Chapter One

Vector-Valued function and Motion in Space.

المتجه $\boxed{\text{Vector}}$ →

يمكن كتابة المتجه بطريقتين

① $(v_1, v_2, v_3) = (2, 1, 3)$ Component

$v_1 = 2,\quad v_2 = 1,\quad v_3 = 3$

② $v_1 i + v_2 j + v_3 k = 2i + 1j + 3k$

$v_1 = 2,\quad v_2 = 1,\quad v_3 = 3$

Vector function →

معادلة المتجه ايضا"

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① $(v_1, v_2, v_3) = (x, y, z) = (f(t), g(t), h(t))$

$$\left.\begin{aligned} v_1 &= x = f(t) \\ v_2 &= y = g(t) \\ v_3 &= z = h(t) \end{aligned}\right\} \quad t \in I,\ I \text{ interval.}$$

$$r(t) = (x, y, z) = (f(t), g(t), h(t)),\ t \in I$$

② $r(t) = f(t)\, i + g(t)\, j + h(t)\, k.$

for example: $r(t) = (\cos t, \sin t, t)$ <u>Vector function</u>

$v_1 = x = \cos t,\quad v_2 = y = \sin t,\quad v_3 = z = t$

$$r(t) = \cos t\, i + \sin t\, j + t\, k$$

Def: Limit of vector function

let $r(t) = f(t)i + g(t)j + h(t)k$ be a vector function with domain D and $L = L_1 i + L_2 j + L_3 k$ a vector, then $r(t)$ has a limit L, as t approaches t_0 and write

$\boxed{\lim_{t \to t_0} r(t) = L}$ if $\forall \epsilon > 0, \ \exists \delta > 0$ s.t

$|r(t) - L| < \epsilon$ whenever $0 < |t - t_0| < \delta \quad \forall t \in D$

Ex: find a limit of $r(t) = (\cos t)i + (\sin t)j + tk$ where $t_0 = \pi/4$

Sol: $\lim_{t \to t_0} r(t) = (\lim_{t \to \pi/4} \cos t)i + (\lim_{t \to \pi/4} \sin t)j + (\lim_{t \to \pi/4} t)k$

$$= \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j + \frac{\pi}{4} k = L$$

Def: Continuous

A vector function $r(t)$ is Continuous at a Point $t = t_0$ in its domain if $\boxed{\lim_{t \to t_0} r(t) = r(t_0)}$.

The function is Continuous if it Continuous at every point in a domain.

Ex: The Vector function $r(t) = (\cos t)i + (\sin t)j + tK$ is Continuous at $t_0 = \pi/4$

Sol: ① $r(t_0) = (\cos \pi/4)i + (\sin \pi/4)j + (\pi/4)K$

$$= \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j + \frac{\pi}{4} K.$$

②

$$\lim_{t \to t_0} r(t) = (\lim_{t \to \pi/4} \cos t) i + (\lim_{t \to \pi/4} \sin t) j + (\lim_{t \to \pi/4} t) K$$

$$= \frac{\sqrt{2}}{2} i + \frac{\sqrt{2}}{2} j + \frac{\pi}{4} K$$

③ $\boxed{\therefore \lim_{t \to t_0} r(t) = r(t_0)}$

$\therefore$ $r(t)$ is Continuous at $t_0 = \pi/4$.

Def: derivative:

The vector function $r(t) = f(t)i + g(t)j + h(t)K$ has a derivative at t if $f, g,$ and h has derivatives at t.

The derivatives is the vector function.

$$r'(t) = \frac{dr}{dt} = \lim_{\Delta t \to 0} \frac{r(t+\Delta t) - r(t)}{\Delta t} = \boxed{\frac{df}{dt} i + \frac{dg}{dt} j + \frac{dh}{dt} K}$$

Ex: find $\frac{dr}{dt}$ of $r(t) = \underbrace{(t+1)}_{f(t)}i + \underbrace{(t^2-1)}_{g(t)}j + \underbrace{2t^2}_{h(t)}K$, $t=1$

Sol:

$$\frac{dr}{dt} = r'(t) = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}K$$

$$r'(t) = 1i + 2tj + 4tK$$

$$r'(t)\Big|_{t=1} = i + 2*1j + 4*1K$$

$$r'(t)\Big|_{t=1} = i + 2j + 4K$$

Ex: find $r'(t)$ of the vector function

$r(t) = (\cos 2t)i + (3\sin 2t)j$ at $t=0$

Sol: $r'(t) = \frac{df}{dt}i + \frac{dg}{dt}j$

$$= (-2\sin 2t)i + (6\cos 2t)j$$

$$r'(t)_{t=0} = (-2\sin 0)i + (6\cos 0)j = 6j$$

Ex: find the derivative of $r(t) = e^{2t}i + (\sin 3t)j + (\cos t)k$ at $t=0$

Sol $r'(t) = \frac{df}{dt}i + \frac{dg}{dt}j + \frac{dh}{dt}K$

$$= (2e^{2t})i + (3\cos 3t)j + (-\sin t)K$$

$$r'(t)\Big|_{t=0} = (2e^{0})i + (3\cos 0)j - (\sin 0)K = 2i + 3j$$

Def: If r is aposition vector of a particle moving along asmooth curve in space, then the Velocity vector is $v(t) = \frac{dr}{dt}$:

tangent to the curve. At any time t, the direction of v is the direction of motion, the magnitude of v is the particle's speed, and the derivative $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$

when it exist, is the Practicle's acceleration vector.

