Operations Research I

First Semester

For the 3rd class students / Mathematics Department / College of Science for Women

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Preface

These lecture notes are for the course “Operations Research I” for the 3rd grade- first semester in Mathematics Department / College of Science for Women / University of Baghdad.

The author claims no originality. These lecture notes are collected from references listed in the “Bibliography”.

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Ch.1: Introduction to Operations Research (OR)

1.1 Development of Operations Research (OR)

1.1.1 Before World War II

No science has ever been born on a specific day. Operations research is no exception. Its roots extend to even early 1800; it was in 1885 when F.W. Taylor (1856 - 1915) emphasized the application of scientific analysis to methods of production and management. He is considered as the father of "industrial engineering". H.L. Gantt (1861 – 1919) has a great contribution in production and management and introduced the Gantt chart in 1910 to represent production schedules. In 1917, A. K. Erlang (1878-1929) published his work on the problem of congestion of telephone traffic which contains his formulae for call loss and waiting time. He is considered as the inventor of "queuing theory".

1.1.2 World War II

The modern field of OR arose during the World War II. The military management in England called on a team of scientists to study the strategic and tactical problem of air and land defense. The objective was to determine the most effective utilization of limited military resources. The application included the effective use of newly invented radar, allocation of British Air Force planes to missions and the determination of best patterns for searching submarines. This group of scientists formed the first OR team.

The name ***operations research*** (or ***operational research***) was because the team was searching out research on (military) operations. The encouraging results of those efforts led to the formation of more such teams in British armed services and the use of such scientific teams soon spread to the Western Allies: the US, Canada, and France.

1.1.3 After World War II

Immediately after the war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their problems. As the industrial boom following the war was running its course, the problems caused by the increasing complexity and specialization in organizations was again coming to forefront. By 1950s, OR is used to a variety of organizations in business, industry, and government. The rapid spread of OR soon followed.

At least two other factors that played a key role in the rapid growth of OR during this period can be identified. One was the substantial progress that was made early in improving the techniques to OR. Many of the standard tools of OR, such as linear programming, dynamic programming, queuing theory, and inventory theory were relatively well developed before the end of 1950s.

A second factor that gave a great impetus to the growth of the field was the onslaught of the computer revolution. A large amount of computations is usually required to deal most effectively with the complex problems typically considered by OR. Doing this by hand would often be out of the question. Therefore the development of electronic, digital computers, especially after the 1980s, with their ability to perform arithmetic calculations thousands or even millions of times faster than a human being can, was a tremendous boon to OR. In 1980s and afterward, several good software packages for doing OR was developed.

Today, OR is recognized worldwide as a modern decision- aiding science that has proved to be of great value to management, business, and industry.

1.2 Definition of Operation Research

Many definitions of OR have been suggested from time to time. We can define OR as follows:

Definition (1.1):

Operations research, in the most general sense, can be characterized as the application of scientific methods, techniques, and tools to problems involving the operations of systems so as to provide those in control of the operations with optimum solutions to the problem.

By a ***system***, we mean an organization of independent components that work together to accomplish the goal of the system. For example a car manufacturing company.

1.3 Prescriptive or Optimization Models

Most of the models in OR are ***prescriptive*** or ***optimization*** models. A ***prescriptive model*** " prescribes" behavior for an organization that will enable it to meet its goal(s). The components of a prescriptive model include:

- ***Objective function:*** it is a function we wish to maximize or minimize.
- ***Decision variables:*** the variables whose values are under our control and influence the performance of the system.
- ***Constraints:*** restrictions on the values of decision variables.

In short, an ***optimization model*** seeks to find values of the decision variables that optimize (minimize or maximize) an objective function among the set of all values for the decision variables that satisfy the given constraints.

1.4 Phases in Solving OR Problems

Any OR analyst has to follow certain sequential steps to solve the problems on hand. The steps are described below:

1.4.1 Formulation of the Problem

This step involves defining the scope of the problem under investigation. The OR team should identify three principal elements of the decision problem:

1- Description of the decision alternatives.
2- Determination of the objective of the study.
3- Specification of the limitations under which the modeled system operates.

1.4.2 Model Construction

This step is to translate the problem definition into mathematical relationships, i.e. to construct a model. In OR study, it is usually a ***mathematical model***. A model helps to analyze a system, make the problem more meaningful, and clarifies important relationships among the variables. It also tells us to which of the variables are more important than the others. It must not be forgotten that a model is only an approximation of the reality (real situation). Hence it may not include all the variables.

1.4.3 Solve the Model

This step entails the use of well-defined optimization algorithm to find optimal, or best, solution. That is to find the values of decision variables.

Since a model is an approximation of the real system or problem, the optimum solution for the model does not guarantee an optimum solution for the real problem. However, if the model is well formulated and tested, solution from the model will provide a good approximation to the solution of the real problem.

1.4.4 Testing the Model and the Solution Derived from it

As already discussed, a model is never a perfect representation of reality. But, if properly formulated and correctly manipulated, it may be useful in predicting the effect of changes in control variables on the overall system effectiveness. The usefulness of a model is tested by determining how well it predicts the effect of these changes. Such an analysis is usually called ***sensitivity analysis***. Sensitivity analysis is particularly needed when the parameters of the model cannot be estimated accurately. In these cases, it is important to study the behavior of the optimum solution in the neighborhood of the estimated parameters.

1.4.5 Establishing Controls Over the Solution

Since life is not static, a solution which we felt was optimum today may not be so tomorrow, since the values of the variables (parameters) may change, new parameters may emerge and the structural relationship between the variables may undergo a change.

The solution derived from a model goes out of control if the values of one or more uncontrolled variables vary or relationship between variables undergoes a change. Therefore, controls must be established to indicate the limits within which the model and its solution can be considered as reliable. Also tools must be developed to indicate as to how and when the model or its solution will have to be modified to take the changes into account.

1.4.6 Implementation

The solution obtained above should be translated into operating procedures which can be easily understood and applied by those who control the operations. Necessary changes should be implemented by OR team.

Ch. 2: Linear Programming Models Solution

2.1 History

The problem of solving a system of linear inequalities dates back at least as far as Fourier (1768–1830). The linear programming method was first developed by Leonid Kantorovich (1912-1986) in 1939. Leonid Kantorovich developed the earliest linear programming problems in 1939 for use during World War II. The method was kept secret until 1947 when George B. Dantzig (1914–2005) published the simplex method and John von Neumann (1903–1957) developed the theory of duality as a linear optimization solution, and applied it in the field of game theory. A larger theoretical and practical breakthrough in the field came in 1984 when Narendra Karmarkar(1957-) introduced a new interior-point method for solving linear-programming problems.

2.2 Linear Programming

Let us start by considering optimization problem

Definition (2.1):

An ***optimization problem*** (or ***mathematical programming problem***) is that branch of mathematics dealing with techniques for maximizing or minimizing an objective function subject to linear, nonlinear, and integer constraints. In other words, it is the problem of minimizing or maximizing the objective function $z = f(x_1, x_2, \dots, x_n)$ subject to the constraints:

$$\begin{aligned} &g_1\ (x_1, x_2, \dots, x_n) \le l_1 \\ &\vdots \\ &g_m\ (x_1, x_2, \dots, x_n) \le l_2 \end{aligned}$$

Definition (2.2):

Linear programming (***LP*** or ***linear optimization***) is a mathematical technique for the optimization (maximization or minimization) of a linear objective function subject to a set of linear constraints.

The objective function may be profit, cost, production capacity or any other measure of effectiveness.

2.3 Conditions for a Linear Programming Problem

1- There must be a well-defined objective function which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

2- There must be constraints or limitations on the available resources. These constraints must be capable of being expressed as linear equations or inequalities in terms of decision variables.
3- There must be alternative course of action. For example, many grades of row material may be available.
4- Since the negative values of (any) physical quantity has no meaning, therefore all the variables must assume non-negative values.
5- Linear programming assumes the presence of a finite number of activities and constraints without which it is not possible to obtain the best or optimal solution.

2.4 The General Linear Programming Problem

The general linear programming problem can be expressed as follows:

Find the values of variables $x_1, x_2, \dots, x_n$ which maximize (or minimize) an objective function Z, i.e.

$$
\left.
\begin{array}{l}
\mathit{Optimize} \qquad Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad \dots(1) \\
\mathit{Subject\ to}: \\
\qquad \left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ (\le\ ,=,\ge)\ b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \ (\le\ ,=,\ge)\ b_2 \\
\qquad\qquad\qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ (\le\ ,=,\ge)\ b_m
\end{array}\right\} \quad \dots\ (2\,) \\
\qquad x_1, x_2, \dots, x_n \ \ge 0 \qquad\qquad \dots\ \ (3)
\end{array}
\right\} \dots LPP
$$

The above formulation may be put in the following compact form by using the summation sign:

$$
\left.
\begin{array}{llll}
 & max.(\,or\ min.) \quad Z = \sum_{j=1}^{n}\ c_jx_j & & \dots\ (1\,a) \\
S.t & \sum_{j=1}^{n}\ a_{ij}x_j\ (\le\ ,=,\ge)\ b_i & i = 1,2,\dots,m & \dots(2\,a) \\
 & x_j \ge 0 \qquad j = 1, 2, \dots, n & & \dots.(3\,a)
\end{array}
\right\} \dots LPP
$$

The variables $x_j\ \ (j = 1, \dots, n)$ are called ***decision variables***, c_j , a_{ij} , and $b_i\ (i = 1, \dots, m\,; j = 1, \dots, n)$ are constants determined from the statement of the problem. The constants $c_j\ (j = 1, \dots, n)$ are called ***cost coefficients***, constants $b_i\ (\,i = 1, \dots, m)$ are called ***stipulations***, and constants $a_{ij}\ (\,i = 1, \dots, m; j = 1, \dots, n)$ are called ***structural coefficients***. The system consisting of the ***objective function*** (1), the ***constraints*** (2), and the ***non-negativity condition*** (3) is called ***linear programming problem*** (***LPP***).

2.5 Standard Form of LPP

After formulating LPP, the next step is to obtain its solution. But before any analytic method is used to obtain the solution, the problem must be available in a form. One of these forms is the standard form.

The standard form is used to develop the general procedure for solving any LPP. The main characteristics of the standard from are:

1- All variables are non-negative.
2- The right-hand side of each constraint is non-negative.
3- Objective function may be of maximization or minimization type.
4- All constraints are expressed as equations. The inequality constraint can be changed to equality by adding or subtracting a non-negative variable from the left-hand side of such constraint. These new variables are called ***slack variables*** or simple ***slack***. They are added if the constraint is ($\leq$) and subtracted if the constraint is ($\geq$). Since in the case of ($\geq$) constraints the subtracted variables represent the surplus of left-hand side over right-hand side, it is commonly known as ***surplus variables***. Both decision variables as well as the slack and surplus variables are called the ***admissible variables***.

Remark (2.1)

1- The minimization of a function, $f(x)$, is equivalent to the maximization of $-f(x)$.
2- An inequality constraint of ($\geq$) type can be changed to an inequality constraint of ($\leq$) type by multiplying both sides of the inequality by-1.
3- An equation may be replaced by two weak inequalities in opposite directions.
4- It is possible, in actual practice, that a variable may be unconstrained (unrestricted) in sign, i.e. it may be positive, zero or negative. If a variable is unconstrained, it is expressed as the difference between two non-negative variables.

Example (2.1):

$min\ Z = 3x_1 - 4x_2 + 9x_3$ is equivalent to: $max\ G = -Z = -3x_1 + 4x_2 - 9x_3$

Example (2.2):

The equation $x_1 + 3x_2 = 5$ is equivalent to the two simultaneous constraints: $x_1 + 3x_2 \leq 5$ and $x_1 + 3x_2 \geq 5$

Or $x_1 + 3x_2 \leq 5$ and $-x_1 - 3x_2 \leq -5$.

Example (2.3):

If x is unconstrained variable, then it can be expressed as: $x = x' - x''$ where $x', x'' \geq 0$.

Example (2.4):

Express the following LPP in the standard form

$$\begin{aligned} &min && Z = 7x_1 + 5x_2 \\ &S.t. && 3x_1 + 4x_2 \leq 17 \\ & && x_1 + x_2 \geq 20 \\ & && x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the above LPP is:

$$\begin{aligned} &min && Z = 7x_1 + 5x_2 \\ &S.t. && 3x_1 + 4x_2 + s_1 = 17 \\ & && x_1 + x_2 - s_2 = 20 \\ & && x_1 , x_2 , s_1 , s_2 \geq 0 \end{aligned}$$

Example (2.5):

Express the following LPP in the standard form

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_3 \\ &S.t. && 2x_1 + 7x_2 \leq 9 \\ & && x_1 + 3x_2 + 4x_3 \geq 7 \\ & && 5x_1 - x_3 \leq 2 \\ & && x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

Here x_3 is unrestricted, so let $x_3 = x_4 - x_5$, where $x_4 , x_5 \geq 0$ .Thus the above problem will be:

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_4 - 3x_5 \\ &S.t. && 2x_1 + 7x_2 \leq 9 \\ & && x_1 + 3x_2 + 4x_4 - 4x_5 \geq 7 \\ & && 5x_1 - x_4 + x_5 \leq 2 \\ & && x_1 , x_2 , x_4 , x_5 \geq 0 \end{aligned}$$

Introducing the slack variables, the standard form is:

$$\begin{aligned} &max && Z = 2x_1 + 5x_2 + 3x_4 - 3x_5 \\ &S.t. && 2x_1 + 7x_2 + s_1 = 9 \\ & && x_1 + 3x_2 + 4x_4 - 4x_5 - s_2 = 7 \\ & && 5x_1 - x_4 + x_5 + s_3 = 2 \end{aligned}$$

$$x_1 , x_2 , x_4 , x_5, s_1 , s_2 , s_3 \geq 0$$

Remark (2.2):

The slack and surplus variables yielding zero profit (or incurring zero cost).

Remark (2.3):

Linear programs are problems that can be expressed in matrix form:

$$\begin{aligned} &max.(\,or\ min.) && Z = cX \\ &S.t. && AX\ (\leq, =, \geq) b \\ &And && X \geq \mathbf{0} \end{aligned}$$

Where $X_{n\times 1}$ represents the vector of variables (to be determined), $c_{1\times n}$ and $b_{m\times 1}$ are vectors of known coefficients, $A_{m\times n}$ is a (known) matrix of coefficients, $\mathbf{0}_{n\times 1}$ is the zero vector.

