University of Baghdad

College of Sciences for Women

Mathematics Department

Third Class

Semester two

Module theory

By

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In this semester, we shall study the following four chapters:

Chapter one: Definitions and Preliminaries.

Chapter two: Modules homomorphism

Chapter three: Sequences

Chapter four: Noetherian and Artinian modules

Chapter one (Definitions and Preliminaries.)

Definition. ***(Modules)***

Let R be a ring. A (left) R-module is an additive abelian group M together with a function f : R x M → M defined by: f(r,a)=ra such that for all r,s ∈R and a,b ∈ M :

1. r(a + b) = ra + r b.
   }(distributive laws)
2. (r + s)a = ra + sa.
3. r(sa) = (rs)a. (associative law)

If R has an identity element 1_R and

4. $1_R a$ = a for all a ∈ M,

then M is said to be a ***unitary*** left R-module.

Remarks.

1. A (unitary) right R-module is defined similarly by a function f:MxR→M denoted by (a,r) → ar and satisfying the obvious analogues of (1)-(4).
2. If R is commutative, then every left R-module M can be given the structure of a right R-module by defining ar = ra for r ∈ R, a ∈ M.

3. Every module M over a commutative ring R is assumed to be both a left and a right module with ar = ra for all $r \in R$, $a \in M$.
4. We shall refer to left R-module by R- module. Also, in this course, all R-modules are unitary.

Remarks.

1. If 0_M is the additive identity element of M and 0_R is the additive identity element of a ring R (where M is an R-module), then for all $r \in R$, $a \in M$: $r\, 0_M = 0_M$ and $0_R.\, a = 0_M$.
2. $(-r)a = -(ra) = r(-a)$ and $n(ra) = r(na)$ for all $r \in R$, $a \in M$ and $n \in \mathbb{Z}$(ring of integers).

Examples.

1. Every commutative ring is an R-module.

   Proof. Define $f: R \times R \to R$ by $f(r_1, r_2) = r_1 r_2$ for all $r_1, r_2 \in R$.then
   a. $(r_1+r_2)r = r_1 r + r_2 r$
   b. $r(r_1+ r_2) = rr_1+ rr_2$
   c. $(r_1 r_2)r = r_1(r_2 r)$
2. Every additive abelian group G is a unitary $\mathbb{Z}$ -module.

   Proof. Define $\alpha: \mathbb{Z} \times G \to G$ by: $\alpha(n, m) = nm$ for all $n \in \mathbb{Z}$ and $m \in G$.

   i.e $\alpha(n, m) = \underbrace{m + m + \cdots + m}_{n-times} = nm$

   since G is group and $m \in G$, then there is $-m \in G$ such that

   $(-nm) = -\underbrace{m - m - \cdots - m}_{n-times}$

   Now,

   i. $(n_1+n_2)m = n_1 m + n_2 m$

   ii. $n(m_1+ m_2) = \underbrace{(m_1 + m_2) + (m_1 + m_2) + \cdots + (m_1 + m_2)}_{n-times}$

   $= nm_2 + nm_2$

   iii. $(n_1\, n_2)m = n_1(n_2 m)$

   also, since $\mathbb{Z}$ has identity element, then

   iv. $1.\, m = m$

3. Every ideal in a ring R is an R- module
4. Every vector space V over a field F is F-module.
5. If Q is the set of rational numbers, then Q is ℤ-module.
   Proof. Define β: ℤ x Q → Q by:
   $\beta(m, \frac{n}{t}) = m\frac{n}{t} = \frac{mn}{t}$ for all m∈ ℤ and $\frac{n}{t}$ ∈ Q.
6. If $\mathbb{Z}_n$ is the group of integers modulo n, then $\mathbb{Z}_n$ is ℤ-module.
   Proof. define α: ℤ x $\mathbb{Z}_n$ → $\mathbb{Z}_n$ by: $\alpha(n, \bar{a}) = n\bar{a}$ for all n∈ ℤ, $\bar{a} \in \mathbb{Z}_n$.
7. Let A be an abelian group and
   S= end_R(A) = Hom_R(A, A)= {f: A→ A; f is a group homomorphism}
   Define " + " on S by: for all f, g ∈ S and a ∈ A,
   (f+g)(a) = f(a) + g(a)
   Then
   1. (S, +) is an abelian group:
      i. S is closed under "+"
      ii. 0(a) = 0 (zero function 0 : A→ A)
      iii. (-f(a)) = -(f(a)) (additive inverse)

      (f+ (-f)(a) = f(a) + -(f(a)) = 0

      iv. "+" is an associative operation

      iv."+" is an abelian:

      (f+g)(a) = f(a) + g(a) = g(a) + f(a) = (g+f)(a)

      (S, +) is an abelian group

   2. Define " . " on S by: for all f, g ∈ S and a ∈ A,
      f.g ≡ fog and (fog)(a) = (f(g(a))
   (S, +, .) is a ring with identity I: A→ A (where foI = Iof = f)
   3. Now, one can consider A as a unitary S-module:
   with α : S x A → A, α(f, a) = f(a) f ∈ S and a ∈ A

8. If R is a ring, every abelian group can be consider as an R-module with trivial module structure by defining ra =0 for all r ∈ R and a ∈ A.
9. ***The R-module $M_{n.}(R)$***. let

$$M_{n.}(R) = \text{the set of nxn matrices over R}$$
$$=\{(a_{ij})_{nxn} \mid a \in R\}$$

$M_n(R)$ is an additive abelian group under matrix addition. If $(a_{ij})\in M_{n.}(R)$ and $a\in R$, then the operation $a.(a_{ij}) = (a.a_{ij})$ makes $M_{n.}(R)$ into an R-module. $M_{n.}(R)$is also a left R-module under the operation $a.(a_{ij})= (a.a_{ij})$.

10. **The Module R[X]**. If R[X] is the set of all polynomials in X with their coefficients in R,

i.e $R[X] = \{(a_0 ,a_1 ,\ldots ,a_n)\mid a_i\in R, i = 1,2,\ldots,n,\}$

then (R[X], +) is an additive abelian group under polynomial addition on R[X] is an R-module via the function R x R[X]→ R[X] defined by : $a.(a_0 + x.a_1 +\ldots + x^n.a_n) = (a.a_0) +(a.a_1).x + \ldots + (a.a_n).x^n$

Definition. Let R be a ring, A an R-module and B a nonempty subset of A. B is a ***submodule*** of A provided that B is an additive subgroup of A and rb ∈ B for all r ∈ R and b ∈ B.

Remark. Let R be a ring, A an R-module and B a nonempty subset of A. B is a submodule iff:

1. for all a, b ∈ B, a+b ∈ B

2. for all r ∈ R and a ∈ B, ra ∈B.

Another characterization for a submodule concept

Remark. A nonempty subset B of an R-module A a submodule iff: ax + by ∈ B, for all a, b ∈ R and x, y ∈ B.

