Functional Analysis

$4^{Th.}$ Class / 2021-2022

What is Functional Analysis?

Functional analysis represents one of the most important branches of mathematical sciences. Together with abstract algebra and mathematical logics it serves as a foundation of many other branches of mathematics.

Functional analysis is, in particular, widely used in probability theory and random functions theory and their numerous applications. Functional analysis serves also as a powerful tool in modern control and information sciences. The main subject of mathematical analysis represents scalar and finite-dimensional vector functions of scalar or finite-dimensional vector variables. Functional analysis is studying more general functions whose arguments and values may be the elements of any sets. While studying functions in mathematical analysis and linear algebra geometrical presentations are widely used; a function is considered as the mapping of one finite-dimensional space into another finite-dimensional space.

For instance, the scalar function of one scalar variable represents the mapping of the real axis R into the real axis R. The scalar function of two (three) scalar variables represents the mapping of the plane R^2 (the three-dimensional space R^3 respectively) into R. While studying more general functions whose arguments and values may be the elements of any sets wonderful analogies appear between many properties of functions and the visual geometric properties of more simple functions.

You meet such analogies in linear algebra where the spaces of any finite dimensions are considered (the n-dimensional spaces R^n at any finite n). In particular, the properties of linear functions in R^n are absolutely identical with the properties of linear functions in one-, two-- and three-dimensional spaces. These properties of functions caused the generalization of the notion of a space and wide application of intuitive geometrical presentations and geometrical terminology while studying any functions.

Functional analysis was born in the works of Italian mathematician Vito Volterra (Volterra 1913, Volterra and Peres 1935). He was the first who considered functions as the points of some space. The spaces whose points are functions are called function spaces.

Volterra defined also a real function whose argument represents the set of all the values of a continuous function in the interval [a, b]. Such a function he called a functional. This was the reason to call the branch of mathematics studying functionals a functional analysis.
It is worthwhile to recall that long before Volterra some functionals where considered by great Euler who created calculus of variations, though he did not use the term "functional".

Primarily functional served as the main object of study in functional analysis. In further development the notion of a function was essentially generalized. Respectively the range of interests of functional analysis was considerably extended. So, the object of functional analysis represents now the study of functions whose arguments and values may be the elements of any sets which are usually called spaces.

In this course we studied the following subjects:

1- Vector Spaces: Finite and Infinite Dimentional, Metric Spaces, Norms & Normed Spaces.
2- Banach Spaces: Some Important Inequalities(Cauchy, Holder and Minkowski's inequalities), Examples of Banach Spaces, Quotient Space of a Normed Linear Space, Continuous and Bounded Linear Transformations, Norm of Bounded Linear Transformations, Linear Operator on a Normed Space. Equivalent Norms, Continuous Linear Functional, Dual Spaces, The Hahan-Banach Theorem.
3- Hilbert Spaces: Definitions, Pre-Hilbert Spaces, Chauchy- Schwarz Inequality, orthogonal, Gram- Schmidt Theorem.

<u>References:</u>

1- Introductionary Functional Analysis and Application, By E. Kreyzig, 1978.
2- Introduction to Hilbert Space, by S. K. Berberian, 1976.

Chapter One: Vector Space

Definition 1.1.

A *vector space over* F is a non-empty set V together with two functions, one from V x V to V, and the other from F x V to V, denoted by $x + y$ and αx respectively, for all $x, y \in V$ and $\alpha \in$F, such that, for any $\alpha,\beta \in$ F and any $x, y, z \in V$,

(a) $x+y=y+x$, $x+(y+z)=(x+y)+z$;

(b) there exists a unique $0 \in V$ (independent of x) such that $x + 0 = x$;

(c) there exists a unique $-x \in V$ such that $x + (-x) = 0$;

(d) $1x = x$, $\alpha(\beta x) = (\alpha\beta)x$;

(e) $\alpha(x + y) = \alpha x + \alpha y$, $(\alpha + \beta)x = \alpha x + \beta x$.

If F = R (respectively, F = C) then V is a *real* (respectively, *complex)* vector space. Elements of F are called *scalars,* while elements of V are called *vectors.* The operation $x + y$ is called *vector addition,* while the operation ax is called *scalar multiplication.*

Some important inequalities

1- *Holder's inequality* : if $p, q \in$ IR such that $\frac{1}{p}+\frac{1}{q}=1$, then

$$\sum_{i=1}|x_iy_i| \le (\sum_{i=1}|x_i|^p)^{1/p}(\sum_{i=1}|y_i|^q)^{1/q}$$

2- If p=2 then q=2 and:

$$\sum_{i=1}|x_iy_i| \le (\sum_{i=1}|x_i|^2)^{1/2}(\sum_{i=1}|y_i|^2)^{1/2}$$

and is called *Cauchy - Schwar's inquality*.

3- *MinKowsk's inquality*: if $p \ge 1$, then:

$$(\sum_{i=1}|x_i+y_i|^p)^{1/p} \le (\sum_{i=1}|x_i|^p)^{1/p} + (\sum_{i=1}|y_i|^p)^{1/p}$$

Example 1.2. [H.W.2-6]

[1] S={x= $\alpha_n)_{n=1}^{\infty} : \alpha_n \in R$ or $C, \forall n$}is a vector space over R or C (sequence space).

[2] $l_p = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R$ or $C, \forall n$ s.t. $\sum_{n=1}^{\infty}|\alpha_n|^p < \infty\}$, l_p is a vector space over R or C $(1 \le p \le \infty)$

[3] $l_\infty = \{x = (\alpha_n)_{n=1}^{\infty} : \alpha_n \in R \text{ or } C, \forall n \text{ s.t. } \sum_{n=1}^{\infty} |\alpha_n|^p \leq m\}$ is a vector space over R or C.

[4] C[a, b]={f: [a, b]→R : f is continuous and C[a, b]} is a vector space over R or C.

[5] L^p[a, b]= {f:[a, b] →R , f is Lebesgue integrable on [a, b] s.t. $\int_a^b |f(x)|\,dx < \infty$ } is a vector space over R or C.

[6] Let V be the set M(m, n)(C) of complex{valued m x n matrices, with usual addition of matrices and scalar multiplication.

Sol.

[1] Let $x = (\alpha_n)_{n=1}^{\infty}$, $y = (\beta_n)_{n=1}^{\infty}$ ∈S, λ is a scalar, then

1. $x + y = (\alpha_n)_{n=1}^{\infty} + (\beta_n)_{n=1}^{\infty} = (\alpha_n + \beta_n)_{n=1}^{\infty} \in S$
2. $\lambda(\alpha_n)_{n=1}^{\infty} = (\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n, ...) = (\lambda\alpha_n)_{n=1}^{\infty} \in S$

Definition 1.3

Let V be a vector space. A non-empty set U ⊂ V is a *linear subspace* of V if U is itself a vector space (with the same vector addition and scalar multiplication as in V*).* This is equivalent to the condition that:

$\alpha x + \beta y \in$ U*,* for all $\alpha, \beta \in$F and $x, y \in$U

(which is called the *subspace test).*

Example 1.4.

[1]The set of vectors in R^n of the form (x_1, x_2, x_3, 0,…, 0) forms a three-dimensional linear subspace.

[2] The set of polynomials of degree ≤ r forms a linear subspace of the set of polynomials of degree ≤ n for any $r \leq n$.

Definition 1.5. Linear independence and dependence of a given set M of vectors $x_1,...,x_r$ (r ≥1) in a vector space V are defined by means of the equation

$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_r x_r = 0$ …. (*)

where $\alpha_1, \alpha_2, \ldots, \alpha_r$ are scalars. Clearly, equation (*) holds for $\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0$. If this is the only r-tuple of scalars for which (*) holds, the set M is said to be *linearly independent*. M is said to be *linearly dependent* if M is not linearly independent, that is , if (*) also holds for some r-tuple of scalars, not all zero.

Definition 1.6.: Let V be a vector space over a field F, $x\in$V is called linear combination of $x_1,x_2,\ldots,x_n \in$V if $x=\lambda_1x_1+\lambda_2x_2+\ldots+\lambda_nx_n=\sum_{i=1}^{n}\lambda_i\alpha_i,\ \lambda_i\in F,\quad 1\le i\le m.$

Definition 1.7.: Let V be a vector space over a field F, and let S={ $x_1,x_2,\ldots,x_n$}$\subseteq$V, S is said to be *generated* V if $x=\sum_{i=1}^{n}\lambda_i\alpha_i,\ \forall x_i\in S, \lambda_1\in \mathrm{F},\ 1\le i\le m.$

Definition 1.8.: Let V be a vector space over a field F, and A be a non-empty subset of V ($\phi\neq$A$\subseteq$ V), A is said to be basis of V if :

1- A linearly independent set.
2- A generated V.

Definition 1.9. A vector space V is said to be *finite dimensional* if there is a positive integer n such that X contains a linearly independent set of n vectors whereas any set of n+1 or more vectors of X is linearly dependent. n is called the dimension of X, written n=dim X. By definition, X={0} is finite dimensional and dim X=0. If X is not finite dimensional, it is said to be infinite dimensional.

Examples 1.10.: dim R=1, dim R^2=2, dim R^n=n .

Remarks

1- Let V(F) be a finite dimensional V.S. over a field F, and let w subspace of V(F), then dim W $\le$ dim V, If dim W=dim V then W=V.
2- Let ($\phi\neq$S$\subseteq$ V) then if 0 $\in$S then S is linear dependent subspace.
3- The singleton {x} is linear dependent iff x$\neq$0.
4- Any subset of linear dependent set is linear dependent.
5- Any set containing a linearly dependent subset is linearly dependent too.

Definition 1.10 : A ***metric space*** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that is, a function defined on X x X such that for all $x, y, z \in$ X, we have:

(1) d is real-valued, finite and nonnegative function.

(2) $d(x, y)=0$ if and only if $x=y$

(3) $d(x, y) = d(y, x)$ (Symmetry).

(4) $d(x, y) \leq d(x, z)+d(z, y)$ (Triangle inequality).

Examples (H.W. 2-6)

1) **Real line IR**: this is the set of all real numbers, taken with the usual metric defined by:

$$d(x, y) = | x-y | \qquad \forall x,y \in \text{IR}$$

2) **Euclidean plane IR^2**: The metric space IR^2, with Euclidean metric:

if $x=(x_1, x_2)$, $y=(y_1,y_2)$, then:

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$$

3) **Euclidean Space IR^n:** If $x=(x_1, x_2 , \ldots, x_n)$, $y=(y_1,y_2, \ldots,y_n)$, then:

$$d(x,y)=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\ldots+(x_n-y_n)^n}$$

4) **Function space C[a, b]:** As a set X we take the set of all real-valued functions x, y, ... which are functions of an independent real variable t and are defined and continuous on a given closed interval J = [a, b]. Choosing the metric defined by

$$d(x, y)=\max_{t\in J} | x(t)-y(t) |$$

5) **Discrete metric space**: We take any set X and on it the so-called discrete metric for X, defined by:

$d(x, x) = 0$, $\quad d(x,y)=1 \quad (x\neq y)$.

This space (X, d) is called a discrete metric space.

6) **Space B(A) of bounded functions**: By definition, each element $x \in$ B(A) is a function defined and bounded on a given set A, and the metric is defined by:

$$d(x, y)=\sup_{t\in A} | x(t)-y(t) |$$

Sol.

[1] 1- d is real, finite & $d=| x-y | \geq 0$

2) $d(x,y)=0 \leftrightarrow | x-y | =0 \leftrightarrow x-y=0 \leftrightarrow x=y$ $\forall\, x, y \in IR$

3) $d(x,y)= | x-y | = |-(y-x)| = | y-x | = d(y, x)$ $\forall\, x, y \in IR$

4) $d(x,y)= | x-y |= | x-z + z - y| \leq | x-z| + | z-y| = d(x, z)+d(z, y)$ $\forall\, x, y, z \in IR$

Then (IR, d) is a metric space.

A ***norm*** on a vector space is a way of measuring distance between vectors.