Example (2.6):

Express the following LPP in the standard matrix form

$$\begin{aligned} &min && Z = 3x_1 + 4x_2 - 2x_3 \\ &S.t. && 4x_1 + 5x_2 - x_3 \geq 5 \\ & && 2x_1 - 7x_2 + x_3 \leq 9 \\ & && x_1 + 5x_3 = 8 \\ & && x_1 , x_2 , x_3 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP is:

$$\begin{aligned} &min && Z = 3x_1 + 4x_2 - 2x_3 + 0.x_4 + 0.x_5 \\ &S.t. && 4x_1 + 5x_2 - x_3 - x_4 = 5 \\ & && 2x_1 - 7x_2 + x_3 + x_5 = 9 \\ & && x_1 + 5x_3 = 8 \\ & && x_1 , x_2 , x_3, x_4, x_5 \geq 0 \quad (x_4 , x_5 \text{ are slack variables}) \end{aligned}$$

Thus, the given problem in the matrix form is:

$$\begin{aligned} &\text{min} && Z = cX \\ &S.t. && AX = b \\ & && X \geq \mathbf{0} \end{aligned}$$

Where: $X = [x_1 , x_2 , x_3 , x_4 , x_5]^T$, $c = [3 , 4 , -2 , 0 , 0]$, $b = [5,9,8]^T$,and

$$A = \begin{bmatrix} 4 & 5 & -1 & -1 & 0 \\ 2 & -7 & 1 & 0 & 1 \\ 1 & 0 & 5 & 0 & 0 \end{bmatrix}.$$

2.6 Some Important Definitions

Consider the general LPP defined in (2.4):

Definition (2.3):

$x_j\ (j = 1, 2, \dots, n)$ is a ***solution*** of the general linear programming problem if it satisfies the constraints (2).

Definition (2.4):

$x_j\ (j = 1, 2, \dots, n)$ is a ***feasible solution*** of the general linear programming problem if it satisfies the conditions (2) and (3) .

Definition (2.5):

The solution obtained by setting $n - m$ variables equal to zero and solving for the values of the remaining m variables is called a ***basic solution***. These m variables (some of them may be zero) are called ***basic variables*** (***BV***) and each of the remaining $n - m$ that have been put equal to zero is called ***non-basic variable*** (***NBV***).

Definition (2.6):

A ***basic feasible solution*** (***BFS***) is a basic solution that satisfies the non-negativity restriction (3).

Definition (2.7):

A basic feasible solution is said to be ***non-degenerate*** if it has exactly m positive (non-zero) x_j . The solution, on the other hand, is ***degenerate*** if one or more of the m basic variables are equal to zero.

Definition (2.8):

A basic feasible solution is said to be ***optimal*** (or ***optimum***) if it is also optimize the objective function (1).

Definition (2.9):

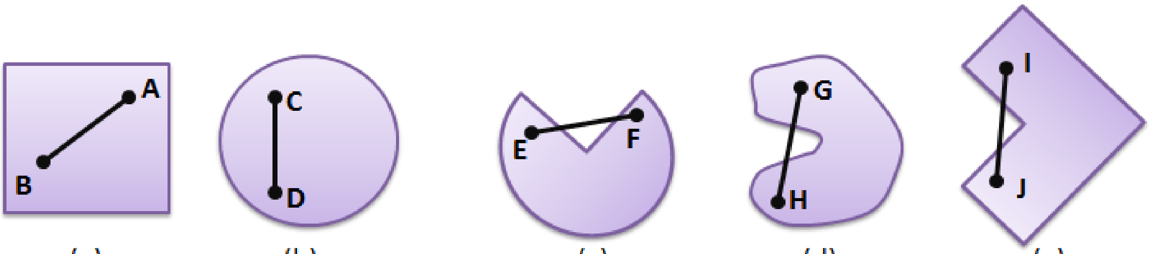
If the value of the objective function can be increased or decreased indefinitely, the solution is called ***unbounded solution***.

Definition (2.10):

A set (of points) $\boldsymbol{S}$ is said to be a ***convex set*** if for any two points in the set, the line segment joining these points lies entirely in the set.

Example (2.7):

Figure (2.1) represents convex and non-convex sets.



(a) (b) (c) (d) (e)

Convex sets Non-convex sets

Figure (2.1)

2.7 Formation of LPP

First, the given problem must be presented in LP form. This requires defining the variables of the problem, establishing inter-relationships between them, and formulating the objective function and constraints. A model which approximates as closely as possible to the given problem is then developed. If some constraints happen to be non-linear, they are approximated the appropriate linear functions to fit the LP format. To formulate an LP model:

Step 1: From the study of the situation find the key-decision to be made.

Step 2: Assume symbols for variable quantities noticed in step 1.

Step 3: Express the objective function as a linear function of variables in step 2.

Step 4: List down all the constraints.

Step 5: Presenting the problem.

Example (2.8):

A furniture manufacturer produces tables and chairs. Both the products must be processed through two machines M1 and M2. The total machine hours available are: 200 hours of M1 and 400 hours of M2 respectively. Time in hours required for producing a chair and a table on both machines is as follows:

Time in hours		
Machine	**Table**	**Chair**
M1	7	4
M2	5	9

Profit from the sale of table is 20 $ and that from a chair is 15 $. Formulate LP model for the problem to maximize the total profit.

Solution:

For this example:

Step 1: The key-decision is to decide the number of tables and chairs produced to maximize the total profit.

Step 2: Let x_1=no. of tables produced

x_2=no. of chairs produced

Step 3: The objective function is to maximizing the profit. Since profit per unit from a table is 20 $ and a chair is 15 $, then the objective function is:

$$max \quad Z = 20x_1 + 15x_2$$

Step 4: The constraints are:

i) Total time on machine M1 cannot exceed 200 hours. Since it takes 7 hours to produce a table and 4 hours to produce a chair on machine M1, then:

$$7x_1 + 4x_2 \leq 200$$

ii) Total time on machine M2 cannot exceed 400 hours. Since it takes 5 hours to produce a table and 9 hours to produce a chair on machine M1, then:

$$5x_1 + 9x_2 \leq 400$$

Step 5: The LP model is:

$$\begin{aligned} max \quad & Z = 20x_1 + 15x_2 \\ \text{S.t} \quad & 7x_1 + 4x_2 \leq 200 \\ & 5x_1 + 9x_2 \leq 400 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

Since if $x_1 \leq 0$ and $x_2 \leq 0$ it means that negative quantities of products are being manufactured which has no meaning.

Example (2.9) (Diet problem):

A person wants to decide the components of a diet which will fulfill his daily requirements of proteins, fats, and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given in the following table:

Food type	**Yield per unit**			**Cost per unit (ID)**
	Proteins	**Fats**	**Carbohydrates**	
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Minimum requirement	800	200	700	

Formulate linear programming model for the problem.

Solution:

Let $x_1, x_2, x_3,$ and x_4 denote the number of units of food of type 1, 2, 3, and 4 respectively.

The LP model is:

$$\begin{aligned} min \quad & Z = 45x_1 + 40x_2 + 85x_3 + 65x_4 \\ S.t. \quad & 3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800 \\ & 2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200 \\ & 6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700 \\ & x_1 , x_2, \ x_3, \ x_4 \geq 0 \end{aligned}$$

Example (2.10):

An advertising company wishes to plan its advertising strategy in three different media-television, radio and magazines. The purpose of advertising is to reach as large as potential customers as possible. Following data have been obtained from market survey (cost in ID):

	Television	**Radio**	**Magazine I**	**Magazine II**
Cost of an advertising unit	30000	20000	15000	10000
No. of potential customers reached per unit	200000	600000	150000	100000
No. of female customers reached per unit	150000	400000	70000	50000

The company wants to spend no more than 450000 ID on advertising. Following are the further requirements that must be met:

i) At least 1 million exposures take place among female costumers.

ii) Advertising on magazines be limited to 150000 ID.

iii) At least 3 advertising units are bought on magazine I and 2 units on magazine II.

iv) The number of advertising units on television and radio should each be between 5 and 10.

Formulate an LP model for the problem.

Solution:

Step 1: The company wants to maximize the number of potential customers reached.

Step 2: Let $x_1, x_2, x_3,$ and x_4 denote the number of advertising units to be bought on television, radio, magazine I, and magazine II.

Step 3: The objective function is:

$$max \quad Z = 10^5(2x_1 + 6x_2 + 1.5x_3 + x_4)$$

Step 4: The constraints are:

On advertising:

Budget: $30000x_1 + 20000x_2 + 15000x_3 + 10000x_4 \leq 450000$

Or $30x_1 + 20x_2 + 15x_3 + 10x_4 \leq 450$

On number of females:

Customers reached by the advertising

Campaign: $150000x_1 + 400000x_2 + 70000x_3 + 50000x_4 \geq 1000000$

Or $15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100$

On expenses:

Magazine advertising: $15000x_3 + 10000x_4 \leq 150000$

Or $15x_3 + 10x_4 \leq 150$

On no. of units on magazine: $x_3 \geq 3$

$x_4 \geq 2$

On no. of units on television: $5 \leq x_1 \leq 10$ or $x_1 \geq 5$, $x_1 \leq 10$

On no. of units on radio: $5 \leq x_2 \leq 10$ or $x_2 \geq 5$, $x_2 \leq 10$

Where $x_1 , x_2, \ x_3, \ x_4 \geq 0$

Step 5: The LP model is:

$$max \quad Z = 10^5(2x_1 + 6x_2 + 1.5x_3 + x_4)$$

$$\begin{aligned} S.t. \quad & 30x_1 + 20x_2 + 15x_3 + 10x_4 \leq 450 \\ & 15x_1 + 40x_2 + 7x_3 + 5x_4 \geq 100 \\ & 15x_3 + 10x_4 \leq 150 \\ & x_1 \geq 5 \\ & x_1 \leq 10 \\ & x_2 \geq 5 \\ & x_2 \leq 10 \\ & x_3 \geq 3 \\ & x_4 \geq 2 \\ & x_1 , x_2, \ x_3, \ x_4 \geq 0 \end{aligned}$$

Example (2.11):

A company has two grades of inspectors, I and II to undertake quality control inspection. At least 1500 pieces must be inspected in an 8-hour day. Grade I inspector can check 20 pieces in an hour with an accuracy of 96%. Grade II inspector can check 14 pieces in an hour with an accuracy of 92%. Wages of grade I inspector is 5 \$ per hour while those of grade II inspector is 4 \$ per hour. Any error made by an inspector cost 3 \$ to the company. If there are, in all, 10 grade I inspectors and 15 grade II inspectors in the company, formulate an LP model to minimize the daily inspection cost.

Solution:

Let x_1=no. of grade I inspectors

x_2=no. of grade II inspectors

Objective is to minimize the daily inspection cost. The company has to incur two types of costs: wages paid to the inspectors and the cost of their inspection error.

The cost of grade I inspector= 5+3x0.04x20 =7.40 \$

The cost of grade II inspector= 4+3x0.08x14 =7.36 \$

∴ The objective function is:

$$min \quad Z = 8(7.40x_1 + 7.36x_2) = 59.20x_1 + 58.88x_2$$

Constraints are:

On the number of grade I inspector: $x_1 \leq 10$

On the number of grade II inspector: $x_2 \leq 15$

On the number of pieces to be inspected daily: $20 \times 8x_1 + 14 \times 8x_2 \geq 1500$

Or $160x_1 + 112x_2 \geq 1500$

Where $x_1, x_2 \geq 0$

The LP model is:

$$\begin{aligned} min \quad & Z = 59.20x_1 + 58.88x_2 \\ S.t. \quad & 160x_1 + 112x_2 \geq 1500 \\ & x_1 \leq 10 \\ & x_2 \leq 15 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Example (2.12) (Blending problem):

A firm produces an alloy having the following specifications:

i) Specific gravity ≤ 0.98 ii) chromium ≥ 8% iii) melting point ≥ 450° C.

Raw materials A, B, and C having the properties shown in the following table can be used to make the alloy:

property	Properties of raw material		
	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium	7%	13%	16%

Melting point	440°C	490°C	480°C

Costs of the various raw materials per ton are: 90 $ for A, 280 $ for B, and 40 $ for C. Formulate the LP model to find the properties in which A, B, and C be used to obtain an alloy of desired properties while the cost of raw materials is minimum.

Solution:

Let $x_1, x_2,$ and x_3 denote the percentage contents of raw materials A, B, and C to be used for making the alloy.

The LP model is:

$$\begin{aligned} min \quad & Z = 90x_1 + 280x_2 + 40x_3 \\ S.t \quad & 0.92x_1 + 0.97x_2 + 1.04x_3 \leq 0.98 \\ & 7x_1 + 13x_2 + 16x_3 \geq 8 \\ & 440x_1 + 490x_2 + 480x_3 \geq 450 \\ & x_1 + x_2 + x_3 = 100 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Example (2.13):

An investment company wants to invest up to 10 million ID into various bonds. The management is currently considering four bonds, the detail on return and maturity of which are as follows:

Bond	Type	Return	Maturity time
A	Government	22%	15 years
B	Government	18%	5 years
C	Industrial	28%	20 years
D	Industrial	16%	3 years

The company has decided to put less than half of its investment in the government bonds and that the average age of the portfolio should not be more than 6 years. The investment should maximize the return on investment, subject to the above restrictions. Formulate the LP model.

Solution:

Let $x_1, x_2, x_3,$ and x_4 denote the amount to be invested in bonds A, B, C, and D respectively.