Examples.

1. let M an R-module and x ∈M, the set

$R_x = \{rx|\ r \in R\}$ is a submodule of M such that

a. $r_1x - r_2x = r_1x + (-r_2)x \in R_x$.

b. $r_1(r_2x) = (r_1r_2)x$

2. let R be a commutative ring with identity and S be a set. Consider the set

$$X = R^s = \{f : S \rightarrow R;\ f \text{ is a function}\}.$$

The two operation "+" and "." on X denoted by

$(f+g)(s) = f(s) + g(s)$ and $(f.g)(s) = f(s) . g(s)$ for $s \in S$ and $f, g \in X$

Then (X, +) is an abelian group (H.W).

The function $\alpha : R \times X \rightarrow X$ denoted by $\alpha(r, f) = rf$ since $(rf)(s)= r(f(s))$ for all $s \in S$, $r \in R$ and $f \in X$, then X is an R-module(H. W)

And $Y = \{f : \in X : f(s) = 0$ for all but at most a finite number of $s \in S\}$, the Y is a submodule of an R-module X. (H.W)

3. ***Finite Sums of Submodules***. If $M_1, M_2, \ldots, M_n$ are submodules of an R-module M, then $M_1 + M_2 + \ldots + M_n = \{x_1 + x_2 + \ldots + x_n|\ x_i \in M_i$ for $i=1,2,\ldots,n\}$ is a submodule of M for each integer $n \geq 1$.
4. If one take n=2 in (3) then

   $$N+K=\{x+y\ |x \in N,\ y \in K\}$$

   is a submodule of M for each submodule N and K of M

   Proof. let w_1 , $w_2 \in N+K$. Then

   i. $w_1 = x_1 + y_1$ and $w_2 = x_2 + y_2$ for $x_1, x_2 \in N$ and $y_1, y_2 \in K$. Now, $w_1 + w_2 = (x_1 + y_1) + (x_2 + y_2) = (x_1 + x_2) + (y_1 + y_2) \in N+K$.

   ii. let $w = x + y \in N+K$, $r \in R$. so, $rw = r(x+y) = rx + ry \in N+K$.
5. let N_α; $\alpha \in I$(I is the index set), be a family of submodules of an R-module M, then $\bigcap_{\alpha \in I} N_\alpha$ is also a submodule of M.

   Proof. H.W.

6. let N be a submodule of an R-module M and $\frac{M}{N}$ = {m+N| m∈M}. clearly that ($\frac{M}{N}$,+) is an abelian group where for each m, m_1, m_2∈ M, r ∈ R:
   i. (m_1+N) + (m_2+N) = (m_1+ m_2) +N
   ii. and r.(m_2+N) = (r. m_2)+ N.

   then $\frac{M}{N}$ is an R-module, which is called the ***quotient module*** of M by N.

Remark. ***(Modular Law)*.**

There is one property of modules that is often useful. It is known as the modular law or as the modularity property of modules. If N , L and K are modules, then N∩(L+K) = (N∩L)+(N∩K).

If N , L and K are submodules of an R-module M and L ≤ N, then N∩(L+K) = L + (N∩K).

Definition. Let M be an R-module. If there exists x_1, x_2, …,x_n ∈ M such that M= Rx_1+ Rx_2+ …+Rx_n. M is said to be ***finitely generated*** module. If M = Rx=<x> = {rx | r ∈ R} is said to be ***cyclic*** module.

Examples.

1. $\mathbb{Z}_n = \langle \bar{1} \rangle$ is cyclic ℤ-module for all n ∈ ℤ.
2. nℤ = $\langle n \rangle$ is cyclic ℤ-module for all n ∈ ℤ.
3. If F is any field, then the ring F[x,y] has the submodule(ideal) <x,y> which is not cyclic.
4. Q is not finitely generated ℤ-module.

Direct sums and products

Definition. Let R be a ring and {M_i| i ∈I } be an arbitrary (possibly infinite) of a nonempty family of R-modules. $\prod_{i\in I} M_i$ is the ***direct product*** of the abelian groups M_i, and $\bigoplus_{i\in I} M_i$ the ***direct sum*** of the of the abelian groups M_i, where

$$\prod_{i\in I} M_i = \{f: I \to \bigcup_{i\in I} M_i \mid f(i) \in M_i, \text{ for all } i\in I\}$$

Define a binary operation "+" on the direct product (of modules) $\prod_{i\in I} M_i$ as follows: for each f,g $\in \prod_{i\in I} M_i$ (that is, f,g : I $\to \bigcup_{i\in I} M_i$ and f(i),g(i) $\in M_i$ for each i), then f+g : I $\to \bigcup_{i\in I} M_i$ is the function given by i $\to$ f(i)+g(i).

i.e $\quad$ (f+g)(i) = f(i)+g(i) $\quad$ for each i $\in$I.

Since each M_i is a module, f(i)+g(i) $\in M_i$ for every i, whence f+g $\in \prod_{i\in I} M_i$. So ($\prod_{i\in I} M_i$, +) is an abelian group

Now, if r $\in$ R and f $\in \prod_{i\in I} M_i$, then rf : I $\to \bigcup_{i\in I} M_i$ as (rf)(i) = r(f(i)).

1. $\prod_{i\in I} M_i$ is an ***R-module*** with the action of R given by r(f(i)) = (rf(i)) (i.e define α: R $x \prod_{i\in I} M_i \to \prod_{i\in I} M_i$ by α(r,f) = rf)
2. $\bigoplus_{i\in I} M_i$ is a ***submodule*** of $\prod_{i\in I} M_i$. (H.W.)

Remark. $\prod_{i\in I} M_i$ is called the (external) direct product of the family of R-modules $\{M_i \mid i \in I\}$ and $\bigoplus_{i\in I} M_i$ is (external) direct sum. If the index set is finite, say i = { 1 ,2, … , n}, then the direct product and direct sum coincide and will be written $M_1 \oplus M_2 \oplus \ldots \oplus M_n$.

Definition. ((internal) direct sum) Let R be a ring and N, K submodules of an R-module M such that:

1. M = N + K
2. N ∩ K = 0

Then N and K is said to be ***direct summand*** of M and M = N⊕ K ***internal direct sum*** of N and K.

Definition. Let R be an integral domain. An element x of an R-module M (x∈ M) is said to be ***torsion*** element of M if ∃(0≠) r ∈ R with rx = 0.

Example.