In Summary :

1- Velosity is the derivative of Position: $v = \frac{dr}{dt}$

2- speed is the magnitude of velocity: speed $= |v|$

3- Acceleration is the derivative of velocity $a = \frac{dv}{dt} = \frac{d^2r}{dt^2}$

4- The unit vector $\frac{v}{|v|}$ is the direction of motion at time t.

Ex: find the velocity, speed, and acceleration of a particle whose motion in space is given by the position vector $r(t) = 2\cos t\, i + 2\sin t\, j + 5\cos^2 t\, k$.

Sol

Velocity $\quad v = \frac{dr}{dt} = -2\sin t\, i + 2\cos t\, j - 10\cos t\sin t\, k$

$= -2\sin t\, i + 2\cos t\, j - 5(\sin 2t)\, k$

speed $|v| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-5\sin 2t)^2}$

$= \sqrt{4\sin^2 t + 4\cos^2 t + 25\sin^2 2t}$

$= \sqrt{4(\cos^2 t + \sin^2 t) + 25\sin^2 2t}$

$= \sqrt{4 + 25\sin^2 2t}$

acceleration:

$a = \frac{dv}{dt} = \frac{d^2r}{dt^2} = -2\cos t\, i - 2\sin t\, j - 10\cos 2t\, k$

When $t = \frac{7\pi}{4}$, we have:

$v = \sqrt{2}\, i + \sqrt{2}\, j - 5k, \quad a(7\pi/4) = -\sqrt{2}\, i - \sqrt{2}\, j$

$|v(\frac{7\pi}{4})| = \sqrt{29}.$

Differentiation Rules for Vector functions:

Let u and v be differentable vector functions of t, c is a constant vector, m any scalar, and f any differentable scalar function.

1. Constant function Rule: $\frac{d}{dt} C = 0$

2. Scalar Multiple Rule: $\frac{d}{dt}[m\, u(t)] = m\, u'(t)$

$$\frac{d}{dt}[f(t)\, u(t)] = f'(t)\, u(t) + f(t)\, u'(t)$$

3. Sum Rule: $\frac{d}{dt}[u(t) + v(t)] = u'(t) + v'(t)$

4 Difference Rule: $\frac{d}{dt}[u(t) - v(t)] = u'(t) - v'(t)$

5. Dot Product Rule: $\frac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$

6. Cross Product Rule: $\frac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$

7. chain Rule: $\frac{d}{dt}[u(f(t))] = f'(t)\, u'(f(t))$

Remark:

If r is a differentable vector function of t of constant Length, then $\quad r \cdot \frac{dr}{dt} = 0$

Integrals of Vector functions:

Def: The indefinite integral of r with respect to t is the set of all antiderivatives of r, denoted by $\int r(t)\,dt$. If R is any antiderivative of r, then

$$\int r(t)\,dt = R(t) + C.$$

where a differentiable vector function $R(t)$ is an antiderivative of a vector function $r(t)$ on an interval I, and C is constant vector.

Def: If the Component of $r(t) = f(t)i + g(t)j + h(t)k$ are integrable over $[a,b]$, then So is r, and the definite integral of r from a to b is

$$\int_a^b r(t)\,dt = \left(\int_a^b f(t)\,dt\right)i + \left(\int_a^b g(t)\,dt\right)j + \left(\int_a^b h(t)\,dt\right)k.$$

Ex: find the integral of a vector function $r(t) = \cos t\, i + j - 2t\, k$

Sol

$$\int r(t)\,dt = \int (\cos t\, i + j - 2t\, k)\,dt$$

$$= \left(\int \cos t\,dt\right)i + \left(\int dt\right)j + \left(\int -2t\,dt\right)k.$$

$$= (\sin t + C_1)i + (t + C_2)j - (t^2 + C_3)k$$

$= (\sin t) i + t j - t^2 K + C \ , \quad C = C_1 + C_2 - C_3$

Ex: find the integral of $r(t) = (\cos t) i + j - 2tK$ on the interval $[0, \pi]$.

Sol

$$\int_a^b r(t)\, dt = \int_0^{\pi} (\cos t\, dt) i + \left(\int_0^{\pi} dt\right) j - \left(\int_0^{\pi} 2t\, dt\right) K$$

$$= \left[\sin t\right]_0^{\pi} i + \left[t\right]_0^{\pi} j - \left[t^2\right]_0^{\pi} K$$

$$= \left[\sin \pi - \sin 0\right] i + \left[\pi - 0\right] j - \left[\pi^2 - 0^2\right] K$$

$$= [0 - 0] i + \pi j - \pi^2 K$$

$$= \pi j - \pi^2 K$$

Ex: find the vactor function $r(t)$ where the acceleration vector $a(t) = -(3\cos t) i - 3(\sin t) j + 2K$, and velocity $v(0) = 3j$ from the Point $(3, 0, 0)$.