The objective function is: $max \quad Z = 0.22x_1 + 0.18x_2 + 0.28x_3 + 0.16x_4$

Subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 \leq 10^7$$

$$x_1 + x_2 \leq 5 \times 10^6$$

$\frac{15x_1+5x_2+20x_3+3x_4}{x_1+x_2+x_3+x_4} \leq 6 \quad \Longrightarrow \quad 15x_1 + 5x_2 + 20x_3 + 3x_4 \leq 6x_1 + 6x_2 + 6x_3 + 6x_4 \Longrightarrow 15x_1 - 6x_1 + 5x_2 - 6x_2 + 20x_3 - 6x_3 + 3x_4 - 6x_4 \leq 0 \Longrightarrow 9x_1 - x_2 + 14x_3 - 3x_4 \leq 0$

Then the LP model will be:

$$\max \quad Z = 0.22x_1 + 0.18x_2 + 0.28x_3 + 0.16x_4$$

S.t.

$$x_1 + x_2 + x_3 + x_4 \leq 10^7$$
$$x_1 + x_2 \leq 5 \times 10^6$$
$$9x_1 - x_2 + 14x_3 - 3x_4 \leq 0$$
$$x_1\,, x_2,\; x_3,\; x_4 \geq 0$$

Exercises 2.1 (In addition to the text book exercises)

1. A paper mill produces rolls of papers used in making cash registers. Each roll of paper is 100 m length and can be used in width of 3,4,6, and 10 cm. The company's production process results in rolls that are 24 cm in width. Thus, the company must cut its 24 cm roil to the desired width. It has six basic cutting alternatives as follows:

Cutting alternatives	Width of rolls (cm)				Waste (cm)
	3	4	6	10	
1	4	3	---	---	---
2	---	3	2	---	---
3	1	1	1	1	1
4	---	---	2	1	2
5	---	4	1	---	2
6	3	2	1	---	1

The minimum demand for the four rolls is as follows:

Roll width (cm)	Demand
2	2000
4	3600
6	1600
10	500

The paper mill wishes to minimize the waste resulting from trimming to size. Formulate the LP model.

2. A manufacturer of biscuits is considering four types of gift-packs containing three types of biscuits: orange cream (o.c.), chocolate cream (c.c.) and wafers (w.). Market research conducted to assess the preferences of the consumers shows the following types of assortments to be in good demand:

Assortment	Contents	Selling price/kg(thousands of ID)
A	Not less than 40% of o.c., not more than 20% of c.c.	20
B	Not less than 20% of o.c., not more than 40% of c.c	25
C	Not less than 50% of o.c., not more than 10% of c.c	22
D	No restriction	12

For the biscuits, the manufacturing capacity and costs are given below:

Biscuit variety	Plan200t capacity200(kg/day)	Manufacturing cost(thousands of ID)
o.c.	200	8
c.c.	200	9
w	150	7

Formulate the LP model to find the production schedule which maximizes the profit assuming that there are no market restrictions.

3. A farmer has 1000 acres (1 acre≈4047 m^2) of land on which he can grow corn, wheat or soya bean. Each acre of corn costs 100 $ for preparation, requires 7 man-days of work and yields a profit of 30 $. An acre of wheat costs 120 $ to prepare, requires 10 man-days of work and yields a profit of 40 $. An acre of soya bean costs 70 $ to prepare, requires 8 man-days of work and yields a profit of 20 $. If the farmer has 100000 $ for preparation, and can count 8000 man-days work, formulate the LP model to allocate the number of acres to each crop to maximize the total profit.

Methods of Solutions for LPP:

2.8 Graphical Solution of Linear Programming Models

The solution of a LPP with only two variables can be derived using a graphical method. The graphical procedure consists of the following steps:

Step 1: represent the given problem in mathematical form.

Step 2: Draw the x_1 and the x_2 –axes. The non-negativity restrictions imply that the values of the variables x_1 and x_2 can lie only in the first quadrant.

Step 3: Plot each of the constraints on the graph. First replace each inequality with an equation and then graph the resulting straight line by locating two distinct points on it. Simply we can take the points of intersection with the x_1 and the x_2 –axes.

Step 4: Each inequality (constraint) will divide the $(x_1,\ x_2)$- plane into two half-spaces, one on each side of the graphed plane. Only one of these two halves satisfy the inequality. To determine the correct side, choose (0, 0) as a ***reference point*** (you can choose any other point). If it satisfies the inequality, then the side in which it lies is the feasible half-space, otherwise the other side is. Find the feasible region of each constraint, the ***feasible solution space*** of the problem represents the area in the first quadrant in which all the constraints are satisfied simultaneously.

Step 5: Determine the optimal solution. For this, plot the objective function by assuming $Z = 0$. This will be a line passing through the origin (drawn as a dotted line). As the value of Z is increased from zero, the dotted line moves to the right, parallel to itself until it passes through a corner of the feasible solution space in which the value of the objective function is optimized.

Alternatively use the ***extreme point enumeration approach***. For this, find the coordinates of each extreme point (or corner point or vortex) of the feasible region. Find the value of the objective function at each extreme point. The point at which objective function is maximum/ minimum is the optimal point and its coordinates give the optimal solution.

Example (2.14):

A firm has two bottling plants, one located at Baghdad and, other at Erbil. Each plant produces three drinks, Coca-Cola, Fanta, and Seven-up, named A, B, and C respectively. The number of bottles produced per day is, as follows:

	Plant at	
	Baghdad (Bg)	**Erbil (E)**
Coca-Cola	15000	15000
Fanta	30000	10000
Seven-up	20000	50000

A market survey indicates that, during the month of April, there will be a demand of 200000 bottles of Coca-Cola, 400000 bottles of Fanta, and 440000 bottles of Seven-up. The operating cost per day for plants at Baghdad and Erbil is 600 and 400 monetary units respectively. For how many days each plant is run in April to minimize the production cost, while still meeting the market demand.

Solution:

Let x_1= the number of running days of the planet at Baghdad.

x_2= the number of running days of the planet at Erbil.

The LP model is:

$$\left.\begin{array}{ll} min & Z = 600x_1 + 400x_2 \\ S.t. & 15000x_1 + 15000x_2 \geq 200000 \\ & 30000x_1 + 10000x_2 \geq 400000 \\ & 20000x_1 + 50000x_2 \geq 440000 \\ & x_1, x_2 \geq 0 \end{array}\right\} \text{Or} \quad \begin{array}{ll} min & Z = 600x_1 + 400x_2 \\ S.t. & 15x_1 + 15x_2 \geq 200 \\ & 3x_1 + x_2 \geq 40 \\ & 2x_1 + 5x_2 \geq 44 \\ & x_1, x_2 \geq 0 \end{array}$$

Then:

For $15x_1 + 15x_2 = 200 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{200}{15} = 13.3 \Rightarrow$ (0, 13.3) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{200}{15} = 13.3 \Rightarrow$ (13.3, 0) is the intersection point with the $x_1 -$ axis.

For $3x_1 + x_2 = 40 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{40}{1} = 40 \Rightarrow$ (0, 40) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{40}{3} = 13.3 \Rightarrow$ (13.3, 0) is the intersection point with the $x_1 -$ axis.

For $2x_1 + 5x_2 = 44 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{44}{5} = 8.8 \Rightarrow$ (0, 8.8) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{44}{2} = 22 \Rightarrow$ (22, 0) is the intersection point with the $x_1 -$ axis.

The graphical representation is:

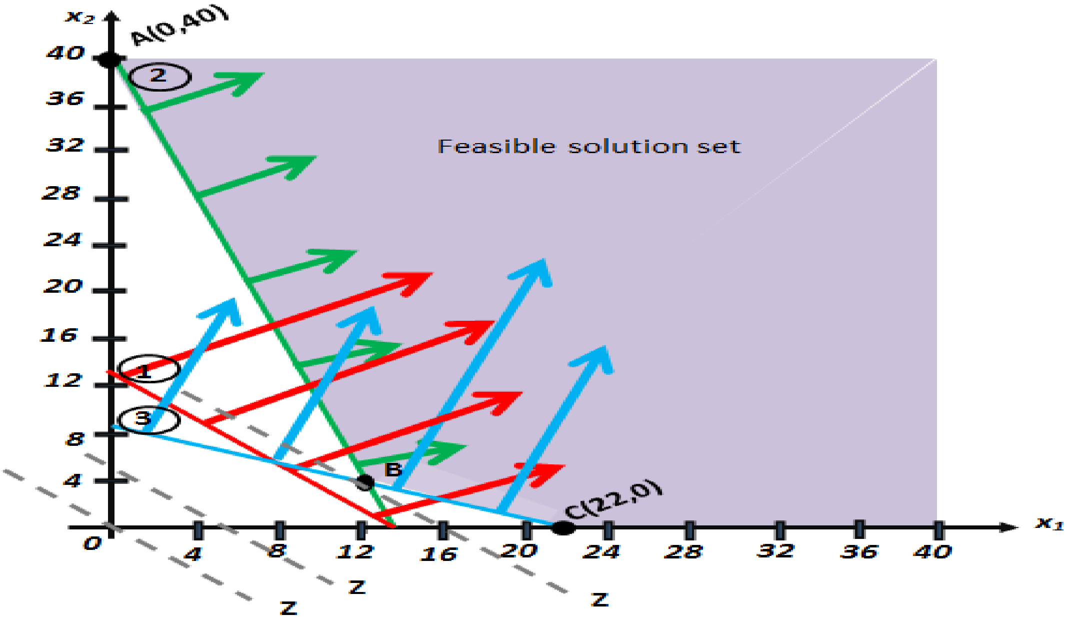


Figure (2.2)

The point B is resulting from the intersection of the lines representing the second and the third constraints, so we use these constraints to find the coordinates of B.

$$\left.\begin{array}{l}3x_1 + x_2 = 40\\ 2x_1 + 5x_2 = 44\end{array}\right\} \xrightarrow{5\times(2)} - \begin{array}{l}15x_1 + \cancel{5x_2} = 200\\ \mp 2x_1 \mp \cancel{5x_2} = \mp 44\end{array}$$

$$\Rightarrow 13x_1 = 156 \Rightarrow x_1 = 12 \overset{(2)}{\Rightarrow} x_2 = \frac{400-360}{1} = 4$$

The feasible solution region space is the shaded area in figure (2.2) whose corners are the points $A(0{,}40)$, $B(12{,}4)$, and $C(22{,}0)$ (i.e. any point in or at the boundary of the shaded region).

Corner	Value of Z
$A(0{,}40)$	$Z = 600 \times 0 + 400 \times 40 = 16000$
$B(12{,}4)$	$Z = 600 \times 12 + 400 \times 4 = 8800$ *
$C(22{,}0)$	$Z = 600 \times 22 + 400 \times 0 = 13200$

From the table, we see that the optimum occurs at corner B, then $x_1 = 12$ days, $x_2 = 4$ days, and $Z_{min} = 8800$ monetary units.

Note that the first constraint: $15000x_1 + 15000x_2 \geq 200000$ does not affect the solution space, such a constraint is called a ***redundant constraint***.

Example (2.15):

In one of the stages of production, a carpets company cuts lengths of carpet after its production in another department of the company. After cutting lengths by special machines to certain lengths, the lengths are folded in the form of rolls and then wrapped by certain substances for the purpose of selling in the markets. The following table shows the data for the two types of carpets A and B:

Production department	Product A	Product B	Available time
Cutting	8	6	2200
Folding	4	9	1800
Wrapping	1	2	400
Profit per length unit	12	8	

Each product must pass through the three mentioned stages. Determine the number of units produced to maximize the profit.

Solution:

Let x_1= the number units produced of A.

x_2= the number units produced of B.

The LP model is:

$$
\begin{aligned}
&max \quad Z = 12x_1 + 8x_2 \\
&S.t. \quad 8x_1 + 6x_2 \leq 2200 \\
&\qquad\ \ 4x_1 + 9x_2 \leq 1800 \\
&\qquad\quad x_1 + 2x_2 \leq 400 \\
&\qquad\quad x_1, x_2 \geq 0
\end{aligned}
$$

Then:

For $8x_1 + 6x_2 = 2200 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{2200}{6} = 366.7 \Rightarrow$ (0, 366.7) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{2200}{8} = 275 \Rightarrow$ (275, 0) is the intersection point with the $x_1 -$ axis.

For $4x_1 + 9x_2 = 1800 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{1800}{9} = 200 \Rightarrow$ (0,200) is the intersection point with the $x_2 -$ axis.

And if $x_2 = 0$ then $x_1 = \frac{1800}{4} = 450 \Rightarrow$ (450, 0) is the intersection point with the $x_1 -$ axis.

For $x_1 + 2x_2 = 400 \Rightarrow$ if $x_1 = 0$ then $x_2 = \frac{400}{2} = 200 \Rightarrow$ (0, 200) is the intersection point with the $x_2 -$ axis.And if $x_2 = 0$ then $x_1 = 400 \Rightarrow$ (400,0) is the intersection point with the $x_1 -$ axis.The graphical representation is:

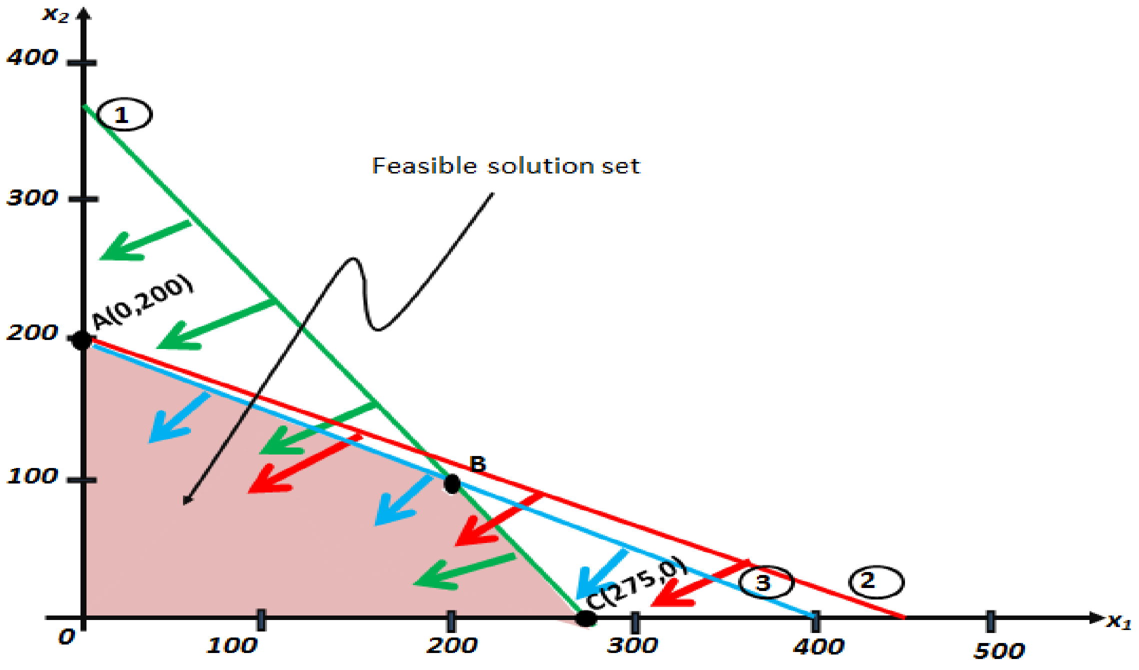


Figure (2.3)

The point B is resulting from the intersection of the lines representing the first and the third constraints, so we use these constraints to find the coordinates of B.

$$\left.\begin{aligned} 8x_1+6x_2&=2200\\ x_1+2x_2&=400 \end{aligned}\right\}\xrightarrow{3\times(3)} - \begin{array}{r} 8x_1+\cancel{6x_2}=2200\\ \mp 3x_1\mp\cancel{6x_2}=\mp 1200 \\ \hline \end{array}$$

$$\Rightarrow 5x_1=1000\Rightarrow x_1=200\overset{(3)}{\Rightarrow} x_2=\frac{400-200}{2}=100$$

The feasible solution region space is the shaded area OABC in figure (2.3) whose corners are the points $O(0,0)$, $A(0,200)$, $B(200,100)$, and $C(275,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,200)$	$Z=12\times 0+8\times 200=1600$
$B(200,100)$	$Z=12\times 200+8\times 100=3200$
$C(275,0)$	$Z=12\times 275+8\times 0=3300$ *

From the table, we see that the optimum occurs at corner C, then $x_1=275$ units, $x_2=0$ units, and $Z_{max}=3300$ monetary units.