1. Let M = $\mathbb{Z}_6$ as $\mathbb{Z}$-module. Then every element in $\mathbb{Z}_6$ is torsion:

$\bar{3} \in \mathbb{Z}_6$, $\exists\ 2 \in \mathbb{Z}$ such that $2.\ \bar{3} = \bar{0}$
$\bar{2} \in \mathbb{Z}_6$, $\exists\ 3 \in \mathbb{Z}$ such that $3.\ \bar{2} = \bar{0}$
$\bar{1} \in \mathbb{Z}_6$, $\exists\ 6 \in \mathbb{Z}$ such that $6.\ \bar{1} = \bar{0}$
$\bar{4} \in \mathbb{Z}_6$, $\exists\ 3 \in \mathbb{Z}$ such that $3.\ \bar{4} = \bar{0}$
$\bar{5} \in \mathbb{Z}_6$, $\exists\ 6 \in \mathbb{Z}$ such that $6.\ \bar{5} = \bar{0}$

2. Every element in $\mathbb{Z}_n$ as $\mathbb{Z}$-module is torsion.
3. The only torsion element in M = Q as $\mathbb{Z}$-module is zero (if $(0\neq)\ x \in Q$, then $\nexists\ (0\neq)\ r \in \mathbb{Z}$ such that $rx = 0$.

Remark. Let M be an R-module where R is an integral domain, then the set of all torsion elements of M, denoted by $\tau(M)$ is a submodule of M

$(\tau(M) = \{x \in M \mid \exists (0\neq)\ r \in R \text{ such that } rx = 0\})$

Proof. 1. $\tau(M) \neq \varphi$ $(0 \in \tau(M))$

2. if $x, y \in \tau(M)$, then $\exists (0\neq)\ r_1, r_2 \in R$ such that $r_1x = 0$ and $r_2y = 0$. Since R is an integral domain, $r_1 \neq 0$ and $r_2 \neq 0$, so $r_1.\ r_2 \neq 0$. Hence

$r_1.r_2(x+y) = r_1.r_2\ x + r_1.r_2y = r_2.r_1\ x + r_1.r_2y = 0 + 0 = 0$. Thus $x+y \in \tau(M)$

3. let $(0\neq)\ r \in R$ $w \in \tau(M)$, $\exists (0\neq)\ r_1 \in R$ with $r_1w = 0$. Now, $r_1(rw) = 0$ implies $rw \in \tau(M)$.

$\therefore\ \tau(M)$ is a submodule of M.

Remark. In general, If R is not integral domain, then $\tau(M)$ may not submodule of M in general.

Definition. Let M be a module over integral domain R. If $\tau(M) = 0$, Then M is said to be ***torsion free*** module. If $\tau(M) = M$, then M is said to be ***torsion*** module.

Examples. 1. The $\mathbb{Z}$-module Q, is torsion free module.

2. The $\mathbb{Z}$-module $\mathbb{Z}_n$, is torsion module.

Remark. Let M be a module over an integral domain R, then $\frac{M}{\tau(\text{M})}$is torsion free R-module. (i.e $\tau(\frac{M}{\tau(\text{M})}) = \tau(\text{M}))$

Proof. Let m+ τ(M) ∈ $\tau(\frac{M}{\tau(\text{M})})$, ∃(0≠) r ∈ R such that r(m + τ(M)) = τ(M). → rm + τ(M) = τ(M) → rm ∈ τ(M)

→ ∃(0≠) s ∈ R such that s(rm) = (sr)m = 0

∵ sr ≠ 0 → m ∈ τ(M) → m + τ(M) = τ(M) → $\tau(\frac{M}{\tau(\text{M})}) = \tau(\text{M})$.

Exercises.

1. Every submodule of torsion module over integral domain is torsion module.
2. Every submodule of torsion free module over integral domain is torsion free module.

Definition. Let M be a module over an integral domain R. An element x∈M is said to be ***divisible*** element if for each (0≠) r ∈ R ∃y ∈ M such that ry = x.

Examples.

1. 0 is divisible element in every module M.
2. Every element in a ℤ-module Q is divisible element.
3. 0 is the only divisible element in 2ℤ as ℤ-module.

Remark. Let M be a module over an integral domain R. the set of all divisible element of M denoted by ∂(M) = {m ∈ M| ∀ (0≠) r ∈ R, ∃ y ∈ M such that m = ry}

Definition. Let M be a module over an integral domain R. M is said to be ***divisible*** module if ∂(M) = M.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is not divisible.
2. The module Q over the ring $\mathbb{Z}$ is divisible.
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not divisible.

Proposition. Let R be an integral domain and M be an R-module. Then:

1. ∂(M) is a submodule of M.
2. If M is divisible module, then so is $\frac{M}{N}$ for all submodule N of M.
3. M is divisible module iff M = rM for all $0 \neq r \in R$.
4. If $M = M_1 \oplus M_2$, then $\partial(M) = \partial(M_1) \oplus \partial(M_2)$.

Proof. 1. Let $x, y \in \partial(M)$, then

$\forall\ 0 \neq r \in R,\ \exists\ x_1 \in M$ such that $x = rx_1$

$\forall\ 0 \neq r \in R,\ \exists\ y_1 \in M$ such that $y = ry_1$

i) $x + y = r(x_1 + y_1)$, for all $0 \neq r \in R$. implies $x+y \in \partial(M)$.

ii) let $x \in \partial(M)$ and $0 \neq s \in R$, then $\forall\ 0 \neq r \in R$, $\exists\ y \in M$ such that $x = ry$. Since R is an integral domain, $r \neq 0$ and $s \neq 0$, then $rs \neq 0$.

So $sx = s(ry) = (sr)y$. implies that $sx \in \partial(M)$.

$\therefore\ \partial(M)$ is a submodule of M

2. Let $x + N \in \frac{M}{N}$ where $x \in M$. Since M is divisible and $x \in M$, then for $\forall\ 0 \neq r \in R$, $\exists\ y \in M$ such that $x + N = ry + N = r(y+N)$.

$\therefore\ \frac{M}{N}$ is divisible module

3. →)Suppose that M is divisible module. To prove M = Rm, must prove that: a. $M \leq rM$ b. $rM \leq M$

for that :

a. Let $m \in M$. Since $M = \partial(M)$ (M is divisible), so $m \in \partial(M)$.

For all $0 \neq r \in R$, $\exists\, n \in M$ such that $m = rn \in rM$. Hence $M \leq rM$.

b. Since M is a module then $rM \leq M$.

$$\therefore M = rM$$

$\leftarrow$) Suppose that $M = rM$ for all $0 \neq r \in R$. if $m \in M = rM$, then $m = rn$ for $n \in M$ and all $0 \neq r \in R$. implies that $m \in \partial(M)$. Thus $M \leq \partial(M)$.

let $x \in \partial(M)$, $\forall\, 0 \neq r \in R$, $\exists\, y \in M$ such that $x = ry$. Thus $\partial(M) \leq M$. Hence $M = \partial(M)$. So M is divisible module.

Remark. Point (2) in the previous proposition means: the quotient of divisible module is divisible.

Exercise. Is every submodule of divisible module divisible?

Definition. Let M be an R-module and $x \in M$. Then the set

$$\mathbf{ann_R(x)} = \{r \in R \mid rx = 0\}$$

is said to be **annihilator of the element x in R**.

Remarks.