Sol

$$a(t) = \frac{d^2 r}{dt^2} = -(3\cos t) i - 3(\sin t) j + 2K$$

$v(0) = 3j$ with $(3, 0, 0)$, i.e. $r(0) = 3i + 0j + 0K$

$$a(t) = \frac{d^2 r}{dt^2} = \frac{dv}{dt} = -(3\cos t) i - 3(\sin t) j + 2K$$

$$\int \frac{dv}{dt} dt = \left(\int -(3\cos t) dt\right) i - \left(3 \int \sin t\, dt\right) j + \int 2\, dt\, K$$

$$\int dv = v(t) = -3(\sin t) i + (3\cos t) j + 2tK + C_1$$

use $v(0) = 3j$ to find C_1 in the eq.

$v(t) = -(3\sin t)i + (3\cos t)j + 2tK + C_1$

$3j = (\underbrace{-3\sin 0}_{0})i + (3\underbrace{\cos 0}_{1})j + 0K + C_1$

$3j = 3j + C_1 \Rightarrow C_1 = 3j - 3j = 0$

$v(t) = (-3\sin t)i + (3\cos t)j + 2tK + 0$

$\therefore v(t) = \frac{dr}{dt} = (-3\sin t)i + (3\cos t)j + 2tK$

$\therefore \int \frac{dr}{dt}\, dt = \left(\int -3\sin t\, dt\right)i + \left(\int 3\cos t\, dt\right)j + \left(\int 2t\, dt\right)K$

$r(t) = (3\cos t)i + (3\sin t)j + t^2K + C_2$

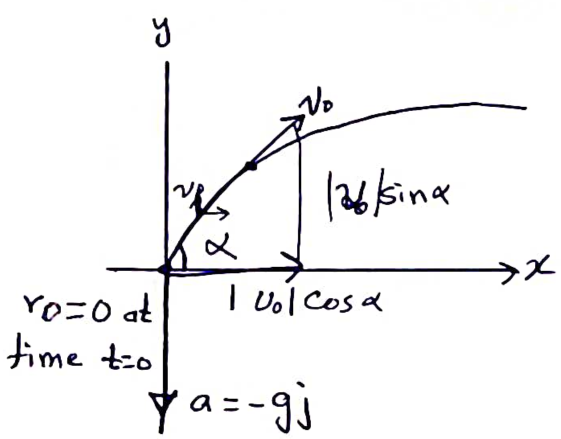
use initial Condition $r(0) = 3i$ in $r(t)$ to find C_2

$3i = (3\underbrace{\cos 0}_{1})i + (\underbrace{3\sin 0}_{0})j + 0K + C_2$

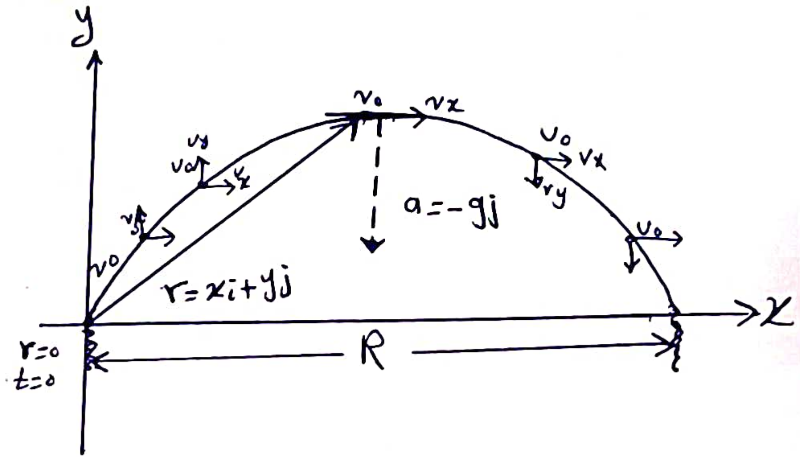
$3i = 3i + C_2 \Rightarrow C_2 = 0$

$\therefore$ vector function $r(t) = (3\cos t)i + (3\sin t)j + t^2K$

The Vector and Parametric Equations for Ideal Projectile Motion



a



b.

assume that the Projectile is launched from the origin at time $t=0$ into the first quadrant with the initial velocity v_0. If v_0 makes an angle α, then

$$v_0 = (|v_0|\sin\alpha)i + (|v_0|\cos\alpha)j$$

use v_0 for the initial speed $|v_0|$, then

$$v_0 = (v_0 \sin\alpha)\, i + (v_0 \cos\alpha)\, j \quad --- \; (*)$$

The Projectile's initial Position is

$$r_0 = 0 = 0i + 0j \quad (**)$$

use Newoton's law of motion the force acting on the Projectile is Projectile's mass (m) times its acceleration (d^2r/dt^2) i.e.

$$f = m * \left(\frac{d^2r}{dt^2}\right) \quad \cdots (1)$$

If r is Projectile's Position vector, and t is time.