Exercises 2.2 (In addition to the text book exercises)

1. The canonical form is the form in which the objective function is of maximization type and the constraints are of the ($\leq$) type (except the non-negativity restriction which is of ($\geq$) $type$), i.e. it has the form:

$$max. \quad Z=\sum_{j=1}^{n} c_j x_j$$

$$\text{S.t.} \quad \sum_{j=1}^{n} a_{ij}x_j \leq b_i \qquad i=1,2,\dots,m$$

$$x_j\geq 0 \qquad j=1,2,\dots,n$$

Write the following LPP in canonical form then find the optimal solution of the canonical form (remember remark (2.1)):

$$\begin{aligned} min \quad & Z=-4x_1-2x_2\\ S.t. \quad & -2x_1-4x_2\geq -20\\ & -2x_1-2x_2\geq -12\\ & -2x_1+2x_2\geq -4\\ & -2x_1+4x_2\geq -2\\ & x_1,x_2\geq 0 \end{aligned}$$

2. Find the optimal solution for the following LPP:

$$min \quad Z=40x_1+20x_2$$

$$
\begin{aligned}
S.t. \quad & 2x_1 + 4x_2 \le 80 \\
& 6x_1 + 2x_2 \ge 60 \\
& 8x_1 + 6x_2 \ge 120 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

2.9 The Simplex Method

The graphical method cannot be applied when the number of variables in the LPP is more than three, or rather two, since even with three variables the graphical solution becomes tedious as it involves intersection of planes in three dimensions. The simplex method can be used to solve any LPP (for which the solution exists) involving any number of variables and constraints.

The computational procedure in the simplex method is based on a fundamental property that the optimal solution to an LPP, if it exists, occurs only at one of the corner points of the feasible region. The simplex method is an iterative method starts with initial basic feasible solution at the origin, i.e. Z=0. If the solution is not optimal, we move to the adjacent corner, until after a finite number of trials, the optimal solution, if it exists, is obtained.

The steps of the simplex method are as follows:

Step 1: Convert the given problem into the standard form. The Right Hand Side (RHS) of each constraint must be non-negative. Write the objective function in the form: $Z - \sum_{j=1}^{n} c_j x_j \ = 0$

Step 2: Set $x_1 = x_2 = \ldots = x_n = 0$, i.e. $x_1, x_2, \ldots, x_n$ are non-basic variables, thus $s_1, s_2, \ldots, s_m$ are the basic-variables.

Step 3: Construct the initial ***simplex table*** (or ***tableau***) with all slack variables in the BVS. The simplex table for the general LPP (in 2.4) is:

Basic variables	x_1	x_2	…	x_n	s_1	s_2	…	s_m	**Solution**
s_1	a_{11}	a_{12}	…	a_{1n}	1	0	…	0	b_1
s_2	a_{21}	a_{22}	…	a_{2n}	0	1	…	0	b_2
⋮	⋮	⋮	…	⋮	⋮	⋮	…	⋮	⋮
s_m	a_{m1}	a_{m2}	…	a_{mn}	0	0	…	1	b_m
Z	$-c_1$	$-c_2$	…	$-c_n$	0	0	…	0	*0*

Table (2.1)

The coefficients a_{ij} in the constraints (written under non-basic variables $x_1, x_2, \ldots, x_n$) is called the ***body matrix*** (or ***coefficient matrix***). The last column of the table (2.1) is called ***solution value column*** (briefly ***solution column***) or ***quantity column*** or ***b-column*** or ***RHS column***.

Step 4: Check the optimality of the current solution: In maximization (minimization) problem the simplex table is optimal, if in the Z-row there are non-negative (non-positive) coefficients in any NBV's. If the table is optimal the algorithm terminates, and the optimal value and decision can be read from the BV and RHS columns.

Step 5: If the current solution is not optimal, then determine which non-basic variable should become a basic variable (***entering variable***) and which basic variable should become a non-basic variable (***leaving variable***) to find a new BFS with a better objective function value.

i) In maximization (minimization) problem, the entering variable will correspond to the variable with the most negative (positive) coefficient in the objective function. The column of this variable is called the ***pivot*** (***key***) ***column***.

ii) The mechanics of determining the leaving variable from the simplex table calls for computing the non-negative ratio of the b-column to the corresponding coefficients in the pivot column (since solutions must satisfy the non-negativity condition). The minimum non-negative ratio identifies the leaving variable; its row is called the ***pivot*** (***key***) ***row***. The rule associated with this ratio is called the ***feasibility condition***.

iii) Update the solution by preparing the new simplex table. This is done by performing ***Gauss-Jordan row operations***. The intersection of the pivot row and the pivot column is called the ***pivot*** (***key***) ***element***.

The Gauss-Jordan computations needed to produce the new BFS includes:

a) Pivot row:
   1- Replace leaving variable in the Basic variables column with the entering variable.
   2- New pivot row=Current pivot row ÷ Pivot element
b) All other rows, including Z:
   New row= Current row – its pivot column coefficient × New pivot row

Step 6: Repeat steps 4 and 5 until, after a finite number of steps, an optimal solution, if it exists, is reached.

Example (2.16):

Find the optimal solution of the following LPP:

$$\begin{aligned} max \quad & Z = 12x_1 + 15x_2 + 14x_3 \\ S.t. \quad & -x_1 + x_2 \le 0 \\ & -x_2 + 2x_3 \le 0 \\ & x_1 + x_2 + x_3 \le 100 \end{aligned}$$

$x_1, x_2, x_3 \geq 0$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$$max \quad Z - 12x_1 - 15x_2 - 14x_3 = 0$$

$$S.t. \quad -x_1 + x_2 + s_1 = 0$$

$$-x_2 + 2x_3 + s_2 = 0$$

$$x_1 + x_2 + x_3 + s_3 = 100$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Set $x_1 = x_2 = x_3 = 0$in the constraints yield the following initial basic feasible solution: $s_1 = 0, s_2 = 0, s_3 = 100, Z = 0$. The simplex table is:

Basic variables	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
s_1	−1	$\boxed{1}$	0	1	0	0	0	0/1=0
s_2	0	−1	2	0	1	0	0	
s_3	1	1	1	0	0	1	100	100/1=100
Z	− 12	−15	−14	0	0	0	0	

Since some elements in Z row are negative then the initial solution is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_2	−1	1	0	1	0	0	0
s_2	−1	0	2	1	1	0	0
s_3	$\boxed{2}$	0	1	−1	0	1	100
Z	−27	0	−14	15	0	0	0

$x_1 = 0, s_2 = 0, s_3 = 100, Z = 0$ and it is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution	
x_2	0	1	1/2	1/2	0	1/2	50	50/(1/2)=100
s_2	0	0	$\boxed{5/2}$	1/2	1	1/2	50	50/(5/2)=20
x_1	1	0	1/2	−1/2	0	1/2	50	50/(1/2)=100
Z	0	0	−1/2	3/2	0	27/2	1350	

$x_1 = 50, s_2 = 50, x_1 = 50, Z = 1350$ and it is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
x_2	0	1	0	2/5	−1/5	2/5	40
x_3	0	0	1	1/5	2/5	1/5	20
x_1	1	0	0	−3/5	−1/5	2/5	40
Z	0	0	0	8/5	1/5	68/5	1360

Optimal solution is: $x_1 = 40, x_2 = 40, x_3 = 20, Z_{max} = 1360.$

Example (2.17):

Find the optimal solution of the following LPP:

$$\begin{aligned} min \quad & Z = x_1 - 3x_2 + 3x_3 \\ S.t. \quad & 3x_1 - x_2 + 2x_3 \le 7 \\ & 2x_1 + 4x_2 \ge -12 \\ & -4x_1 + 3x_2 + 8x_3 \le 10 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Solution:

The RHS of the second constrain is negative, it is made positive by multiplying both side of the constraint by -1. Thus, the constraint takes the form:

$$-2x_1 - 4x_2 \le 12$$

The standard form of the LPP (with modification of the objective function) is:

$$\begin{aligned} min \quad & Z - x_1 + 3x_2 - 3x_3 = 0 \\ S.t. \quad & 3x_1 - x_2 + 2x_3 + s_1 = 7 \\ & -2x_1 - 4x_2 + s_2 = 12 \\ & -4x_1 + 3x_2 + 8x_3 + s_3 = 10 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \ge 0 \end{aligned}$$

Set $x_1 = x_2 = x_3 = 0$ in the constraints yield the following initial basic feasible solution: $s_1 = 7, s_2 = 12, s_3 = 10, Z = 0$. This solution and further improved solutions are given in the following tables:

B.V.	x_1	x_2	x_3	s_1	s_2	s_3	Solution
s_1	3	−1	2	1	0	0	7
s_2	−2	−4	0	0	1	0	12
s_3	−4	$\boxed{3}$	8	0	0	1	10
Z	−1	3	−3	0	0	0	0
s_1	$\boxed{5/3}$	0	14/3	1	0	1/3	31/3
s_2	−22/3	0	32/3	0	1	4/3	76/3
x_2	−4/3	1	8/3	0	0	1/3	10/3
Z	3	0	−11	0	0	−1	−10
x_1	1	0	14/5	3/5	0	1/5	31/5
s_2	0	0	156/5	22/5	1	14/5	354/5
x_2	0	1	32/5	12/5	0	3/5	58/5
Z	0	0	−97/5	−9/5	0	−8/5	−143/5

Optimal solution is: $x_1 = \frac{31}{5}, x_2 = \frac{58}{5}, x_3 = 0, Z_{min} = -143/5$.

Exercises 2.3 (In addition to the text book exercises)

Solve the following problems by the simplex method:

$$\mathbf{1.}\quad max \quad Z = 6x_1 + 3x_2$$
$$\begin{aligned} S.t.\quad & 3x_1 + 6x_2 \leq 30 \\ & 3\,x_1 + 3x_2 \leq 18 \\ & 3x_1 - 3x_2 \leq 6 \\ & 3x_1 - 6x_2 \leq 3 \\ & x_1\,, x_2, \geq 0 \end{aligned}$$

$$\mathbf{2.}\quad min \quad Z = 2x_1 + x_2 - 3x_3 + 5x_4$$
$$\begin{aligned} S.t.\quad & x_1 + 7x_2 + 3x_3 + 7x_4 \leq 46 \\ & 3x_1 - x_2 + x_3 + 2x_4 \leq 8 \\ & 2x_1 + 3x_2 - x_3 + x_4 \leq 10 \\ & x_1\,, x_2,\ x_3,\ x_4 \geq 0 \end{aligned}$$

2.10 The M-method (Big M-method)

If the LPP has any contain of ($\geq$) or ($=$) type, then the slack variables cannot provide an initial basic feasible solution. In such cases, we introduce another type of variables called ***artificial variables***. These variables have no physical meaning; they are only a device to get the starting BFS so that the simplex algorithm is applied as usual to get optimal solution. This method consists of the following steps:

Step 1: Express the LPP in standard form, add slack variables to the constraints of (≤) type and subtract them to the constraints of (≥) type.

Step 2: Add non-negative variables to the left-hand-side of all constraints of ($\geq$) or ($=$) type. These variables are called ***artificial variables***. In order to get rid of the artificial variables in the final optimum iteration, we assign a very large penalty −M (M) in maximization (minimization) problem to the artificial variables.

Step 3: Solve the modified LPP by simplex method. While making iterations by this method, one of the following three cases may arise:

1. If no artificial variable remains in the basis, and the optimal condition is satisfied, then the current solution is an optimal BFS.
2. If at least one artificial variable appears in the basis zero level (with zero value in the solution column), and the optimality condition is satisfied, then the current solution is optimal BFS (though degenerate).

3. If at least one artificial variable appears in the basis zero level (with positive value in the solution column), and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function because it contains a very large penalty M and is termed as the ***pseudo optimal solution***.

While applying the simplex method, whenever an artificial variable happens to leave the basis, we drop artificial variable, and omit all the entries corresponding to its column from the simplex table.

Step 4: Application of simplex method until, either an optimal BFS is obtained or there is an indication of the existence of an unbounded solution to the given LPP.

Remark (2.4):

1. For computer solutions, some specific value must be assigned to M.
2. Variables, other than the artificial variables, once driven out in an iteration, may re-enter in a subsequent iteration. But, an artificial variable, once driven out, can never re-enter because of the large penalty coefficient M associated with it.