1. Let M be an R-module. Then the set

$$\mathbf{ann_R(M)} = \{r \in R \mid rM = 0\}$$

$$= \{ r \in R \mid rm = 0 \text{ for all } m \in M\}$$

   is said to be **annihilator of the module M in R**.

2. Let M be an R-module. If $ann_R(M) = 0$, then M is said to be **faithful** module.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}$ is faithful ($ann_{\mathbb{Z}}(\mathbb{Z}) = 0$)
2. The $\mathbb{Z}$-module Q is faithful ($ann_{\mathbb{Z}}(Q) = 0$)
3. The $\mathbb{Z}$-module $\mathbb{Z}_n$ is not faithful ($ann_{\mathbb{Z}}(\mathbb{Z}_6) = 6\,\mathbb{Z}$)

4. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{3}\}) = \{\bar{0}, \bar{2}, \bar{4}\}$
5. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{3}\}) = 2\mathbb{Z}$
6. $ann_{\mathbb{Z}}(\{\bar{0}, \bar{2}, \bar{4}\}) = 3\mathbb{Z}$
7. $ann_{\mathbb{Z}_6}(\{\bar{0}, \bar{2}, \bar{4}\}) = \{\bar{0}, \bar{3}\}$
8. $ann_{\mathbb{Z}}(\mathbb{Z}_n) = n\mathbb{Z}$

Definition. Let N and K be submodules of an R-module M. The set

$$(N: K) = \{r \in R \mid rK \leq N\}$$

is an ideal of R which is called residual.

Remark.

1. If N = 0, then
$$(0: K) = \{r \in R \mid rK = 0\} = ann_R(K)$$
2. If N = 0 and K = M, then
$$(0: M) = \{r \in R \mid rM = 0\} = ann_R(M)$$

Chapter two (Module homomorphisms)

Definition. Let M and N be modules over a ring R . A function f : M → N is an ***R-module homomorphism*** (simply homomorphism) provided that for all x, y ∈ M and r ∈ R :

1. f(x+y) = f(x) + f(y)
2. f(rx) = rf(x).

If R is a field, then an R-module homomorphism is called a ***linear transformation***.

Remarks.

1. if f is injective and homomorphism, then is said to be monomorphism.
2. if f is surjective and homomorphism, then is said to be epimorphism.

3. if f is injective, surjective and homomorphism, then is said to be isomorphism (and written $M \approx N$) .

Examples.

1. $2\,\mathbb{Z}_{\mathbb{Z}} \approx 3\,\mathbb{Z}_{\mathbb{Z}}$.

Proof. Define $g: 2\mathbb{Z} \to 3\mathbb{Z}$ as $g(2n) = 3n$ for all $n \in \mathbb{Z}$.

i. g is well-define.

ii. g is homomorphism : for $2n, 2n_1, 2n_2 \in 2\mathbb{Z}$, $r \in \mathbb{Z}$

$g(2n_1+ 2n_2) = g(2(n_1+ n_2)) = 3(n_1+n_2) = 3n_1+ 3n_2 = g(2n_1)+g(2n_2)$

$g(r(2n)) = g(2rn) = 3rn = r(3n) = rg(2n)$

iii. g is one – to – one. If $g(2n_1) = g(2n_2)$, then

$\to\ 3n_1 = 3n_2 \ \to\ n_1 = n_2 \ \to\ 2n_1 = 2n_2$.

iv. g is onto. for all $y = 3n \in 3\mathbb{Z}$, there is $x = 2n \in 2\mathbb{Z}$ such that $g(2n) = 3n$.

Hence $2\mathbb{Z} \approx 3\mathbb{Z}$(i.e g is an isomorphism).

2. Let R be a ring and $\{ M_i \mid i \in I \}$ a family of submodules of an R-module M such that:

i. M is the sum of the family $\{ M_i \mid i \in I \}$

ii. for each $k \in I$, $M_k \cap \sum_{i\in I, i \neq k} M_i = 0$

Then $M \approx \bigoplus_{i\in I} M_i$

(Hint : define $\beta: \bigoplus_{i\in I} M_i \to M$ by $\beta(f) = \sum_{i\in I} f(i)$)

3. Let $\{ M_i \mid i \in I \}$be family of R-modules.

i. For each $k \in I$, the canonical projection $\rho_k: \prod_{i\in I} M_i \to M_k$ defined by $\rho_k(f) = f(k)$ is an R- module epimorphism .

ii. For each $k \in I$, the canonical injection $J_k: M_k \to \prod_{i\in I} M_i$ defined by for $x \in M_k$, $(J_k(x))i = \begin{cases} x & if\ i = k \\ 0 & otherwise (i \neq k) \end{cases}$ is an R-module monomorphism.

iii. $\rho_k \circ J_k = I_{M_k}$.

Proof. $\rho_k o J_k : M_k \to M_k$ with $(\rho_k o J_k)(x) = \rho_k (J_k(x)) = J_k(x)(k) = x$

iv. J_k o $\rho_k \neq I_{M_k}$.

4. Let K be a submodule of a module M. the function $\pi: M \to \frac{M}{K}$ defined by $\pi(x) = x+K$ for all $x \in M$, is an R-homomorphism and onto. This homomorphism is called the natural epimorphism.

Exercises. Prove :

1. If R is a ring, the map $R[x] \to R[x]$ given by $f \to f(x)$(for example, $(x^2 + 1) \to x(x^2 + 1)$) is an R-module homomorphism, **but not** a ring homomorphism (prove that).
2. Hom(R, M) ≈ M
3. for each $n \in \mathbb{Z}$, $\frac{\mathbb{Z}}{n\mathbb{Z}} \approx \mathbb{Z}_n$.

Theorem. Let $f : M \to N$ be a homomorpism, then

1. ***kernel of f*** (kerf = {x ∈M| f(x) = 0}) is a submodule of M.
2. ***Image of f*** (Imf={n ∈N| n = f(m) for some m ∈ M}) is a submodule of N.
3. f is a monomorpism iff kerf = 0.
4. $f : M \to N$ is an R-module isomorphism if and only if there is A homomorphism $g : N \to M$ such that $gf = I_M$ and $fg = I_N$.

Proof. H.W.

Proposition. Let R be an integral domain and M be an R-module, then:

1. If $f : M \to \acute{M}$ be a module homomorphism, then $f(\tau(M)) \leq \tau(\acute{M})$.
2. If $M = M_1 \oplus M_2$, then $\tau(M) = \tau(M_1) \oplus \tau(M_2)$.

Definition. An R-module, M is called ***simple*** if M ≠ {0} and the only submodules of M are M and {0}

Proposition. Every simple module M is cyclic (i.e M = Rm for every nonzero m ∈ M).

