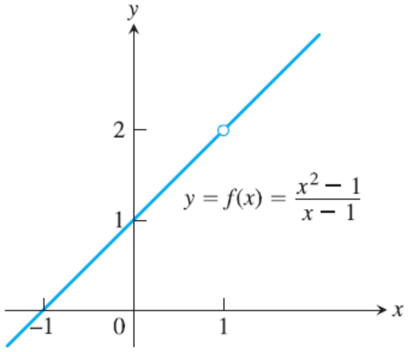
EXAMPLE 1:

How does the function $f(x) = \dfrac{x^2 - 1}{x - 1}$ behave near $x=1$?

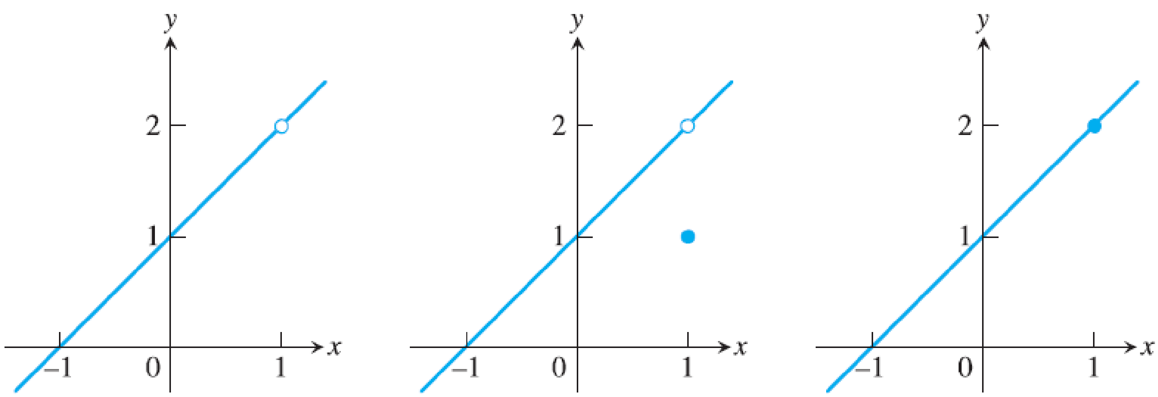
Solution: The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and cancelling common factors:

$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \quad \text{for} \quad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point (1, 2) *removed.* This removed point is shown as a "hole" in the figure.



EXAMPLE 2:



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

The limits of *f(x), g(x),* and *h(x)* all equal 2 as *x* approaches 1. However, only *h(x)* has the same function value as its limit at *x*=1.

Some limits:

(a) If *f* is the identity function *f(x)* = *x,* then for any value of x_o,

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} x = x_0$$

(h) If *f* is the constant function *f(x)* = *k* (function with the constant value *k),* then:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k.$$

For example $\lim_{x \to 3} x = 3$ and $\lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$

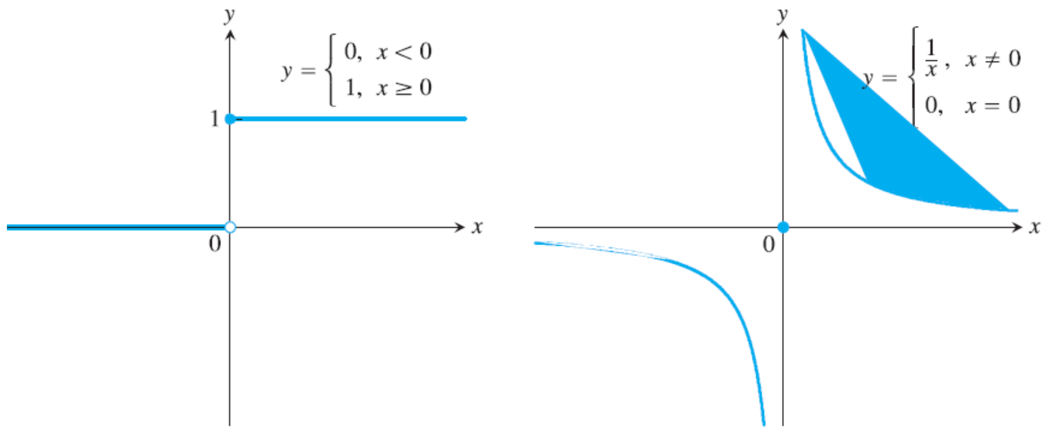
Note: Some ways that limits can fail to exist as described in the next example.