Definition 1.11.: A ***norm*** on a linear space V over F is a function $\| . \| : V \to R$ with the properties that :

(1) $\| x \| \geq 0$ & $\| x \| = 0 \leftrightarrow x = 0$ (positive definite);

(2) $\| x + y\| \leq \| x \| + \| y \|$ for all $x, y \in V$ (triangle inequality);

(3) $\|\alpha x\| = |\alpha|\, \|x\|$ for all $x \in V$ and $\alpha \in F$.

In Definition 1.11(3) we are assuming that F is R or C and $| . |$ denotes the usual absolute value. If $\| . \|$ is a function with properties (2) and (3) only it is called a ***semi-norm***.

Definition 1.12. A ***normed linear space*** is a linear space V with a norm $\| . \|$ (sometimes we write $\| . \|_V$).

Theorem 1.13. If V is a normed space then:

1) $\| 0 \|=0$
2) $\| x \| = \| -x \|$ for every $x \in V$.
3) $\| x-y \| = \| y-x \|$ for every $x \in V$.
4) $| \| x \| - \| y \| | \leq \| x-y \|$ for every $x \in V$.

Proof :

Properties (1), (2) and (3) conclude directly from the definition, to prove property (4):

$x = (x-y)+y$

$\| x \| = \| (x-y)+y \| \leq \| x-y \| + \| y \| \to \| x \| - \| y \| \leq \| x-y \|$ ….(1)

Similarly:

$\| y \| - \| x \| \leq \| x-y \|$

$-(\| x \| - \| y \|) \leq \| x-y \| \rightarrow (\| x \| - \| y \|) \geq - \| x-y \|$ ....(2)

From (1) & (2), we get:

$-\| x-y \| \leq \| x \| - \| y \| \leq \| x-y \| \rightarrow | \| x \| - \| y \| | \leq \| x-y \|$

Examples 1.14.:- [H.W.6,7]

[1] The vector space V is normed v.s. with the norm $\| x \| = | x |$ for all $x \in$ V.

Proof:

1) Since $| x | \geq 0 \rightarrow \| x \| \geq 0$.
2) $\| x \| = 0 \leftrightarrow | x | = 0 \leftrightarrow x=0$
3) Let $x \in$ V, $\alpha \in$ F , then

   $\| \alpha x \| = | \alpha x | = |\alpha| \, | x | = | \alpha | \, \| x \|$
4) Let $x,y \in$ V, then:

   $\| x+y \| = | x+y | \leq | x | + | y | = \| x \| + \| y \|$

[2] Let V = R^n with the usual Euclidean norm

$\| x \| = \| x \|_2 = (\sum_{j=1}^{n} | x_j |^2)^{1/2}$

proof:

1) Since $x_j^2 \geq 0$ for all $j = 1,2,\ldots,n \rightarrow \| x \| \geq 0$
2) $\| x \| = 0 \leftrightarrow (\sum_{j=1}^{n} | x_j |^2)^{1/2} = 0 \leftrightarrow \sum_{j=1}^{n} | x_j |^2 = 0$

   $\leftrightarrow x_j^2 = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x_j = 0$ for all $j = 1,2,\ldots,n \leftrightarrow x=0$
3) Let $x \in IR^n$, $\alpha \in$ IR:

   $\alpha x = \alpha (x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$

   $\| \alpha x \| = (\sum_{j=1}^{n} | \alpha x_j |^2)^{1/2} = | \alpha | (\sum_{j=1}^{n} | x_j |^2)^{1/2} = | \alpha | \, \| x \|$.
4) Let $x,y \in IR^n$:

   $x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

   $\| x+y \| = (\sum_{j=1}^{n} | x_j + y_j |^2)^{1/2}$

By using *MinKowski's inquality* where p=2, we have:

$$\| x+y \| = (\sum_{i=1} | x_i + y_i |^2)^{1/2} \le (\sum_{i=1} | x_i |^2)^{1/2} + (\sum_{i=1} | y_i |^2)^{1/2} = \| x \| + \| y \|$$

[3] There are many other norms on R^n, called the p-norms. For $1 \le p < \infty$ defined by:

$$\| x \|_p = (\sum_{j=1}^{n} | x_j |^p)^{1/p}$$

Then $\| \,.\, \|_p$ is a norm on V (to check the triangle inequality use *MinKowski's Inequality*)

$$(\sum_{j=1}^{n} | x_j + y_j |^p)^{1/p} \le (\sum_{j=1}^{n} | x_j |^p)^{1/p} + (\sum_{j=1}^{n} | y_j |^p)^{1/p}$$

[4] There is another norm corresponding to $p = \infty$, defined by:

$$\| x \|_\infty = \max_{1 \le j \le n} \{ | x_j | \}$$

where $\| \,.\, \| : IR^n \to IR$ and $x = (x_1, \ldots, x_n)$.

proof:

1) Since $| x_i | \ge 0$ for all i=1, … , $n \to \| x \| \ge 0$.
2) $\| x \| = 0 \leftrightarrow \max \{| x_1|, \ldots , | x_n |\}=0 \leftrightarrow | x_i | = 0$ for all i=1, … , n

   $\leftrightarrow x_i = 0$ for all i=1, … , $n \leftrightarrow x=0$
3) Let $x \in IR^n$ and $\alpha \in IR$, then

   $\alpha x = \alpha(x_1, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$

   $\| \alpha x \| = \max \{ | \alpha x_1 |, \ldots ,| \alpha x_n |\}$

   $= \max \{ | \alpha | | x_1 |, \ldots , | \alpha | | x_n |\}$

   $= | \alpha | \max \{ | x_1 |, \ldots , | x_n |\}$

   $= | \alpha | \| x \|$
4) Let $x, y \in IR^n$

   $x + y = (x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (, \ldots,)$

   $\| x + y \| = \max \{ | x_1+y_1|, \ldots , | x_n+y_n | \}$

   $\le \max \{ | x_1| + | y_1|, \ldots , | x_n |+| y_n | \}$

   $\le \max \{ | x_1|, \ldots , | x_n | \} + \max \{ | y_1|, \ldots , | y_n | \}$

   $= \| x \| + \| y \|$

[5] Let X = C[a; b], and put $\|f\| = \sup_{t\in[a,b]} |f(t)|$. This is called the uniform or supremum norm.

proof :

1) Since $|f(t)| \geq 0$ for all $t \in$ [a, b] $\rightarrow \|f\| \geq 0$.
2) $\|f\| = 0 \leftrightarrow \sup_{t\in[a,b]} |f(t)| = 0 \leftrightarrow |f(t)| = 0$ for all $t \in$ [a, b]

   $\leftrightarrow f(t) = 0$ for all $t \in$ [a, b] $\leftrightarrow f = 0$.
3) Let $f \in$ X, $\alpha \in$ IR, then:

   $\| \alpha f \| = \sup\{ |\alpha f(t)| : t \in [a, b]\}$

   $= \sup\{ |\alpha| |f(t)| : t \in [a, b]\}$

   $= |\alpha| \sup\{ |f(t)| : t \in [a, b]\}$

   $= |\alpha| \|f\|$.
4) $\|f + g\| = \sup\{ |(f+g)(t)| : t \in [a, b]\} = \sup\{ |(f(t) + g(t)| : t \in [a, b]\}$

   $\leq \sup\{ |f(t)| + |g(t)| : t \in [a, b]\}$

   $\leq \sup\{ |f(t)| : t \in [a, b]\} + \sup\{ |g(t)| : t \in [a, b]\} = \|f\| + \|g\|$.

[6] Let X = C[a; b], and choose $1 \leq p < \infty$. Then (using the integral form of Minkowski's inequality) we have the p-norm

$$\|f\|_p = \left(\int_a^b |f|^p \right)^{1/p}$$

[7] Let V be the set of Riemann-integrable functions $f : (0; 1) \rightarrow$ R which are square-integrable. Let $\|f\|_2 = \left(\int_0^1 |f(x)|^2 \, dx\right)^{1/2} < \infty$. Then V is a normed linear space.

Definition 1.15. A set C in a linear space is *convex* if for any two points $x, y \in$ C,

$tx + (1 - t)y \in$ C for all t $\in$ [0; 1].

Definition 1.16. A norm $\| . \|$ is *strictly convex* if $\|x\| = 1$, $\|y\| = 1$, $\|x+y\| = 2$ together imply that $x = y$.

Definition 1.17. :If (X; $\| . \|_X$) and (Y; $\| . \|_Y$) are normed linear spaces, then the *product*

X x Y = { $(x, y) \setminus x \in$ X; $y \in$ Y }

is a linear space which may be made into a normed space in many different ways, a few of which follow.

Example 1.18.

[1] $\|(x, y)\| = \max\{\| x \|_X , \| y \|_Y \}$.

proof:

1) $\| (x,y) \| = 0 \leftrightarrow \max\{\| x \|_X , \| y \|_Y\} = 0 \leftrightarrow \| x \|_X = 0 , \| y \|_Y = 0 \leftrightarrow x=0, y=0 \leftrightarrow (x, y)=0$

2) let (x_1,y_1) , $(x_2,y_2) \in XxY$, then

$(x_1,y_1) + (x_2,y_2) = (x_1+ x_2, y_1+ y_2)$

$\| (x_1+ x_2, y_1+ y_2)\| = \max \{\| x_1+ x_2\|_X, \| y_1+ y_2\|_Y)\} \leq \max\{\| x_1\|_X + \| x_2\|_X , \| y_1\|_Y + \| y_2\|_Y)$

$\leq \max\{\| x_1\|_X, \| y_1\|_Y \} + \max\{\| x_2\|_X , \| y_2\|_Y\} = \|(x_1, y_1)\| + \|(x_2, y_2)\|$

3) let $(x, y) \in XxY$ and $\alpha \in F$, then

$\| \alpha(x, y)\| = \max\{ \|\alpha x\|_X, \|\alpha y\|_Y\} = \max \{|\alpha| \|x\|_X, |\alpha| \|y\|_Y\} = |\alpha| \max \{\|x\|_X, \|y\|_Y\} = |\alpha| \|(x, y)\|$

[2] H.W. $\|(x, y)\| = (\|x\|_X + \|y\|_Y)^{1/p}$;

Theorem 1.19. Every normed linear space is metric space.

proof:

let (X, ||.||) is a normed space. We define the function d: XxX $\rightarrow$ IR as:

$d(x,y)=\|x-y\|$ for all $x,y \in X$, since this function satisfies all the conditions of metric :

1)let $x,y \in X \rightarrow x-y \in X$ (since X is vector space) $\rightarrow \| x-y \| \geq 0 \rightarrow d(x, y) \geq 0$.

2) $d(x, y) = 0 \leftrightarrow \| x-y \| =0 \leftrightarrow x-y = 0 \leftrightarrow x=y$

3) $d(x, y) = \| x-y \| = \| y-x \| = d(y, x)$

4) let $x,y,z \in X$:

$\| x-y \| = \| (x-z)+(z-y) \| \leq \| x-z \| + \| z-y \| \rightarrow d(x, y) \leq d(x, z) + d(z, y)$

Remark : The converse may be not true, for example:

If X be a v.s., define d: X x X $\rightarrow$ IR as:

$$d(x,y) = \begin{cases} 0 & x = y \\ 2 & x \neq y \end{cases}$$

And define $\| . \|$: X $\rightarrow$ IR as $\| x \| = d(x, 0)$

(X, || . ||) fails to be normed space.

Since if $x \neq 0 \rightarrow \| x \| = d(x, 0) = 2$

$\| 2x \| = d(2x, 0) \rightarrow |2| \| x \| =2 \rightarrow 2.2=2 \rightarrow 4=2$ C!

Definition 1.20.: Let X = (X; $\| . \|_X$) be a normed linear space. A sequence of vectors (x_n) in X is said to ***convergent*** if:

$\exists\, x\in X$ s.t. , $\forall \varepsilon >0$ $\exists$ $k(\varepsilon)\in Z^+$ s.t. $\| x_n - x\| < \in$ $\forall\, n > k$.