Proof. Let M be a simple R-module and m ∈ M. Both Rm and

B = { c ∈ M| Rc = 0 } are submodules of M. Since M is simple, then each of them is either 0 or M. But RM ≠ 0 implies B ≠ M. Consequently B = 0, whence Ra = M for all nonzero m ∈ M. Therefore M is cyclic

Remark. The converse is not true in general: that is a cyclic module need not be simple for example, the cyclic Z-module Z_6.

Examples.

1. The $\mathbb{Z}$-module $\mathbb{Z}_3$ is simple.
2. The $\mathbb{Z}$-module $\mathbb{Z}_p$is simple for each prime integer's p.
3. The $\mathbb{Z}$-module $\mathbb{Z}_4$ is not simple, since the submodule$\{\bar{0}, \bar{2}\} \neq 0$ and $\{\bar{0}, \bar{2}\} \neq \mathbb{Z}_4$.
4. The $\mathbb{Z}$-module $\mathbb{Z}$ is not simple.(why?)
5. Every division ring D is a simple ring and a simple D-module

Lemma. (**Schur's lemma**)

1. Every R-homomorphism from a simple R-module is either zero or monomorphism.
2. Every R-homomorphism into a simple R-module is either zero or epimorphism.
3. Every R-homomorphism from a simple R-module into simple R-module is either zero or isomorphism.

Proof 1. Let M be a simple module and f: M→ N be an R-module homomorphism. Then kerf is a submodule of M . But M is simple.

So either kerf = {0} , implies f is one-to-one

or kerf = M, implies f is zero homomorphism.

Proof 2. Let N be a simple module and f: M→ N be an R-module homomorphism. Then Imf is a submodule of N . But N is simple.

So either Imf = {0} , implies f zero homomorphism or Imf = N, implies f is onto.

Proof 3. as a consequence to (1) and (2), the proof of (3) holds.

Examples. 1. An R-module homomorphism f: $\mathbb{Z}_4 \rightarrow \mathbb{Z}_5$ is zero.

2. An R-module homomorphism f: $\mathbb{Z}_3 \rightarrow \mathbb{Z}_5$ is zero.

Exercise. Let $M \neq \{0\}$ be an R-module. Prove that:

If N_1, N_2 are submodules of M, with N_1 simple and $N_1 \cap N_2 \neq 0$, then $N_1 \leq N_2$

Remark. Let A, B be two simple R-module, then Hom(A, B) is either zero or for all f ∈ Hom(A, B) is an isomorphism, where Hom(A, B) = {f:A→B| f is homomorphism}

Isomorphism theorems

First isomorphism theorem. Suppose f: M→ N is an R-module homomorphism. Then $\frac{M}{kerf} \approx f(M)$.

Proof. Define h : $\frac{M}{kerf} \rightarrow f(M)$ by: h(m + kerf) = f(m) for all m ∈ M.

1. h is well define: Let m_1+ kerf , m_2 + kerf $\in \frac{M}{kerf}$ such that

$$m_1 + kerf = m_2 + kerf \text{ implies } m_1 - m_2 \in kerf$$

and so

$$f(m_1 - m_2) = f(m_1) - f(m_2) = 0 \rightarrow f(m_1) = f(m_2)$$

Hence

$$h(m_1 + kerf) = h(m_2 + kerf)$$

$$\therefore \text{ h is well define}$$

2. h is a homomorphism since f is homomorphism.

3. h is a monomorphism: for that suppose that

$$h(m_1 + kerf) = h(m_2 + kerf).$$

from definition of h, $f(m_1) = f(m_2)$ implies $f(m_1) - f(m_2) = f(m_1 - m_2) = 0$

so $m_1 - m_2 \in kerf \rightarrow m_1 + kerf = m_2 + kerf$

4. h is an epimporphism: let $y \in f(m) \in f(M)$, $\exists$ $m + kerf \in \frac{M}{kerf}$ such that $h(m + kerf) = f(m) = y$

$$\therefore \text{ h is an epimorphism}$$

So h is an isomorphism and by this, $\frac{M}{kerf} \approx f(M)$

Remark. If f is an epimorphism, then $\frac{M}{kerf} \approx N$

<u>Second isomorphism theorem.</u> Let N and K be submodules of an R-module M, then $\frac{K+N}{N} \approx \frac{K}{N \cap K}$

Proof. Define α: $K \rightarrow \frac{K+N}{N}$ by $\alpha(x) = x + N$ for each $x \in K$.

1. α is well-define (prove)
2. α is homomorphism (prove)
3. α is epimorphism (prove)
4. $ker\alpha = \{ x \in K | \alpha(x) = 0\}$
   $= \{ x \in K \mid x + N = N\}$
   $= \{ x \in K \mid x \in N\}$
   $= N \cap K$

   Then by the first isomorphism theorem, $\frac{K}{N \cap K} \approx \frac{K+N}{N}$

<u>Third isomorphism theorem</u>. Let N, K be submodules of M, and $K \leq N$, then $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Proof. Define g: $\frac{M}{K} \rightarrow \frac{M}{N}$ by :$g(m + K) = m + N$ for all $m \in M$.

1. g is well-define:
   suppose $m_1 + k = m_2 + K$ iff $m_1 - m_2 \in K \leq N$ iff $m_1 + N = m_2 + N$

   $\therefore$ g is well defined
2. g is a homomorphism (prove)
3. g is an epimorphism (prove)
4. kerg = {m+K| g(m+ k) = N}
   ={m+K| m+ N = N}
   = {m+K| m∈ N}
   = $\frac{N}{K}$ (where $K \leq N$ and $m \in N$)

   $\therefore$ kerg = $\frac{N}{K}$

Then by the first isomorphism theorem, $\frac{\frac{M}{K}}{\frac{N}{K}} \approx \frac{M}{N}$.

Exercise. Let M be a cyclic R-module, say M=Rx. Prove that M≈R/ ann(x), where ann(x) = {r ∈ R| rx = 0}.

[Hint: Define the mapping f: R→M by f(r) = rx]

Chapter three (Sequence)

Short exact sequence

Definition. A sequence $M_1 \overset{f}{\rightarrow} M \overset{g}{\rightarrow} M_2$ of R-modules and R-module homomorphismsis said to be ***exact** at* M Im f = ker g while a sequence of the form

$$\partial: \quad \ldots \rightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_{n+1}} M_{n+1} \rightarrow \cdots$$

n∈ ℤ, is said to be an *exact sequence* if it is exact at M_n for each n ∈ ℤ. A sequence such as

$$0 \rightarrow M_1 \overset{f}{\rightarrow} M \overset{g}{\rightarrow} M_2 \rightarrow 0$$

that is exact at M_1, at M and at M_2 is called a ***short exact sequence***.

Remarks.

1. If an exact sequence $0 \to M_1 \xrightarrow{f} M \xrightarrow{g} M_2 \to 0$ is short exact then
   i. f is a monomorphism
   ii. g is an epimorphism
2. A sequence $0 \to M_1 \xrightarrow{f} M$ is exact iff f is monomorphism
3. A sequence $M \xrightarrow{g} M_2 \to 0$ is exact iff g is epimorphism
4. If the composition(between two homomorphisms f and g) gof = 0, then Imf ≤ kerg.

