Chapter 1: Functions

Functions and their graph

Definition:

Let A and B be a nonempty sets, then the function from A to B is a rule that assigns to each element of A exactly one element of B:

($\forall$ x $\in$ A, $\exists!$ y $\in$ B such that f(x) = y).

i.e we say that :

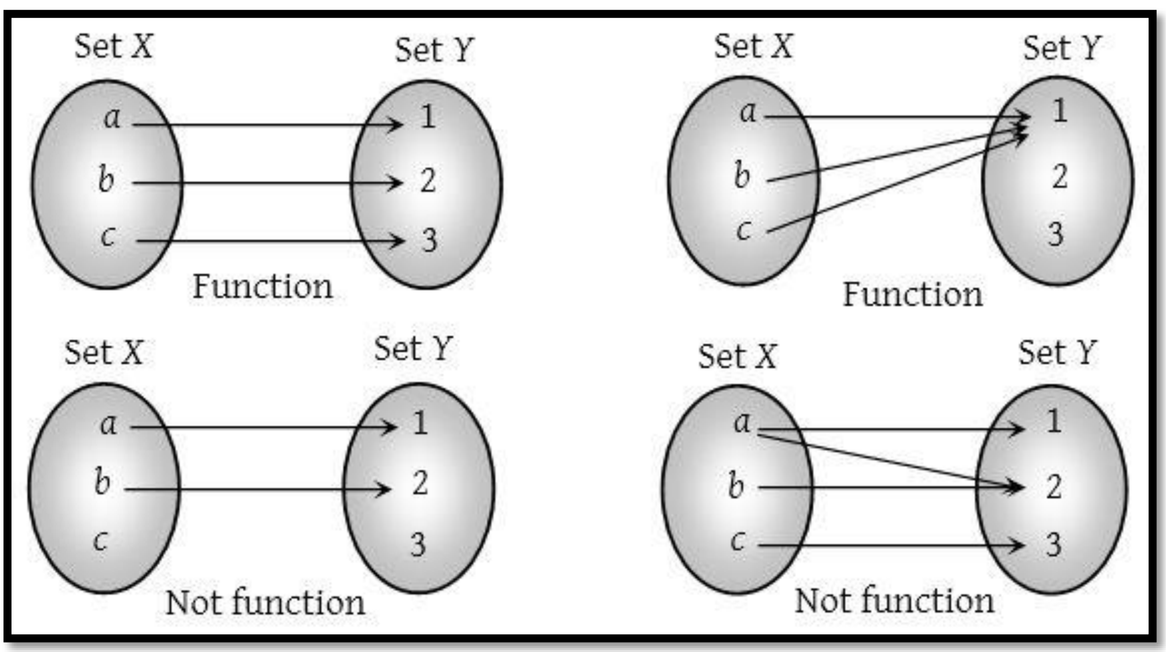
f = { (x,y) | x $\in$ A and y $\in$ B , A, B nonempty sets} is a function if whenever:

$(x,y_1) \in f$ and $(x,y_2) \in f \Longrightarrow y_1=y_2$.

Remarks:

The set A is called Domain f (Dom f) and B is co- domain (Co-Domf) and f(A) is the range of the function f.

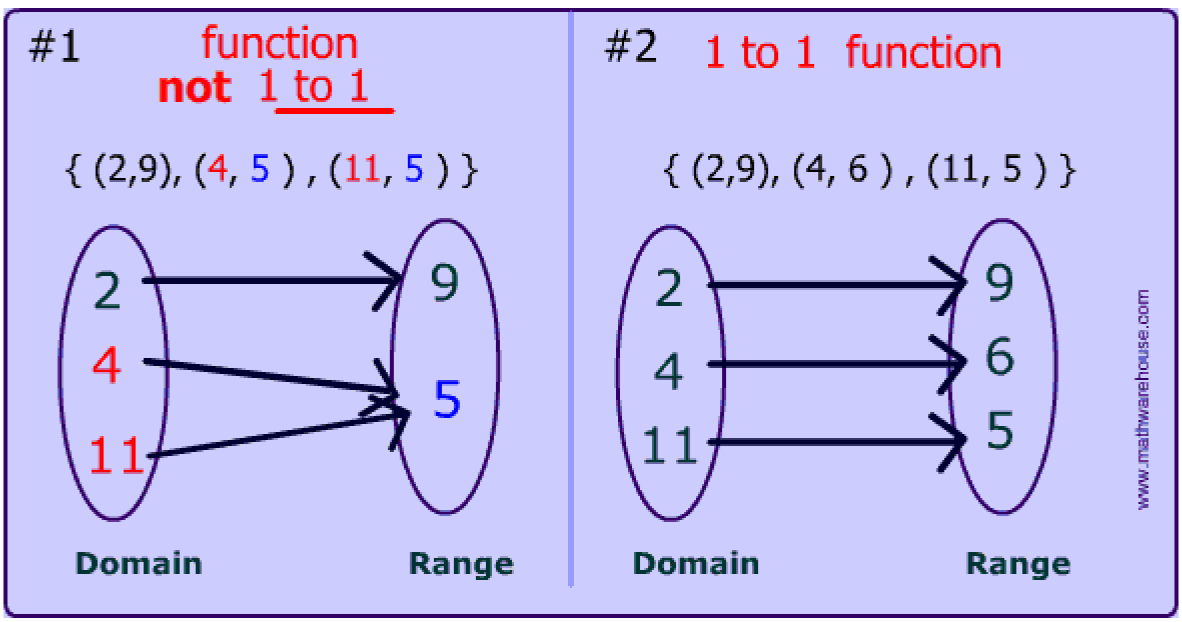
Examples:



Some types of functions:

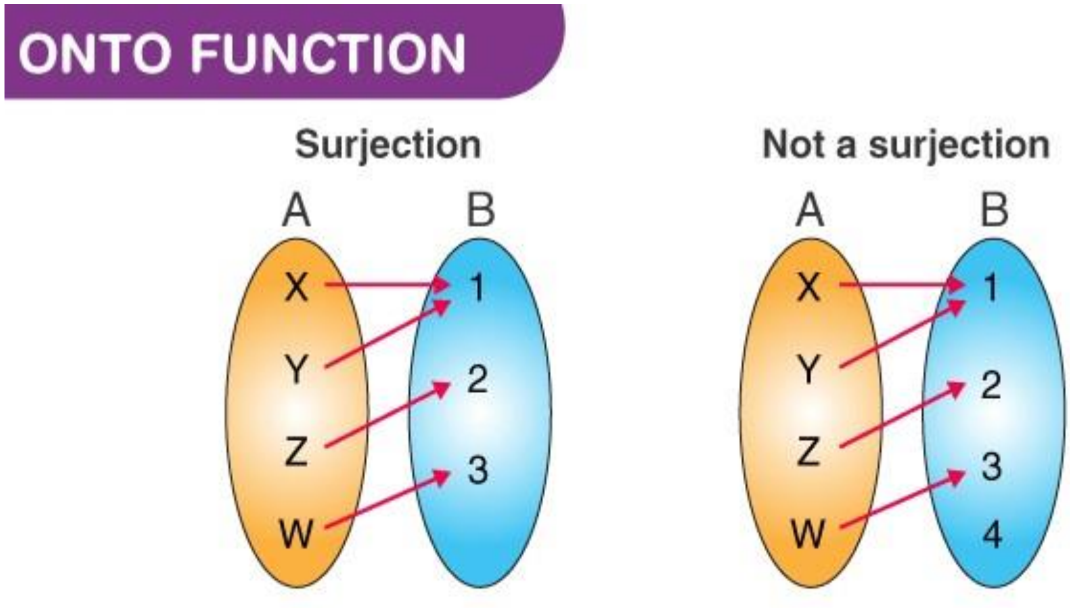
1- One to One function (injective):

A function f is one to one (1 to 1) if no two elements in the domain of f correspond to the same element in the range of f.
In other words, each x in the domain has exactly one image in the range. And no y in the range is the image of more than one x in the domain.



2- On to function (surjective):

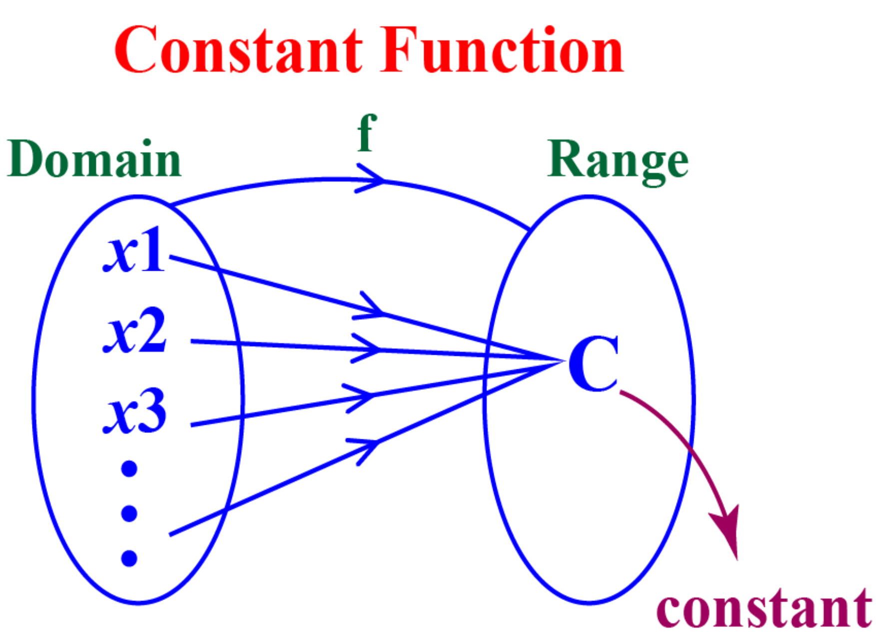
An on to function is such that for every element in the codomain there exists an element in domain which maps to it i.e f(A) = B.