Example (2.18):

Find the optimal solution of the following LPP:

$$\begin{aligned} max \quad & Z = 3x_1 - x_2 \\ S.t. \quad & x_1 - 2x_2 \geq 8 \\ & x_1 + x_2 \leq 16 \\ & x_1 \geq 8 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP after adding the artificial variables is:

$$\begin{aligned} max \quad & Z = 3x_1 - x_2 - MR_1 - MR_2 \\ S.t. \quad & x_1 - 2x_2 - s_1 + R_1 = 8 \\ & x_1 + x_2 + s_2 = 16 \\ & x_1 - s_3 + R_2 = 8 \\ & x_1 , x_2, s_1, s_2, s_3, R_1, R_2 \geq 0 \end{aligned}$$

From the first and the third constraints:

$$R_1 = 8 - x_1 + 2x_2 + s_1$$

$$R_2 = 8 - x_1 + s_3$$

Substitute R_1 and R_2 in Z-equation:

$Z = 3x_1 - x_2 - M(8 - x_1 + 2x_2 + s_1) - M(8 - x_1 + s_3)$

$Z = 3x_1 - x_2 - 8M + Mx_1 - 2Mx_2 - Ms_1 - 8M + Mx_1 - Ms_3$

$Z - (3 + 2M)x_1 + (1 + 2M)x_2 + Ms_1 + Ms_3 = -16M$

The standard form of LPP (with modification of the objective function) is:

$$
\begin{aligned}
\max \quad & Z - (3 + 2M)x_1 + (1 + 2M)x_2 + Ms_1 + Ms_3 = -16M \\
S.t. \quad & x_1 - 2x_2 - s_1 + R_1 = 8 \\
& x_1 + x_2 + s_2 = 16 \\
& x_1 - s_3 + R_2 = 8 \\
& x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0
\end{aligned}
$$

Let $x_1 = x_2 = s_1 = s_3 = 0$,then $R_1 = 8, s_2 = 16, R_2 = 8, Z = -16M$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	s_3	R_1	R_2	**Solution**	
R_1	$\boxed{1}$	−2	−1	0	0	1	0	8	8/1=8
s_2	1	1	0	1	0	0	0	16	16/1=16
R_2	1	0	0	0	−1	0	1	8	8/1=8
Z	−3−2M	1+2M	M	0	M	0	0	−16M	

The current solution is not optimal, then:

B.V.	x_1	x_2	s_1	s_2	s_3	R_2	**Solution**	
x_1	1	−2	−1	0	0	0	8	
s_2	0	3	1	1	0	0	8	8/3=2.7
R_2	0	$\boxed{2}$	1	0	−1	1	0	0/2=0
Z	0	−5−2M	−3−M	0	M	0	24	

$x_1 = 8, s_2 = 8, R_2 = 0, Z = 24$, the current solution is not optimal, further improved solutions are given in the following tables

B.V.	x_1	x_2	s_1	s_2	s_3	**Solution**
x_1	1	0	0	0	−1	8
s_2	0	0	−1/2	1	$\boxed{3/2}$	8
x_2	0	1	1/2	0	−1/2	0
Z	0	0	−1/2	0	−5/2	24
x_1	1	0	−1/3	2/3	0	40/3
s_3	0	0	−1/3	2/3	1	16/3
x_2	0	1	$\boxed{1/3}$	1/3	0	8/3
Z	0	0	−4/3	5/3	0	112/3

x_1	1	1	0	1	0	16
s_3	0	1	0	1	1	8
s_1	0	3	1	1	0	8
Z	0	4	0	3	0	48

The optimal solution is: $x_1 = 16, x_2 = 0, Z_{max} = 48$

Example (2.19):

Find the optimal solution of the following LPP:

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 \\ S.t. \quad & 6x_1 + 2x_2 + 6x_3 \geq 6 \\ & 6x_1 + 4x_2 = 12 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_1 , x_2, \ x_3 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP after adding the artificial variables is:

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 + MR_1 + MR_2 \\ S.t. \quad & 6x_1 + 2x_2 + 6x_3 - s_1 + R_1 = 6 \\ & 6x_1 + 4x_2 + R_2 = 12 \\ & 2x_1 - 2x_2 + s_2 = 2 \\ & x_1 , x_2, s_1, s_2, R_1, R_2 \geq 0 \end{aligned}$$

From the first and the second constraints:

$$R_1 = 6 - 6x_1 - 2x_2 - 6x_3 + s_1$$

$$R_2 = 12 - 6x_1 - 4x_2$$

Substitute R_1 and R_2 in Z-equation:

$$Z = 3x_1 + 8x_2 + x_3 + M(6 - 6x_1 - 2x_2 - 6x_3 + s_1) + M(12 - 6x_1 - 4x_2)$$

$$Z = 3x_1 + 8x_2 + x_3 + 6M - 6Mx_1 - 2Mx_2 - 6Mx_3 + Ms_1 + 12M - 6Mx_1 - 4Mx_2$$

$$Z + (-3 + 12M)x_1 + (-8 + 6M)x_2 + (-1 + 6M)x_3 - Ms_1 = 18M$$

The LPP (with modification of the objective function) is:

$$\begin{aligned} min \quad & Z + (-3 + 12M)x_1 + (-8 + 6M)x_2 + (-1 + 6M)x_3 - Ms_1 = 18M \\ S.t. \quad & 6x_1 + 2x_2 + 6x_3 - s_1 + R_1 = 6 \\ & 6x_1 + 4x_2 + R_2 = 12 \\ & 2x_1 - 2x_2 + s_2 = 2 \\ & x_1 , x_2, s_1, s_2, R_1, R_2 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = x_3 = s_1 = s_3 = 0$,then $R_1 = 6, s_2 = 2, R_2 = 12, Z = 18M$. The simplex table is:

B.V.	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution	
R_1	$\boxed{6}$	2	6	−1	0	1	0	6	6/6=1
R_2	6	4	0	0	0	0	1	12	12/6=2
s_2	2	−2	0	0	1	0	0	2	2/2=1
Z	−3+12M	−8+6M	−1+6M	−M	0	0	0	18M	

The current solution is not optimal, then:

B.V.	x_1	x_2	x_3	s_1	s_2	R_2	Solution	
x_1	1	1/3	1	−1/6	0	0	1	1/(1/3)=3
R_2	0	$\boxed{2}$	−6	1	0	1	6	6/2=3
s_2	0	−8/3	−2	1/3	1	0	0	
Z	0	−7+2M	2 − 6M	(−1/2)+M	0	0	3+6M	

$x_1 = 1, s_2 = 0, R_2 = 6, Z = 3 + 6M,$ the current solution is not optimal, further improved solutions are given in the following tables:

B.V.	x_1	x_2	x_3	s_1	s_2	Solution	
x_1	1	0	2	−1/3	0	0	
x_2	0	1	−3	1/2	0	3	3/(1/2)=6
s_2	0	0	−10	$\boxed{5/3}$	1	8	8/(5/3)=4.8
Z	0	0	−19	3	0	24	
x_1	1	0	0	0	1/5	8/5	
x_2	0	1	0	0	−3/10	3/5	
s_1	0	0	−6	1	3/5	24/5	
Z	0	0	−1	0	−9/5	48/5	

The optimal solution is: $x_1 = \frac{8}{5}, x_2 = \frac{3}{5}, x_3 = 0, Z_{min} = 48/5$

Exercises 2.4 (In addition to the text book exercises)

Find the optimal solution of the following LPP:

1. $$\begin{aligned} min \quad & Z = 9x_1 + 6x_2 + 3x_3 \\ S.t. \quad & 3x_1 + 12x_2 + 9x_3 \geq 150 \\ & 6x_1 + 3x_2 + 3x_3 \geq 90 \\ & -9x_1 - 6x_2 - 3x_3 \leq -120 \\ & x_1 , x_2, \ x_3 \geq 0 \end{aligned}$$

2. $$\begin{aligned} max \quad & Z = -12x_1 - 3x_2 \\ S.t. \quad & 9x_1 + 3x_2 = 9 \\ & 12x_1 + 9x_2 \geq 18 \\ & 3\,x_1 + 6x_2 \leq 9 \end{aligned}$$

$$x_1\ ,x_2 \geq 0$$

2.11 Definition of the Dual Problem

The ***dual*** problem is an LPP defined directly and systematically from the ***primal*** (or ***original***) LP model. The two problems are so closely related that the optimal solution of one problem automatically provides the optimal solution to the other. If the primal problem contains a large number of constraints and a smaller number of variables, the computational procedure can be considerably reduced by converting it into dual and then solving it.

2.12 Dual Problem Characteristics

1. If the primal contains *n* variables and *m* constraints, the dual will contain m variables and n constraints.
2. The maximization problem in the primal becomes a minimization problem in the dual and vice versa.
3. Constraints of $(\leq)\ type$ in the primal become of $(\geq)\ type$ in the dual and vice versa.
4. The coefficient matrix of the constraints of the dual is the transpose of the coefficient matrix in the primal and vice versa.
5. A new set of variables appear in the dual.
6. The constants $c_1, c_2, \ldots, c_n$ in the objective function of the primal appear in the right-hand-side of the constraints of the dual.
7. The constants $b_1, b_2, \ldots, b_m$ in the constraints of the primal appear in the objective function of the dual.
8. The variables of both problems are non-negative.
9. For each constraint in the primal there is an associated variable in the dual.

Example (2.20):

Construct the dual of the primal problem

$$\begin{aligned} min \quad & Z = 3x_1 - 2x_2 + 4x_3 \\ S.t. \quad & 3x_1 + 5x_2 + 4x_3 \geq 7 \\ & 6x_1 + x_2 + 3x_3 \geq 4 \\ & 7x_1 - 2x_2 - x_3 \leq 10 \\ & x_1 - 2x_2 + 5x_3 \geq 3 \\ & 4x_1 + 7x_2 - 2x_3 \geq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

All the constraints must be of the same type. Multiplying the third constraint by (-1) on both sides, we get:

$$-7x_1 + 2x_2 + x_3 \geq -10$$

The dual of the given problem is:

$$\begin{aligned} max \quad & W = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5 \\ S.t. \quad & 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3 \\ & 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2 \\ & 4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Example (2.21):

Construct the dual of the primal problem

$$\begin{aligned} max \quad & Z = 3x_1 + 5x_2 \\ S.t. \quad & 2x_1 + 7x_2 = 12 \\ & -9x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Solution:

The first constraint is of equality form, which is equivalent to:

$$2x_1 + 7x_2 \leq 12 \ \ and \ \ 2x_1 + 7x_2 \geq 12$$

The primal problem can be expressed as:

$$\begin{aligned} max \quad & Z = 3x_1 + 5x_2 \\ S.t. \quad & 2x_1 + 7x_2 \leq 12 \\ & -2x_1 - 7x_2 \leq -12 \\ & -9x_1 + x_2 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Let y_1', y_1'' and y_2 be the dual variables associated with the first, second, and third constraints. Then the dual problem is:

$$\begin{aligned} min \quad & W = 12y_1' - 12y_1'' + 4y_2 \\ S.t. \quad & 2y_1' - 2y_1'' - 9y_2 \geq 3 \\ & 7y_1' - 7y_1'' + y_2 \geq 5 \\ & y_1', y_1'', y_2 \geq 0 \end{aligned}$$

Or equivalently:

$$\begin{aligned} min \quad & W = 12(y_1' - y_1'') + 4y_2 \\ S.t. \quad & 2(y_1' - y_1'') - 9y_2 \geq 3 \\ & 7(y_1' - y_1'') + y_2 \geq 5 \\ & y_1', y_1'', y_2 \geq 0 \end{aligned}$$

If we put $y_1 = {y_1}' - {y_1}''$, then the new variable y_1, which is the difference between two non-negative variables, become unrestricted in sign and the dual problem becomes:

$$\begin{aligned} &min && W = 12y_1 + 4y_2 \\ &S.t. && 2y_1 - 9y_2 \geq 3 \\ & && 7y_1 + y_2 \geq 5 \\ & && y_1 \text{ unrestricted}, y_2 \geq 0 \end{aligned}$$

This example leads to the following remark.

Remark (2.5):

The dual variable which corresponds to an equality constraint must be unrestricted in sign. Conversely, when a primal variable is unrestricted in sign, its dual constraint must be in equality form.

2.13 Some Duality Theorems

Theorem (2.1):

If either the primal or the dual problem has an unbounded solution, then the solution to the other problem is infeasible.

Theorem (2.2) (Fundamental Theorem of Duality):

If both the primal and the dual problems have feasible solutions, then both have optimal solutions and $max\, Z = min\, W$ (and $min\, Z = max\, W$).

Remark (2.6):

Values of the decision variables of the primal are given by the Z-row of the solution under the slack variables (if there are any) in the dual, neglecting the –ve sign if any.

Example (2.22):

Use duality to solve the following LPP:

$$\begin{aligned} &min && Z = 36x + 60y + 45z \\ &S.t. && x + 2y + 2z \geq 40 \\ & && 2x + y + 5z \geq 25 \\ & && x + 4y + z \geq 50 \\ & && x, y, z \geq 0 \end{aligned}$$

Solution:

The dual problem of the LPP is:

$$\begin{aligned} &max && W = 40y_1 + 25y_2 + 50y_3 \\ &S.t. && y_1 + 2y_2 + y_3 \leq 36 \\ & && 2y_1 + y_2 + 4y_3 \leq 60 \end{aligned}$$

$$2y_1 + 5y_2 + y_3 \le 45$$
$$y_1, y_2, y_3 \ge 0$$

Adding slack variables s_1, s_2, and s_3, we get:

$$max \quad W - 40y_1 - 25y_2 - 50y_3 = 0$$
$$S.t. \quad y_1 + 2y_2 + y_3 + s_1 = 36$$
$$2y_1 + y_2 + 4y_3 + s_2 = 60$$
$$2y_1 + 5y_2 + y_3 + s_3 = 45$$
$$y_1, y_2, y_3, s_1, s_2, s_3 \ge 0$$

The initial basic feasible solution of the dual is: $y_1 = y_2 = y_3 = 0, s_1 = 36, s_2 = 60, s_3 = 45, W = 0$. This solution and further improved solutions are given in the following tables:

B.V.	y_1	y_2	y_3 ↓	s_1	s_2	s_3	**Solution**	
s_1	1	2	1	1	0	0	36	36/1=36
← s_2	2	1	$\boxed{4}$	0	1	0	60	60/4=15 ←
s_3	2	5	1	0	0	1	45	45/1=45
W	−40	−25	−50	0	0	0	0	
s_1	1/2 ↓	7/4	0	1	−1/4	0	21	21/(1/2)=42
y_3	1/2	1/4	1	0	1/4	0	15	15/(1/2)=30
← s_3	$\boxed{3/2}$	19/4	0	0	−1/4	1	30	30/(3/2)=20 ←
W	−15	−25/2	0	0	25/2	0	750	
s_1	0	1/6	0	1	−1/6	−1/3	11	
y_3	0	−4/3	1	0	1/3	−1/3	5	
y_1	1	19/6	0	0	−1/6	2/3	20	
W	0	35	0	0	10	10	1050	

The optimal solution of the primal is $x = 0, y = 10, z = 10, and\ Z_{min} = W_{max} = 1050$.