Examples.

1. If N is a submodule of M, then $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$ is a short exact sequence, where i is the canonical injection and π is the natural epimorphism. for example : since kerf is a submodule of M, then $0 \to kerf \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{kerf} \to 0$ is a short exact sequence.
2. Consider the sequence

   $$\mu: \quad 0 \to M_1 \xrightarrow{J_1} M_1 \oplus M_2 \xrightarrow{\rho_2} M_2 \to 0$$

   ImJ_1 = $M_1 \oplus \{0\}$; $J_1(x) = (x, 0)$
   kerρ_2 = $M_1 \oplus \{0\}$; $\rho_2(x, y) = (0, y)$
   for any $x \in M_1$, $y \in M_2$ and $(x,y) \in M_1 \oplus M_2$
   J_1 is a monomorphism and ρ_2 is an epimorphism

   $\therefore$ μ is short exact sequence
3. The sequence $0 \to 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0$ of $\mathbb{Z}$-modules is a short exact sequence

Remark. ***Commutative Diagrams***

The following diagram

$$A \xrightarrow{f_1} B$$

$$g_1\downarrow \qquad \downarrow g_2$$

$$C \xrightarrow{f_2} D$$

is said to be ***commutative*** if $g_2 o f_1 = f_2 o g_1$.Similarly, for a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & h \searrow & \swarrow g \\ & C & \end{array}$$

is commutative if gof = h and we say that g ***completes the diagram commutatively*.**

Theorem. **(*The short five lemma*).** Let R be a ring and

$$\begin{array}{ccccccccc} 0 & \to & A & \xrightarrow{f_1} & B & \xrightarrow{g_1} & C & \to & 0 \\ & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\ 0 & \to & \acute{A} & \xrightarrow{f_2} & \acute{B} & \xrightarrow{g_2} & \acute{C} & \to & 0 \end{array}$$

a commutative diagram of R-modules and R-module homomorphisms such that each row is a short exact sequence. Then

1. If α and γ are monomorphisms, then β is a monomorphism.
2. If α and γ are epimorphisms, then β is an epimorphism.
3. if α and γ are isomorphisms, then β is an isomorphism.

Proof 1.

To show that β is a monomorphism, must prove ker β = 0.

Let b ∈ ker β → β(b) = 0 → g_2(β(b)) = g_2(0) = 0. Since the diagram is commutative, then:

γog_1(b) = γ(g_1(b)) = 0 → g_1(b) ∈ kerγ = {0}(γ is a monomorphism) → g_1(b) = 0 → b ∈ kerg_1= Imf_1= f_1(A). There is a ∈ A such that

f_1(a) = b → β(f_1(a)) = β(b).

Since

βof_1 = f_2oα → f_2oα(a) = β(b) → f_2(α(a)) = 0 → α(a) ∈ kerf_1= {0}(f_2 is a monomorphism), so

α(a) = 0 → a ∈ kerα = {0} (α is a monomorphism) → a =0.

But f_1(a)=b and a=0 → b=f_1(a)= f_1(0)= 0 → b=0.

kerβ = {0} → β is a monomorphism

Proof 2.

Let $\acute{b}$ ∈ $\acute{B}$ → $g_2(\acute{b})$ ∈ $\acute{C}$ → $g_2(\acute{b}) = \acute{c}$. Since γ is an epimorphism, there is c ∈ C such that

γ(c) = $\acute{c}$ → $g_2(\acute{b})$ = γ(c).

But g_1 is an epimorphism, then there is b ∈ B such that

g_1(b) = c → $g_2(\acute{b})$ = γ(c) = γ(g_1(b)) = γog_1(b) = g_2oβ(b)

so

$g_2(\acute{b})$ = g_2(β(b)) → g_2(β(b) - $\acute{b}$) = 0 (g_2 is homomorphism).

and

β(b) - $\acute{b}$ ∈kerg_2= Imf_2 →β(b) - $\acute{b}$ ∈ Imf_2.

There is $\acute{a} \in \acute{A}$ such that $f_2(\acute{a})$ = β(b) - $\acute{b}$. But α is an epimorphism, there is a ∈ A such that α(a) = $\acute{a}$. Since β∘f_1 = f_2∘α (the diagram is commutative).

Then

$$\beta\,(f_1(a)) = f_2(\alpha(a)) = f_2(\acute{a}) = \beta(b) - \acute{b}$$

so

$$\acute{b} = \beta(b) - \beta\,(f_1(a)) = \beta(b - f_1(a))\ (\beta \text{ is homomorphism})$$

i.e there is b – f_1(a) ∈ B such that β(b – f_1(a)) = $\acute{b}$

Hence β is an epimorphism.

Proof 3. is an immediate consequence of (1) and (2).

<u>Exercise.</u> Consider the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \rightarrow & 0 \\ h\downarrow & & & & & & \\ D & & & & & & \end{array}$$

where the row is exact and h∘f = 0. Prove that, there exact a unique homomorphism k: C → D such that k∘g = h.

<u>Definition</u>. Let $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ be a short exact sequence. This sequence is said to be ***splits*** if Imf is a direct summand of B.
(i.e there is D ≤ B such that B = Imf ⊕ D).

Example. The sequence $0 \rightarrow 2\mathbb{Z} \xrightarrow{i} \mathbb{Z} \xrightarrow{\pi} \frac{\mathbb{Z}}{2\mathbb{Z}} \rightarrow 0$ of ℤ-modules and ℤ-homomorphism is a short exact sequence which is not split (where Imi = 2 ℤ is not direct summand of ℤ).

Theorem. Let R be a ring and

$$\mathcal{F}: \quad 0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

a short exact sequence of R-module homomorphisms. Then the following conditions are equivalent

1. $\mathcal{F}$ splits.
2. f has a left inverse (i.e $\exists$ h: B $\rightarrow$ A homomorphism with hof = I_A).
3. g has a right inverse(i.e $\exists$ k:C$\rightarrow$B a homomorphism with gok = I_C).

Proof. (1 $\rightarrow$ 2) since $\mathcal{F}$ splits, then Imf is a direct summand of B.

(i.e. $\exists$ $B_1 \leq B$ such that B = Imf $\oplus$ B_1).

Define h: B $\rightarrow$ A by h(x) = h(a_1+b_1) = a for x = a_1+b_1 $\in$ Imf $\oplus$ B_1.

where $a_1$$\in$ Imf (i.e $\exists$ a $\in$ A such that f(a) = a_1) and $b_1 \in B_1$.

a. Since f is one-to-one, then h is well-define.
b. h is a homomorphism
c. let w $\in$ A, hof(w) = h(f(w)) = h(f(w)+ 0) = w (by definition of h)

$\therefore$ h is a left inverse of f.