And we say x is the convergent point for the sequence (x_n) and write $x_n\to x$ when $n\to\infty$, this means $x_n\to x$ $\leftrightarrow$ $\| x_n- x\| \to 0$. If (x_n) not convergent is called ***divergent***.

<u>Theorem 1.21.</u>: Let X be a normed space, (x_n), (y_n) be a sequence in X such that $x_n\to x_0$, $y_n\to y$, then:

1- $x_n\pm y_n\to x_0\pm y_0$

2- $\| x_n \| \to \| x_0 \|$

3- $\| x_n - y_n \| \to \| x_0- y_0 \|$

4- $\alpha x_n \to \alpha x_0$ $\forall\alpha\in F$

Proof:

1- Since $x_n\to x_0$, $y_n\to y$, then:

if $\varepsilon > 0$

$\exists\ k_1(\varepsilon) \in Z^+$ s.t. $\| x_n- x_0 \| < \varepsilon / 2$, $\forall\ n > k_1(\varepsilon)$

$\exists\ k_2(\varepsilon) \in Z^+$ s.t. $\| y_n- y_0 \| < \varepsilon / 2$, $\forall\ n > k_2(\varepsilon)$

Define $k_3(\varepsilon)= \max \{ k_1(\varepsilon),k_2(\varepsilon)\}$

$$\| (x_n+ y_n) - (x_0+ y_0) \| = \| x_n+ y_n - x_0- y_0 \|$$

$$\leq \| x_n- x_0\| + \| y_n- y_0\|$$

$$< \varepsilon / 2 +\varepsilon / 2 =\varepsilon\ ,\ \forall\ n > k_3(\varepsilon)$$

$\to x_n+ y_n\to x_0+ y_0$

2- Since $x_n\to x_0$ T.P. $\| x_n \| \to \| x_0 \|$ i.e. T.P. $| \| x_n \| - \| x_0 \| | \to 0$

By Theorem (1.13.)-4 : $| \| x_n \| - \| x_0 \| | \leq \| x_n - x_0 \|$ ….. (1)

Since $x_n\to x_0 \to \| x_n - x_0 \| \to 0$ …..(2)

By (1) & (2) we get: $| \| x_n \| - \| x_0 \| | \to 0$

Then $\| x_n \| \to \| x_0 \|$

3- T.P. $\| x_n - y_n \| \to \| x_0- y_0 \|$, i.e. T.P. $| \| x_n - y_n \| - \| x_0- y_0\| | \to 0$

Since $x_n\to x_0 \Rightarrow \| x_n - x_0 \| \to 0$

& $y_n\to y_0 \Rightarrow \| y_n - y_0 \| \to 0$

$$| \| x_n - y_n \| - \| x_0- y_0\| | \leq \| x_n - y_n - x_0+ y_0 \|$$

$$\leq \| x_n - x_0 \| + \| y_n- y_0\|$$

$\Rightarrow |\, \| x_n - y_n \| - \| x_0 - y_0 \| \,| \to 0 \quad \Rightarrow \quad \| x_n - y_n \| \to \| x_0 - y_0 \|$

4- $\|\alpha x_n - \alpha x_0\| = \|\alpha(x_n - x_0)\| = |\alpha| \, \| x_n - x_0\|$

since $\| x_n - x_0\| \to 0$ where $n \to \infty \Rightarrow \|\alpha x_n - \alpha x_0\|$ where $n \to \infty \Rightarrow \alpha x_n \to \alpha x_0$

Theorem 1.22.: If the sequence (x_n) is convergent in the normed space X then its convergent point is unique.

proof:

suppose that $x_n \to x$ and $x_n \to y$ s.t. $x \neq y$, and let $\| x-y \| = \varepsilon \to \varepsilon > 0$

since $x_n \to x \Rightarrow \exists k_1 \in Z^+$ s.t. $\| x_n - x\| < \varepsilon/2$, $\forall n > k_1$

and $x_n \to y \Rightarrow \exists k_2 \in Z^+$ s.t. $\| x_n - y\| < \varepsilon/2$, $\forall n > k_2$

put $k = \max\{ k_1, k_2\}$. Then $\| x_n - x\| < \varepsilon/2$, $\| x_n - y\| < \varepsilon/2 \quad \forall n > k$.

$\varepsilon = \| x-y \| = \| (x - x_n) + (x_n - y) \| \leq \| (x_n - x) \| + \| (x_n - y) \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$!

and this contradiction then $x=y$.

Definition 1.23. A sequence (x_n) in a normed space X is a ***Cauchy convergent sequence*** if:

$\forall \varepsilon > 0 \ \exists \ k(\varepsilon) \in Z^+$ such that $\| x_n - x_m \| < \varepsilon \quad \forall n,m > k(\varepsilon)$

Theorem 1.24.: Every convergent sequence is a Cauchy convergent sequence.

proof:

Suppose that (x_n) is a convergent sequence in the normed space X, then $\exists\, x \in X$ s.t. $x_n \to x$

Let $\varepsilon > 0$, since $x_n \to x \Rightarrow \exists\, k \in Z^+$ s.t. $\| x_n - x \| < \varepsilon/2 \ \ \forall n > k$

If $n,m \geq k$,then $\| x_n - x_m \| = \| (x_n - x) + (x - x_m) \| \leq \| x_n - x\| + \| x - x_m \| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Then (x_n) is a Cauchy sequence.

Remark:

The converse to above theorem may not be true. For example:

Let X= IR-{0}, $(x_n) = (1/n)$

(x_n) Cauchy convergent sequence in IR

Since IR complete $\Rightarrow (x_n) = (1/n) \to 0$ convergent in IR

But (x_n) not convergent in IR-{0} , since $0 \notin$ IR-{0}.

Definition 1.25.: Let X be a normed space, $x_0 \in X$, a function f is said to be ***continuous*** at x_0 if:

$\forall \ \varepsilon > 0, \exists \ \delta(x_0, \varepsilon) > 0$ s.t. $\| f(x) - f(x_0) \| < \varepsilon$ whenever $\| x - x_0 \| < \delta$.

Theorem 1.26. : Let X, Y be two Normed space, a function f : X $\to$ Y continuous at $x_0 \in$X iff for each sequence (x_n) in X such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Definition 1.27.: Let X be a normed space, a function f : X $\to$ IR is called ***bounded*** if:

$\exists$ M > 0 s.t. $\| f(x) \| \leq$ M, $\forall x \in$X.

Definition 1.28.: Let (x_n) be a sequence in a normed space X, say (x_n) is ***bounded sequence*** in X if : $\exists$ M > 0 s.t. $\| x_n \| \leq$ M, $\forall n \in Z^+$.

Theorem 1.29.: If (x_n) is Cauchy convergent sequence in a normed space X then it is bounded.

proof:

Let (x_n) be a Cauchy sequence in X

Given ε=1, $\exists$ k$\in Z^+$ s.t. $\| x_n - x_m \| < 1$, $\forall n, m >$ k.

Let m = k+1 $\Rightarrow \| x_n - x_{k+1} \| <1$

Since $| \| x_n\| - \| x_{k+1} \| | \leq \| x_n - x_{k+1} \| < 1$

$\Rightarrow | \| x_n\| - \| x_{k+1} \| | < 1 \Rightarrow \| x_n\| < 1 + \| x_{k+1} \|$, $\forall n$>k

Put M = max $\{ \| x_1\|, \| x_2\|, \dots , \| x_k\|, \| x_{k+1}\|\} \Rightarrow \| x_n \| \leq$ M , $\forall n \in Z^+$.

Theorem1.30. : Every convergent sequence in the normed space X is bounded.

proof:

Let (x_n) be a convergent sequence in X $\Rightarrow (x_n)$ a Cauchy convergent sequence in X

$\Rightarrow (x_n)$ bounded .

Definition 1.31. : Let X is a normed space, $x_0 \in$X, r >0, define:

1) $B_r(x_0)=\{ x \in X: \| x - x_0 \| < r \}$ is called ***open ball*** of center x_0 and radius r.
2) $D_r(x_0)=\{ x \in X: \| x - x_0 \| \leq r \}$ is called ***closed ball*** of center x_0 and radius r.
3) $B_1(0)=\{ x \in X: \| x \| < 1 \}$ is called ***open unite*** of center 0 and radius 1.
4) $D_1(0)=\{ x \in X: \| x \| \leq 1 \}$ is called ***closed unite*** of center 0 and radius 1.

Definition 1.32.: Let $\| . \|_1$, $\| . \|_2$ be two norms on vector space X, $\| . \|_1$ is said to be ***equivalent*** to $\| . \|_2$ ($\| . \|_1 \sim \| . \|_2$) if there exist a and b positive real numbers such that:

$$a \| . \|_2 \leq \| . \|_1 \leq b \| . \|_2$$

Example: Let X=IR^n,

$\| x \| = \sum_{i=1}^{n} | x_i |$, $\forall x \in \mathrm{IR}^n$

$\| x \|_e = \sum_{i=1}^{n} | x_i^2 |^{\frac{1}{2}}$, $\forall x \in \mathrm{IR}^n$

Then $\| x \| \sim \| x \|_e$

proof:

$\sum_{i=1}^{n} | x_i y_i | \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} y_i^2)^{\frac{1}{2}}$, $\forall x_i, y_i \in \mathrm{IR}^n$ (by using Cauchy – Schwars inquality)

Put $y_i=1$, $\forall\ i = 1, 2, \ldots, n$.

$\Rightarrow \sum_{i=1}^{n} | y_i | \leq (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} (\sum_{i=1}^{n} 1)^{\frac{1}{2}}$

$\| x\| \leq \| x \|_e . \sqrt{n}$

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e$ (i.e. $a= \frac{1}{\sqrt{n}}$) ….. (1)

But $\| x \|_e \leq \| x\|$ (i.e. b=1)

From (1) & (2), we have:

$\frac{1}{\sqrt{n}} \| x\| \leq \| x \|_e \leq \| x\|$

Then $\| x \| \sim \| x \|_e$

Theorem 1.33.: On a finite dimensional normed space, all norms are equivalent.

Examples:

1- $X=\mathrm{IR}^2$, $\| . \|_e$, $\| . \|_2$, $\| . \|_3$ are equivalent.

2- $X= \mathrm{IR}^n$, $\| . \|_e$, $\| . \|_2$, $\| . \|_3$ are equivalent.

Chapter Two: Banach spaces

Definition 2.1. A normed linear space X is said to be ***complete*** if all Cauchy convergent sequences in X are convergent in X. The complete normed space is called ***Banach space***.

Examples 2.2.

[1] The space F^n with the norm $\| x \| = (\sum_{i=1}^{n} | x_i |^2)^{\frac{1}{2}}$, $\forall x=(x_1,x_2,\ldots,x_n) \in F^n$ is a Banach space.

Proof: F^n is a normed space ,

let $\{ x_n\}$ is Cauchy sequence in $F^n \Rightarrow x_m \in F^n \Rightarrow x_m=(x_{1m},x_{2m},\ldots,x_{nm})$

let $\varepsilon >0 \Rightarrow \exists\, k\in Z^+$ s.t. $\| x_m-x_l\| < \varepsilon \quad \forall\, m.l >k$

$\Rightarrow \| x_m-x_l\|^2 < \varepsilon^2 \quad \forall\, m.l >k$ ….. (1)

$x_m - x_l = (x_{1m}-x_{1l} ,x_{2m}-x_{2l} ,\ldots,x_{nm}-x_{nl})$

$\| x_m- x_l\|^2 = \sum_{i=1}^{n} | x_{im} - x_{il} |^2$ …… (2)

from (1) & (2) , we get:

$\sum_{i=1}^{n} | x_{im} - x_{il} |^2 < \varepsilon^2 \quad \forall\, m.l \geq k$

then

$| x_{im}- x_{il}|^2 < \varepsilon^2 \quad \forall\, m.l \geq k \Rightarrow | x_{im}- x_{il}| < \varepsilon \quad \forall\, m.l \geq k$

$\Rightarrow \forall i$, $\{x_{im}\}$ is a Cauchy sequence in F

Since F is complete (because F is IR or C)

$\Rightarrow \forall i, \exists\, x_i\in$ F s.t. $x_{im}\rightarrow x_i$

Put $x=(x_1,x_2, \ldots ,x_n) \Rightarrow x\in F$, T.P. $x_m\rightarrow x$.