3- <u>Constant function:</u>

A function f: X⟶Y is called constant function if f(x) = c, for all x ∈X and c∈R.

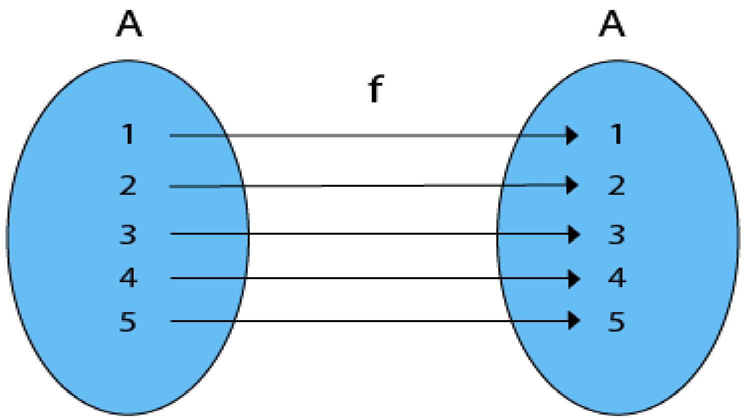
(R the set of real numbers).



Example: f : R → R defined as f(x) = 4 , for all x ∈ R.

4- Identity function:

The function f is an identity function as each element of A is mapped onto itself. Mathematically it can be expressed as: $f(a) = a, \ \forall a \in A$ (Dom f = Co-dom f). The function f is a one-one and onto.



5- Polynomail function:

A polynomial function is a function that can be expressed in the form of:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots\ldots + a_2 x^2 + a_1 x + a_0$$

Where $a_n \neq 0$ and $a_0, a_1, \ldots, a_n \in R$. The highest power of the variable of f(x) is known as its degree. The domain of a polynomial function is entire real numbers (R).

Types of Polynomial Functions

There are various types of polynomial functions based on the degree of the polynomial. The most common types are:

- Zero Polynomial Function: $f(x) = a = ax^0$, for Ex: f(x) = 4
- Linear Polynomial Function: $f(x) = ax + b$, for Ex: f(x) = 2x+3
- Quadratic Polynomial Function: $f(x) = ax^2+bx+c$, for Ex: $f(x) = x^2+2x-3$
- Cubic Polynomial Function: $f(x) = ax^3+bx^2+cx+d$,
  for Ex: $f(x) = 2x^3 + 4x^2+x+7$

Remarks:

1. The polynomial function is called a Constant function if the degree is zero.
2. The polynomial function is called a Linear if the degree is one.
3. The polynomial function is Quadratic if the degree is two.
4. The polynomial function is Cubic if the degree is three.

6- Even and Odd functions:

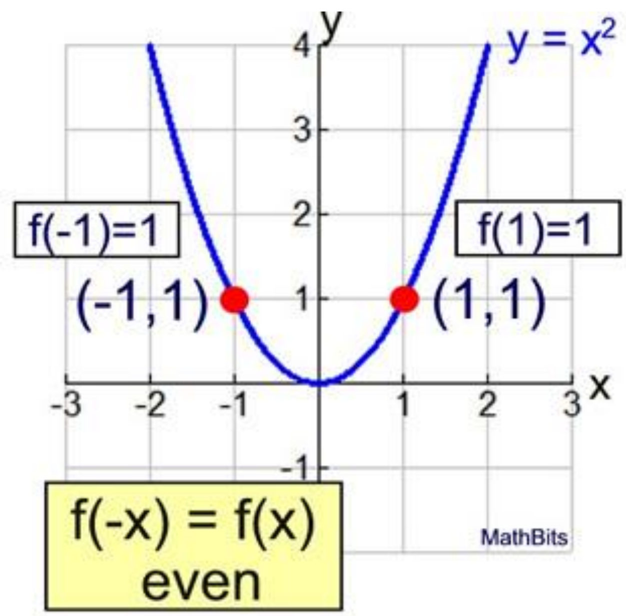
DEFINITIONS A function $y = f(x)$ is an

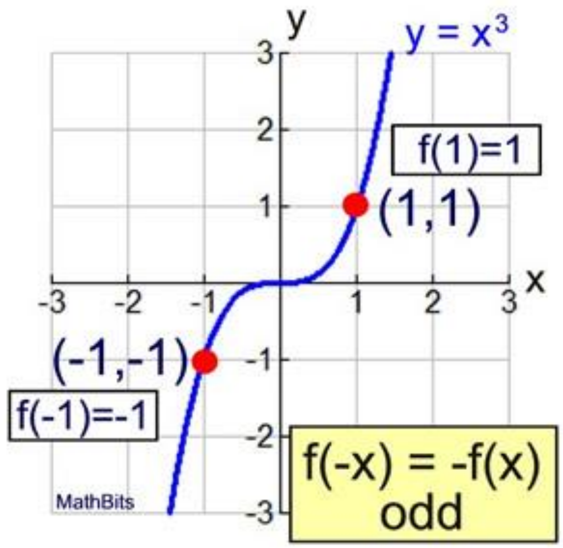
even function of $\boldsymbol{x}$ if $f(-x) = f(x)$,
odd function of $\boldsymbol{x}$ if $f(-x) = -f(x)$,

for every x in the function's domain.

The graph of an even function is symmetric about they-axis. Since *f(-x) = f(x)* , a point (x,y) lies on the graph if and only if the point (-x, y) lies on the graph . A reflection across the y-axis leaves the graph unchanged.

The graph of an odd function is symmetric about the origin. Since *f(-x) = - f(x)* , a point *(x,y)* lies on the graph if and on1y if the point *(-x, -y)* lies on the graph.





Function	Even, Odd, or Neither?
$f(x)=3x^2+8$	$\boldsymbol{f}(-\boldsymbol{x})=3(-x)^2+8=3x^2+8=\boldsymbol{f}(\boldsymbol{x})$ **Even!**
$f(x)=x^5-4x$	$\boldsymbol{f}(-\boldsymbol{x})=(-x)^5-4(-x)=-x^5+4x$ $=-(x^5-4x)=-\boldsymbol{f}(\boldsymbol{x})$ **Odd!**
$f(x)=2x^2-x-1$	$\boldsymbol{f}(-\boldsymbol{x})=2(-x)^2-(-x)-1=2x^2+x-1$ $-f(x)=-(2x^2-x-1)=-2x^2+x+1$ $\boldsymbol{f}(-\boldsymbol{x})\neq \boldsymbol{f}(\boldsymbol{x})\neq -\boldsymbol{f}(\boldsymbol{x})$ **Neither!**

7- <u>Rational function:</u>

A rational function is any function which can be written as the ratio of two polynomial functions.

$f(x) = \frac{P(x)}{Q(x)}$, where $Q(x) \neq 0$.

8- Square root function:

The principal square root function f(x) = $\sqrt{x}$ (usually just referred to as the "square root function") is a function that maps the set of nonnegative real numbers onto itself (f: $R^+ \cup \{0\} \rightarrow R^+ \cup \{0\}$).

For example:

- f(x) = $\sqrt{x+9}$
- f(x) = $\sqrt{16-x^2}$

Piecewise-Defined Functions

Sometimes a function is described by using different formulas on different parts of its domain. One example is the **absolute value function ($|x| = \sqrt{x^2}$).**

Example1:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

Example 2: the function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

How to find the domain and range of function:

Example: Find the domain and range of the following functions:

1- $y = x^3$
2- $y = x^2$
3- $y = \sqrt{x}$
4- $y = \frac{1}{x}$
5- $y = \sqrt{4 - x}$
6- $y = \sqrt{1 - x^2}$
7- $y = 2$

Solution:

1- $y = x^3$

Dom f = $\mathbb{R}$, Ran f = $\mathbb{R}$

2- $y = x^2$

Dom f = $\mathbb{R}$, The range of $y = x^2$ is $[0, \infty)$ because the square of any real number is nonnegative and $x = \sqrt{y}$, x to be real $y \geq 0$.

3- $y = \sqrt{x}$

Dom f = $R^+ \cup \{0\}$

Ran f = $R^+ \cup \{0\}$, (The range of $y = \sqrt{x}$ can be found by $y \geq 0$ and $x = y^2$ so range = $[0, \infty)$)

4- $y = \frac{1}{x}$

Dom f = $\mathbb{R} \setminus \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Rang f = $\mathbb{R} \setminus \{0\}$

5- $y = \sqrt{4 - x}$

$4 - x \geq 0 \Longrightarrow x \leq 4$

Dom f = $(-\infty, 4]$

Rang f: first $y \geq 0$, second $x = 4 - y^2 \rightarrow$ range $= [0, \infty)$.

6- $y = \sqrt{1-x^2}$

$1-x^2 \geq 0 \implies x \leq \pm 1$

Dom f = [-1,1] , Rang f = [0,1]

7- y = 2

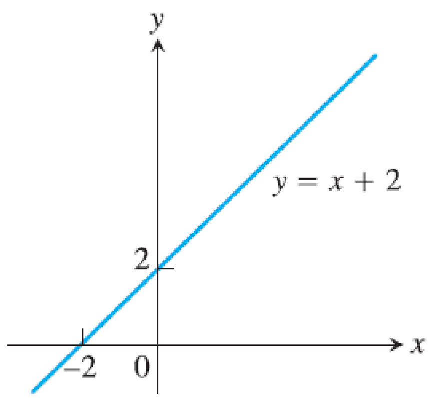
Dom f = R , Rang f = {2}.