(2 $\rightarrow$ 3) suppose f has a left inverse say h(i.e. hof = I_A).

Define k: C $\rightarrow$ B by: k(y) = b - foh(b) where g(b) = y with b $\in$ B_1.

a. k is well define:
   let y, y_1 $\in$ C such that y = y_1 with g(b) = y and g(b_1) = y_1 for b, b_1 $\in$$B_1$.
   Now,
   $$g(b) = g(b_1) \rightarrow b_1 - b \in \ker g = \text{Imf}$$
   so, b_1-b $\in$ Imf $\rightarrow$ $\exists$ a $\in$ A such that f(a) = b_1 – b.
   Then h(f(a)) = h(b_1- b) . But hof = I_A,
   so a = hof(a) = h(f(a)) = h(b_1- b) = h(b_1) – h(b)

$\therefore$ a = h(b_1) – h(b) $\rightarrow$ f(a) = f(h(b_1)) – f(h(b)) = b_1 – b

$\therefore$ b - f(h(b)) = b_1- f(h(b_1)) $\rightarrow$ k(y) = k(y_1) $\rightarrow$ k is well define.

b. k is homomorphism (why?)
c. $gok = I_C$. for that
   let $y \in C$, $gok(y) = g(k(y)) = g(b\text{-}foh(b))$ where $g(b) = y$.
   $\rightarrow gok(y) = g(b) + gofoh(b)$. But Im f = kerg. So, $gof = 0$.
   $\rightarrow gok(y) = g(b) + 0 = y$

$$\therefore gok = I_C$$

(3 → 1) suppose that g has a right inverse say $k: C \rightarrow B$ such that $gok = I_C$

Let $B_1 = \{b \in B \mid kog(b) = b\}$

a. $B_1 \neq \varphi$ ($0 \in B_1$ where $g(0) = k(g(0)) = k(0) = 0$)
b. B_1 is a submodule of B. (prove?)
c. $B = Imf \oplus B_1$, for that:
   i. Let $w = Imf \cap B_1 \rightarrow w = f(a) \in B_1$ for some $a \in A$ with $kog(w)=w \rightarrow k(g(f(a))) = k(0) = 0$. But $k(g(f(a))) = k(g(w)) = w$.

   Thus $w = 0$ and so $Imf \cap B_1 = 0$.
   ii. Let $b \in B \rightarrow b = b - kog(b) + kog(b)$.

   Since $kog(kog(b)) = kog(b)$, then $kog(b) \in B_1$ and $g(b\text{-}kog(b)) = g(b) - gokog(b) = g(b) - Iog(b) = g(b) - g(b) = 0$ (where $gok = I_C$).

   $\rightarrow b\text{-}kog(b) \in kerg = Imf$

   $\therefore b = b\text{-}kog(b) + kog(b) \in Imf + B_1$

   $\therefore B = Imf \oplus B_1 \rightarrow$ Imf is a direct summand of B which implies $\mathcal{F}$ splits.

Exercise If the short exact sequence

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

splits, then $B \approx Imf \oplus Img$

Chapter four (Noetherian and Artinian modules)

Ascending and Descending chain condition

Definition. An R-module M is said to be satisfy the ascending chain condition (resp. descending chain condition) if for every ascending (resp. descending) chain of submodules

$$M_1 \leq M_2 \leq M_3 \leq \ldots \leq M_n \leq \ldots$$

$$(\text{resp.} \quad M_1 \geq M_2 \geq M_3 \geq \ldots \geq M_n \geq \ldots)$$

there exists $m \in \mathbb{Z}_+$ such that $M_n = M_m$ whenever $n \geq m$.

Definition. A module which satisfies the ascending chain condition is said to be ***Noetherian***.

Definition. A module which satisfies the descending chain condition is said to be ***Artinian***.

Remark. A ring R is said to be *Noetherian* (*Artinian)* if it is *Noetherian* (*Artinian*) as an R-module. i.e., if it satisfies a.c.c. (d.c.c.) on ideals.

Example. Every simple module is both Noetherian and Artinian.

Theorem 1. Let M be an R-module. Then the following statements are equivalent:

1. M satisfies the ascending (descending) chain condition.
2. For any nonempty family $\{M_\alpha\}_{\alpha\in I}$ of submodules of M, there exist a maximal (minimal) element M_0 satisfies the maximal condition (resp. minimal condition)
   (i.e $\exists M_0 \in \{M_\alpha\}_{\alpha\in I}$ such that whenever $M_0 \leq M_\beta$, then $M_0 = M_\beta$)
   (resp. i.e $\exists M_0 \in \{M_\alpha\}_{\alpha\in I}$ such that whenever $M_\beta \leq M_0$, then $M_0 = M_\beta$)

Proof. (1→2) consider the set

$$\mathcal{F} = \{M_i | \ M_i \leq M\}$$

$\mathcal{F} \neq \varphi$

Suppose $\mathcal{F}$ has no maximal element.

Let $M_1 \in \mathcal{F}$ implies M_1 is not maximal element.

$\exists$ $M_2 \in \mathcal{F}$ such that $M_1 \leq M_2$. Since M_2 is not max. element, then there is $M_3 \in \mathcal{F}$ such that $M_2 \leq M_3$.

Continuing in this way, we get

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

A chain of submodules of M. if this sequence is an infinite, then it does not satisfy the ACC. C!

$$\therefore\ \mathcal{F} \text{ has maximal element}$$

$(2 \rightarrow 1)$ suppose M satisfies the maximal condition for submodules, and let

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

be ascending chain of submodules of M.

Let $\mathcal{H} = \{M_\alpha\}_{\alpha \in I}$ be a family of the submodules of M. Then $\mathcal{H} \neq \varphi$ and has maximal element M_m. implies whenever $n \geq m$, $M_m = M_n$.

$$\therefore \mathcal{H} \text{ satisfies the ascending chain condition.}$$

Theorem 2. Let M be an R-module. Then the following statements are equivalent:

1. M is Noetherian.
2. Every submodule of M is finitely generated.

Proof. $(1 \rightarrow 2)$ suppose M is Noetherian module and K be submodule of M. Let $\mathcal{F} = \{A \mid A \text{ is finitely generated submodule of } K\}$

$\mathcal{F} \neq \varphi$ (the zero submodule of A is in $\mathcal{F}$)

Since M is Noetherian module , so $\mathcal{F}$ has maximal element say K_0.

Hence K_0 is finitely generated submodule of K

$$\text{i.e } K_0 = Rk_1 + Rk_2 + \ldots + Rk_n$$

Suppose $K_0 \neq K \rightarrow \exists\ a \in K$ and $a \notin K_0$ and so

$$K_0 + Ra = K_0 = Rk_1 + Rk_2 + \ldots + Rk_n + Ra$$

$\because K_0 + Ra$ is a finitely generated submodule of K, then $K_0 + Ra \in \mathcal{F}$ is a contradiction with the maximalist of K_0. Hence $K_0 = K$

$\therefore$ K is a finitely generated

$(2 \rightarrow 1)$ suppose that every submodule of M is finitely generated.