Let $\varepsilon > 0$, $\forall\, m>k$, we get:

$\| x_m-x\|^2 = \sum_{i=1}^{n} | x_{im} - x_i |^2 < \varepsilon^2 \Rightarrow \| x_m- x\| < \varepsilon \quad \forall m > k \Rightarrow \{x_m\}$ convergent $\Rightarrow F^n$ is complete

Since F^n is normed space $\Rightarrow F^n$ is a Banach space

[2] H.W. The space $l^p (1\leq p < \infty)$ with the norm $\| x \| = (\sum_{i=1}^{n} | x_i |^p)^{\frac{1}{p}}, x = (x_1,x_2,\ldots) \in l^p$, is a Banach space.

[3] The space l^∞ with the norm $\| x \| = \sup_i | x_i |$ is a Banach space.

Proof:

l^∞ is a normed space

Let $\{ x_m \}$ is a Cauchy sequence in $l^\infty \Rightarrow x_m \in l^\infty \Rightarrow x_m = (x_{1m}, x_{2m}, \ldots, x_{nm}, \ldots)$

Let $\varepsilon > 0$, $\exists$ k$\in Z^+$ s.t.

$\| x_m - x_l \| < \varepsilon$, $\forall m,l >$k …….(1)

$x_m - x_l = (x_{1m} - x_{1l}, \ldots, x_{nm} - x_{nl}, \ldots)$

$\| x_m - x_l \| = \sup_i | x_{im} - x_{il} |$ ……..(2)

From (1) and (2), we have:

$\sup_i | x_{im} - x_{il} | < \varepsilon$, $\forall m , l >$ k

then for all i , $| x_{im} - x_{il} | < \varepsilon$, $\forall m , l >$ k …… (3)

$\Rightarrow \forall i$, then $\{ x_{im} \}$ is Cauchy sequence in F

Since F is complete $\Rightarrow \{ x_{im} \}$ is convergent $\Rightarrow \exists\, x_i \in$ F s.t. $x_{im} \to x_i$

Put $x = (x_1, x_2, \ldots)$, we must prove that $x \in l^\infty$, $x_m \to x$

From (3) , we get:

$| x_{im} - x_i | < \varepsilon$, $\forall\, m >$ k ….. (4)

Since $x_m \in l^\infty \Rightarrow \exists\, k_m \in$IR s.t.: $| x_{im} | \leq k_m$, $\forall\, i$

$x_i = (x_i - x_{im}) + x_{im}$

$| x_i | \leq | x_i - x_{im} | + | x_{im}$

[4] Let X=C[a, b], $\|x\|_1 = \sup\{| f(x)| : a \leq x \leq b\}$, $\forall x \in [a,b]$ is a Banach space.

Proof:

T.P. (C[a, b], $\|.\|_1$) is Banach space

1. C[a, b] is v.s. over IR

2. (C[a, b], $\|.\|_1$) is normed space

3. T.P. (C[a, b], $\|.\|_1$) is complete

Let (f_m) be a Cauchy seq. in C[a, b]

Given $\varepsilon>0$, $\exists$ k$\in Z^+$ s.t. $\| f_m - f_n \|_1 < \varepsilon$, $\forall\, m, n >$ k

$\| f_m - f_n \|_1 = \sup\{| (f_m - f_n)(x)| : a \leq x \leq b\} = \sup\{| f_m(x) - f_n(x)| : a \leq x \leq b\} < \varepsilon$, $\forall\, m, n >$ k

$\Rightarrow | f_m(x) - f_n(x) | < \varepsilon$, $\forall\, x \in$ [a, b], $\forall\, m, n >$ k

Since (f_m) is Cauchy seq. in IR, IR is complete

Then (f_m) is convergent

i.e. $\exists f \in$ IR (f cont's & bounded) s.t. $f_m \to f$

$\Rightarrow$ (C[a, b], $\|.\|_1$) is complete n.s.

$\Rightarrow$ (C[a, b], $\|.\|_1$) is Banach space

[5] Let X=C[0, 1], $\| \, . \, \|_2$: C[0, 1] $\to$ IR defined by

$\|f\|_2 = \int_0^1 | f(x) | \, dx, \forall f \in C[0,1]$

Then (C[0, 1], $\|f\|_2$) is not Banach space because it is normed space but not complete

Proof:

Let (f_n) is Cauchy seq. in C[0, 1], where:

$$f_n = \begin{cases} 1 & 0 \le x \le \frac{1}{2} \\ -nx + \frac{1}{2}n + 1 & \frac{1}{2} < x \le \frac{1}{2} + \frac{1}{n} \\ 0 & \frac{1}{2} + \frac{1}{n} < x \le 1 \end{cases}$$

let $m, n > 3$, then:

$$\|f_m - f_n\| = \int_0^1 | (f_m - f_n)(x) | \, dx = \int_0^1 | f_m(x) - f_n(x) | \, dx$$

$$= \int_0^{1/2} | (f_m(x) - f_n(x) | \, dx + \int_{1/2}^1 | f_m(x) - f_n(x) | \, dx$$

$$\le \int_0^{1/2} |1-1| dx + \int_{1/2}^1 | (f_m(x) | \, dx + \int_{1/2}^1 | f_n(x) | \, dx$$

$$\le \int_{1/2}^{\frac{1}{2}+\frac{1}{m}} \left| -mx + \frac{1}{2}m + 1 \right| dx + \int_{1/2}^{\frac{1}{2}+\frac{1}{n}} \left| -nx + \frac{1}{2}n + 1 \right| dx = [-m\frac{1}{2}x^2 + \frac{1}{2}mx + x]_{1/2}^{\frac{1}{2}+\frac{1}{m}} + [-n\frac{1}{2}x^2 + \frac{1}{2}nx + x]_{1/2}^{\frac{1}{2}+\frac{1}{n}}$$

Since $-mx + \frac{1}{2}m + 1 \ge 0$ when $\frac{1}{2} \le x \le \frac{1}{2} + \frac{1}{m}$

$\Rightarrow \|f_m - f_n\| \le \frac{1}{2m} + \frac{1}{2n}$ as $m, n \to \infty$

$\Rightarrow$ (f_n) is Cauchy convergent seq.

T.P. (f_n) is not convergent.

Suppose (f_n) is convergent

$\exists f \in$ C[0, 1] s.t. $f_n \to f$

i.e. $\lim_{m\to\infty} f_n(x) = f(x), \forall x \in [0,1]$

$$\Rightarrow f(x) = \begin{cases} 1 & , 0 \le x \le 1 \\ 0 & , \frac{1}{2} < x \le 1 \end{cases} \qquad C!$$

Since f is not continuous at $x = 1/2$

$\Rightarrow$ (C[0, 1], $\|f\|_2$) is not complete $\Rightarrow$ not Banach space·

Lemma (linear combination) 2.3.: Let X be a normed space, { $x_1, x_2, \ldots, x_n$} linearly independent set in X, then $\exists$c >0 s.t.:

$$\|\sum_{i=1}^{n} \lambda_i x_i\| \ge c \sum_{i=1}^{n} |\lambda_i|, \quad \forall \lambda_i \in F, \quad 1 \le i \le n$$

Theorem 2.4.: If X is finite dimension normed space then X is complete.

Proof: Let dim X=$n > 0$ and $\{x_1, x_2, \ldots, x_n\}$ is a base to X.

T.P. X is complete space we must prove every Cauchy sequence in X is convergent.

Suppose that $\{y_n\}$ is Cauchy sequence,

$\| y_m - y_l\| \to 0$ when $m,l \to \infty$ ….. (1)

since y_m , $y_l \in$ X then:

$$y_m = \sum_{i=1}^{n} \lambda_{im} x_i \quad , \quad \lambda_{im} \in F$$

$$y_l = \sum_{i=1}^{n} \lambda_{il} x_i \quad , \quad \lambda_{il} \in F$$

$$\Rightarrow \quad y_m - y_l = \sum_{i=1}^{n} (\lambda_{im} - \lambda_{il}) x_i \quad , \quad \lambda_{im} \in F$$

Since the set $\{x_1, x_2, \ldots, x_n\}$ is linear independent , then $\exists$ c > 0 such that

$$\| y_m - y_l \| = \| \sum_{i=1}^{n} (\lambda_{im} - \lambda_{il}) x_i \| \ge c \sum_{i=1}^{n} (\lambda_{im} - \lambda_{il}) x_i \qquad \text{….. (2)}$$

From (1) & (2) , we get $\sum_{i=1}^{n} |\lambda_{im} - \lambda_{il}| \to 0$ when $m,l \to \infty$, then:

$|\lambda_{im} - \lambda_{il}| \to 0$ when $m,l \to \infty$, $\forall i$.

$\therefore$ $\forall i = 1, \ldots, n$, $\{\lambda_{im}\}$ is Cauchy sequence in F.

Since F is IR or C and both of them is complete

Then $\forall i$, $\exists\ \lambda_i \in F$ s.t. $\lambda_{im} \to \lambda_i$

Put $y = \sum_{i=1}^{n} \lambda_i x_i \ \Rightarrow y \in$ X , $y_m \to y \ \Rightarrow$ X is complete.

<u>Theorem 2.5.:</u> Let X be a Banach space, M subspace of X, M is a Banach space iff M is closed in X.

Proof:

$\to$) Suppose M is Banach space $\Rightarrow$ M is complete

T.P. M is closed (i.e. $\overline{\text{M}}$=M)

Let $x \in \overline{\text{M}}$

$\Rightarrow \exists$ (x_n) sequence in M s.t. $x_n \to x$

$\Rightarrow$ (x_n) is a Cauchy seq. in M

Since M is complete

$\Rightarrow \exists\ y \in$ M s.t. $x_n \to y$

But the limit point is unique

$\Rightarrow x=y \Rightarrow x \in$M $\Rightarrow$ M is closed

<u>Corollary 2.6.:</u> Let X be a normed space, if M is finite dimension subspace in X then M is closed.

Proof:

M is a normed space (Every subspace of normed space is normed space)

Since M is finite dimension $\Rightarrow$ M is complete (From theorem 2.4.)

By using theorem 2.5. $\Rightarrow$ M is closed.

<u>Definition 2.7.:</u> Let X be a normed space, A be a subset in X, A is said to be ***bounded subspace*** in X if there exist M >0 such that $\| x \| \leq$ M, $\forall x \in$X.

<u>Theorem 2.8.:</u> Let X be a normed space, A subspace in X, then the two following statements are equivalent.

1- A is bounded.
2- If (x_n) seq. in X and (λ_n) seq. in F such that $\lambda_n \to 0$ as $n \to \infty$, then $x_n\lambda_n \to 0$ as $n \to \infty$.

Proof:

T.P. 1 $\to$ 2

$\| x_n\lambda_n - 0\| = \| x_n\lambda_n \| = | \lambda_n| \, \| x_n \| \to 0$

T.P. $2 \to 1$

Suppose A unbounded

i.e. $\exists\, x_n \in A$ s.t. $\| x_n \| > M$, $\forall\, n \in Z^+$

put $\lambda_n = 1/n \to 0$ as $n \to \infty$

but $\lambda_n x_n \to 0$ C!

then A is bounded.

2. Quotient spaces

Definition 2.9.: The linear vector space X /Y is called quotient or factor space formed as follows:

The elements of X / Y are cosets of Y {sets of the form x + Y for $x \in$ X. The set of cosets is a linear v. space under the operations:

$(x_1 + Y) \oplus (x_2 + Y) = (x_1 + x_2) + Y;$

$\lambda(x + Y) = \lambda x + Y.$

So for example Y + Y = Y and λY = Y for $\lambda \neq 0$. Two cosets x_1 + Y and x_2 + Y are equal if assets $x_1 + Y = x_2 + Y$, which is true if and only if $x_1 + x_2 \in Y$.

Definition 2.10.: A quotient vector space X/Y is called quotient normed space if there exists norm define on X/Y.