Graphs of Functions

If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for *f*. In set notation, the graph is $\{(x, f(x)) / x \in D\}$.

EXAMPLE 1*:* The graph of the function f(x) = x + 2

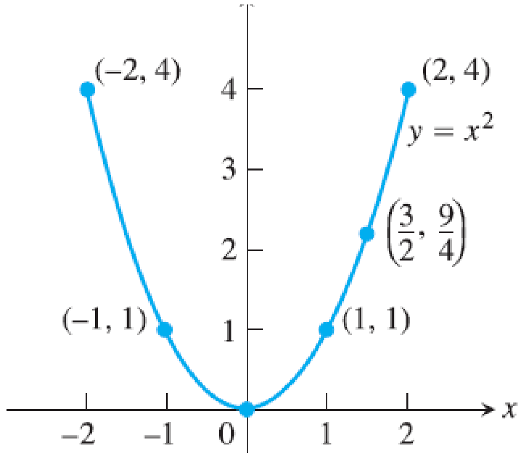
Dom f = R



x	y
1	3
2	4
0	2
-1	1
-2	0

EXAMPLE 2: Graph the function $y = x^2$ over the interval $[-2, 2]$.

Dom f = R



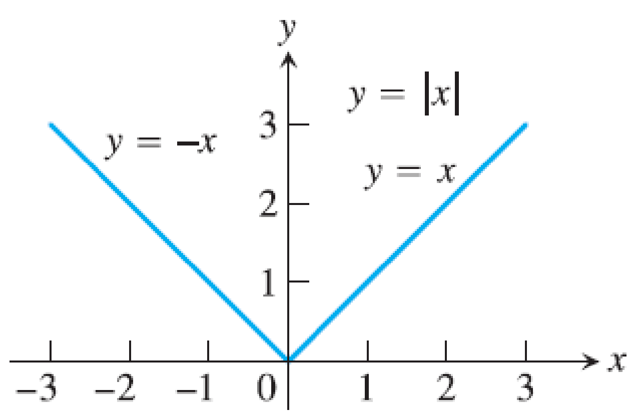
x	y
1	1
2	4
0	0
-1	1
-2	4

Example 3: Graph the function:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

Dom f = R

Rang f = [0, ∞)



$x \geq 0$

x	y
0	0
1	1
2	2

$x < 0$

x	y
-1	1
-2	2
-3	3

Example 4: Graph the function:

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$x > 1$

$y = 1$

x	y
2	1
3	1

$0 \leq x \leq 1$

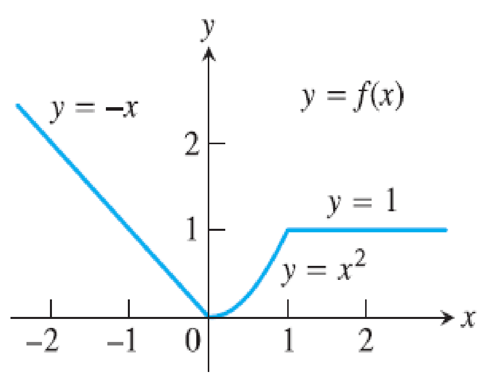
$y = x^2$

x	y
0	0
$\frac{1}{2}$	$\frac{1}{4}$
1	1

$x < 0$

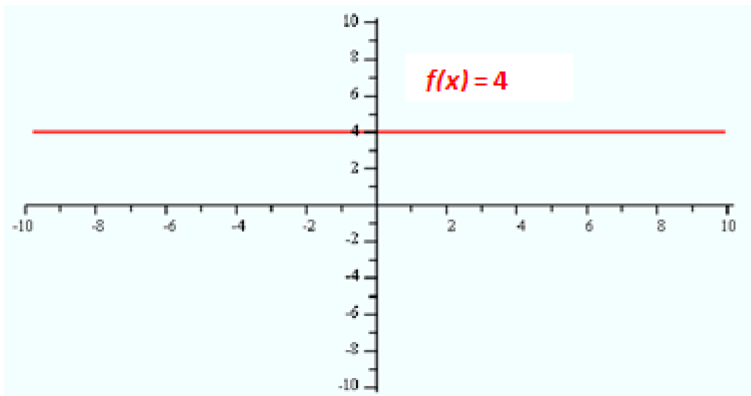
$y = -x$

x	y
-1	1
-2	2

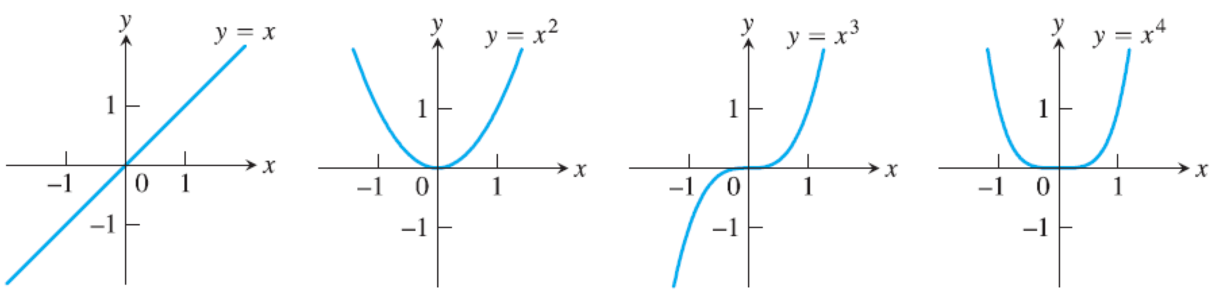


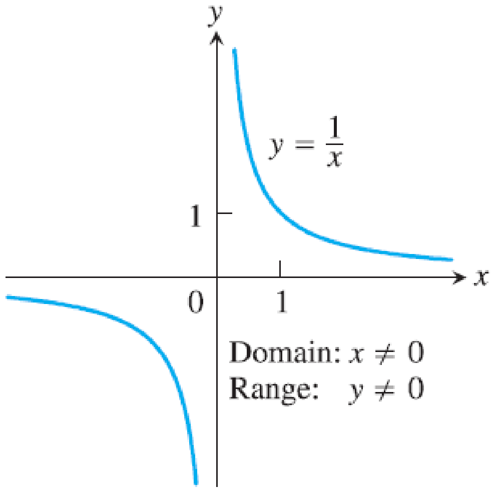
Example 5: Graph the function y = 4

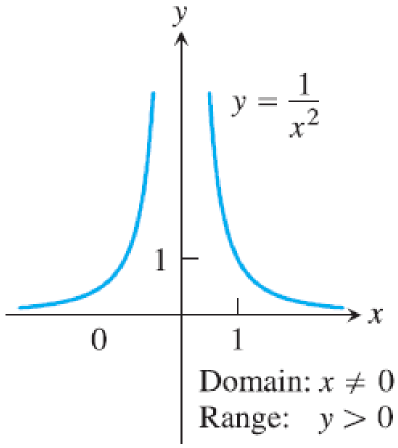
A Constant Function Graph

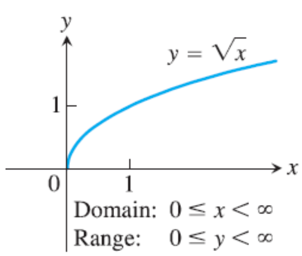


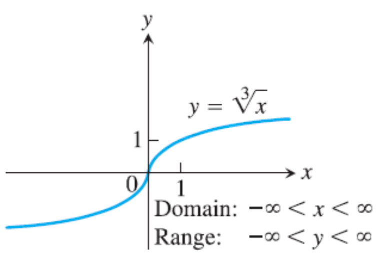
Graphs of special function:

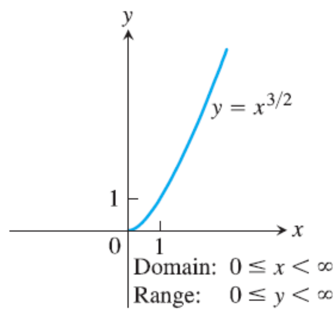












Shifting a Graph of a Function:

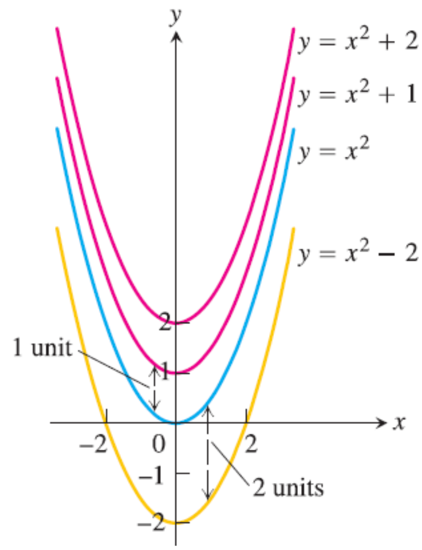
Vertical Shifts

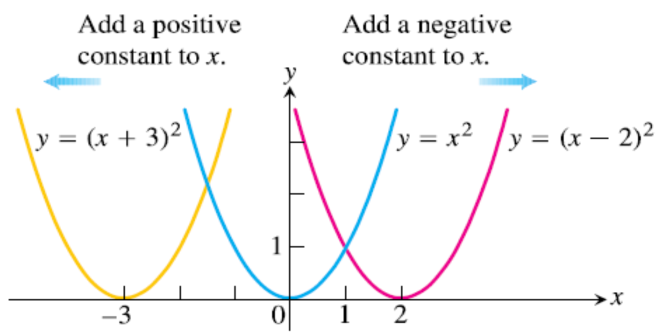
$y = f(x) + k$ — Shifts the graph of f *up* k units if $k > 0$
Shifts it *down* $|k|$ units if $k < 0$