Example (2.23):

Use the duality to find the optimal solution of the LPP in example (2.19).

$$min \quad Z = 3x_1 + 8x_2 + x_3$$
$$S.t. \quad 6x_1 + 2x_2 + 6x_3 \ge 6$$
$$6x_1 + 4x_2 = 12$$
$$2x_1 - 2x_2 \le 2$$
$$x_1, x_2, x_3 \ge 0$$

Solution:

Multiplying the third constraint by (−1) on both sides, we get:

$$-2x_1 + 2x_2 \ge -2$$

The dual problem of the LPP is:

$$\begin{aligned} max \quad & W = 6y_1 + 12y_2 - 2y_3 \\ S.t. \quad & 6y_1 + 6y_2 - 2y_3 \leq 3 \\ & 2y_1 + 4y_2 + 2y_3 \leq 8 \\ & 6y_1 \leq 1 \\ & y_1, y_3 \geq 0, y_2 \text{ is unrestrected} \end{aligned}$$

Adding slack variables $s_1, s_2,$ and s_3, we get:

$$\begin{aligned} max \quad & W - 6y_1 - 12y_2 + 2y_3 = 0 \\ S.t. \quad & 6y_1 + 6y_2 - 2y_3 + s_1 = 3 \\ & 2y_1 + 4y_2 + 2y_3 + s_2 = 8 \\ & 6y_1 + s_3 = 1 \\ & y_1, y_3, s_1, s_2, s_3 \geq 0, y_2 \text{ is unrestrected} \end{aligned}$$

The initial basic feasible solution of the dual is: $y_1 = y_2 = y_3 = 0, s_1 = 3, s_2 = 8, s_3 = 1, W = 0$. This solution and further improved solutions are given in the following tables:

B.V.	y_1	y_2	y_3	s_1	s_2	s_3	**Solution**	
s_1	6	$\boxed{6}$	−2	1	0	0	3	3/6=1/2
s_2	2	4	2	0	1	0	8	8/4=2
s_3	6	0	0	0	0	1	1	
W	−6	−12	2	0	0	0	0	
y_2	1	1	−1/3	1/6	0	0	1/2	
s_2	−2	0	$\boxed{10/3}$	−2/3	1	0	6	
s_3	6	0	0	0	0	1	1	
W	6	0	−2	2	0	0	6	
y_2	4/5	1	0	1/10	−1/10	0	11/10	
y_3	−3/5	0	1	−1/5	3/10	0	9/5	
s_3	6	0	0	0	0	1	1	
W	24/5	0	0	8/5	3/5	0	48/5	

The optimal solution of the primal is $x_1 = 8/5, x_2 = 3/5, x_3 = 0, and\ Z_{min} = W_{max} = 48/5$.

Exercises 2.5 (In addition to the text book exercises)

Use the duality to solve the following LPP:

$$\begin{aligned} \textbf{1.}\ min \quad & Z = 10x_1 + 15x_2 + 30x_3 \\ S.t. \quad & x_1 + 3x_2 + x_3 \geq 90 \\ & 2x_1 + 5x_2 + 3x_3 \geq 120 \end{aligned}$$

$$x_1 + x_2 + x_3 \geq 60$$
$$x_1, x_2, x_3 \geq 0$$

2. $max \quad Z = 10x_1 + 24x_2 + 8x_3$

$S.t. \quad 2x_1 + 4x_2 + 2x_3 \leq 10$

$$4x_1 - 2x_2 + 6x_3 = 4$$
$$x_1, x_2, x_2 \geq 0$$

2.14 The Dual Simplex Method

The dual simplex method starts with a solution that satisfies the optimality condition but infeasible. To start the LP optimal and infeasible, two requirements must be met:

1. The objective function must satisfy the optimality condition of the regular simplex method.
2. All the constraints must be of the type ($\leq$).

The dual simplex method consists of the following steps:

Step 1: Convert the ($\geq$) type constraint to a ($\leq$) type constraint by multiplying both sides by (-1). If the LPP includes an equality constraint, the equation can be replaced by two inequalities, then convert the constraint of ($\geq$) type into a constraint of ($\leq$) type.

Step 2: Convert the LPP into the standard form and express the problem information in the form of a table known as the ***dual simplex table***.

Step 3: Three cases arises:

a) If the Z- row satisfies the optimality condition and all $b_i \geq 0$, then the current solution is optimal basic feasible solution.

b) If at least one element in the Z-row doesn't satisfy the optimality condition, the method fails.

c) If the Z- row satisfies the optimality condition and at least one $b_i \leq 0$, then proceed to step 4.

Step 4: Select the row that contains the most negative b_i. Ties are broken arbitrarily. This row is called the ***pivot*** (***key***) ***row***. The corresponding variable leaves the basis. This is called the ***dual feasibility condition***.

Step 5: Look at the elements of the pivot row:

a) If all elements are non-negative, the problem does not have a feasible solution.

b) If at least one element is negative, divide the elements of the Z-row to the corresponding negative elements in the pivot row. Choose the smallest of these ratios. Ties are broken arbitrarily. The corresponding column is the ***key column*** and the associated variable is the ***entering variable***. This is called ***dual optimality condition***. Mark the ***pivot*** (***key***) element.

Step 6: Make the key element unity. Perform as in regular simplex method and repeat iterations until an optimal feasible solution is obtained in a finite number of steps or there is an indication of the non-existence of a feasible solution.

Example (2.24):

Find the optimal solution of the following LPP

$$\begin{aligned} max \quad & Z = -3x_1 - 2x_2 - x_3 \\ S.t. \quad & 2x_1 + x_2 + x_3 \geq 4 \\ & 3x_1 + x_2 + 3x_3 \geq 10 \\ & -x_1 + 2x_2 - x_3 \geq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

First we convert constraints of (≥) type into constraints of (≤) type , so the LPP will be:

$$\begin{aligned} max \quad & Z = -3x_1 - 2x_2 - x_3 \\ S.t. \quad & -2x_1 - x_2 - x_3 \leq -4 \\ & -3x_1 - x_2 - 3x_3 \leq -10 \\ & x_1 - 2x_2 + x_3 \leq -1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The standard form (with modification in the objective function) is:

$$\begin{aligned} max \quad & Z + 3x_1 + 2x_2 + x_3 = 0 \\ S.t. \quad & -2x_1 - x_2 - x_3 + s_1 = -4 \\ & -3x_1 - x_2 - 3x_3 + s_2 = -10 \\ & x_1 - 2x_2 + x_3 + s_3 = -1 \\ & x_1, x_2, x_3, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = x_3 = 0,$ then $s_1 = -4, s_2 = -10,$ and $s_3 = -1.$ Since $s_1, s_2,$ and s_3 are negative, then solution is infeasible. The dual simplex table is:

BV	x_1	x_2	x_3	s_1	s_2	s_3	b
s_1	-2	-1	-1	1	0	0	-4
← s_2	-3	-1	$\boxed{-3}$	0	1	0	$\boxed{-10}$
s_3	1	-2	1	0	0	1	-1
Z	3	2	1	0	0	0	0

$(\left|\frac{3}{-3}\right| = 1, \left|\frac{2}{-1}\right| = 2, \left|\frac{1}{-3}\right| = 1/3)$

BV	x_1	x_2	x_3	s_1	s_2	s_3	b
s_1	−1	−2/3	0	1	−1/3	0	−2/3
x_3	1	1/3	1	0	−1/3	0	10/3
← s_3	0	$\boxed{-7/3}$	0	0	1/3	1	$\boxed{-13/3}$
Z	2	5/3	0	0	1/3	0	−10/3
s_1	−1	0	0	1	−3/7	−2/7	4/7
x_3	1	0	1	0	−2/7	1/7	19/7
x_2	0	1	0	0	−1/7	−3/7	13/7
Z	2	0	0	0	4/7	5/7	−45/7

The optimal solution is : $x_1 = 0, x_2 = \frac{13}{7}, x_3 = \frac{19}{7}$, and $Z_{max} = -45/7$

Example (2.25):

Use the dual simplex method to find the optimal solution of the LPP in example (2.19).

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 \\ S.t. \quad & 6x_1 + 2x_2 + 6x_3 \geq 6 \\ & 6x_1 + 4x_2 = 12 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution:

Replace the second constraint by the following two constraints:

$6x_1 + 4x_2 \leq 12$ and $6x_1 + 4x_2 \geq 12$

Then convert each constraint of ($\geq$) type into a constraint of ($\leq$) type. The LPP will be:

$$\begin{aligned} min \quad & Z = 3x_1 + 8x_2 + x_3 \\ S.t. \quad & -6x_1 - 2x_2 - 6x_3 \leq -6 \\ & 6x_1 + 4x_2 \leq 12 \\ & -6x_1 - 4x_2 \leq -12 \\ & 2x_1 - 2x_2 \leq 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

The standard form (with modification in the objective function) is:

$$\begin{aligned} min \quad & Z - 3x_1 - 8x_2 - x_3 = 0 \\ S.t. \quad & -6x_1 - 2x_2 - 6x_3 + s_1 = -6 \\ & 6x_1 + 4x_2 + s_2 = 12 \end{aligned}$$

$$-6x_1 - 4x_2 + s_3 = -12$$
$$2x_1 - 2x_2 + s_4 = 2$$
$$x_1\,, x_2,\ x_3, s_1, s_2, s_3, s_4 \geq 0$$

Let $x_1 = x_2 = x_3 = 0$, then $s_1 = -6, s_2 = 12,$ and $s_3 = -12, s_4 = 2$. Since s_1 and s_3 are negative, then solution is infeasible. The dual simplex table is:

BV	x_1	x_2	x_3	s_1	s_2	s_3	s_4	b
s_1	−6	−2	−6	1	0	0	0	−6
s_2	6	4	0	0	1	0	0	12
← s_3	$\boxed{-6}$	−4	0	0	0	1	0	$\boxed{-12}$
s_4	2	−2	0	0	0	0	1	2
Z	−3	−8	−1	0	0	0	0	0
s_1	0	2	−6	1	0	−1	0	6
s_2	0	0	0	0	1	1	0	0
x_1	1	2/3	0	0	0	−1/6	0	2
← s_4	0	$\boxed{-10/3}$	0	0	0	1/3	1	$\boxed{-2}$
Z	0	−6	−1	0	0	−1/2	0	6
s_1	0	0	−6	1	0	−4/5	3/5	24/5
s_2	0	0	0	0	1	1	0	0
x_1	1	0	0	0	0	−1/10	1/5	8/5
x_2	0	1	0	0	0	−1/10	−3/10	3/5
Z	0	0	−1	0	0	−11/10	−9/5	48/5

The optimal solution is : $x_1 = \frac{8}{5}, x_2 = \frac{3}{5}, x_3 = 0,$ and $Z_{min} = 48/5$

Exercises 2.6 (In addition to the text book exercises)

Use dual simplex method to find the optimal solution of the following LPP:

1. $min \quad Z = 3x_1 + 6x_2 + 9x_3$

$S.t. \quad 6x_1 - 3x_2 + 3x_3 \geq 12$

$3x_1 + 3x_2 + 6x_3 \leq 24$

$3x_2 - 6x_3 \geq 6$

$x_1\,, x_2,\ x_3 \geq 0$

2. $max \quad Z = -6x_1 - 3x_3$

$S.t. \quad 3x_1 + 3x_2 - 3x_3 \geq 15$

$3\,x_1 - 6x_2 + 12x_3 \geq 24$

$x_1\,, x_2,\ x_3 \geq 0$

Ch.3: Advanced Topics in Linear Programming

3.1 Special Cases in Linear Programming

There are some special cases that arise in the application.

3.1.1 Tie in the Choice of the Entering Variable

The non-basic variable that enters the basis is the one that gives the largest per unit improvement in the objective function. They are variable having minimum (maximum) negative (positive) value in a maximization (minimization) problem in Z-row is the entering variable. A tie in the choice of entering variable exists when more than one variable has the same largest negative (positive) value. To break this tie, select any one of them arbitrarily as the entering variable. There is no method to predict which of them is better. If there is a tie between a decision variable and a slack/surplus variable, select the decision variable.

3.1.2 Unbounded Solution

In some LP models, the values of the variables may be increased indefinitely without violating any of the constraint- meaning that the solution space is unbounded in at least one variable. As a result, the objective function value may increase (maximization case) or decrease (minimization case) indefinitely. In this case, both the solution space and the optimum objective value are unbounded. In simplex technique, this happens when all the constraint coefficients of the non-basic variable that is to enter the basis are negative or zero so that there is no minimum in the non-negative ratio. That it is not possible to determine the basic variable that should leave the basis.

Example (3.1):

Discuss the following LPP:

$$
\begin{aligned}
&max \quad Z = x_1 + 2x_2 \\
&S.t \quad x_1 - x_2 \leq 10 \\
&\qquad\; 2x_1 \leq 40 \\
&\qquad\;\; x_1, x_2 \geq 0
\end{aligned}
$$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$$
\begin{aligned}
&max \quad Z - x_1 - 2x_2 = 0 \\
&S.t \quad x_1 - x_2 + s_1 = 10 \\
&\qquad\; 2x_1 + s_2 = 40 \\
&\qquad\;\; x_1, x_2, s_1, s_2 \geq 0
\end{aligned}
$$

Let $x_1 = x_2 = 0$, then $s_1 = 10$ and $s_2 = 40, Z = 0$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	Solution
s_1	1	−1	1	0	10
s_2	2	0	0	1	40
Z	−1	−2	0	0	0

The current solution is not optimal. In the starting table, the x_2-column is the pivot column. But, all the constraint coefficients under the x_2 are negative or zero. This means that there is no leaving variable and that x_2 can increase indefinitely without violating any of the constraints. Because each unit increase in x_2 will increase Z by 2, an infinite increase in x_2 leads to an infinite increase in Z. Thus, the problem has no bounded solution. We can see this graphically:

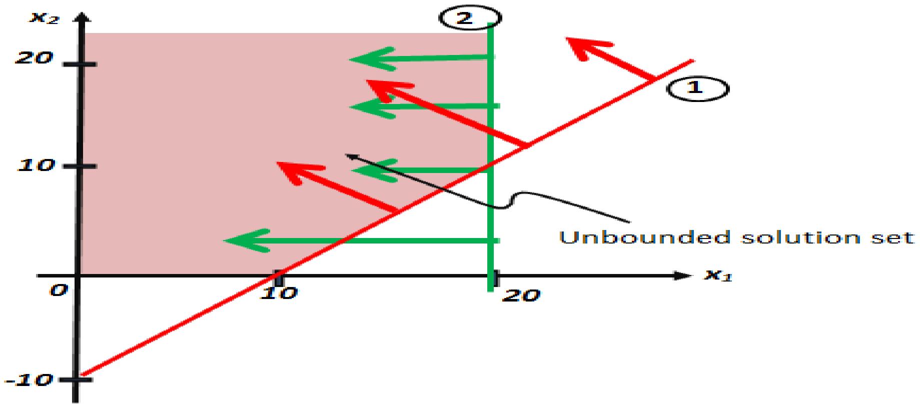


Figure (3.1)

3.1.3 Alternative Optima

This happens when there are multiple optimal solutions. Graphically, this happens when the objective function is parallel to a non-redundant constraint. In the optimal simplex table, if a non-basic variable has zero coefficients in the Z-row, there exists an alternate optimal solution. It is because that non-basic variable can enter the basis without changing the value of Z, but causing a change in the value of the basic variables. These variables may be a decision or slack or surplus variable.