Theorem 2.11.: If X is a normed space, and Y is a normed linear subspace, then X/Y is a normed space under the norm:

$\| x + Y \| = \inf\{\| x+y \| : y \in Y\}$

Theorem 2.12.: Let X be a normed space and **M** closed subspace of X, if X is Banach space then X/M is Banach space.

3. Linear Transformations

Definition 2.13.: Let X and Y are vector spaces on F. The function $T : X \to Y$ is called ***linear transformation*** if satisfy the following conditions:

1) $T(x+y) = T(x) + T(y)$, $\forall\, x,y \in X$
2) $T(\lambda x) = \lambda T(x)$, $\forall\, x,y \in X$.

i.e. $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$, $\forall x,y \in X$, α,β scalars.

The linear transformation $f: X \to F$ is called ***linear functional*** on X.

Remarks:

1- $D(T)$ = ***Domain*** T

2- $R(T)=\{T(x) : x\in X\} \subset Y$ = ***Range*** T, $R(T)$ is a vector space.

3- $N(T)=\{x\in D(T): T(x) = 0\}$ = ***Null space***, $N(T)$ is a vector space.

4- If Y=X, then $T: X\to X$ is called ***linear operator***.

Examples:

1- Zero Transformation

$O: X\to Y$, $O(x)=0$, $\forall x\in X$

Let $x,y \in X$, α, β scalars

$O(\alpha x + \beta y) = 0 = 0 + 0 = \alpha\, O(x)+\beta\, O(y)$

2- Identity Transformation

$I : X\to X$, $I(x)=x$, $\forall x\in X$

Let $x,y \in X$, α,β scalars

$I(\alpha x + \beta y) = \alpha x + \beta y = \alpha\, I(x)+ \beta\, I(y)$

3- Differential Transformation

Let X is a space of all polynomials on [a, b]

$P_n(x)=a_0 +a_1 x+ \dots + a_n x^n$, $\forall x \in [a, b]$, $\forall n$

$D: X \to X$, $D(P_n(x)) = P_n'(x)$, $\forall\, P_n(x) \in X$

Let $P_n(x)$, $B_n(x) \in X$, α, β are scalars

$D(\alpha P_n(x) + \beta B_n(x)) = (\alpha P_n(x) + \beta B_n(x))'$

$= \alpha P'_n(x) + \beta B'_n(x)) = \alpha D(P_n(x)) + \beta D(B_n(x))$

4- **Integrable Transformation (H.W.)**

Let X=C[a, b] , T: C[a, b] $\to$ C[a, b]

$T(f(x)) = \int_0^x f(t)dt, \forall f \in C[a,b]$

T is linear transformation

5- **Bilateral shift Transformation**

Let X=l_2= { x= $(x_i)_{i=1}^{\infty} : x_i \in R$ or C s.t. $\sum_{i=1}^{\infty} |x_i|^2 < \infty$}

B= $l_2 \to l_2$, B($z_1, z_2, \ldots, z_n, \ldots$) = ($z_2, \ldots, z_n, \ldots$) , $\forall$ z= ($z_1, z_2, \ldots, z_n, \ldots$)$\in l_2$

Let z= ($z_1, z_2, \ldots, z_n, \ldots$) , w= ($w_1, w_2, \ldots, w_n, \ldots$) $\in l_2$, α, β scalars

B($\alpha z + \beta w$)= B($\alpha z_1 + \beta w_1, \alpha z_2 + \beta w_2, \ldots, \alpha z_n + \beta w_n, \ldots$)=($\alpha z_2 + \beta w_2, \ldots, \alpha z_n + \beta w_n, \ldots$) $= \alpha$ ($z_2, \ldots, z_n, \ldots$)$+\beta$($w_2, \ldots, w_n, \ldots$) $= \alpha B(z) + \beta B(w)$

6- **Unilateral Shift Transformation (H.W.)**

U : $l_2 \to l_2$, U($z_1, z_2, \ldots, z_n, \ldots$) = (0, $z_1, z_2, \ldots, z_n, \ldots$) , $\forall$ z= ($z_1, z_2, \ldots, z_n, \ldots$)$\in l_2$

Definition 2.14.: Let T: X$\to$ Y be a linear transformation, T is said to be ***bounded linear transformation*** if: there exists a real number M > 0 s.t. $\| Tx \|_Y \le M \| x \|_X$, $\forall$ $x \in$ X.

Definition 2.15.: Let T : X$\to$ Y be a bounded linear transformation:

$||| T ||| = l.u.b.\{ \frac{\| T(x) \|_y}{\| x \|_x} : x \neq 0, x \in X\}$ is the norm of the bounded linear transformation.

Remarks:

1- $||| T ||| \ge \frac{\| T(x) \|_Y}{\| x \|_X}$, $\forall x \neq 0 \in$ X, $\| T_x \|_Y \le ||| T ||| \, \| x \|_X$

2- If $T = 0 \Rightarrow ||| T ||| = 0$

3- If $\| x \|_X = 1 \Rightarrow ||| T ||| = l.u.b.\{ \| T(x) \|_Y : x \in X\}$

Examples:

1- O: X$\to$Y, O(x)=0, $\forall x \in$X, O is bounded linear transformation, $\| 0 \|$=0.

2- I: X$\to$X, I(x)=x, $\forall x \in$X, I is bounded linear transformation , $\| I \|$ =1.

3- Let X be a normed space of all polynomial on [0, 1]

D: X→X, $D(P_n(x))=P'_n(x)$, $\forall P_n(x) \in X$

D unbounded linear transformation

Proof:

Let $P_n(x)=x^n$, $x\in[0, 1]$, $\forall n$

$\| P_n \| =1$

$D(P_n(x))= D(x^n)= n\, x^{n-1}$

$\| D(P_n(x)) \| = \| n\, x^{n-1}\| = n \| x^{n-1}\| \geq n \| x^n\|$

$\geq n \| P_n(x)\| = n$

⇒ D unbounded linear transformation

(because there is not exist $M > 0$ s.t. $\| D(P_n(x)) \| \leq M \| P_n(x)\|$)

Definition 2.16.: Let T : X→ Y be a linear transformation, T is said to be ***continuous linear transformation*** at $x_0\in X$ if:

$$\forall \varepsilon > 0, \exists\, \delta(\varepsilon, x_0) > 0 \quad \text{s.t.} \quad \text{if } \| x- x_0 \| < \delta \text{, then } \| T(x) - T(x_0) \| < \varepsilon$$

- If T is continuous at every $x_0\in X$, then we say that T is continuous on X.
- If X=Y, T is called ***continuous linear operator***.

Theorem 2.17: If T is linear transformation from normed space X into normed space Y then T is bounded if and only if T is continuous.

Proof:

→) suppose that T is bounded ⇒ $\exists\, k > 0$ s.t. $\| T(x)\| \leq k\|x\|$, $\forall x\in X$.

T.P. T is continuous in $x_0\in X$, let $\forall \varepsilon >0$, we choose $\delta=\varepsilon/k$ s.t. $\|x-x_0\| < \delta$

$\Rightarrow \| T(x) - T(x_0)\| = \| T(x-x_0)\| \leq k\| x-x_0\| \quad \Rightarrow \quad \| T(x) - T(x_0)\| \leq k\delta$

Then T is continuous on x_0, since x_0 is arbitrary point in X ⇒ T is continuous.

←) let T is continuous T.P. T is bounded

Suppose that T is not bounded ⇒ $\forall n \in Z^+$, $\exists\, x_n\in X$ s.t. $\| T(x_n)\| > n\| x_n\|$

$$\frac{1}{n\| x_n \|}\| T(x_n) \| > 1 \qquad \Rightarrow \qquad \| T(\frac{x_n}{n\| x_n \|}) \| > 1$$

put $y_n = \dfrac{x_n}{n\| x_n \|} \qquad \Rightarrow \qquad \| y_n \| = \dfrac{1}{n}$

$\|y_n\| \to 0$ when $n\to\infty$ $\Rightarrow$ $y_n \to 0$ when $n\to\infty$

since T is continuous $\Rightarrow T(y_n) \to T(0)=0$ then $\| T(y_n)\| \to 0$ C! because $\| T(y_n)\| >1$

$\Rightarrow T$ is bounded.

Theorem 2.18.: Let T is linear transformation from normed space X into normed space Y. If X is finite dimensional then T is bounded (continuous).

proof :

Let dim X = n and $\{x_1,\ldots,x_n\}$ is a base of X

$\forall\, x\in X$, $x=\sum_{i=1}^{n}\lambda_i x_i$, $\lambda_i \in F$

$$T(x)=\sum_{i=1}^{n}\lambda_i T(x_i) \Rightarrow \|T(x)\|=\|\sum_{i=1}^{n}\lambda_i T(x_i)\|\le\sum_{i=1}^{n}|\lambda_i|\,\|T(x_i)\|$$

put k = max{ $\|T(x_1)\|, \ldots , \|T(x_n)\|$} , we get:

$$\|T(x)\|= k\sum_{i=1}^{n}|\lambda_i| \quad \ldots. (1)$$

by using linear composition property (Lemma 2.3.) : $\exists$ C > 0 s.t. $\|x\|=\|\sum_{i=1}^{n}\lambda_i x_i\|\ge C\sum_{i=1}^{n}|\lambda_i|$

$$\Rightarrow \sum_{i=1}^{n}|\lambda_i|\le\frac{1}{C}\|x\| \qquad \ldots(2)$$

From (1) & (2), we have:

$\|T(x)\|\le\frac{k}{C}\|x\|$ then T is bounded by theorem 2.17.

4. The space of Bounded Linear Transformation

Definition 2.19.: Let X and Y are normed spaces on F, The set of all bounded linear transformations from X to Y denoted by B(X, Y):

B(X, Y) = {T : X→Y: T bounded linear transformation}

If X=Y , we write B(X).

Theorem 2.20.: Let X and Y are normed spaces on F, then B(X, Y) is vector space on F with respect to standard addition and multiplication.

proof:

T.P. B(X, Y) is a vector space over F

Let $T, S \in$ B(X, Y) , $\alpha, \beta \in$ F then to prove that $\alpha T + \beta S \in$ B(X, Y)

It is easy prove that $\alpha T + \beta S$ is linear transformation. (H.W.)

T.P. $\alpha T + \beta S$ is bounded

Since T and S are bounded linear transformations

$\Rightarrow \exists\ k_1>0, k_2>0$ such that $\| T(x)\| \leq k_1 \|x\|\ \forall x\in X$ and $\| S(x)\| \leq k_2 \|x\|\ \forall x\in X$

$(\alpha T + \beta S)(x)=\alpha\, T(x)+\beta\, S(x)$

$\| (\alpha T + \beta S)(x) \| = \|\alpha\, T(x)+\beta\, S(x)\|$

$\leq |\alpha|\, \| T(x) \|+| \beta |\, \| T(x) \|$

$\leq (|\alpha|\, k_1+|\beta|\, k_2) \|x\|$

then $\alpha T + \beta S$ is bounded $\Rightarrow \alpha T + \beta S \in$ B(X,Y)

Definition 2.21.: Let X and Y are normed spaces on F and T: X →Y is linear transformation.

The norm T is defined by: $||| T ||| = \sup \{ \| T(x) \|_Y : x\in X, \|x\|_X \leq 1\}$

Which is equivalent to: $$||| T ||| = l.u.b.\{ \frac{\| Tx \|_y}{\| x \|_x} : x \neq 0, x \in X\}$$

Theorem 2.22.: Let X and Y are normed spaces on F and T :X $\rightarrow$Y is linear transformation.