Horizontal Shifts

$y = f(x + h)$ — Shifts the graph of f *left* h units if $h > 0$
Shifts it *right* $|h|$ units if $h < 0$

Example: $y = x^2$

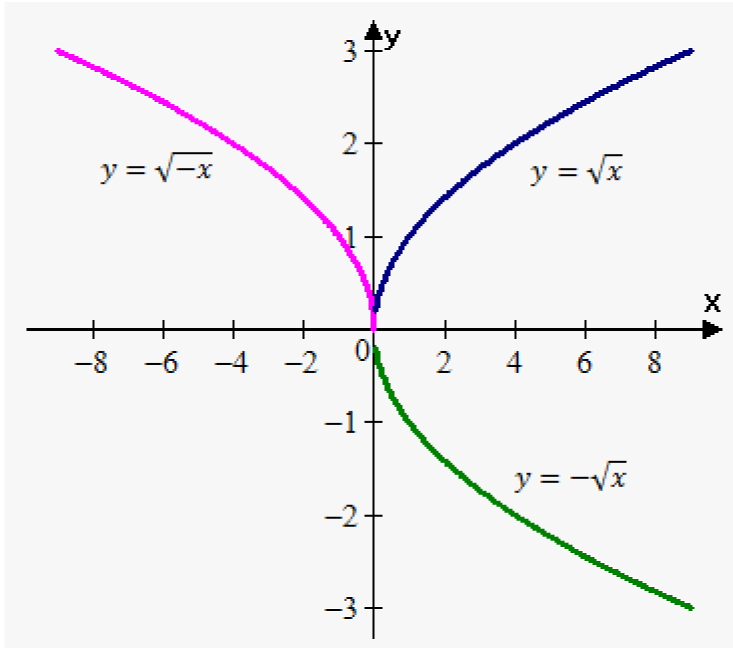




Reflecting a graph of a function:

1- $y = -f(x)$ Reflects the graph of f across the x- axis.
2- $y = f(-x)$ Reflects the graph of f across the y- axis.

Example: The graph of $y = -\sqrt{x}$ is a reflection of $y = \sqrt{x}$ across the x-axis and $y = \sqrt{-x}$ is a reflection of $y = \sqrt{x}$ across the y- axis.



Sums, Differences, Products, and Quotients

$$(f + g)(x) = f(x) + g(x).$$

$$(f - g)(x) = f(x) - g(x).$$

$$(f\,g)(x) = f(x)g(x).$$

At each of these functions the domain = domain (f) ∩ $domain(g)$

At any point of domain (f) ∩ $domain(g)$ at which g(x) ≠ 0, we can also define the function f/g by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad (\text{where } g(x) \neq 0)$$

Functions can also be multiplied by constants: If c is a real number, then the function

cf is defined for all x in the domain of f by (cf)(x) = cf(x).

EXAMPLE 1 The functions defined by the formulas

$$f(x) = \sqrt{x} \qquad \text{and} \qquad g(x) = \sqrt{1 - x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

The following table summarizes the formulas and domains for the various algebraic combinations of the two functions. We also write $f \cdot g$ for the product function fg.

Function	Formula	Domain
$f + g$	$(f + g)(x) = \sqrt{x} + \sqrt{1 - x}$	$[0, 1] = D(f) \cap D(g)$
$f - g$	$(f - g)(x) = \sqrt{x} - \sqrt{1 - x}$	$[0, 1]$
$g - f$	$(g - f)(x) = \sqrt{1 - x} - \sqrt{x}$	$[0, 1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1 - x)}$	$[0, 1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1 - x}}$	$[0, 1)$ ($x = 1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1 - x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Operations with Functions

Add Functions

Add $f(x) = 4x - 5$ and $g(x) = 2x$

$(f + g)(x)$
$= f(x) + g(x)$
$= 4x - 5 + 2x$
$= 6x - 5$

Subtract Functions

Subtract $f(x) = 4x - 5$ and $g(x) = 2x$

$(f - g)(x)$
$= f(x) - g(x)$
$= 4x - 5 - 2x$
$= 2x - 5$

Multiply Functions

Multiply $f(x) = 4x - 5$ and $g(x) = 2x$

$(f \bullet g)(x)$
$= f(x) \bullet g(x)$
$= (4x - 5)\ 2x$
$= 8x^2 - 10x$

Divide Functions

Divide $f(x) = 4x - 5$ and $g(x) = 2x$

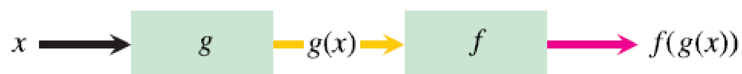
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$
$$= \frac{4x - 5}{2x}$$

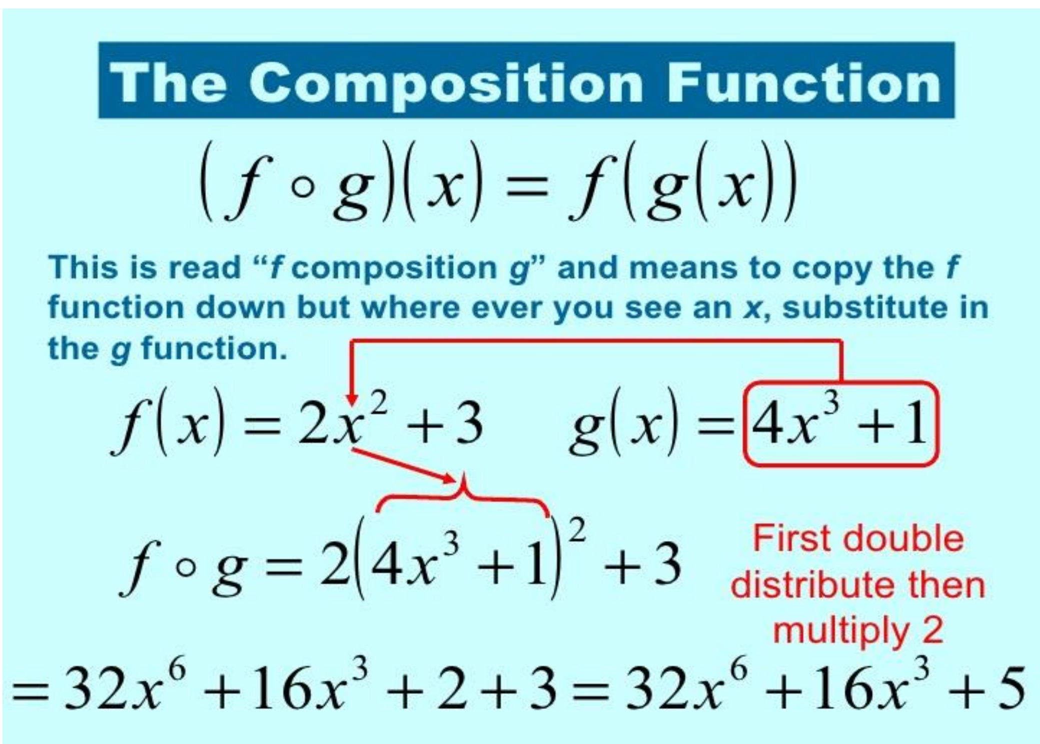
Composite Functions

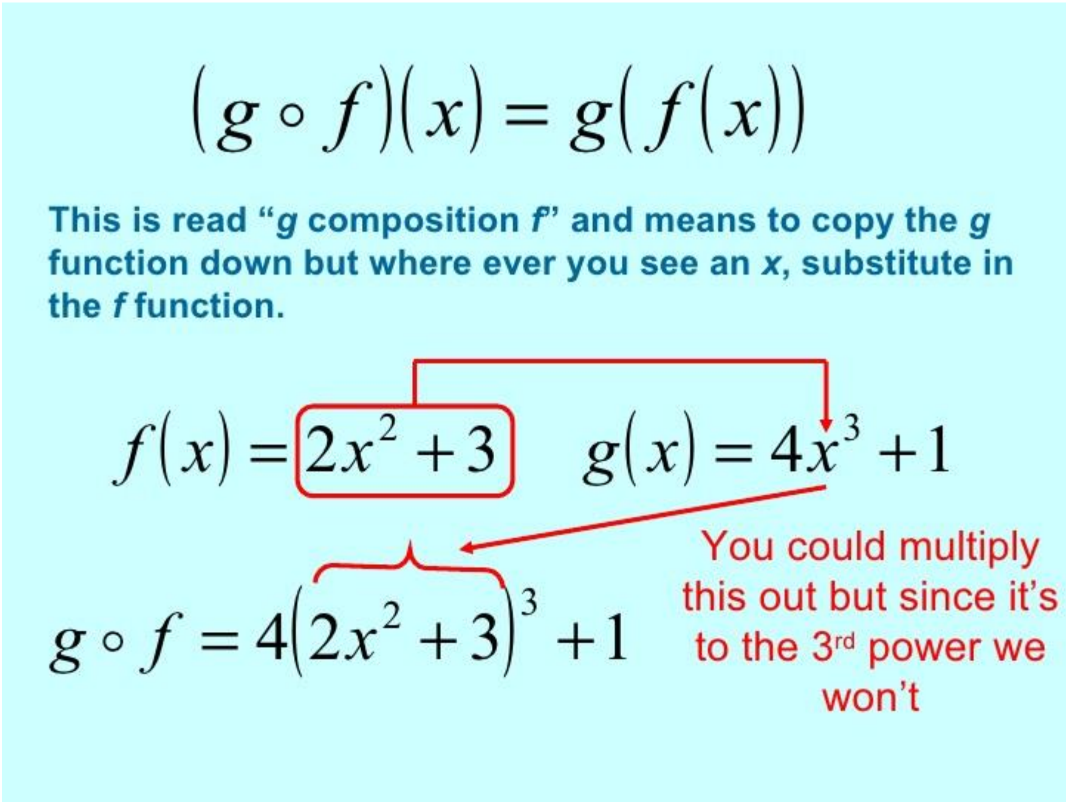
Definition: If f and g are functions, the **composite** function fog is defined by

(f 0 g)(x) = f(g(x)).