EXAMPLE 3: Discuss the behaviour of the following functions as x →0

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution:



(a) Unit step function $U(x)$ (b) $g(x)$

a) *U(x)* has no limit as *x* →0 because its values jump at *x* = 0. For negative values of *x* arbitrarily close to zero, *U(x)* = 0. For positive values of *x* arbitrarily close to zero, *U(x)*= 1. There is no *single* value *L* approached by *U(x)* as *x* → 0.

b) *g(x)* has no limit as *x* → 0 because the values of g grow arbitrarily large in positive value as *x* →0 and go to very small in negative value. There is no *fix*ed real number.

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 4:

(a) $$\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$$
$$= c^3 + 4c^2 - 3$$

(b) $$\lim_{x \to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$$
$$= \frac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

EXAMPLE 5:

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0$$

EXAMPLE 6:

$$\lim_{x\to 1}\frac{x^2+x-2}{x^2-x} \qquad \lim_{x\to 1}\frac{(x-1)(x+2)}{x(x-1)} \qquad \lim_{x\to 1}\frac{x+2}{x}=\frac{1+2}{1}=3$$

THEOREM 4—The Sandwich Theorem Suppose that $g(x)\le f(x)\le h(x)$ for all x in some open interval containing c, except possibly at $x=c$ itself. Suppose also that

$$\lim_{x\to c} g(x)=\lim_{x\to c} h(x)=L.$$

Then $\lim_{x\to c} f(x)=L$.

EXAMPLE 7:

Given that: $1-\frac{x^2}{4}\le u(x)\le 1+\frac{x^2}{2}$ for all $x\neq 0$,

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution $\lim_{x\to 0}(1-(x^2/4))=1$ and $\lim_{x\to 0}(1+(x^2/2))=1$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x)=1$

EXAMPLE 8:

Find $\lim_{x\to 0} x^2\sin\frac{1}{x}$, $x\neq 0$

Solution $-1\le sin\frac{1}{x}\le 1$

$$-x^2\le x^2\sin\frac{1}{x}\le x^2$$

$$\lim_{x\to 0}(-x^2)\le \lim_{x\to 0} x^2\sin\frac{1}{x}\le \lim_{x\to 0} x^2$$

$$0\le \lim_{x\to 0} x^2\sin\frac{1}{x}\le 0$$

EXAMPLE 9: Given $|g(x)-4|\le 5(x-2)^2$ $\forall x$, $find$ $\lim_{x\to 2} g(x)$

Solution: $-5(x-2)^2\le g(x)-4\le 5(x-2)^2$

$$-5(x-2)^2+4 \le g(x) \le 5(x-2)^2+4$$

$$\lim_{x\to 2} -5(x-2)^2+4 \le \lim_{x\to 2} g(x) \le \lim_{x\to 2} 5(x-2)^2+4$$

$$4 \le \lim_{x\to 2} g(x) \le 4$$

$\therefore \lim_{x\to 2} g(x) = 4$ (by Sandwich theorem)

EXAMPLE 10: For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.