Example (3.2) (Infinite number of solutions):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = 2x_1 + 4x_2 \\ S.t \quad & x_1 + 2x_2 \le 5 \\ & x_1 + x_2 \le 4 \\ & \ x_1, x_2 \ge 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification of the objective function) is:

$$\begin{aligned} &max \quad Z - 2x_1 - 4x_2 = 0 \\ &S.t \quad x_1 + 2x_2 + s_1 = 5 \\ &\qquad\; x_1 + x_2 + s_2 = 4 \\ &\qquad\; x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = 0$, then $s_1 = 5$ and $s_2 = 4, Z = 0$. The simplex table is:

B.V.	x_1	x_2	s_1	s_2	Solution	
s_1	1	2	1	0	5	5/2=2.5
s_2	1	1	0	1	4	4/1=4
Z	−2	−4	0	0	0	
x_2	½	1	½	0	5/2	
s_2	½	0	−1/2	1	3/2	
Z	0	0	2	0	10	
x_2	0	1	1	−1	1	
x_1	1	0	−1	2	3	
Z	0	0	2	0	10	

The first iteration gives the optimum solution $x_1 = 0, x_2 = 5/2,$ and $Z_{max} = 10$, which coincides with point B in the graphical representation of the problem. The coefficient of the non-basic variable x_1 in the Z-equation is zero, indicating that x_1 can enter the basic solution without changing the value of Z, but causing a change in the values of variables. In second iteration: $x_1 = 3, x_2 = 1,$ and $Z_{max} = 10$. This solution occurs at the corner point C (3, 1). Any point in the line segment BC represents an alternative optimum with $Z_{max} = 10$. The simplex method determines only the two corners B and C. Mathematically; we can determine all the points (x_1, x_2) on the line segment BC as a non-negative weighted average of the points B and C. Thus given

$$B: x_1 = 0, x_2 = 5/2 \quad and \quad C: x_1 = 3, x_2 = 1$$

Then all the points on the line segment BC are given by:

$$\left.\begin{aligned} \widehat{x_1} &= \alpha(0) + (1-\alpha)(3) = 3 - 3\alpha \\ \widehat{x_2} &= \alpha\left(\frac{5}{2}\right) + (1-\alpha)(1) = 1 + \frac{3}{2}\alpha \end{aligned}\right\} \quad 0 \leq \alpha \leq 1$$

When $\alpha = 0, (\widehat{x_1}, \widehat{x_2}) = (3,1)$ which is the point C. When $\alpha = 1, (\widehat{x_1}, \widehat{x_2}) = (0,5/2)$ which is the point B. For values of $\alpha (0 \le \alpha \le 1)$, $(\widehat{x_1}, \widehat{x_2})$ lies between B and C.

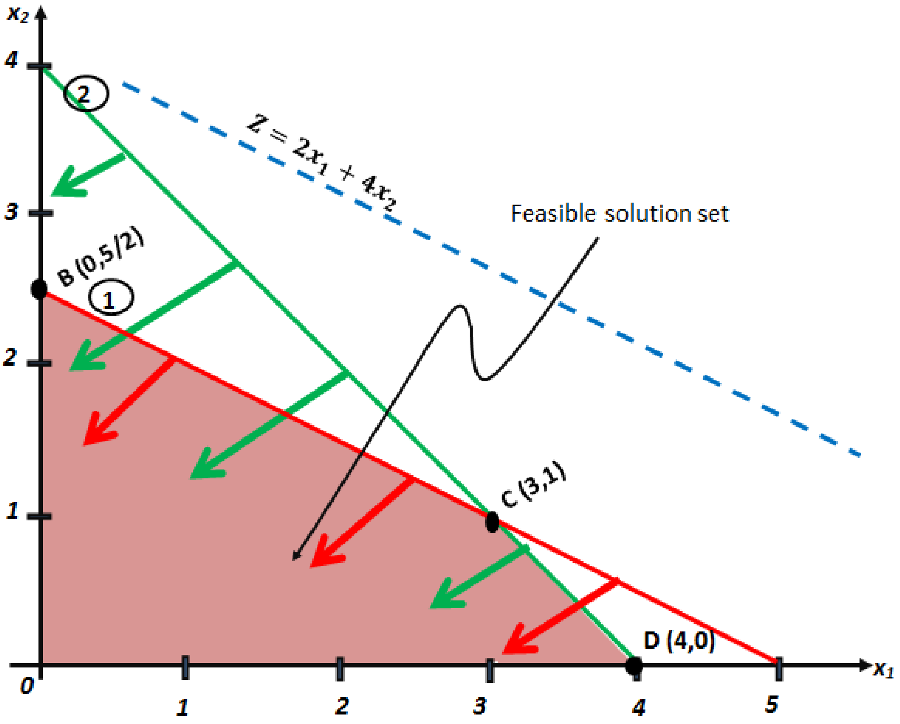


Figure (3.2)

3.1.4 No Feasible Solution (Infeasible Solution)

In this case, there is no feasible solution in LPP that satisfies all the constraints and non-negativity restrictions. It means that the constraints in the problem are conflicting and inconsistent. As an example, see examples (2.18) and (2.19).

Example (3.3):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = 3x_1 + 2x_2 \\ S.t. \quad & -2x_1 + 3x_2 \le 9 \\ & 3x_1 - 2x_2 \le -20 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification in Z-equation) is:

$$\begin{aligned} max \quad & Z + (-3 + 3M)x_1 + (-2 - 2M)x_2 + Ms_2 = -20M \\ S.t. \quad & -2x_1 + 3x_2 + s_1 = 9 \\ & -3x_1 + 2x_2 - s_2 + R_1 = 20 \\ & x_1, x_2, s_1, s_2, R_1 \ge 0 \end{aligned}$$

Let $x_1 = x_2 = s_2 = 0$, then $s_1 = 9, R_1 = 20, \text{and } Z = -20M$. The simplex iteration of the LP model is:

B.V.	x_1	x_2	s_1	s_2	R_1	Solution	
s_1	−2	3	1	0	0	9	9/3=3
R_1	−3	2	0	−1	1	20	20/2=10
Z	−3+3M	−2−2M	0	M	0	−20M	
x_2	−2/3	1	1/3	0	0	3	
R_1	−5/3	0	−2/3	−1	1	14	
Z	$-\frac{13}{3}+\frac{5}{3}M$	0	$\frac{2}{3}+\frac{5}{3}M$	M	0	6+4M	

Optimum iteration shows that the artificial variable R_1 is positive, which indicates that the problem is infeasible. The result is what we may call a ***pseudo-optimal solution***. The graphic representation of the problem shows clearly that the absence of feasible solution.

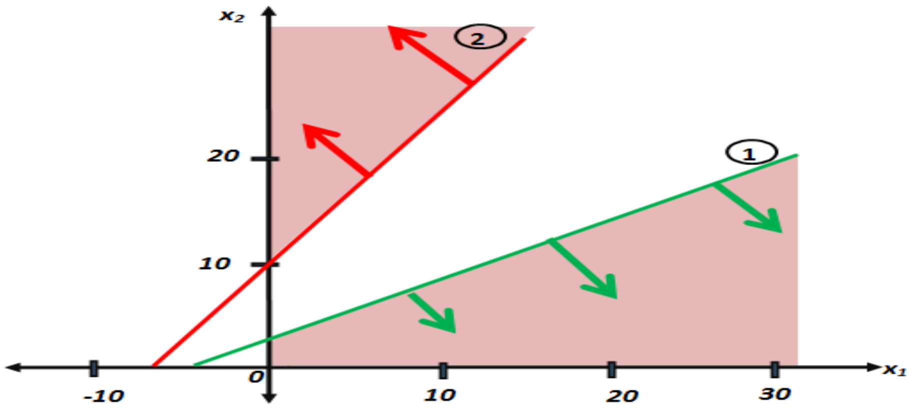


Figure (3.3)

3.1.5 Degeneracy (Tie in the Choice of the Leaving Variable)

Degeneracy in Linear Programming is said to occur when one or more basic variables have zero value. If the minimum ratio is zero for two or more basic variables, degeneracy may result and the simplex routine will cycle indefinitely. That is, the solution which we have obtained in one iteration may repeat after few iterations and therefore no optimum solution may be obtained. This concept is known as ***cycling*** or ***circling***.

To resolve degeneracy, we follow the following method which is called the ***perturbation method*** by A. Charnes:

1. Divide each element in the tied rows by positive coefficients of the pivot (key) column in that row.

2. Compare the resulting ratio, column by column, first in the identity and then in the body of the simplex table, from left to right.
3. The row which first contains the smallest algebraic ratio contains the outgoing variable. The simplex method is then continued to reach the optimal solution.

If any artificial variable is one of the tied variables, it should be immediately selected to leave the basis without following the above rules.

Example (3.4):

Discuss the following LPP:

$$\begin{aligned} max \quad & Z = 2x_1 + x_2 \\ S.t. \quad & 4x_1 + 3x_2 \leq 12 \\ & 4x_1 + x_2 \leq 8 \\ & 4x_1 - x_2 \leq 8 \\ & x_1 , x_2 \geq 0 \end{aligned}$$

Solution:

The standard form of the LPP (with modification in the Z-equation) is:

$$\begin{aligned} max \quad & Z - 2x_1 - x_2 = 0 \\ S.t. \quad & 4x_1 + 3x_2 + s_1 = 12 \\ & 4x_1 + x_2 + s_2 = 8 \\ & 4x_1 - x_2 + s_3 = 8 \\ & x_1 , x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = 0$, then: $s_1 = 12, s_2 = 8, s_3 = 8$, and $Z = 0$.

B.V.	x_1 ↓	x_2	s_1	s_2	s_3	Solution	
s_1	4	3	1	0	0	12	12/4=3
s_2	4	1	0	1	0	8	8/4=2
← s_3	$\boxed{4}$	-1	0	0	1	8	8/4=2 ←
Z	-2	-1	0	0	0	0	

In the above table x_1 is the entering variable, as s_2 and s_3 are the tied rows, perturbation method is used to determine the outgoing variable. The first column of the identity has the elements 0 and 0 in the tied rows. Dividing them by the corresponding elements of the key column, the resulting ratios are 0 and 0. Hence first column of the identity fails to identify the outgoing variable. The second column of the identity has the elements 1 and 0 in the tied rows. Dividing them by the corresponding elements of the key column, the resulting ratios are 1/4 and 0. As s_3-row yields the smaller ratio, the s_3 is the leaving variable.

Performing iterations to get the optimal solution results in the following tables:

B.V.	x_1	x_2	s_1	s_2	s_3	Solution	
s_1	0	4	1	0	−1	4	4/4=1
s_2	0	2	0	1	−1	0	0/2=0
x_1	1	−1/4	0	0	1/4	2	
Z	0	−3/2	0	0	1/2	4	
s_1	0	0	1	−2	1	4	4/1=4
x_2	0	1	0	1/2	−1/2	0	
x_1	1	0	0	1/8	1/8	2	2/(1/8)=16
Z	0	0	0	3/4	−1/4	4	
s_3	0	0	1	−2	1	4	
x_2	0	1	1/2	−1/2	0	2	
x_1	1	0	−1/8	3/8	0	3/2	
Z	0	0	1/4	1/4	0	5	

Then the optimal solution is: $x_1 = \frac{3}{2}, x_2 = 2, and\ Z_{max} = 5$.

Exercises 3.1 (In addition to the text book exercises)

Discuss the following LPP's:

1. $min \quad Z = 6x_1 + 10x_2$
$S.t. \quad 6x_1 + 10x_2 \geq 30$
$x_1 \leq 4$
$x_2 \leq 2$
$x_1, x_2 \geq 0$

2. $max \quad Z = 16x_1 + 2x_2$
$S.t. \quad 16x_1 + 2x_2 \leq 16$
$4x_1 + 2x_2 \leq 12$
$6x_1 + 2x_2 \leq 12$
$2\ x_1 + 12x_2 \leq 16$
$x_1, x_2 \geq 0$

3. $max \quad Z = 12x_1 + 15x_2$
$S.t. \quad 3x_1 + 3x_2 \geq 3$
$-6x_1 + 3x_2 \leq 3$
$12x_1 - 3x_2 \leq 3$

$x_1 , x_2 \geq 0$

4. $max \quad Z = 8x_1 + 105x_2 + 4x_3$

$S.t. \quad 4\,x_1 + 2x_2 + 2x_3 \leq 20$

$2\,x_1 + 6x_2 + 2x_3 \leq 24$

$2x_1 + 2x_2 + 2x_3 = 12$

$x_1 , x_2, x_3 \geq 0$

5. $max \quad Z = 3x_1 - 6x_2 - 9x_3$

$S.t. \quad 6\,x_1 + 3x_2 + 9x_3 = 6$

$6x_1 + 9x_2 + 12x_3 = 3$

$x_1 , x_2, x_3 \geq 0$

3.2 Sensitivity Analysis

In LPP, the parameters (input data) of the model can change within certain limits without causing the optimum solution to change. This is referred to as ***sensitivity analysis***. In LPP models, the parameters are usually not exact; we can ascertain the impact of this uncertainty on the quality of the optimum solution. The changes in (discrete) parameters of an LPP include changes in the values of few $b_i's$ or c_j or a_{ij} or addition/deletion of some constraints/ variables. Generally, these parameter changes result in one of the following three cases:

1- The optimal solution remains unchanged, i.e., the basic variables and their values remain unchanged.
2- The basic variables remain unchanged but their values change.
3- The basic variables as well as their values are changed.