The domain of *fog* consists of the numbers x in the domain of g for which g(x) lies in the domain of f.







EXAMPLE 2 If $f(x) = \sqrt{x}$ and $g(x) = x + 1$, find

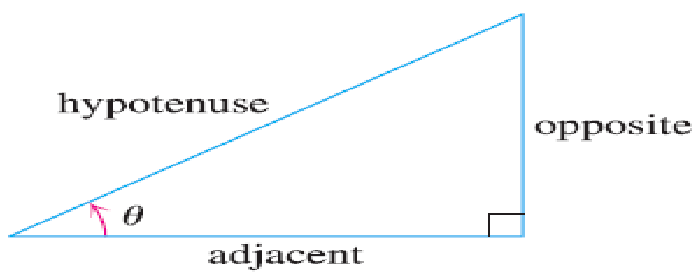
(a) $(f \circ g)(x)$ **(b)** $(g \circ f)(x)$ **(c)** $(f \circ f)(x)$ **(d)** $(g \circ g)(x)$.

	Composite	Domain
(a)	$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x + 1}$	$[-1, \infty)$
(b)	$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
(c)	$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
(d)	$(g \circ g)(x) = g(g(x)) = g(x) + 1 = (x + 1) + 1 = x + 2$	$(-\infty, \infty)$

Chapter Two

2.1 Trigonometric Functions:

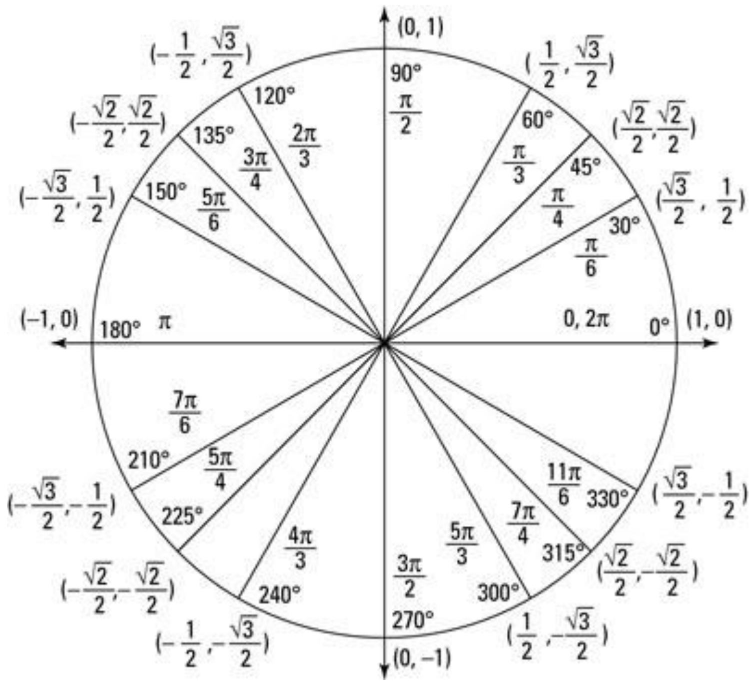
Trigonometric functions (also called circular functions, angle functions or goniometric functions) are real functions which relate an angle of a right-angled triangle to ratios of two side lengths (**defined as ratios of two sides of a right triangle containing the angle)** and can equivalently be defined as the lengths of various line segments from a unit circle (a circle with radius of one).



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



$$1 \text{ radian} = \frac{180}{\pi} (\approx 57.3) \text{ degrees} \qquad \text{or} \qquad 1 \text{ degree} = \frac{\pi}{180} (\approx 0.017) \text{ radians.}$$

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

Notice, also that whenever the quotients are defined:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Some properties of trigonometric functions:

1.

Even

$\cos(-x) = \cos x$

$\sec(-x) = \sec x$

Odd

$\sin(-x) = -\sin x$

$\tan(-x) = -\tan x$

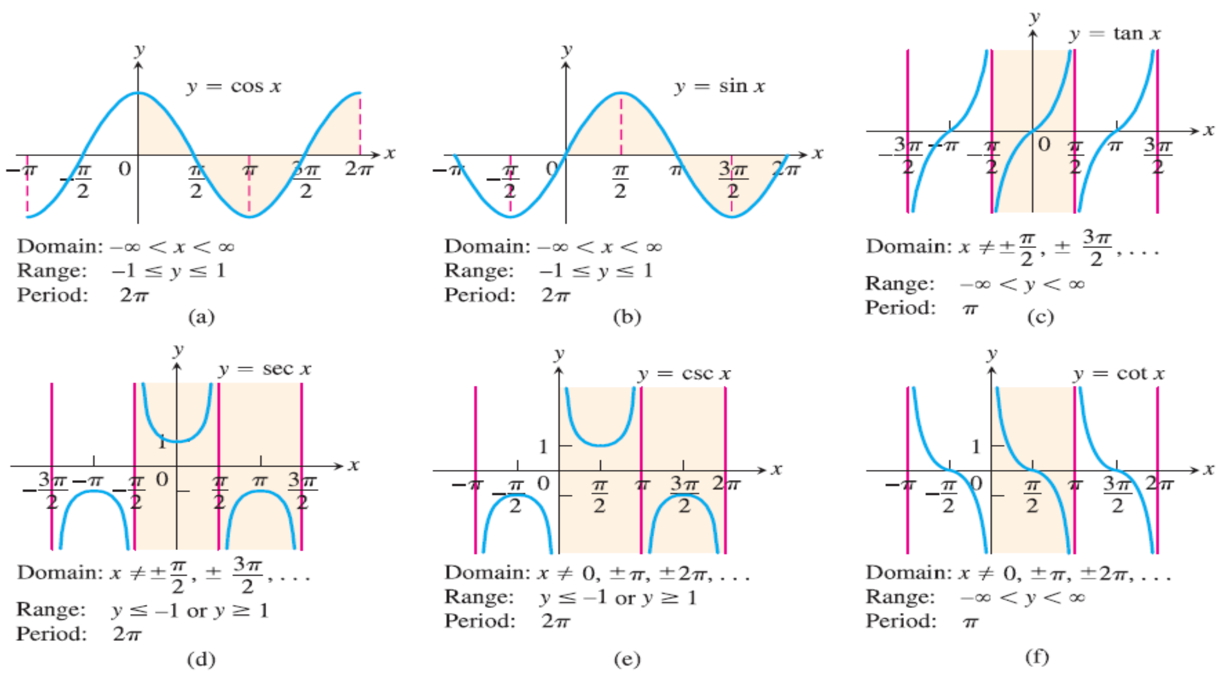
$\csc(-x) = -\csc x$

$\cot(-x) = -\cot x$

Definition: A function f(x) is said to be periodic function if there is a positive number p such that f(x+p) = f(x) for every value of x.

2. The six basic trigonometric functions are periodic:

$$\tan(x+\pi) = \tan x$$
$$\cot(x+\pi) = \cot x$$
$$\sin(x+2\pi) = \sin x$$
$$\cos(x+2\pi) = \cos x$$
$$\sec(x+2\pi) = \sec x$$
$$\csc(x+2\pi) = \csc x$$



Some Laws:

$$\cos^2\theta + \sin^2\theta = 1.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$
$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

2 Limit of a Function and Limit Laws:

The limit of a function is the behavior of that function near a particular input.

Definition 1: Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say that the limit of f(x) as x approaches a is L, and we write:

$$\lim_{x\to a} f(x) = L$$

Definition 2: The limit of the function f(x) exists if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \text{L}$$

Some limits:

(a) If *f* is the identity function *f(x)* = *x*, then for any value of x_o,

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} x = x_0$$

(b) If *f* is the constant function *f(x)* = *k* (function with the constant value *k)*, then:

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} k = k.$$

For example:

$$\lim_{x\to 3} x = 3 \quad \text{and} \quad \lim_{x\to -7}(4) = \lim_{x\to 2}(4) = 4.$$

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c} (f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c} (k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c} (f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x \to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE:

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$

$$= c^3 + 4c^2 - 3$$

(b) $\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)}$

$$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

EXAMPLE:

$$\lim_{x\to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

EXAMPLE:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x\to 1} \frac{(x - 1)(x + 2)}{x(x - 1)} = \lim_{x\to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3$$

Example: Find 1. $\lim_{x\to 0} |x|$ 2. $\lim_{x\to 0} \frac{|x|}{x}$

Solution: 1.

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

$\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0 = L_1$

$\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} -x = 0 = L_2$, $L_1 = L_2 \rightarrow \lim_{x\to 0} |x| = 0$.