If force is the gravitational force, i.e $f = -mgj$ ---②

subsitituting ② in ①, to get

$$m \frac{d^2 r}{dt^2} = -mgj \Rightarrow \boxed{\frac{d^2 r}{dt^2} = -gj} \quad ----(3)$$

where g is the acceleration due to gravity.

to find r as a function of t

$$\frac{d^2 r}{dt^2} = -gj \quad ---(3)$$ Differential Equation with

$r = r_0$ and $\frac{dr}{dt} = v_0$ (when $t=0$) initial Conditions

The first integration gives

$$\frac{dr}{dt} = -(gt)j + v_0$$

A second integration gives

$$r = \frac{-1}{2}(gt^2)j + v_0 t + r_0$$

subsitituting the value of v_0 and r_0 in eq. (*) and (**), to get

$$r = \frac{-1}{2} g t^2 j + \underbrace{(v_0 \cos\alpha)t i + (v_0 \sin\alpha) t j}_{v_0 t} + 0$$

$$\boxed{\therefore r = (v_0 \cos\alpha) t i + (v_0 \sin\alpha) t - \frac{1}{2} g t^2) j} \quad --- 4$$

this Ideal Projectile Motion Equation

Equation (4) is the Vector equation for ideal Projectile motion, the angle α is the Projectile's launch angle (firing angle, angle of elevation), v_0 is the Projectile initial speed.

The Component of r give the Parametric equation

$$\boxed{x = (v_0 \cos\alpha)t \quad \text{and} \quad y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2} \quad \text{---}(5)$$

where x is down range and y is hight of projectile.

Ex: A Projectile is fired from the origin over horizontal ground at an initial speed of 500 m/sec and a launch angle of 60°. where will the Projectile be 10 sec later where g=9.8?

Sol: $v_0 = 500$, $\alpha = 60°$, $t = 10$, $g = 9.8$

to find Projectile's Components use eq-(4).

$$r = (v_0 \cos\alpha)t\, i + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right) j$$

$$= (500 \cos 60)(10)\, i + \left((500 \sin 60)(10) - \frac{1}{2}(9.8)(10)^2\right) j$$

$$= \left(500 * \frac{1}{2} * 10\right) i + \left(500 * \frac{\sqrt{3}}{2} * 10 - \frac{1}{2} * 9.8 * 100\right) j$$

$$= 2500\, i + 3840\, j$$

$\therefore$ 2500 m down range of Projectile from origin

3840 m the Projectile above ground.

– Hight, Flight Time, and Range for Ideal Projectile Motion.

For ideal Projectile motion when an object is launched from the origin over a horizontal surface with initial speed v_0 and Launch angle α:

Maximum height: $y_{max} = \dfrac{(v_0 \sin\alpha)^2}{2g}$

Flight Time: $t = \dfrac{2 v_0 \sin\alpha}{g}$

Range: $R = \dfrac{v_0^2}{g} \sin 2\alpha$

Ex: A Projectile is fired at a speed of 840 m/sec at an angle of 60°. How long ~~to~~ will it take to get 21 km drown range? and find flight time

Sol $v_0 = 840$ m/see , $\alpha = 60$, down range $= 21$ km $= x$

$t = ?$, $g = 9.8$

downrange $x = (v_0 \cos\alpha) t$

$\because x = 21$ km $\Rightarrow x = 21000$ m

$x = (v_0 \cos\alpha) t \Rightarrow 21000 = 840 (\cos 60) t$

$21000 = \frac{840}{2} t \Rightarrow 21000 = 420 t \Rightarrow t = \frac{21000}{420} = 50$ sec

$t = \dfrac{2 v_0 . \sin\alpha}{g} = \dfrac{(2)(840 \sin 60)}{9.8} = 148$ sec

Arc Length in Space

Def:. The Length

The length of a smooth curve $r(t) = x(t)i + y(t)j + z(t)k$,

$a \leq t \leq b$, is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt .$$

Arc Length formula:

$$L = \int_a^b |v| \, dt$$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_{t_0}^{t} |v(\tau)| \, d\tau$$

Ex: A glider is soaring upward along the helix

$r(t) = (\cos t)i + (\sin t)j + tK$.

How Long is the glider's path from $t=0$ to $t=2\pi$?