Let $K_1 \leq K_2 \leq K_3 \leq \ldots$ be an ascending chain of submodules of M.

Put $K = \bigcup_{i=1}^{\infty} K_i$ $\rightarrow$ K is submodule of M.

$\rightarrow$ K is a finitely generated submodule of M

$\rightarrow K = Rk_1 + Rk_2 + \ldots + Rk_n$

$\rightarrow$ each K_j is in K_i's

$\rightarrow \exists$ m such that $k_1, k_2, \ldots, k_r \in K_m \qquad \forall\ n \geq m$

$\therefore$ M is Noetherian module.

Examples.

1. The $\mathbb{Z}$- module $\mathbb{Z}$ is Noetherian module (every submodule of the $\mathbb{Z}$- module $\mathbb{Z}$ (= $n\mathbb{Z}$ cyclic) is finitely generated) which is not Artinian ($2\mathbb{Z} > 4\mathbb{Z} > 8\mathbb{Z} > \ldots > 2^n\mathbb{Z} > \ldots$. is a chain of ideals of Z that does not terminate)
2. The ring of integers $\mathbb{Z}$ is Noetherian (every principal ideal ring is Noetherian).

3. Q is not Noetherian module (since the ℤ- module Q is not finitely generated).
4. A division ring D is Artinian and Noetherian since the only right or left ideals of D are 0 and D.
5. Every finite module is an Artinian module.

Remark. Every nonzero Artinian module contains a simple submodule.

Proof. let $0 \neq M$ be an Artinian module.

If M is a simple module, then we are done.

If not, $\exists$ $0\neq M_1$ submodule of M. If M_1 is a simple, then we are done.

If not, $\exists$ $0\neq M_2$ submodule of M_1. If M_2 is a simple, then we are done.

If not, $\exists$ $0\neq M_3$ submodule of M_2. If M_3 is a simple, then we are done.

So there is a descending chain

$$M \geq M_1 \geq M_2 \geq M_3 \geq \dots$$

of submodules of M. Since M is an Artinian module, then the family $\{M_i\}_{i\in I}$ of the chain has minimal element and this element is the simple submodule.

Proposition. Let $0 \to N \xrightarrow{i} M \xrightarrow{\pi} \frac{M}{N} \to 0$be a short exact sequence of R-modules and module homomorphism. Then M is Noetherian (resp. Artinian) iff both N (Artinian) and $\frac{M}{N}$ are Noetherian (Artinian) (resp. Artinian).

Proof.→)Suppose that M is a Noetherian module and N submodule of M . So every submodule of N is a submodule of M. so N is Noetherian. Let

$$\frac{M_1}{N} \leq \frac{M_2}{N} \leq \frac{M_3}{N} \leq \dots$$

be an ascending chain of submodules of $\frac{M}{N}$, where

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

is an ascending chain of submodules of M which contain N. But M Noetherian, ∃m such that $M_n = M_m$ for all $n \geq m$.

$$\therefore \frac{M}{N} \text{ is Noetherian module.}$$

⟵) Suppose that N and $\frac{M}{N}$ are Noetherian modules. Let

$$M_1 \leq M_2 \leq M_3 \leq \dots$$

be an ascending chain of submodules of M. Then

$$M_1 \cap N \leq M_2 \cap N \leq M_3 \cap N \leq \dots$$

is an ascending chain of submodules of N, so there is an integer $m_1 \geq 1$ such that $M_n \cap N = M_{m_1} \cap N$ for all $n \geq m_1$. Also,

$$\frac{M_1+N}{N} \leq \frac{M_2+N}{N} \leq \frac{M_3+N}{N} \leq \dots$$

is an ascending chain of submodules of $\frac{M}{N}$ and there is an integer $m_2 \geq 1$ such that $\frac{M_n+N}{N} = \frac{M_{m_2}+N}{N}$ for all $n \geq m_2$. Let $m = \max.\{m_1, m_2\}$. Then for all $n \geq m$,

$$M_n \cap N = M_m \cap N \quad \text{and} \quad \frac{M_n+N}{N} = \frac{M_m+N}{N}$$

If $n \geq m$ and $x \in M_n$, then $x + N \in \frac{M_n+N}{N} = \frac{M_m+N}{N}$, so there is a $y \in M_m$ such that $x + N = y + N$ implies that $x - y \in N$ and since $M_m \leq M_n$ we have $x - y \in M_n \cap N = M_m \cap N$ when $n \geq m$ If $x - y = z \in M_m \cap N$, then $x = y + z \in M_m$, so $M_n \leq M_m$. Hence, $M_n = M_m$ whenever $n \geq m$, so M is Noetherian.

Remark. In general, if the sequence $0 \to A \overset{f}{\to} B \overset{g}{\to} C \to 0$ is a short exact, then B is Noetherian (Artinian) if and only if each of A and C is Noetherian (Artinian).

Example. Let M_1 and M_2 be R-modules. Then $M_1 \oplus M_2$ is Noetherian (Artinian) iff each of M_1 and M_2 is Noetherian (Artinian). (i.e every finite direct sum of Noetherian (Artinian)is Noetherian (Artinian)

(The proof is done using the short exact sequence

$$0 \to M_1 \overset{J_1}{\to} M_1 \oplus M_2 \overset{\rho_2}{\to} M_2 \to 0)$$

Theorem. Let $\alpha : M \to \acute{M}$ be an epimorphism. If M is Noetherian (Artinian), then so is $\acute{M}$.

Proof. Since $ker\alpha$ is a submodule of M, then the sequence

$$0 \to ker\alpha \overset{i}{\to} M \overset{\pi}{\to} \frac{M}{ker\alpha} \to 0$$

is a short exact sequence. By hypothesis, M is Noetherian, implies that $\frac{M}{ker\alpha}$is Noetherian. But $\frac{M}{ker\alpha} \approx \acute{M}$(first isomorphism theorem) and $\frac{M}{ker\alpha}$ is Noetherian, so $\acute{M}$ is a Noetherian.

Theorem. The following are equivalent for a ring R.

1. R is right Noetherian.
2. Every finitely generated R-module is Noetherian.

Proof.(1→ 2) let M be a finite generated over a Noetherian ring R.

$\exists\ x_1, x_2, \ldots, x_n \in M$ such that $M = Rx_1 + Rx_2 + \ldots + Rx_n$. since R is Noetherian, then so is the finite direct sum of copies of R. Define

$\alpha : R^{(n)} \to M$ by : $\alpha(r_1, r_2, \ldots, r_n) = r_n x_1 + r_n x_2 + \ldots + r_n x_n$.

It's clear that α is a well-define, homomorphism and onto. So, Imα = M is Noetherian.

(2 → 1)Since R = <1>, so R is finitely generated and hence R is Noetherian.

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