2. $\frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1 & x \geq 0 \\ \frac{-x}{x} = -1 & x < 0 \end{cases}$

$\frac{|x|}{x} = \begin{cases} 1 & x \leq 0 \\ -1 & x > 0 \end{cases}$

$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} 1 = 1 = L_1$

$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} = -1 = L_2$

$L_1 \neq L_2$

∴ f(x) has no limit at x = 0.

Exercise: Evaluate the following:

1. $\lim_{x \to 1} \frac{x^2-1}{x-1}$
2. $\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$
3. $\lim_{x \to 0} \frac{\sqrt{x^2+100}-10}{x^2}$

2.3 Continuity of Functions:

Definition of Continuity

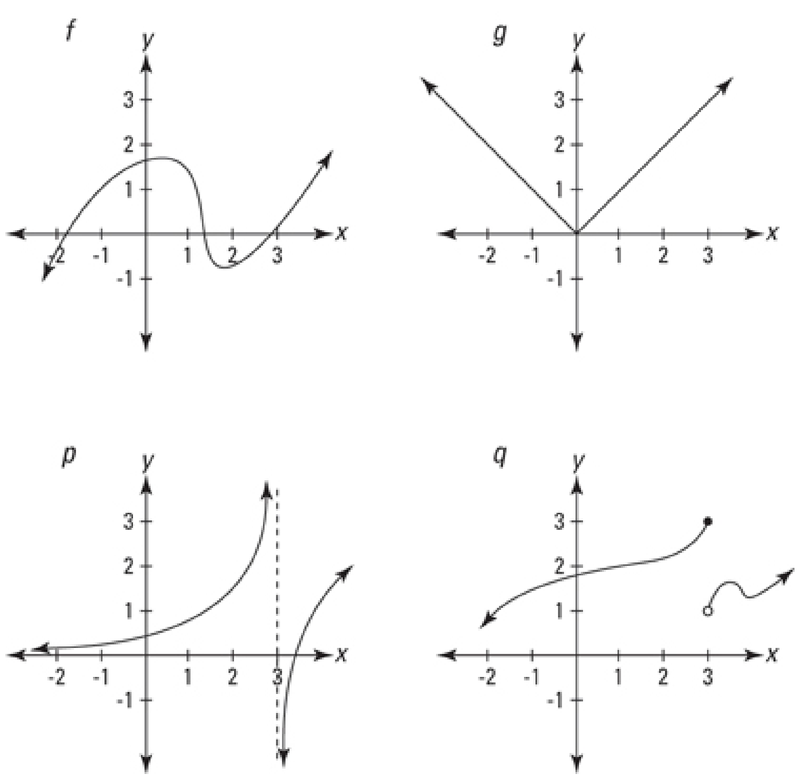
A function f is continuous at $x = a$ when:

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x)$ exist
3. $\lim_{x \to a} f(x) = f(a)$

If any one of the conditions is not met, the function is not continuous at $x = a$.

A *continuous* function is simply a function with no gaps. A function that you can draw without taking your pencil off the paper. Consider the four functions in this figure.

The first two functions in this figure $f(x)$ and $g(x)$ have no gaps, so they're continuous. The next two $p(x)$ and $q(x)$ have gaps at $x = 3$, so they're not continuous.



Example 1:

Is f(x) = x^2 is continuous at x = 3?

Solution:

1. f(3) = 9
2. $\lim_{x\to 3} f(x)$ = $\lim_{x\to 3} x^2$ = 9
3. f(3) = $\lim_{x\to 3} f(x)$

∴ f(x) is continuous at x = 3.

In general f(x) = x^2 continuous at all x ∈ ℝ.

Example 2:

Is f(x) = $\begin{cases} 4x-2 & x>2 \\ 2 & x=2 \\ 3x & x<2 \end{cases}$ continuous at x = 2?

Solution:

1. f(2) = 2
2. $\lim_{x\to 2^+} f(x)$ = $\lim_{x\to 2^+} 4x-2 = 4(2)-2 = 6$.
   $\lim_{x\to 2^-} f(x)$= $\lim_{x\to 2^-} (3x) = 3.2 = 6$.
   $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) \to \lim_{x\to 2} f(x)$ = 6.
3. f(2) ≠ $\lim_{x\to 2} f(x)$,
4. ∴ f is not continuous at x=2.

Example 3:

Is g(x) = $\begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 4 & x = 2 \end{cases}$

Continuous at x=2?

Solution:

1. g(2) = 4
2. $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = 4$
3. g(2) = $\lim_{x \to 2} g(x)$ = 4

∴ g(x) is continuous at x=2.

Chapter Three

DIFFERENTIATION

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Example: Use definition of derivative to find the derivative of f(x) = x^2.

Solution:

$$\frac{dy}{dx} = y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} 2x+h = 2x.$$

Example: Use definition of derivative to find the derivative of f(x) = $\sqrt{x}$.

Solution:

$$\frac{dy}{dx} = y' = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} * \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{(\sqrt{x+h} + \sqrt{x})} = \frac{1}{(2\sqrt{x})}.$$

Example: Use definition of derivative to find the derivative of f(x) = |x| is not differentiable at x = 0.

Solution:

$$\frac{dy}{dx} = y' = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$y'(0) = \lim_{h \to 0} \frac{|0+h|-|0|}{h} = \lim_{h \to 0} \frac{|h|}{h} = \begin{cases} 1 & h \geq 0 \\ -1 & h < 0 \end{cases}$, The limit does not exist, so the function f(x)=|x| is not differentiable at x = 0.

Basic Derivatives Rules

Constant Rule: $\frac{d}{dx}(c) = 0$

Constant Multiple Rule: $\frac{d}{dx}[cf(x)] = cf'(x)$

Power Rule: $\frac{d}{dx}(x^n) = nx^{n-1}$

Sum Rule: $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

Difference Rule: $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Product Rule: $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

Example 1 – *Using the Power Rule*

(a) If $f(x) = x^6$, then $f'(x) = 6x^5$.

(b) If $y = x^{1000}$, then $y' = 1000x^{999}$.

(c) If $y = t^4$, then $\frac{dy}{dt} = 4t^3$.

(d) $\frac{d}{dr}(r^3) = 3r^2$

Example 2

Let $y = \frac{x^2 + x - 2}{x^3 + 6}$. Then

$$y' = \frac{(x^3 + 6)\frac{d}{dx}(x^2 + x - 2) - (x^2 + x - 2)\frac{d}{dx}(x^3 + 6)}{(x^3 + 6)^2}$$

$$= \frac{(x^3 + 6)(2x + 1) - (x^2 + x - 2)(3x^2)}{(x^3 + 6)^2}$$

$$= \frac{(2x^4 + x^3 + 12x + 6) - (3x^4 + 3x^3 - 6x^2)}{(x^3 + 6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{(x^3 + 6)^2}$$

Examples: Find $\frac{dy}{dx}$ for the following functions:

1. $y = 2x^3 + \sqrt{5x^2 + 2}$ 2. $y = (4x^2-1)(7x^3+x)$ 3. $y = \frac{x^2-1}{x^4+1}$

Second- and Higher-Order Derivatives

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

Example: The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

Derivatives of trigonometric functions:

Original Rule	Generalized Rule (Chain Rule)
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\sin \mathbf{u} = \cos \mathbf{u}\,\frac{du}{dx}$
$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\cos \mathbf{u} = -\sin \mathbf{u}\,\frac{du}{dx}$
$\frac{d}{dx}\tan x = \sec^2 x$	$\frac{d}{dx}\tan \mathbf{u} = \sec^2\mathbf{u}\,\frac{du}{dx}$
$\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\cot \mathbf{u} = -\operatorname{cosec}^2\mathbf{u}\,\frac{du}{dx}$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\sec \mathbf{u} = \sec \mathbf{u} \tan \mathbf{u}\,\frac{du}{dx}$
$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot$	$\frac{d}{dx}\operatorname{cosec} \mathbf{u} = -\operatorname{cosec} \mathbf{u} \cot \mathbf{u}\,\frac{du}{dx}$

Example: Find f′(x) where f(x) = $\frac{sinx\ secx}{1+x\ tanx}$

Solution:

f′(x) = $\frac{(1+x\ tanx)(sinx\ .\sec x \tan x + secx.cosx)-(sinx\ secx)(x.\ sec^2x + tanx\ .1)}{(1+x\ tanx)^2}$

The Chain rule:

1. If y = f(u) and u = g(x), then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Or $\frac{dy}{dx}$ = f′ (g(x)) g′(x)

2. If y = f(t) and x = g(t) , then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Example: Let y = u^{10} and u = x^3 +7x+1, find $\frac{dy}{dx}$.

Solution:

$\frac{dy}{du} = 10\ u^9$, $\frac{du}{dx} = 3x^2 + 7 \rightarrow \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (10\ u^9).\ (3x^2 + 7)$

$= [10(x^3 + 7x + 1)^9].\ (3x^2 + 7)$

Example:

$y = (x^3+7x+1)^{10} \rightarrow \frac{dy}{dx} = [10(x^3+7x+1)^9].\ (3x^2+7)$

Example:

Let y= t^2 and x = 5t +2, find $\frac{dy}{dx}$.

Solution:

$\frac{dy}{dt} = 2t$, $\frac{dx}{dt} = 5 \rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{5} = \frac{2}{5}\left(\frac{x-2}{5}\right)$.