Sol

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$r(t) = \underbrace{(\cos t)}_{x(t)}\, i + \underbrace{(\sin t)}_{y(t)}\, j + \underbrace{t}_{z(t)}\, K \quad , \quad a=0, \quad b=2\pi$$

$$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{\underbrace{\sin^2 t + \cos^2 t}_{1} + 1}\, dt$$

$$= \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big]_0^{2\pi} = 2\pi\sqrt{2}.$$

Ex: Find the arc Length Parametirization of a curve if $t_0=0$ and the helix $r(t) = (\cos t)i + (\sin t)j + tK$

Sol

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau$$

$$= \int_0^t \sqrt{(-\sin \tau)^2 + (\cos \tau)^2 + (1)^2}\, d\tau$$

$$S(t) = \int_0^t \sqrt{\underbrace{\sin^2\tau + \cos^2\tau}_{=1} + 1}\; d\tau$$

$$= \int_0^t \sqrt{2}\; d\tau = \sqrt{2}\,\tau\Big]_0^t = \sqrt{2}\, t$$

$$\therefore S = \sqrt{2}\, t \implies t = S/\sqrt{2}$$

Subsitituting $t = S/\sqrt{2}$ in the Position vector r gives the arc length Parametrization for the helix.

$$r(t) = (\cos t)\, i + (\sin t)\, j + t\, K$$

$$r(t(s)) = \left(\cos \frac{S}{\sqrt{2}}\right) i + \left(\sin \frac{S}{\sqrt{2}}\right) j + \frac{S}{\sqrt{2}}\, K.$$

Remark:

The unit tangent vector is $\boxed{T = \frac{v}{|v|}}$ where v is the velocity vector $v = \frac{dr}{dt}$ is tangent to the curve $r(t)$..

Ex: find the unit tangent vector of the curve

$$r(t) = (3\cos t)\, i + (3\sin t)\, j + t^2 K$$

Sol: $T = \frac{v}{|v|}$

$$v = dr/dt$$

$V = -(3\sin t)i + (3\cos t)j + 2t\,k$

$|V| = \sqrt{9\sin^2 t + 9\cos^2 t + 4t^2}$

$|v| = \sqrt{9(\underbrace{\sin^2 t + \cos^2 t}_{=1}) + 4t^2}$

$|v| = \sqrt{9 + 4t^2}$

$$T = \frac{V}{|V|} = \frac{-3\sin t}{\sqrt{9+4t^2}}\, i + \frac{3\cos t}{\sqrt{9+4t^2}}\, j + \frac{2t}{\sqrt{9+4t^2}}\, k$$

Curvature and Normal Vectors of a Curve.

Curvature of a plane curve

Def:

If T is the unitvector of a smooth curve, the curvature function of the curve is:

$$\kappa = \left|\frac{dT}{ds}\right|$$

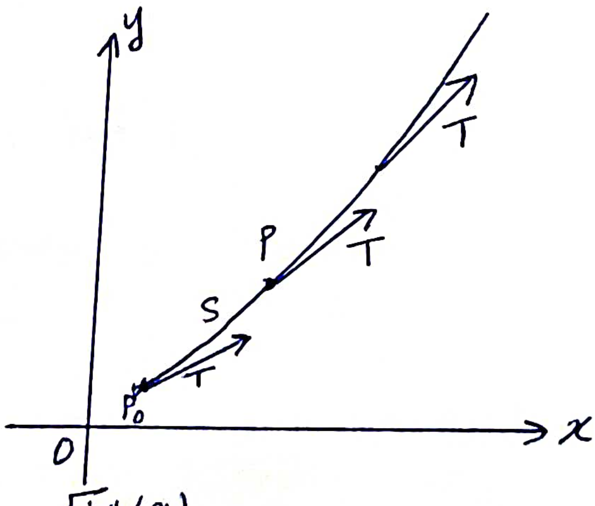


Fig (a)

As P moves along the curve in the direction of increasing arc value of $|dT/ds|$ at P is called the curvature

Formula for Calculating Curvature:

If $r(t)$ is a smoth curve, then the curvature is

$$\boxed{K = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|}$$

$K = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt} \cdot \frac{dt}{ds}\right|$ chain rule

$= \left|\frac{dt}{ds}\right| \left|\frac{dT}{dt}\right| = \left|\frac{1}{ds/dt}\right| \left|\frac{dT}{dt}\right|$

where $T = \frac{v}{|v|}$ is the unit tangent vector. $= \frac{1}{|v|} \cdot \frac{dT}{dt}$

Ex: find the curvature of a circle $r(t) = (a\cos t)i + (a\sin t)j$

Sol: $K = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$

$\therefore v = \frac{dr}{dt} = -(a\sin t)i + (a\cos t)j$

$|v| = \sqrt{(-a\sin t)^2 + (a\cos t)^2} = \sqrt{a^2(\underbrace{\sin^2 t + \cos^2 t}_{1})} = a$

$T = \frac{v}{|v|} = \frac{-\cancel{a}\sin t}{\cancel{a}} i + \frac{\cancel{a}\cos t}{\cancel{a}} j = (-\sin t)i + (\cos t)j$

$\frac{dT}{dt} = (-\cos t)i - (\sin t)j$

$\left|\frac{dT}{dt}\right| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\therefore K = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right| = \frac{1}{a} \cdot 1 = \frac{1}{a}$