Solution :

Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$. ■

EXAMPLE 11: Find $\lim_{x\to 0} x \sin\frac{1}{x}$

Solution :

$$0 \le |\sin\left(\frac{1}{x}\right)| \le 1$$

$$0 \le |x|\,|\sin\left(\frac{1}{x}\right)| \le |x|$$

$$0 \le |x \sin\left(\frac{1}{x}\right)| \le |x|$$

$$\lim_{x\to 0} 0 \le \lim_{x\to 0} |x \sin\left(\frac{1}{x}\right)| \le \lim_{x\to 0} |x|$$

$$0 \le \lim_{x\to 0} |x \sin\left(\frac{1}{x}\right)| \le 0$$

By Sandwich theorem $\lim_{x\to 0} |x \sin\left(\frac{1}{x}\right)| = 0$, hence $\lim_{x\to 0} x \sin\frac{1}{x} = 0$ (as in example 10)

1.3 The Precise Definition of a Limit

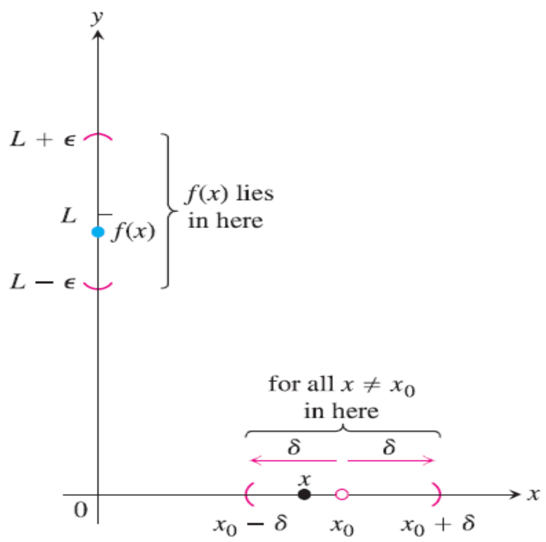
In this section we dose not tell how to find a limit of function but we verify that the limit is correct.

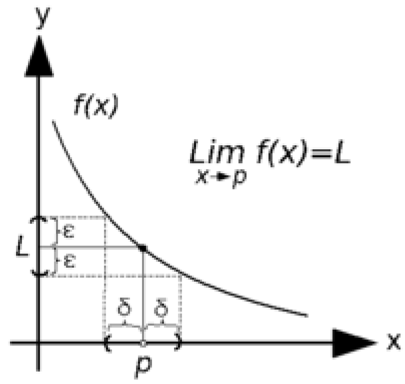
DEFINITION Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$





EXAMPLE 1: Show that

$$\lim_{x \to 1} (5x - 3) = 2$$

Solution Set $x_0 = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\epsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $x_0 = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ϵ of $L = 2$, so

$$|f(x) - 2| < \epsilon.$$

We find δ by working backward from the ϵ-inequality:

$$|(5x - 3) - 2| = |5x - 5| < \epsilon$$
$$5|x - 1| < \epsilon$$
$$|x - 1| < \epsilon/5.$$

Thus, we can take $\delta = \epsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \epsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\epsilon/5) = \epsilon,$$

which proves that $\lim_{x \to 1}(5x - 3) = 2$.

The value of $\delta = \epsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \epsilon$. Any smaller positive δ will do as well. The definition does not ask for a "best" positive δ, just one that will work. ■

EXAMPLE 4 For the limit $\lim_{x \to 5} \sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\epsilon = 1$. That is, find a $\delta > 0$ such that for all x

$$0 < |x - 5| < \delta \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$

Solution We organize the search into two steps, as discussed below.

1. *Solve the inequality* $|\sqrt{x - 1} - 2| < 1$ *to find an interval containing* $x_0 = 5$ *on which the inequality holds for all* $x \neq x_0$.

$$|\sqrt{x - 1} - 2| < 1$$
$$-1 < \sqrt{x - 1} - 2 < 1$$
$$1 < \sqrt{x - 1} < 3$$
$$1 < x - 1 < 9$$
$$2 < x < 10$$

   The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value of* $\delta > 0$ *to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x_0 = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 to make $|\sqrt{x - 1} - 2| < 1$ (Figure 2.22):

$$0 < |x - 5| < 3 \quad \Rightarrow \quad |\sqrt{x - 1} - 2| < 1.$$ ■

Example 3: Show that $\lim_{x \to 4} \sqrt{x} = 2$.