Implicit Differentiation:

1. Differentiate both sides of the equation with respect to x
2. Move all $\frac{dy}{dx}$ terms to the left side, and all other terms to the right side
3. Factor out $\frac{dy}{dx}$ from the left side
4. Solve for $\frac{dy}{dx}$, by dividing

Example: Find $\frac{dy}{dx}$, if $5y^2 + \sin y = x^2$.

Solution:

$10\,y\,\frac{dy}{dx} + \cos y\,\frac{dy}{dx} = 2x$

$(10y + \cos y)\,\frac{dy}{dx} = 2x$

$\frac{dy}{dx} = \frac{2x}{(10y+\cos y)}$

Example: Find $\frac{dw}{dx}$ if $a^2w^2 + b^2\,x^2 = 1$ where a, b are constant.

Solution:

$2\,a^2\,w\,\frac{dw}{dx} + 2\,b^2\,x = 0 \;\rightarrow\; \frac{dw}{dx} = \frac{2-b^2\,x}{2a^2w} = \frac{-b^2\,x}{a^2w}$

The exponential function:

Definition1: A function of the form $\mathbf{y = f(x) = a^x}$, $a > 0$.

Example: $y = 2^{\sin x}$, $y = 3^{3x+1}$.

Properties:

1. $\mathbf{a^x . a^b = a^{x+y}}$
2. $\mathbf{\frac{a^x}{a^y} = a^{x-y}}$
3. $\mathbf{(a^x)^n = a^{nx}}$

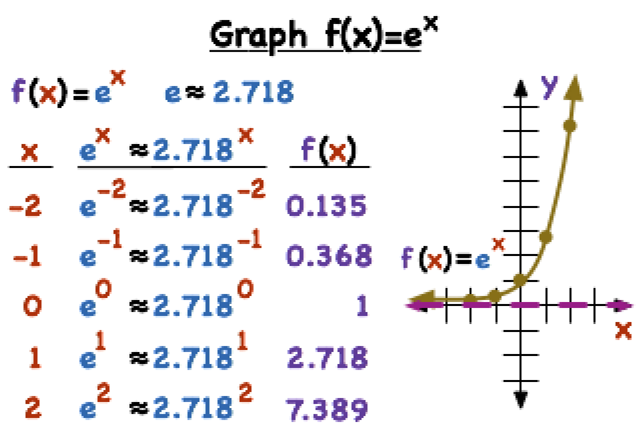
The natural exponential function:

If a = e in definition 1, then $\mathbf{y = e^{u(x)}}$ is called natural exponential function.

$e = 2.7182828 \cong 2.7$

$D_f = R$, $R_f = \{y : y > 0\}$

$\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to-\infty} e^x = 0$



Properties:

1. $e^x \cdot e^y = e^{x+y}$
2. $\frac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^n = e^{nx}$

Logarithms:

A function of the form $y = \log_a u(x)$ is called logarithm function of u(x) with basis a where $0 < a \neq 1$.

Properties:

1. $\log_a(x \cdot y) = \log_a x + \log_a y$
2. $\log_a \frac{x}{y} = \log_a x - \log_a y$
3. $\log_a x^n = n \log_a x$
4. $\log_a a = 1$, $\log_a 1 = 0$.

Example:

Solve for x

$\log_5(5^{2x}) = 8$

$2x \log_5 5 = 8$

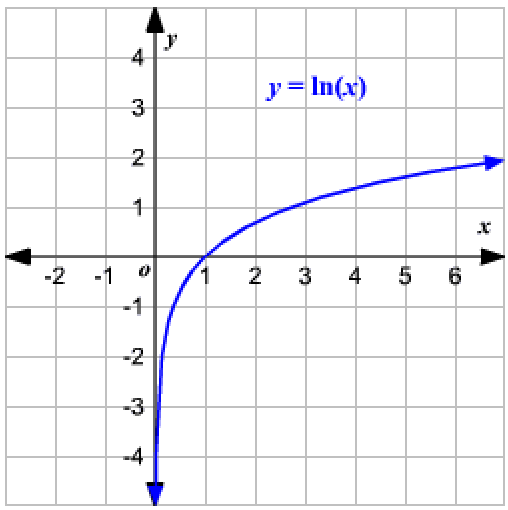
$2x(1) = 8 \Rightarrow x = 4$.

The Natural Logarithms:

If a = e , then $\log_e x = \ln x$

y = f(x) = ln x

$D_f = \mathbb{R}^+$, $R_f = \mathbb{R}$.



Properties:

1. $\ln(x.y) = \ln x + \ln y$
2. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$.
3. $\ln x^n = n \ln x$
4. $\ln 1 = 0$, $\ln e = 1$

Derivative of the exponential function, natural exponential function, Logarithms and natural Logarithms:

1. If $y = a^u$, then $\frac{dy}{dx} = a^u \frac{du}{dx} \ln a$
2. If $y = e^u$, then $\frac{dy}{dx} = e^u \frac{du}{dx}$
3. If $y = \log_a u$, then $\frac{dy}{dx} = \frac{1}{u} \log_a e \frac{du}{dx}$
4. If $y = \ln u$, then $\frac{dy}{dx} = \frac{1}{u} . \frac{du}{dx}$

Examples: Find $\frac{dy}{dx}$ of the following functions:

1. $y = 5^{x^2} \Longrightarrow \frac{dy}{dx} = 5^{x^2} \ln 5.\ 2x$
2. $y = e^{4x^2} \Longrightarrow \frac{dy}{dx} = e^{4x^2}\ .\ 8x$
3. $y = \ln 2\,x^3 \Longrightarrow \frac{dy}{dx} = \frac{1}{2x^3} \cdot 6x^2\ = \frac{3}{x}.$
4. $y = e^{2x} + \ln(5x^2 + 7x + 1) \Longrightarrow \frac{dy}{dx} = e^{2x}.\ 2 + \frac{1}{5x^2+7x+1}(10x+7)$

   $= 2.\ e^{2x} + \frac{10x+7}{5x^2+7x+1}.$

Example: Find the value of x :

$$2 \ln \sqrt{x+1} + \ln(x-3) = \ln 5$$

Solution:

$2 \ln (x+1)^{1/2} + \ln(x-3) = \ln 5$

$2.\frac{1}{2} \ln (x+1) + \ln(x-3) = \ln 5$

$\ln[(x+1).(x-3)] = \ln 5 \Longrightarrow \ln(x^2-2x-3) = \ln 5$

$e^{\ln(x^2-2x-3)} = e^{\ln 5} \Longrightarrow x^2-2x-3 = 5$

$\Longrightarrow x^2-2x-8=0$

$(x-4)(x+2)=0 \Longrightarrow$ either $x = 4$ or $x = -2$ تهمل $\Longrightarrow x = 4$.

Chapter Four

The integration is a method to calculate the areas and volumes of triangles, spheres, cones and other more general shapes and has many applications in statistic, economics, sciences and engineering.

There are two types of integral are definite and indefinite integral.

1. Definite integral:

The symbol for the number of integral in the definite integral is :

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Which is read as the integral from a to b of f(x).

Rules satisfied by definite integrals:

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition
2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — A Definition when $f(a)$ exists
3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any constant k
4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

$$f(x) \geq g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

$$f(x) \geq 0 \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \geq 0 \quad \text{(Special Case)}$$

EXAMPLE:

Let $\int_{-1}^{1} f(x)\,dx = 5, \quad \int_{1}^{4} f(x)\,dx = -2, \quad \text{and} \quad \int_{-1}^{1} h(x)\,dx = 7.$

Then:

1. $\int_4^1 f(x)\,dx = -\int_1^4 f(x)\,dx = -(-2) = 2$

2. $\int_{-1}^{1} [2f(x) + 3h(x)]\,dx = 2\int_{-1}^{1} f(x)\,dx + 3\int_{-1}^{1} h(x)\,dx$

$= 2(5) + 3(7) = 31$

3. $\int_{-1}^{4} f(x)\,dx = \int_{-1}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx = 5 + (-2) = 3$

2. Indefinite integral:

If the function f(x) is a derivative then the set of all antiderivative of f is called the indefinite integral of f denoted by symbols:

$$\int \mathbf{f(x)\,dx = F(x) + C},$$

where F(x) is antiderivative of a function f(x) and C is any arbitrary constant and it is called the constant of integration.

Rules for indefinite integrals:

1. $\int dx$ = x + C , where c is constant.
2. $\int k\ dx$ = k x + C , where k is any number.
3. $\int(dx\ \mp dz) = \int dx \mp \int dz$ = x ∓ z + C.
4. $\int(f(x) \mp g(x))dx = \int f(x)dx \mp \int g(x)dx.$
5. $\int x^n dx = \frac{x^{n+1}}{n+1}$ + C   (n ≠ -1)
6. $\int \frac{dx}{x}$ = ln |x| + C.

Examples:

1. $\int x^2 dx = \frac{x^3}{3}$ + C.
2. $\int_1^3 x^2 dx = \frac{x^3}{3}\Big|_1^3 = \frac{27}{3} - \frac{1}{3} = \frac{26}{3}$
3. $\int(2x^2+1)^3 2x\ dx = \frac{1}{2}\frac{(2x^2+1)^4}{4}$ +C.
4. $\int \frac{1}{x^5}\ dx = \int x^{-5}dx = \frac{x^{-4}}{-4}$ + C.
5. $\int \frac{1}{x+1}\ dx$ = ln|x+1| +C.
6. $\int \frac{2x+1}{x^2+x}\ dx$ = ln $|x^2+x|$ +C.
7. $\int \frac{2x}{(x^2+1)^3}\ dx = \int 2x(x^2+1)^{-3}dx = \frac{(x^2+1)^{-2}}{-2}$ + C.
8. $\int 2x\sqrt{1+x^2}\ dx = \int(1+x^2)^{\frac{1}{2}}$ 2x dx = $\frac{(1+x^2)^{\frac{3}{2}}}{\frac{3}{2}}$ + C = $\frac{2}{3}(1+x^2)^{\frac{3}{2}}$ + C.