Def:

At a point where $\kappa \neq 0$, the principal unit normal vector for a smooth curve in the plane is:

$$\boxed{N = \frac{1}{\kappa} \cdot \frac{dT}{ds}}$$

Formula for Calculating N

$$N = \frac{1}{\kappa} \cdot \frac{dT}{ds}$$

$$= \frac{1}{|dT/ds|} \cdot \frac{dT}{ds} = \frac{dT/ds}{|dT/ds|}$$

$$= \frac{\frac{dT}{dt} \cdot \frac{dt}{ds}}{\left|\frac{dT}{dt}\right| \cdot \left|\frac{dt}{ds}\right|} \qquad \text{chain Rule}$$

$$= \frac{dT/dt}{|dT/dt|} \qquad \left[\frac{dt}{ds} = \frac{1}{ds/dt} > 0 \text{ Cancels}\right]$$

$\therefore$ The formula for Calculating N where $r(t)$ is a smooth curve is

$$\boxed{N = \frac{dT/dt}{|dT/dt|}}$$

where $T = \frac{v}{|v|}$ is the unit tangent vector.

Ex: find T and N for the circular motion

$$r(t) = (\cos 2t)i + (\sin 2t)j$$

sol, ① find $T = \frac{v}{|v|}$

$$v = \frac{dr}{dt} = (-2\sin 2t)i + (2\cos 2t)j$$

$$|v| = \sqrt{4\sin^2 2t + 4\cos^2 2t} = \sqrt{4\underbrace{(\sin^2 2t + \cos^2 2t)}_{=1}}$$

$$= \sqrt{4} = 2$$

$$\therefore T = \frac{-2\sin 2t}{2}i + \frac{2\cos 2t}{2}j = -(\sin 2t)i + (\cos 2t)j$$

② find $N = \frac{dT/dt}{|dT/dt|}$

$$\therefore dT/dt = (-2\cos 2t)i - (2\sin 2t)j$$

$$|dT/dt| = \sqrt{4\cos^2 2t + 4\sin^2 2t} = \sqrt{4} = 2$$

$$\therefore N = \frac{-\cancel{2}\cos 2t}{\cancel{2}}i + \frac{-\cancel{2}\sin 2t}{\cancel{2}}j$$

$$= (-\cos 2t)i + (-2\sin 2t)j$$

Ex:

find the curvature of the Parabola equation $r(t) = t i + t^2 j$ at the origin

sol

$$K = \frac{1}{|v|} \cdot \left| \frac{dT}{dt} \right|$$

$$v = \frac{dr}{dt} = i + 2t j$$

$$|v| = \sqrt{(1)^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

$$T = \frac{v}{|v|} = \frac{1}{\sqrt{1+4t^2}} i + \frac{2t}{\sqrt{1+4t^2}} j$$

$$T = (1+4t^2)^{\frac{-1}{2}} i + 2t(1+4t^2)^{\frac{-1}{2}} j$$

$$\frac{dT}{dt} = \left(\frac{-1}{2}(1+4t^2)^{\frac{-3}{2}} \cdot 8t\right) i + \left(2t \cdot \frac{-1}{2}(1+4t^2)^{\frac{-3}{2}} \cdot 8t + (1+4t^2)^{\frac{-1}{2}} \cdot 2\right) j$$

$$\therefore \frac{dT}{dt} = -4t(1+4t^2)^{\frac{-3}{2}} i + \left[-8t^2(1+4t^2)^{\frac{-3}{2}} + 2(1+4t^2)^{\frac{-3}{2}}\right] j$$

The curvature at the origin i.e. at $t = 0$

$$K(0) = \frac{1}{|v(0)|} \cdot \left| \frac{dT}{dt}(0) \right|$$

$$|v(0)| = \sqrt{1 + 4(0)^2} = 1$$

$$\left| \frac{dT}{dt}(0) \right| = |0i + 2j| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$$

$$\therefore K(0) = \frac{1}{1} \cdot 2 = 2$$

Ex: find the curvature for the helix

$$r(t) = (a\cos t)i + (a\sin t)j + bt\,k, \quad a, b \geq 0, \quad a^2+b^2 \neq 0$$

Sol:

$$T = \frac{v}{|v|}$$

$$v = \frac{dr}{dt} = (-a\sin t)i + (a\cos t)j + b\,k$$

$$|v| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + (b)^2}$$

$$= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2(\sin^2 t + \cos^2 t) + b^2} = \sqrt{a^2+b^2}$$

$$T = \frac{1}{\sqrt{a^2+b^2}}\left[(-a\sin t)i + (a\cos t)j + bk\right]$$

$$K = \frac{1}{|v|}\cdot\left|\frac{dT}{dt}\right|$$

$$\frac{dT}{dt} = \frac{1}{\sqrt{a^2+b^2}}\left[(-a\cos t)i - (a\sin t)j\right]$$

$$\frac{dT}{dt} = \frac{a}{\sqrt{a^2+b^2}}\left[(-\cos t)i - (\sin t)j\right]$$

$$\left|\frac{dT}{dt}\right| = \left|\frac{a}{\sqrt{a^2+b^2}}(-\cos t)i - (\sin t)j\right|$$

$$= \frac{a}{\sqrt{a^2+b^2}}\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1}} = \frac{a}{\sqrt{a^2+b^2}}$$

$$\kappa = \frac{1}{\sqrt{a^2+b^2}} \cdot \frac{a}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}$$