Solution:

Remark (the solution will start later). Before we use the general formal definition, let's ϵ=0.5. How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5<y<2.5$? In this case, we can proceed as follows:

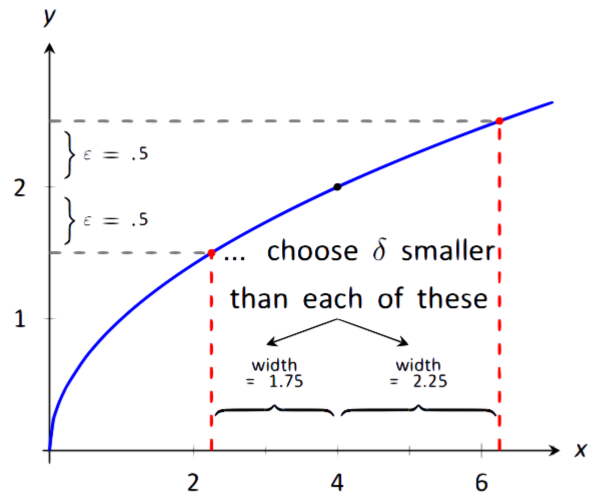
$1.5 < y < 2.5$

$1.5 < \sqrt{x} < 2.5$

$1.5^2 < x < 2.5^2$

$2.25 < x < 6.25$.

So, what is the desired x tolerance? Remember, we want to find a symmetric interval of x values, namely $4-\delta < x < 4+\delta$. The lower bound of 2.25 is 1.75 units from 4; the upper bound of 6.25 is 2.25 units from 4. We need the smaller of these two distances; we must have $\delta \leq 1.75$. See Figure below:



With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Solution start from here:

In general: for all $\epsilon>0$ we need to find $\delta>0$ *s.t.* if $|x-4| < \delta$ *implies* $|f(x)-2|<\epsilon$:

$-\epsilon<\sqrt{x}-2<\epsilon$

$2-\epsilon<\sqrt{x}<2+\epsilon$ (Add 2)

$(2-\epsilon)^2 < x < (2+\epsilon)^2$ (Square all)

$4 - 4\epsilon + \epsilon^2 < x < 4 + 4\epsilon + \epsilon^2$ (Expand)

$4 - (4\epsilon - \epsilon^2) < x < 4 + (4\epsilon + \epsilon^2).$

The form in the last step is "4−something< x <4+something." Since we want this last interval to describe an x around 4, we have that either $\delta \le 4\epsilon - \epsilon 2$ or $\delta \le 4\epsilon + \epsilon 2$, whichever is smaller: $\delta \le \min\{4\epsilon - \epsilon^2,\ 4\epsilon + \epsilon^2\}$.

Since $\epsilon > 0$, the minimum is $\delta \le 4\epsilon - \epsilon^2 2$.

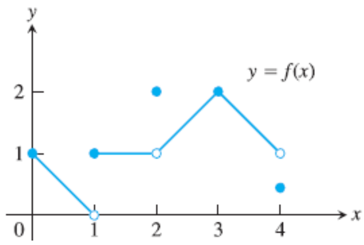
So given any $\epsilon > 0$, set $0 \le \delta \le 4\epsilon - \epsilon 2$. Then if $|x-4| < \delta$, then $|f(x)-2| < \epsilon$, satisfying the definition of the limit.

1.4 One-Sided Limits

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values *f(x)* must approach L as x approaches c from either side. Because of this, ordinary limits are called two-sided.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a right-hand limit. From the left, it is a left-hand limit.

EXAMPLE 1: For the function graphed in Figure



At $x = 0$: $\lim_{x \to 0^+} f(x) = 1$,

$\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x \to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x \to 1^+} f(x) = 1$,

$\lim_{x \to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x \to 2^-} f(x) = 1$,

$\lim_{x \to 2^+} f(x) = 1$,

$\lim_{x \to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x \to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x \to 4^+} f(x)$ and $\lim_{x \to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.