If :

a= sup$\{ \| T(x) \|_Y : x\in X, \|x\|_X \leq 1\}$

b= sup$\{ \frac{\| T(x) \|_Y}{\| x \|_X} : x\in X, x \neq 0\}$

c= inf $\{ \lambda > 0, \| T (x) \|_Y \leq \lambda \|x\|_X : \forall x\in X\}$

$\Rightarrow \| T \| =$ a=b=c and $\| T (x) \| \leq \| T \| \|x\| , \forall x\in X$

***proof* :**

by using the definition of c , we have:

$\| T \| \leq c\|x\| , \forall x\in X$

if $\|x\|\leq 1 \Rightarrow c \|x\|\leq c$

$\Rightarrow \| T (x)\| \leq c , \forall x\in X , \|x\|\leq 1$

$\Rightarrow$ sup $\{ \| T (x) \|: x\in X, \|x\|\leq 1\} \leq c$

$\Rightarrow \| T \| \leq c$ ……………………….(1)

by using the definition of b , we have:

$\| T (x) \| \leq b \| x \| , \forall x\neq 0$

Since c= inf $\{ \lambda >0, \| T (x) \| \leq \lambda \|x\| : \forall x\in X\}$

$\Rightarrow c \leq b$ ………………………(2)

Let $x\in X , x\neq 0$

$$\frac{\| T(x) \|}{\| x \|} = \frac{1}{\| x \|} \| T(x) \|$$

$$= \| T(\frac{x}{\| x \|}) \|$$

Put $y = \frac{x}{\| x \|} \Rightarrow \| y\| = 1, y\in X$

$\Rightarrow b \leq a$ …………………….(3)

Then we can proof $a \leq \| T \|$

$\| T\| = a = b = c$

T.P. $\| T (x) \| \leq \| T \| \|x\| , \forall x\in X$

From b , we get:

b $\geq \frac{\| T(x) \|}{\| x \|}$

$\| T(x) \| \leq b\| x \|$, $\forall x \in X$

but $\| T \| = b \Rightarrow \| T(x) \| \leq \| T \| \| x \|$

<u>Theorem 2.23.:</u> The vector space B(X, Y) is normed space with the norm which defined by:

$\| T \| = \sup\{\| T(x)\| : x\in X, \|x\| \leq 1\}$

***proof*:**

1) Since $\| T(x)\| \geq 0, \forall x\in X \Rightarrow \| T \| \geq 0$
2) $\| T \|=0 \Leftrightarrow \sup\{\| T(x)\| : x\in X, \| x \| \leq 1\}=0$

   $\Leftrightarrow \sup\{ \frac{\| T(x) \|}{\| x \|} : x\in X, x \neq 0\}=0$

   $\Leftrightarrow \frac{\| T(x) \|}{\| x \|}=0$, $x\in X$, $x \neq 0$

   $\Leftrightarrow \| T(x)\| = 0 : x\in X$

   $\Leftrightarrow T(x) = 0 : x\in X$

   $\Leftrightarrow T=0$
3) Let $T \in B(X,Y), \lambda\in F$

   $\|\lambda T \| = \sup \{ \| (\lambda T)(x)\|: x\in X, \| x \| \leq 1\}$

   $= \sup \{ |\lambda| \|T \|)(x)\|: x\in X, \| x \| \leq 1\}$

   $= |\lambda| \sup \{\|T \|)(x)\|: x\in X, \| x \| \leq 1\}$

   $= |\lambda| \|T \|$
4) Let $T,S\in B(X,Y)$

   $\| T+S \| = \sup \{ \| (T+S)(x)\|: x\in X, \| x \| \leq 1\}$

   $= \sup \{ \| T(x)+S(x)\|: x\in X, \| x \| \leq 1\}$

   $\leq \sup \{ \| T(x)\|+\|S(x)\|: x\in X, \| x \| \leq 1\}$

   $\leq \sup \{ \| T(x) \|: x\in X, \| x \| \leq 1\}+ \sup \{ \| T(x)\|: x\in X, \| x \| \leq 1\}$

   $= \| T(x) \|+\| S(x) \|$

Then B(X,Y) is normed space.

<u>Theorem 2.24.:</u> If Y is a Banach space then B(X,Y) is Banach space too.

proof:

B(X,Y) is normed space (from Th.2.23.)

Let $\{T_n\}$ is Cauchy sequence in B(X,Y) $\Rightarrow \| T_n - T_m\| \to 0$ when $n,m\to\infty$

$\Rightarrow \{T_n(x)\}$ is Cauchy sequence in Y for all $x\in$X

Since Y is complete space (because it is Banach space)

$\Rightarrow \exists\, T(x) \in$ Y s.t. $T_n(x)\to T(x)$

$\Rightarrow T\in$ B(X, Y) $\Rightarrow \{T_n\}$ is convergent $\Rightarrow$ B(X,Y) is Banach space.

Definition 2.25. : Let X be a normed space over the field F. The normed space B(X, F) is called dual space to X and denoted by X^*.

i.e. $X^*=\{f : X\to F, f$ is bounded linear functional$\}$.

- If X is normed space then X^* is Banach space.
- If X is finite dimensional vector space then $X' = X^*$.

(X' is the set of all the limit points of X).

Examples:

1- The dual space of IR^n is itself.

i.e. $(IR^n)^* = IR^n$

proof:

since IR^n is finite dimensional $\Rightarrow$ $(IR^n)^* = (IR^n)'$

let $\{x_1, x_2, .., x_n\}$ is a base to IR^n and $x \in IR^n$

$$\to x = \sum_{i=1}^{n} \lambda_i x_i, \qquad \lambda_i \in IR$$

$$\to f(x) = f(\sum_{i=1}^{n} \lambda_i x_i) = \sum_{i=1}^{n} \lambda_i f(x_i) = \sum_{i=1}^{n} \lambda_i y_i, \qquad y_i = f(x_i), \qquad i = 1,2,...,n$$

By using Cauchy-Schwar's inquality, we get:

$$| f(x) \le \sum_{i=1}^{n} | \lambda_i y_i | \le ((\sum_{i=1}^{n} | \lambda_i |^2)^{\frac{1}{2}})((\sum_{i=1}^{n} | y_i |^2)^{\frac{1}{2}})$$

$$\to | f(x) | \le \| x \| (\sum_{i=1}^{n} | y_i |^2)^{\frac{1}{2}}$$

$\| f \| = \sup\{| f(x)|: x\in IR^n, \| x \|=1\}$

$$\to \| f \| \le \sum_{i=1}^{n} | y_i |^2)^{\frac{1}{2}}$$

$\to$ the norm of f is the norm of IR^n

i.e. $\| f \| = \sum_{i=1}^{n} | y_i |^2)^{\frac{1}{2}} \to \|f\| = \| y \|$ s.t. $y=\{ y_1,y_2,\ldots,y_n\} \in IR^n$

then the function $\psi : (IR^n)' \to IR^n$ which is defined by $\psi(f)=y=(y_1,y_2,\ldots,y_n)$ s.t. $y_i=f(x_i)$ is isomorphic linear transformation $\to (IR^n)'=IR^n=(IR^n)^*$

2-The dual space of l_1 is l_∞.(H.W.)

3-The dual space of l_p, $1<p<\infty$ is l_q s.t. $\frac{1}{p}+\frac{1}{q}=1$.(H.W.)

Chapter Three : Hilbert Space

Definition 3.1.

An ***inner product space*** (also known as a ***pre-Hilbert space***) is a vector space X over F (= R or C) together with a map $<.,.>: X \times X \to F$ satisfying (for $x, y, z \in X$ and $\lambda \in F$):

(i) $<x, x> \geq 0$

(ii) $<x, x> = 0 \Leftrightarrow x = 0$

(iii) $\overline{<x,y>} = <y, x>$

(iv) $<\alpha x + \beta y, z> = \alpha<x, z> + \beta<y, z>$

Remark: Every subspace of pre-Hilbert space is pre-Hilbert space.

Examples:

1) Let $X=R^2$, $x=(x_1, x_2)$, $y=(y_1, y_2)$, Which of the following functions should be an inner product space on X and why?

i- $<x, y> = x_1 y_1 + x_2 y_2$

ii- $<x, y> =3 x_1 y_1 + x_2 y_2$ (H.W.)

iii- $<x, y> = x_1^2 y_1^2 + x_2^2 y_2^2$ (H.W.)

Proof **:**

i) 1- $<x, x>= x_1^2 + x_2^2 \geq 0$

2- $<x, x>= 0 \Leftrightarrow x_1^2 + x_2^2=0 \Leftrightarrow x_1^2 = 0,\ x_2^2 = 0 \Leftrightarrow x_1 = 0,\ x_2 = 0 \Leftrightarrow x=0$

3- $<x, y> = x_1 y_1 + x_2 y_2$

4- $\overline{<x,y>} = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = <y, x>$

Let $x=(x_1, x_2)$, $y=(y_1, y_2)$, $z=(z_1, z_2)$, $\alpha, \beta \in R$

$$\alpha x+\beta y = \alpha (x_1, x_2) + \beta (y_1, y_2) = (\alpha x_1+ \beta y_1, \alpha x_2 +\beta y_2)$$

$$(\alpha x+\beta y, z)= (\alpha x_1+ \beta y_1) z_1 +(\alpha x_2 +\beta y_2) z_2$$

$$= \alpha (x_1 z_1+ x_2 z_2) +\beta(y_1 z_1+ y_2 z_2)$$

$$=\alpha < x, z > + \beta < y, z >$$

Then it is an inner product space.

2) Let X= F^n , < , >: XxX → F defined by:

$< x, y >= \sum_{i=1}^{n} x_i \overline{y_i}$, $\forall x, y \in X$ is an inner product on X.

Proof:

1- $< x, x > = \sum_{i=1}^{n} x_i^2 \geq 0$

2- $< x, x > = 0 \Leftrightarrow \sum_{i=1}^{n} x_i^2 = 0 \Leftrightarrow x_i=0 , \forall\ i \Leftrightarrow x = 0$

3- $\overline{< x, y >} = \overline{\sum_{i=1}^{n} x_i \overline{y_i}} = \sum_{i=1}^{n} \overline{x_i} y_i =< y, x >$

$$< \alpha x + \beta y, z >= \sum_{i=1}^{n} (\alpha x_i + \beta y_i)\overline{z_i}$$

$$= \alpha \sum_{i=1}^{n} x_i \overline{z_i} + \beta \sum_{i=1}^{n} y_i \overline{z_i}$$

$$= \alpha < x, z > + \beta < y, z >$$

Then < , > is an inner product on X.

3) Let X=C[a, b], < , > : XxX → R which defined by:

$< f, g >= \int_a^b f(x)\overline{g(x)}dx$ is an inner product on X. (H.W.)

Theorem 3.2.:

If X is a pre-Hilbert space, then :

1) $< x, 0>=< 0, x > =0$

2) $< x, \alpha y + \beta z > = \bar{\alpha} < x, y > + \bar{\beta} < x, z >, \quad \forall x, y, z \in X \ \& \ \forall \alpha, \beta \in F$

Proof:

1- $< 0, x > = < 0.0, x > = 0 < 0, x > = 0$

2- $< x, \alpha y + \beta z > = \overline{< x, \alpha y + \beta z >}$

$$= \overline{\alpha < y,x > + \beta < z,x >}$$

$$= \overline{\alpha < y,x >} + \overline{\beta < z,x >}$$

$$= \overline{\alpha} < y,x > + \overline{\beta} < z,x >$$

Corollary 3.3.:

If X is a pre- Hilbert space , then:

1) $< \sum_{i=1}^{n} \alpha_i x_i , y > = \sum_{i=1}^{n} \alpha_i < x_i , y >$

2) $< x, \sum_{j=1}^{m} \beta_j y_j > = \sum_{j=1}^{m} \overline{\beta_j} < x, y_j >$

3) $< \sum_{i=1}^{n} \alpha_i x_i , \sum_{j=1}^{m} \beta_j y_j > = \sum_{i,j} \alpha_i \overline{\beta_j} < x_i , y_j >$

Theorem 3.4.:(Chauchy- Schwarz Inequality)

Let X be a pre-Hilbert Space and the function $\| . \|: X \to R$ defined by:

$\| x \| = \sqrt{< x,x >}, \forall x \in X$ then $| < x, y > | \leq \| x \| \, \| y \|$, $\forall$ $x,y \in X$

Proof:

If $x = 0$ or $y = 0 \Rightarrow < x,y > =0$.