H-W: find N for the helix
$r(t) = (a\cos t)i + (a\sin t)j + bt\,k \quad , a, b \geq 0$
and $a^2+b^2 \neq 0$

Tangential and Normal Components of Acceleration

Def:
If the acceleration vector is written as $\boxed{a = a_T T + a_N N}$

then $a_T = \dfrac{d^2 s}{dt^2} = \dfrac{d}{dt}|v|$ and

$$a_N = \kappa \left(\frac{ds}{dt}\right)^2 = \kappa |v|^2$$

are the tangential and normal scalar components of acceleration.

Remark:
The formula for Calculate the normal Component of Acceleration is

$$a_N = \sqrt{|a|^2 - a_T^2}$$

Ex: Find the acceleration of the motion

$r(t) = (\cos t + t\sin t)\, i + (\sin t - t\cos t)\, j, \; t>0$

Sol in the form $a = a_T . T + a_N N$.

$a = a_T T + a_N N$

1. find $a_T = \frac{d}{dt}|v|$

$v = \frac{dr}{dt} = (-\cancel{\sin t} + t\cos t + \cancel{\sin t})\, i + (\cancel{\cos t} + t\sin t - \cancel{\cos t})\, j$

$= (t\cos t)\, i + (t\sin t)\, j$

$|v| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = \sqrt{t^2} = |t| = t$

$\therefore a_T = \frac{d}{dt}|v| = \frac{d}{dt}(t) = 1$

2. find $a_N = k|v|^2$ or

$$\boxed{a_N = \sqrt{|a|^2 - a_T^2}}$$

$a = (\cos t + t\sin t)\, i + (\sin t - t\cos t)\, j$

$|a| = \sqrt{(\cos t + t\sin t)^2 + (\sin t - t\cos t)^2}$

$\therefore |a| = \sqrt{t^2+1}$

$|a|^2 = t^2 + 1$

$$a_N = \sqrt{|a|^2 - a_T^2}$$

$$= \sqrt{(t^2+1)-1} = \sqrt{t^2} = t$$

$$\therefore a = a_T T + a_N N = (1)T + (t)N = T + tN$$

Def:

Let $B = T \times N$. The Torsion function of the smooth curve is :

$$\tau = -\frac{dB}{ds} \cdot N$$, where B is Binomial vector s.t $B = T \times N$.

or

$$\tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$ (if $v \times a \neq 0$.)

Computation Formulas for Curves in Space

- Unit Tangent Vector: $T = \frac{v}{|v|}$

- Principal Unit normal Vector: $N = \frac{dT/dt}{|dT/dt|}$

- Binormal Vector: $B = T \times N$

- Curvature: $\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{|v|}\left|\frac{dT}{dt}\right| = \frac{|v \times a|}{|v|^3}$

- Torsion: $\tau = -\frac{dB}{ds} \cdot N$

$$= \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{|v \times a|^2}$$

- Tangential and normal scalar Components of acceleration:

$$a = a_T T + a_N N$$

$$a_T = \frac{d}{dt}|v|$$

$$a_N = \kappa |v|^2 = \sqrt{|a|^2 - a_T^2}$$

Ex: Find T, N, K, B and τ for the space curve

$$r(t) = (3\sin t)\,i + (3\cos t)\,j + 4t\,k$$

Sol

① $T = \dfrac{v}{|v|}$

$$v = \frac{dr}{dt} = (3\cos t)\,i - (3\sin t)\,j + 4\,k$$

$$|v| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (4)^2} = \sqrt{9\cos^2 t + 9\sin^2 t + 16}$$

$$= \sqrt{9(\sin^2 t + \cos^2 t) + 16} = \sqrt{9+16} = \sqrt{25} = 5.$$

$$T = \frac{(3\cos t)i - (3\sin t)j + 4\,k}{5} = \frac{3\cos t}{5}\,i - \frac{3\sin t}{5}\,j + \frac{4}{5}\,k$$