Integration of Trigonometric Functions:

Basic Trig Derivatives and their Corresponding Indefinite Integrals

$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\, dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\, dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\, dx = -\csc x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\, dx = -\cot x + C$

ملاحظة: عند تكامل الدوال المثلثية علينا توفير شرط مشتقة الزاوية.

Examples:

1. $\int \sin(4x^2 - 3)x\, dx = \frac{1}{8}\int \sin(4x^2 - 3)\, 8x\, dx = -\frac{\cos(4x^2-3)}{8} + C.$
2. $\int \frac{\cos x}{1+\sin x}\, dx = \ln|1 + sinx| + C.$
3. $\int \tan x\, dx = \int \frac{\sin x}{\cos x} = -\ln|\cos x| + C.$

Integral of exponential function:

1. $\int a^x\ dx = \frac{a^x}{\ln a} + C$
2. $\int e^x\ dx = e^x + C$

Examples:

1. $\int 2^x \; dx = \frac{2^x}{\ln 2}$ + C.
2. $\int e^{5x} \, dx = \frac{1}{5} e^{5x}$ + C
3. $\int e^{-x} \, dx$ = - e^{-x} + C
4. $\int x^2 e^{x^3} \, dx = \frac{e^{x^3}}{3}$ + C
5. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$ = 2 $e^{\sqrt{x}}$ + C
6. $\int e^{\sin x}$ cos x dx = $e^{\sin x}$
7. $\int \sqrt{e^x}$ dx = $\int e^{x\frac{1}{2}}$ dx = 2 $e^{\frac{1x}{2}}$
8. $\int tan^2 \, x \, dx = \int (sec^2 x - 1) dx = \int sec^2 \, x \, dx$ - $\int 1 \, dx$ = tan x – x+ C

Methods of Integration:

1. Integration by Parts:

Integration by Parts Formula

$$\int u \, dv = uv - \int v \, du$$

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \qquad dv = \cos x \, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

EXAMPLE 2 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\quad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\quad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

Remark: Sometimes we have to use integration by parts more than once as follows:

EXAMPLE 3 Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x\,dx$. Then $du = dx$, $v = e^x$, and

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Using this last evaluation, we then obtain

$$\begin{aligned}\int x^2e^x\,dx &= x^2e^x - 2\int xe^x\,dx \\ &= x^2e^x - 2xe^x + 2e^x + C.\end{aligned}$$

■

EXAMPLE 4 Evaluate

$$\int e^x \cos x\,dx.$$

Solution Let $u = e^x$ and $dv = \cos x\,dx$. Then $du = e^x\,dx$, $v = \sin x$, and

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x\,dx, \qquad v = -\cos x, \qquad du = e^x\,dx.$$

Then

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x\,dx)\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration give

$$\int e^x \cos x\,dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$

2.Tabular Integration:

تستخدم هذة الطريقة في تكامل

$$\int \boldsymbol{f(x)g(x)dx}$$

يجب أن يكون أشتقاق أحد الدالتين يقودنا الى الصفر و الأخرى يمكن أن تكامل عدة مرات.

<u>Example:</u> Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

d		$\int$
x^2	(+)	e^x
$2x$	(−)	e^x
2	(+)	e^x
0		e^x

Then :

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Example: Evaluate $\int x^3 \sin x \, dx$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$

The Integrals of cot x, sec x, and csc x:

$$\int \cot x\,dx = \int \frac{\cos x\,dx}{\sin x} = \int \frac{du}{u} \qquad u = \sin x,\ du = \cos x\,dx$$

$$= \ln|u| + C = \ln|\sin x| + C = -\ln|\csc x| + C.$$

$$\int \sec x\,dx = \int \sec x \frac{(\sec x + \tan x)}{(\sec x + \tan x)}dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}dx$$

$$= \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C \qquad u = \sec x + \tan x,\ du = (\sec x \tan x + \sec^2 x)\,dx$$

$$\int \csc x\,dx = \int \csc x \frac{(\csc x + \cot x)}{(\csc x + \cot x)}dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}dx$$

$$= \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C \qquad u = \csc x + \cot x,\ du = (-\csc x \cot x - \csc^2 x)\,dx$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

The integrals of sin^2x and $cos\ ^2x$:

(a) $$\int \sin^2 x\, dx = \int \frac{1-\cos 2x}{2}\, dx \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2}\int (1-\cos 2x)\, dx$$
$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

(b) $$\int \cos^2 x\, dx = \int \frac{1+\cos 2x}{2}\, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2}$$ ■

Products of Powers of Sines and Cosines:

We begin with integrals of the form: $\int \sin^m x \cos^n x\, dx,$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and *n* being odd or even.

Case 1 If ***m* is odd**, we write m as $2k+1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If ***m* is even and *n* is odd** in $\int \sin^m x \cos^n x\, dx$, we write n as $2k+1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1-\sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If **both *m* and *n* are even** in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1-\cos 2x}{2}, \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

EXAMPLE 1 Evaluate

$$\int \sin^3 x \cos^2 x \, dx.$$

Solution This is an example of Case 1.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^2 x \cos^2 x \sin x \, dx \qquad m \text{ is odd.}$$

$$= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \qquad \sin x \, dx = -d(\cos x)$$

$$= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x$$

$$= \int (u^4 - u^2) \, du \qquad \text{Multiply terms.}$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C. \qquad \blacksquare$$

EXAMPLE 2: Evaluate $\int sin^4x\ cos^5x$ dx

Solution:

$\int sin^4x\ cos^5x$ dx = $\int sin^4x\ cos^4.cosx$ dx

= $\int sin^4x\ (1-sin^2x)^2 cosx\, dx$

Let u = sin x , du = cos x dx

$\int u^4\ (1-u^2)^2\ du$ = $\int u^4(1-2u^2+u^4)\ du$

= $\int u^4-2u^6+u^8)\ du$ = $\frac{u^5}{5} - 2\,\frac{u^7}{7} + \frac{u^9}{9}$ + C.

= $\frac{1}{5}\, sin^5x$ - $\frac{2}{7}\, sin^7x$+ $\frac{1}{9}\, sin^9x$+ C

EXAMPLE 3: Evaluate $\int sin^2 2x\, cos^2 2x\, dx = \int (\frac{1-\cos 4x}{4})\ (\frac{1+\cos 4x}{4}) dx$

$= \frac{1}{4}\int (1 - cos^2 4x)\, dx = \frac{1}{4}\int (1 - \frac{1+\cos 8x}{2}) dx = \frac{1}{4}\int (1 - \frac{1}{2} - \frac{1}{2}\cos 8x) dx$

$= \frac{1}{4}\int (\frac{1}{2} - \frac{1}{2}\cos 8x) dx = \frac{1}{4}[\frac{x}{2} - \frac{1}{16}\sin 8x] + C.$

Integral of sec and tan:

We will use the identity $\tan^2 x = \sec^2 x - 1$ or $\sec^2 x = \tan^2 x + 1$.

Case 1: If $\int \sec^m x\ dx$, $\int \tan^m x\, dx$ and m is even, then we write m as 2k+2.

Example 1: **Evaluate** $\int \tan^4 x\, dx$

Solution:

$\int \tan^4 x\, dx = \int \tan^2 x\ \tan^2 x\, dx$

$= \int (\sec^2 x - 1)\ \tan^2 x\, dx = \int (\sec^2 x \tan^2 x - \tan^2 x)\, dx = = \frac{\tan^3 x}{3} - \tan x + x + C.$

Case 2: If m is odd , then we write m as 2k+1 and use the identity

$\tan^2 x = \sec^2 x - 1$ or $\sec^2 x = \tan^2 x + 1$.

Example 2: **Evaluate** $\int tan^5 x\, dx$

Solution:

$\int tan^5 x\, dx = \int \tan^4 x \tan x\, dx = \int (\sec^2 x - 1)^2\ \tan x\, dx$

$= \int (sec^4 x - 2\, sec^2 x + 1) \tan x\, dx$

$= \int sec^4 x\, tanx\, dx - 2\int sec^2 x\, tanx\, dx + \int tanx\, dx$

$= \int sec^3 x\, secx\, tanx - 2\int sec^2 x\, tanx\, dx + \int tanx\, dx$

$= \frac{sec^4 x}{4} - 2\,\frac{tan^2 x}{2}$ - ln $|cosx|$ +C.

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

$$\sin mx \sin nx = \frac{1}{2}[\cos (m-n)x - \cos (m+n)x],$$

$$\sin mx \cos nx = \frac{1}{2}[\sin (m-n)x + \sin (m+n)x],$$

$$\cos mx \cos nx = \frac{1}{2}[\cos (m-n)x + \cos (m+n)x].$$

Example: Evaluate $\int \sin 7x\ \cos 3x\ dx$

Solution:

$\int \sin 7x\ \cos 3x\ dx = \int \frac{1}{2}\ (\sin 7x - 3x) + \ \sin(7x + 3x))\ dx$

$= \frac{1}{2} \int (\sin 4x + \sin 10x)\ dx = \frac{1}{2}\,(-\frac{1}{4} \cos 4x - \frac{1}{10} \cos 4x) + C.$

Exercise:

Evaluate $\int \sin 3x\ \cos 5x\ dx$