If $y \neq 0$, we put $z = \frac{y}{\| y \|}$

$\Rightarrow \| z \|^2 = < z, z> = < \frac{y}{\| y \|}, \frac{y}{\| y \|} > = \frac{1}{\| y \|^2} < y, y > = \frac{1}{\| y \|^2} \| y \|^2 = 1$

We must prove $| < x, z>| \leq \| x \|$

Let $\lambda \in F$, then:

$< x- \lambda z, x-\lambda z > \geq 0$

$\| x \|^2 - \overline{\lambda} < x, z > - \lambda < z, x > + | \lambda |^2 \| z \|^2 \geq 0$

$\| x \|^2 - \overline{\lambda} < x, z > - \lambda < z, x > + | \lambda |^2 \geq 0$

$\| x \|^2 - < x, z > \overline{< x,z >} + < x, z > \overline{< x,z >} - \overline{\lambda} < x,z > - \lambda < z,x > + \lambda \overline{\lambda} \geq 0$

$\| x \|^2 - |< x,z >|^2 + < x,z > (\overline{< x,z >} - \overline{\lambda}) - \lambda (< z,x > - \overline{\lambda}) \geq 0$

$\| x \|^2 - |< x,z >|^2 + (< x,z > - \lambda)\overline{(< x,z > - \lambda)} \geq 0$

$\| x \|^2 - | < x, z >|^2 + | < x, z > - \lambda^2 | \geq 0$, $\forall$ $\lambda \in F$

Since $< x, z > \in F$, put $\lambda = < x, z >$, then

$\| x - \langle x, z\rangle z\|^2 = \| x \|^2 - |\langle x, z\rangle|^2 + |\langle x, z\rangle - \langle x, z\rangle|^2$

$= \| x \|^2 - |\langle x, z\rangle|^2 \geq 0$

$\Rightarrow |\langle x, z\rangle| \leq \| x \|$

$\Rightarrow |\langle x, \frac{y}{\| y \|}\rangle| \leq \| x \|$

$\Rightarrow |\langle x, y\rangle| \leq \| x \| \, \| y \|$

Theorem 3.5.: Every Pre-Hilbert space is a normed space (metric space).

Proof:

Let X be a Pre-Hilbert space and let the function $\| . \|: X \to R$ such that:

$\| x \| = \sqrt{\langle x, x\rangle}$, $\forall x \in X$

T.P. the space X satisfies the conditions of the norm:

1- Since $\langle x, x\rangle \geq 0$, $\forall x \in X \Rightarrow \| x \| \geq 0, \forall x \in X$.

2-$\| x \| = 0 \Leftrightarrow \sqrt{\langle x, x\rangle} = 0 \Leftrightarrow \langle x, x\rangle = 0 \Leftrightarrow x = 0$

3-let $x \in X$, $\lambda \in F$:

$\| \lambda x \| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{\lambda\bar{\lambda} \langle x, x\rangle} = \sqrt{|\lambda|^2 \langle x, x\rangle} = |\lambda| \sqrt{\langle x, x\rangle} = |\lambda| \, \| x \|$

4- let $x, y \in X$:

$\| x+y \|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \| x \|^2 + \langle x, y\rangle + \overline{\langle x, y\rangle} + \| y \|^2$

Since $\langle x, y\rangle + \overline{\langle x, y\rangle} = 2 \operatorname{Re}(\langle x, y\rangle)$

$\Rightarrow \| x + y \|^2 = \| x \|^2 + 2 \operatorname{Re}(\langle x, y\rangle) + \| y \|^2$

Since $\operatorname{Re}(\langle x, y\rangle) \leq |\langle x, y\rangle|$

$\Rightarrow \| x + y \|^2 \leq \| x \|^2 + 2 |\langle x, y\rangle| + \| y \|^2$

By Cauchy – Schwars inequality $|\langle x, y\rangle| \leq \| x \| \, \| y \|$, we get:

$\Rightarrow \| x + y \|^2 \leq \| x \|^2 + 2 \| x \| \, \| y \| + \| y \|^2 = (\| x \| + \| y \|)^2$

Then $\| x + y \| \leq \| x \| + \| y \|$

Theorem 3.6.: If x, y are vectors on Pre-Hilbert space X, then:

1- $\| x+y \|^2 = \| x \|^2 + 2 \operatorname{Re}(\langle x, y\rangle) + \| y \|^2$ (Polar inequality)

2- $\| x + y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$ (Parallel Law)

3- $< x, y > = \frac{1}{4}[\, \| x + y \|^2 - \| x - y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2 \,]$ (Identical Polarization)

Proof:

1- We get from theorem 3.5.

2- $\| x + y \|^2 = < x+y, x+y > = \| x \|^2 + < x, y > + < y, x > + \| y \|^2$

$\| x - y \|^2 = < x - y, x - y > = \| x \|^2 - < x, y > - < y, x > + \| y \|^2$

$\Rightarrow \| x + y \|^2 + \| x-y \|^2 = 2 \| x \|^2 + 2 \| y \|^2$

3- From (2), we get:

$\| x + y \|^2 - \| x - y \|^2 = 2 < x,y > +2 < x, y >$

$\| x + iy \|^2 = \| x \|^2 - i < x, y > + i < y, x > + \| y \|^2$

$\| x - iy \|^2 = \| x \|^2 + i < x, y > - i < y, x > + \| y \|^2$

$i\| x + iy \|^2 - i\| x - iy \|^2 = 2 < x, y > - 2 < y, x >$

$\| x + y \|^2 - \| x - y \|^2 + i\| x + iy \|^2 - i\| x - iy \|^2 = 4 < x, y >$

$\Rightarrow < x, y > = \frac{1}{4}[\, \| x + y \|^2 - \| x - y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2 \,]$

<u>Theorem 3.7.:</u> Let (X, $\| . \|$) is a normed space such that:

$\| x + y \|^2 + \| x - y \|^2 = 2 \| x \|^2 + 2\| y \|^2 \;, \forall\, x, y \in X$

And let < , > is defined by:

$< x, y > = \frac{1}{4}[\, \| x + y \|^2 - \| x - y \|^2 + i \| x+iy \|^2 - i \| x-iy \|^2 \,]$

Then < , > is an inner product on X (i.e. X is a Pre-Hilbert space).

<u>Remark:</u>

If X is a normed space then it is not necessary X is a Pre-Hilbert space. For example:

Let X = C[a, b] and $\| f \| = \max\{ | f(x) | : a \le x \le b\} \;, \forall f \in X$

Since X is normed space

T.P. it is not Pre-Hilbert space, we need prove that:

$\| f + g \|^2 + \| f - g \|^2 \neq 2 \| f \|^2 + 2 \| g \|^2 \;, f,g \in X$

Let $f(x) = 1, \; g(x) = \frac{x-a}{b-a}, \quad \forall x \in [a, b]$

$\| f \| = 1, \; \| g \| = 1$

$f(x) + g(x) = 1 + \frac{x-a}{b-a} \Rightarrow \| f + g \| = 2$

$f(x) - g(x) = 1 - \frac{x-a}{b-a} \Rightarrow \| f - g \| = 1$

$\Rightarrow \| f + g \|^2 + \| f - g \|^2 = 4+1=5 \neq 2 \| f \|^2 + 2 \| g \|^2 = 2+2=4$

Theorem 3.8.: On Pre-Hilbert space X:

1- If $x_n \to x$, $y_n \to y$ then $< x_n , y_n > \to < x, y >$

2- If $\{ x_n \}$ and $\{ y_n \}$ are Cauchy sequence on X then $\{ < x_n , y_n > \}$ is Cauchy sequence on F.

Proof:

1- $< x_n , y_n > = < x+(x_n - x), y+(y_n - y) >$

$= < x, y > + < x, y_n - y > + < x_n - x, y > + < x_n - x, y_n - y >$

$< x_n , y_n > - < x, y > = < x, y_n - y > + < x_n - x, y > + < x_n - x, y_n - y >$

$| < x_n , y_n > - < x, y > | \leq | < x, y_n - y >| + | < x_n - x, y >| + |< x_n - x, y_n - y >|$

$\leq \| x \| \| y_n - y \| + \| x_n - x \| \| y \| + \| x_n - x \| \| y_n - y \|$

Since $\| x_n - x \| \to 0$, $\| y_n - y \| \to 0$ where $n \to \infty$

$\Rightarrow | < x_n - x > - < y_n - y > | \to 0$ where $n \to \infty$

$\Rightarrow < x_n - x > \to < y_n - y >$

2- Similarly to (1):

$| < x_n , y_n > - < x_m, y_m >| \leq \| x_m \| \| y_n - y_m \| + \| x_n - x_m \| \| y_m \| + \| x_n - x_m \| \| y_n - y_m \|$

Since $\| x_n \|$, $\| y_n \|$ is bounded

Definition 3.9.: The complete Pre-Hilbert space is called ***Hilbert space***. In other words if X a vector space on F with an inner product < , >, then X is Hilbert space if the metric space which is generated by the norm $\| x \|^2 = < x, x >$ complete Hilbert space.

Examples:

1- The space F^n with an inner product which defined by:

$< x, y > = \sum_{i=1}^{n} x_i y_i$, $\forall x, y \in F^n$ is a Hilbert space.

2- The space $l^2 = \{x = (x_1, x_2, \ldots, x_n, \ldots) : x_i \in F, \sum_{i=1}^{\infty} | x_i |^2 < \infty\}$ with an inner product defined by: $< x, y > = \sum_{i=1}^{\infty} x_i \overline{y_i}$ is Hilbert space.

3- The space X=C[-1, 1] with an inner product defined by : $< f, g > = \int_{-1}^{1} f(x)\overline{g(x)}$ is not Hilbert space.

Proof:

The space X with an inner product is not complete space because if we take the sequence $\{ f_n \}$ such that:

$$f_n(x) = \begin{cases} 0 & -1 \le x \le 0 \\ nx & 0 < x < \frac{1}{n} \\ 1 & \frac{1}{n} \le x \le 1 \end{cases}$$

$$\| f_n - f_m \|^2 = < f_n - f_m, f_n - f_m > = \frac{(n-m)^2}{3n^2 m}$$

$\Rightarrow \| f_n - f_m \| \to 0$ where $n, m \to \infty$

i.e. $\{f_n\}$ is Cauchy sequence but it is not convergent in X

if we suppose that $f_n \to f$

$\Rightarrow f \notin$ X because it is not continuous.

4- Every Hilbert space is Banach space but the inverse is not true.

Sol.:

If X is Hilbert space then its Pre-Hilbert space and complete.

Since every Pre-Hilbert space is normed space then X is complete normed space i.e. Banach space.

And the space l_p ($p \neq 2$) is Banach space.

T.P. l_p ($p \neq 2$) not Hilbert space , we prove it is not satisfying Parallel Law.

Let $x = (1, 1, 0, 0, \ldots)$, $y = (1, -1, 0, 0,\ldots)$

$\Rightarrow x, y \in l_p$, $\| x \| = \| y \| = 2^{1/p}$ & $\| x + y \| = \| x - y \| = 2$

$\Rightarrow \| x + y \|^2 + \| x - y \|^2 \neq 2 \| x \|^2 + 2\| y \|^2$

Definition 3.10.:

Let X be Pre-Hilbert space and let $x, y \in$ X, we say x ***orthogonal*** on y if $< x, y >=0$ (write $x \perp y$).

Remarks:

1- The orthogonal is symmetric, i.e. if $x \perp y$ then $y \perp x$.

Since $x \perp y \Rightarrow < x, y >=0 \Rightarrow \overline{< x, y >} = \bar{0} = 0 \Rightarrow < y, x >=0 \Rightarrow y \perp x.$

2- Zero vector orthogonal on all vectors, i.e. $0 \perp x$, $\forall x \in$X, because $<0, x>=0$, $\forall x \in$X.

3- If $x \perp x \Rightarrow x = 0$, because if $x \perp x \Rightarrow < x, x > =0 \Rightarrow x = 0$.