② $N = \dfrac{dT/dt}{|dT/dt|}$

$$dT/dt = \frac{-3\sin t}{5}\,i - \frac{3\cos t}{5}\,j + 0$$

$$|dT/dt| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t} = \frac{3}{5}$$

$$N = \frac{-\cancel{3/5}\,\sin t}{\cancel{3/5}}\,i - \frac{\cancel{3/5}\,\cos t}{\cancel{3/5}}\,j$$

$$N = (-\sin t)\,i - (\cos t)\,j$$

③ $k = \frac{1}{|v|} \cdot \left|\frac{dT}{dt}\right|$

$= \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$

④ $B = T \times N$

$$B = \begin{vmatrix} i & j & k \\ 3/5 \cos t & -3/5 \sin t & 4/5 \\ -\sin t & -\cos t & 0 \end{vmatrix}$$

$$= i \begin{vmatrix} -3/5 \sin t & 4/5 \\ -\cos t & 0 \end{vmatrix} - j \begin{vmatrix} 3/5 \cos t & 4/5 \\ -\sin t & 0 \end{vmatrix}$$

$$+ k \begin{vmatrix} 3/5 \cos t & -3/5 \sin t \\ -\sin t & -\cos t \end{vmatrix}$$

$$= (+4/5 \cos t)\, i - (4/5 \sin t)\, j + \left(\frac{-3}{5} \cos^2 t - \frac{3}{5} \sin^2 t\right) k$$

$$\therefore B = (4/5 \cos t)\, i - (4/5 \sin t)\, j - \frac{3}{5} k$$

$$\tau = \frac{\begin{vmatrix} \dot x & \dot y & \dot z \\ \ddot x & \ddot y & \ddot z \\ \dddot x & \dddot y & \dddot z \end{vmatrix}}{|v \times a|^2}$$

$r(t) = (3\sin t)i + (3\cos t)j + 4t\,k$

$x(t) \quad y(t) \quad z(t)$

$$\begin{vmatrix} \dot x & \dot y & \dot z \\ \ddot x & \ddot y & \ddot z \\ \dddot x & \dddot y & \dddot z \end{vmatrix} = \begin{vmatrix} 3\cos t & -3\sin t & 4 \\ -3\sin t & -3\cos t & 0 \\ -3\cos t & 3\sin t & 0 \end{vmatrix} = 4\begin{vmatrix} -3\sin t & -3\cos t \\ 3\cos t & 3\sin t \end{vmatrix}$$

$= 4(-9\sin^2 t - 9\cos^2 t) = -36(\sin^2 t + \cos^2 t) = -36$

$|v \times a|^2$

$a = a_T T + a_N N$

$a_T T = \frac{d}{dt}|v| = \frac{d}{dt}(5) = 0$

$a_N = k|v|^2 = \frac{3}{25} \cdot 25 = 3$

$a = 0 + 3 = 3$

$\therefore \tau = \frac{-36}{(3 \times 5)^2} = \frac{-36}{225} = \frac{-4}{25}$

H.W

1. Find the Practicle's velocity and acceleration vectors for the motion on the circl $r(t) = (\sin t)i + (\cos t)j$, where $t = \pi/4$ and $\pi/2$.

2. Find the Practicle's velocity and acceleration vector for

$r(t) = (4\cos\frac{t}{2})i + (4\sin\frac{t}{2})j$ where $t = \pi$ and $t = 3\pi/2$.

3- Find the velocity and speed and acceleration for

$r(t) = (t+1)i + (t^2-1)j + 2t\,k$, where $t = 1$.

4. find the Continuity and Limit of

$r(t) = (\sin t)i + (t^2 - \cos t)j + e^t k$, where $t_0 = 0$

5- Find the derivative of $r(t) = (\ln t)i + (\frac{t-1}{t+2})j + (t\ln t)k$

6- Evaluate the integral

$$\int_0^1 [t^3 i + 7j + (t+1)k]\,dt$$

7. Evaluate the integral of

$r(t) = (\sin t)i + (1+\cos t)j + (\sec^2 t)k$ where

$-\pi/4 \leq t \leq \pi/4$.

8. Solve the initial value for r as a vector function of t

differential function: $\frac{dr}{dt} = -t\,i - t\,j - t\,k$

initial Condition: $r(0) = i + 2j + 3k$.

9. Aspring gun at ground level fires a golf ball at an angle 45°. the ball lands 10 m away

a. what wass ball initial speed?

b. for the Same initial speed find the firing angle that make the range 6m? Find flight time and Maximum height.

10. find the curve's unit tangent vector. Also, find the arc length of the curve

$r(t) = (t\cos t)i + (t\sin t)j + ((2\sqrt{2}/3)\, t^{3/2})K, \quad 0 \le t \le \pi$

11. Find the arc length Parametrization of a curve at $t_0 = 0$

For the curve $r(t) = (\cos 4t)i + (\sin 4t)j + 4tK$

12. find the T and N and k for the curves

a. $r(t) = (\ln \sec t)i + tj, \quad -\pi/2 < t < \pi/2$

b. $r(t) = ti + (\ln \cos t)j, \quad -\pi/2 < t < \pi/2$

c. $r(t) = (e^t \cos t)i + (e^t \sin t)j + 2k$

13. write acceleration in the form $a = a_T T + a_N N$ with out finding T and N $r(t) = (1+3t)i + (t-2)j - 3tK$

14. Find τ, T, N, and B at the given value of t

where $r(t) = (\cos t)i + (\sin t)j - k, \quad t = \pi/4$