4- If $x \perp y \Rightarrow \lambda x \perp y$, $\forall \lambda \in$F. because $< \lambda x, y > = \lambda < x, y > = \lambda (0)=0$

Examples:

1- Let X = R^2 with an inner product and x=(1, 2), y=(2, -1), z= (-6, 3)

Since $< x, y >=(1)(2)+(2)(-1)=0 \Rightarrow x \perp y.$

$< x, z > = (1)(-6)+(2)(3)=0 \Rightarrow x \perp z.$

$<y, z > = (2)(-6)+(-1)(3)=-15 \neq 0 \Rightarrow y$ not orthogonal on z.

2- If the vector x is orthogonal on all the vectors $x_1,x_2,\ldots,x_n$ in Pre-Hilbert space X, then x is orthogonal on every linear combination of x_i.

Let $z=\sum_{i=1}^{n}\lambda_i x_i \quad , \lambda_i \in$F

$< x, z > = < x, \sum_{i=1}^{n}\lambda_i x_i > = \sum_{i=1}^{n}\overline{\lambda_i} < x, x_i > = 0$ (because $x \perp x_i$, $\forall i=1, 2, .., n$).

3- Find the values of a which make the vectors $x = (1, 2, a)$, $y = (-1, 3, 5)$ orthogonal in R^3.

$< x, y > = (1)(-1)+(2)(3)+5a = -1+6+5a = 5+5a = 0 \Rightarrow 5a = -5 \Rightarrow a = -1$.

4- Let x, y vectors in Pre-Hilbert space X s.t. $\| x \| = \| y \| =1$, then $x+y$ orthogonal on $x-y$.

$< x+y, x-y > = < x, x > - < x, y > + < y, x > - < y, y >$

$= \| x \|^2 - < x, y > + < x, y > - \| y \|^2 = 1-1= 0.$

Theorem 3.11.:

If x, y are orthogonal vectors in Pre-Hilbert space X, then:

$$\| x + y \|^2 = \| x - y \|^2 = \| x \|^2+\| y \|^2$$

Proof:

Since $x \perp y \Rightarrow < x, y > = < y, x > = 0$

$\Rightarrow \| x + y \|^2 = < x+y, x+y > = < x, x > + < x, y > + < y, x > + < y, y >$

$= \| x \|^2 + 0 + 0 + \| y \|^2$

$= \| x \|^2 + \| y \|^2$

Similarity prove that $\| x - y \|^2 = \| x \|^2+\| y \|^2$.

Corollary 3.12.: If $x_1,x_2,\ldots,x_n$ are orthogonal vectors (i.e. $x_i \perp x_j$, $\forall i\neq j$) in Pre-Hilbert space X, then : $\| \sum_{i=1}^{n} x_i \|^2 = \sum_{i=1}^{n}\| x_i \|^2$.

Definition 3.13.: Let A nonempty subset in Pre-Hilbert space X. The vector $x\in$X is called orthogonal vector on A (write $x \perp$ A) if $x \perp y$, $\forall y\in$A.

Definition 3.14.: Let A & B are nonempty subsets of Pre-Hilbert space X, We say A orthogonal on B (write A $\perp$ B) if $x \perp y$, $\forall$ $x\in$A & $\forall y \in$B.

Remark: If M_1 & M_2 are subspaces of Pre- Hilbert space X such that $M_1 \perp M_2$, then $M_1 \cap M_2 = \{0\}$.

Definition 3.15.: Let A nonempty subset of Pre-Hilbert space X. The Orthogonal complement of A denoted by $A^{\perp}$ and defined by:

$A^{\perp}=\{ x\in X: x\perp y , \forall y\in A\} =\{ x\in X: x\perp A\}$

And define $(A^{\perp})^{\perp} = A^{\perp\perp}=\{x\in X: x\perp y , \forall y\in A^{\perp}\}$.

Theorem 3.16.: Let X be a Pre-Hilbert space, then:

1- $\{0\}^{\perp} =X$, 2- $X^{\perp} =\{0\}$

Proof:

1- $\{0\}^{\perp} = \{ x\in X: x\perp 0\}= X$, 2- $X^{\perp} =\{ x\in X: x\perp x\}=\{0\}$

Theorem 3.17.: Let A and B be two nonempty subsets of Pre-Hilbert space X. Then:

1- $A\cap A^{\perp}\subset \{0\}$

2- $A\subseteq A^{\perp\perp}$

3- If $A\subset B$ then $B^{\perp}\subset A^{\perp}$

4- $A \subseteq B^{\perp} \Leftrightarrow B \subset A^{\perp}$

Proof:

1- Let $x\in A\cap A^{\perp} \Rightarrow x\in A$ & $x\in A^{\perp} \Rightarrow x \perp x \Rightarrow x =0 \Rightarrow A\cap A^{\perp}\subset \{0\}$

2- Let $x \in A \Rightarrow x \perp y , \forall y\in A^{\perp} \Rightarrow x \perp A^{\perp} \Rightarrow x \in A^{\perp\perp} \Rightarrow A\subseteq A^{\perp\perp}$

3- Let $x\in B^{\perp} \Rightarrow x\perp y , \forall y\in B$

Since $A\subset B \Rightarrow x\perp y , \forall y\in A \Rightarrow x \in A^{\perp} \Rightarrow B^{\perp}\subset A^{\perp}$

4- Let $A \subseteq B^{\perp}$ T.P. $B \subset A^{\perp}$

Since $A \subseteq B^{\perp} \Rightarrow B^{\perp\perp}\subset A^{\perp}$ (by part 3)

But $B \subset B^{\perp\perp}$ (by part 2)

$\Rightarrow B \subset A^{\perp}$

Similarity prove if $B \subset A^{\perp} \Rightarrow A \subseteq B^{\perp}$

Theorem 3.18.: If A is nonempty subset of Pre-Hilbert space X, then $A^{\perp}$ is closed subspace of X.

Proof:

Since $0\perp x, \forall x \in A \Rightarrow 0\in A^{\perp} \Rightarrow A^{\perp} \neq \phi$

Let $x,y \in A^{\perp}$, $\alpha, \beta\in F$,

$\forall z\in A$, we have

$< \alpha x+\beta y, z > = \alpha < x, z > + \beta < y, z > = \alpha (0)+\beta (0) =0$

$\Rightarrow \alpha x+\beta y \in A^{\perp} \Rightarrow A^{\perp}$ is subspace of X

T.P. $A^{\perp}$ is closed subspace (i.e. $\overline{A^{\perp}} = A^{\perp}$)

Let $x \in \overline{A^{\perp}}$ $\Rightarrow$ there exist a sequence $\{x_n\}$ in $A^{\perp}$ such that $x_n \to x$

$\forall\, y \in A \Rightarrow \langle x_n , y\rangle =0$, $\forall\, n \in Z^+$

Since $\langle x_n , y\rangle \to \langle x, y\rangle$

$\Rightarrow \langle x, y\rangle =0$, $\forall\, y \in A \Rightarrow x \in A^{\perp} \Rightarrow \overline{A^{\perp}} = A^{\perp} \Rightarrow A^{\perp}$ is closed subspace of X.

<u>Definition 3.19.:</u> Let A be subset of Pre-Hilbert space. The set A is called ***orthogonal*** if $x \perp y$, $\forall x, y \in A$, $x \neq y$, and called A ***orthonormal*** if A is orthogonal and $\| x \| =1$, $\forall x \in A$. In the other word, we say A is orthonormal if :

$$\langle x, y\rangle = \left.\begin{cases} 0 & x \neq y \\ 1 & x = y \end{cases}\right\}, \ \forall\, x,y \in A.$$

The sequence $\{ x_n \}$ is called orthogonal if $x_n \perp x_m$, $\forall n \neq m$, and called orthonormal if:

$$\langle x_n, y_m\rangle = \left.\begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}\right\}$$

<u>Remark:</u> The orthonormal set not contained the zero vector because $\| 0 \| =0 \neq 1$.

<u>Examples:</u>

1- Let X = R^3, and A = {(1, 2, 2), (2, 1, -2), (2, -2, 1)} then A is orthogonal set in R^3.

Sol.: x = (1, 2, 2), y = (2, 1, -2), z = (2, -2, 1)

$\langle x, y\rangle = \sum_{i=1}^{3} x_i y_i$ = (1)(2)+(2)(1)+(2)(-2)= 2+2-4=0

Similarity prove $\langle x, z\rangle =0$ & $\langle y, z\rangle =0$

2- Let X= C[-π, π] and $f_n(x) = \sin (nx)$ then $\{ f_n \}$ is orthogonal sequence.

<u>Sol.:</u>

$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} f_n(x) f_m(x)dx = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)dx = 0$

If $g_n(x)= \cos (nx)$ then the sequence $\{ g_n \}$ is orthogonal.

<u>Theorem 3.20.</u>: Let $x_1, \ldots , x_n$ are orthonormal vectors in Pre-Hilbert space X, $\forall x \in X$:

1- $\| x - \sum_{i=1}^{n} \langle x, x_i\rangle x_i \|^2 = \| x \|^2 - \sum_{i=1}^{n} |\langle x, x\rangle|^2$, $\forall x \in X$

2- $\sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$, $\forall x \in X$

3- $(x- \sum_{i=1}^{n} < x, x_i > x_i) \perp x_j$, $\forall x \in X$ and for all j.

Proof:

Let $\lambda_i = < x, x_i >$

$$\| x - \sum_{i=1}^{n} < x, x_i > x_i \|^2 = \| x - \sum_{i=1}^{n} \lambda_i x_i \|^2 = < x - \sum_{i=1}^{n} \lambda_i x_i , x - \sum_{i=1}^{n} \lambda_i x_i >$$

$$= < x, x > - < x, \sum_{i=1}^{n} \lambda_i x_i > - < \sum_{i=1}^{n} \lambda_i x_i , x > + < \sum_{i=1}^{n} \lambda_i x_i , \sum_{i=1}^{n} \lambda_i x_i >$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{\lambda_i} < x, x_i > - \sum_{i=1}^{n} \lambda_i < x_i, x > + \| \sum_{i=1}^{n} \lambda_i x_i \|^2$$

$$= \| x \|^2 - \sum_{i=1}^{n} \overline{\lambda_i} \lambda_i - \sum_{i=1}^{n} \lambda_i \overline{\lambda_i} + \sum_{i=1}^{n} | \lambda_i |^2 \| x_i \|^2$$

$$= \| x \|^2 - \sum_{i=1}^{n} | \lambda_i |^2 - \sum_{i=1}^{n} | \lambda_i |^2 + \sum_{i=1}^{n} | \lambda_i |^2$$

$$= \| x \|^2 - \sum_{i=1}^{n} | \lambda_i |^2 = \| x \|^2 - \sum_{i=1}^{n} |< x, x_i >|^2$$

2-since $\| x - \sum_{i=1}^{n} < x, x_i > x_i \|^2 \geq 0$

$$\Rightarrow \| x \|^2 - \sum_{i=1}^{n} |< x, x_i >|^2 \geq 0$$

$$\Rightarrow \sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$$

3- $< x - \sum_{i=1}^{n} < x, x_i > x_i , x_j > = < x, x_j > - < \sum_{i=1}^{n} < x, x_i > x_i , x_j >$

$$= < x, x_j > - \sum_{i=1}^{n} < x, x_i > < x_i , x_j >$$

Since $< x_i, x_j > = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

$$\Rightarrow < x - \sum_{i=1}^{n} < x, x_i > x_i , x_j > = < x, x_j > - < x, x_j > = 0$$

Then $x - \sum_{i=1}^{n} < x, x_i > x_i) \perp x_j$, $\forall x \in X$ and for all j.

Corollary 3.21.: Let $\{ x_n \}$ be an orthonormal sequence in Pre-Hilbert space X,

then: $\sum_{i=1}^{n} |< x, x_i >|^2 \leq \| x \|^2$, $\forall x \in$ X.

Theorem 3.22.: (Gram- Schmidt Theorem)

If $\{ y_n \}$ is a sequence of independent linear vectors in Pre-Hilbert space X, then there exist an orthogonal sequence $\{ x_n \}$ in X such that:

$[x_1, x_2, \ldots, x_n] = [y_1, y_2, \ldots, y_n]$ for all n.