3.2.1 Cost Changes

We will, first, consider the changing a cost value by Δ in the original problem. If we are given the original problem and an optimal tableau and If we had done exactly the same calculations beginning with the modified problem, we would have had the same final tableau except that the corresponding cost entry would be Δ lower (this is because we do nothing but to add or subtract scalar multiples of Rows 1 through m to other rows; we never add or subtract Z-row to other rows).

Example (3.5):

Consider the LPP:

$max \quad Z = 3x + 2y$

$S.t. \quad x + y \leq 4$

$$2x + y \leq 6$$
$$x, y \geq 0$$

Suppose that the cost for x is changed to $3 + \Delta$ in the original formulation.

a) What are the limits of Δ so as the solution remains optimal?

b) If the objective function is changed to $max \quad Z = 3.5x + 2y$, what is the optimal solution of the problem?

c) If the objective function is changed to $max \quad Z = x + 2y$, what is the optimal solution of the problem?

Solution:

a) The standard form of the LPP (If the cost of x is changed from 3 to $3 + \Delta$ with modification in the objective function) is:

$$max \quad Z - (3 + \Delta)x - 2y = 0$$
$$S.t. \quad x + y + s_1 = 4$$
$$2x + y + s_2 = 6$$
$$x, y, s_1, s_2 \geq 0$$

Let $x = y = 0$, then $s_1 = 4, s_2 = 6,$ and $Z = 0$. The simplex table is:

B.V.	x	y	s_1	s_2	**b**
s_1	1	1	1	0	4
s_2	2	1	0	1	6
Z	−3−Δ	−2	0	0	0
s_1	0	1/2	1	−1/2	1
x	1	1/2	0	1/2	3
Z	0	−1/2+Δ/2	0	3/2+Δ/2	9+3Δ
y	0	1	2	−1	2
x	1	0	−1	1	2
Z	0	0	1−Δ	1+Δ	10+2Δ

The solution is optimal if the elements of the Z-row are all non-negative. This is true if: $1 - \Delta \geq 0$ and $1 + \Delta \geq 0$ which holds if $-1 \leq \Delta \leq 1 (\Delta \in [-1,1])$. For any Δ in that range, our previous basis (and variable values) is optimal. The objective changes to $10 + 2\Delta$.

The optimal solution for the original problem is: $x = 2, y = 2,$ and $Z_{max} = 10$ and the optimal tableau is:

B.V.	x	y	s_1	s_2	**b**
y	0	1	2	-1	2
x	1	0	-1	1	2
Z	0	0	1	1	10

Note that the table has the same basic variables and the same variable values (except for Z) that the previous solution had.

b) The value of Δ is obtained from subtracting cost coefficients of x in new and old objective functions, thus: $\Delta = 3.5 - 3 = 0.5. 1 - \Delta = 0.5 > 0$ and $1 + \Delta = 1.5 > 0$ then the solution remains optimal. That is: $x = 2, y = 2, Z_{max} = 10 + 2\Delta = 11$

c) $\Delta = 1 - 3 = -2$, then $1 - \Delta = 3 > 0$, but $1 + \Delta = -1 < 0$, then the solution is no longer optimal. To find the optimal solution we use the optimal table, that is:

B.V.	x	y	s_1	s_2 ↓	**b**
y	0	1	2	-1	2
← x	1	0	-1	1	2
Z	0	0	3	-1	6
y	1	1	1	0	4
s_2	1	0	-1	1	2
Z	1	0	2	0	8

Then the optimal solution is: $x = 0, y = 4, Z\_max = 8$

In the previous example, we changed the cost of a basic variable. The next example will show what happens when the cost of a non-basic variable changes.

Example (3.6):

Consider the LPP:

$$\begin{aligned} max \quad & Z = 3x + 2y + 2.5w \\ S.t. \quad & x + y + 2w \leq 4 \\ & 2x + y + 2w \leq 6 \\ & x, y, w \geq 0 \end{aligned}$$

Suppose that the cost for w is changed to $2.5 + \Delta$ in the original formulation. What are the limits of Δ so as the solution remains optimal?

Solution:

The standard form of the LPP (If the cost of w is changed from 2.5 to $2.5+\Delta$ with modification in the objective function) is:

$$\begin{aligned} &max \quad Z-3x-2y-(2.5+\Delta)w=0 \\ &S.t. \quad x+y+2w+s_1=4 \\ &\qquad 2x+y+2w+s_2=6 \\ &\qquad x,y,w,s_1,s_2\geq 0 \end{aligned}$$

Let $x=y=w=0$, then $s_1=4, s_2=6,$ and $Z=0$. The simplex table is:

B.V.	x	y	w	s_1	s_2	b
s_1	1	1	2	1	0	4
s_2	2	1	2	0	1	6
Z	$\boxed{-3}$	-2	$-2.5-\Delta$	0	0	0
s_1	0	1/2	1	1	$-1/2$	1
x	1	1/2	1	0	1/2	3
Z	0	$\boxed{-1/2}$	$0.5-\Delta$	0	3/2	9
y	0	1	2	2	-1	2
x	1	0	0	-1	1	2
Z	0	0	$1.5-\Delta$	1	1	10

In this case, we already have a valid tableau. This will represent an optimal solution if $1.5-\Delta\geq 0$, so $\Delta\leq 1.5$. Any change in the objective coefficient of the non-basic variable will affect only its index row coefficient and not others.

Notice that, the optimal tableau is:

B.V.	x	y	w	s_1	s_2	b
y	0	1	2	2	-1	2
x	1	0	0	-1	1	2
Z	0	0	1.5	1	1	10

3.2.2 Right Hand Side Changes

Example (3.7):

Consider the LPP:

$$\begin{aligned} &max \quad Z=4x+5y \\ &S.t. \quad 2x+3y\leq 12 \\ &\qquad x+y\leq 5 \\ &\qquad x,y\geq 0 \end{aligned}$$

a) Suppose that the value of the right-hand-side of the first constraint from 12 to $12+\Delta$. What are the limits of Δ so as the solution remains feasible?

b) If the value of the right-hand-side of the first constraint is changed to 11, does the solution remains feasible, what is the optimal solution?

c) If the value of the right-hand-side of the first constraint is changed to 25, what is optimal solution?

Solution:

a) The standard form of the LPP (with modification in the objective function and changing the right-hand-side of the first constraint to $12+\Delta$) is:

$$\begin{aligned} &max \quad Z-4x-5y=0 \\ &S.t. \quad 2x+3y+s_1=12+\Delta \\ &\qquad\quad x+y+s_2=5 \\ &\qquad\quad x,y,s_1,s_2\geq 0 \end{aligned}$$

Let $x=y=0,$ then $s_1=12+\Delta$ and $s_2=5$. The simplex table is:

B.V.	x	y	s_1	s_2	b
s_1	2	3	1	0	$12+\Delta$
s_2	1	1	0	1	5
Z	-4	-5	0	0	0
y	2/3	1	1/3	0	$4+\ \Delta/3$
s_2	1/3	0	$-1/3$	1	$1-\Delta/3$
Z	$-2/3$	0	5/3	0	$20+5\Delta/3$
y	0	1	1	-2	$2+\Delta$
x	1	0	-1	3	$3-\Delta$
Z	0	0	1	2	$22+\Delta$

This represents an optimal tableau as long as the right-hand-side is all non-negative. In other words, Δ must be between -2 and 3 in order for the basis not to change (remains feasible). The optimal tableau is:

B.V.	x	y	s_1	s_2	b
y	0	1	1	-2	2
x	1	0	-1	3	3
Z	0	0	1	2	22

The optimal solution is: $x=3, y=2, and\ Z_{max}=22$.

b) $\Delta = 11 - 12 = -1$, then $2 + \Delta = 1 > 0$ and $3 - \Delta = 4 > 0$, thus, the solution remains feasible. The optimal solution is: $x = 4, y = 1, and\ Z_{max} = 21$

c) $\Delta = 25 - 12 = 13$, then $2 + \Delta = 15 > 0$ and $3 - \Delta = -10 < 0$. Thus, the solution is no longer feasible to manage this case we use the optimal table as follows:

B.V.	x	y	s_1	s_2	**b**
y	0	1	1	-2	15
← x	1	0	$\bigcirc\!\!\!\!\!-1$	3	-10
Z	0	0	1	2	35
y	1	1	0	1	5
s_1	-1	0	1	-3	10
Z	1	0	0	5	25

The optimal solution is: $x = 0, y = 5, and\ Z_{max} = 25$.

Exercises 3.2 (In addition to the text book exercises)

1. Consider the following LPP:

$$\begin{aligned} max \quad & Z = 3x_1 + 7x_2 + 4x_3 + 9x_4 \\ S.t. \quad & x_1 + 4x_2 + 5x_3 + 8x_4 \leq 9 \\ & x_1 + 2x_2 + 6x_3 + 4x_4 \leq 7 \\ & x_i \geq 0 \qquad i = 1,2,3,4 \end{aligned}$$

a) Solve this linear program using the simplex method.

b) What are the values of the variables in the optimal solution?

c) What is the optimal objective function value?

d) What would you estimate the objective function would change to if:

- We change the right-hand side of the first constraint to 10.
- We change the right-hand side of the second constraint to 6.5.

2. Solve the problem :

$$\begin{aligned} max \quad & Z = 45x_1 + 100x_2 + 30x_3 + 50x_4 \\ S.t. \quad & 7x_1 + 10x_2 + 4x_3 + 9x_4 \leq 1200 \\ & 3x_1 + 40x_2 + x_3 + x_4 \leq 800 \\ & x_i \geq 0 \qquad i = 1,2,3,4 \end{aligned}$$

Find the effect of:

a) Changing the cost coefficients c_1 and c_4 from 45 and 50 to 40 and 60 respectively.

b) Changing c_1 to 30 and c_2 to 90.

c) Changing c_3 from 30 to 24.

3.3 Integer Programming

Integer linear programming problems (***ILPP***) are linear programming problems with some or all the variables restricted to integer (discrete) values. When all the variables are constrained to be integers, it is called an ***all*** (***pure***) ***integer programming problem***. In case only some of the variables are restricted to have integer values, the problem is said to be a ***mixed integer programming problem***. The ILPP algorithms are based on exploiting the tremendous computational success of LPP. The strategy of these algorithms involves three steps:

Step 1: Relax the solution space of the ILPP by deleting the integer restriction on all integer variables. The result of the relaxation is a regular LPP.

Step 2: Solve the LPP, and identify its optimum.

Step 3: Starting from the optimum point, add special constraints that iteratively modify the LPP solution space in a manner that will eventually render an optimum extreme point satisfying the integer requirements.

Two general methods have been developed for generating the special constraints in step 3:

1. Cutting - plane method.
2. Branch - and – bound (B & B) method.

3.3.1 Gomory's Cutting Plane Method

This systematic procedure for solving pure ILPP was first suggested by R.E. Gomory (1929-) in 1958. Later, he extends the procedure to cover mixed ILPP. The method consists in first solving the ILPP as ordinary continuous LPP and then introducing additional constraints one after the other to cut (eliminate) certain parts of the solution space until an integral solution is obtained.

Definition (3.1):

For all real number x, the ***greatest integer function*** (denoted by $[\ x\ ]$) returns the largest integer less than or equal to *x*. In other words, the greatest integer function rounds down a real number to the nearest integer. The number *x* can be written in the form $x = [x] + e$, where $0 \leq e \leq 1$.We call e the fractional part of x.

Example (3.8):

1) $[0.41] = 0$ since integers less than 0.41 are: $\dots, -2, -1, 0$ and the greatest one of them is 0. The number 0.41 can be written in the form: $0.41 = 0 + 0.41 = [0.41] + 0.41.$
2) $[-0.41] = -1$ since integers less than -0.41 are: $\dots, -3, -2, -1$ and the greatest one of them is -1. The number -0.41 can be written in the form: $-0.41 = -1 + 0.59 = [-0.41] + 0.59$
3) $[9.73] = 9$
4) $[-7.26] = -8$
5) $[3] = 3$
6) $[-5] = -5$

According to definition (3.1), the structural coefficients and the stipulations can be written as:

$a_{ij} = [a_{ij}] + f_{ij},\ b_i = [b_i] + f_i, where\ 0 \le f_{ij} \le 1\ and\ 0 \le f_i \le 1\,;$
$i = 1, \dots, m; j = 1, \dots, n$

The steps of Gomory's cutting plane method for pure ILPP are:

***Step 1:* Integerise the constraints**: Transform the constraints so that all the coefficients are whole numbers. For example, the constraint equation: $\frac{7}{4}x_1 + \frac{1}{5}x_2 + \frac{3}{4}x_3 = \frac{17}{5}$ can be expressed as: $35x_1 + 4x_2 + 15x_3 = 68.$

***Step 2:* Solve the problem**: Ignoring integrality restriction, find the optimal solution to the problem. If the solution is all integers, it is an optimal basic feasible integer solution. If not, proceed to step 3. Ignore non-integer values for slack variables since they represent unused resources only.

Step 3:* Develop a cutting plane:** From the final table select the constraint with the largest fractional cut. The selected row is called the ***source row. In case of a tie, choose the constraint having the highest contribution (maximization problem) or the lowest cost (minimization problem). Alternatively select the constraint with $max\ \frac{f_i}{\sum_{j=1}^{n} f_{ij}}$ $\dots..(1)$. Construct the Gomory's constraint:

$s_i = \sum_{j=1}^{n} f_{ij} y_j - f_i$ .....(2) (y_j may be decision or slack variable)

And add it to the final table. Add an additional column for s_i also.

***Step 4:* Solve using the dual simplex method:** Solve the augmented ILPP obtained above by the dual simplex method so that the outgoing variable is s_i . If the optimal solution thus obtained has all integral values, it is an optimal solution for the given ILPP. If not, repeat step 3 until an optimal feasible integer solution is obtained.

Remark (3.1):

In mixed ILPP, only constraints corresponding to integer variables are used to construct the cut.

Example (3.9):

Find the optimal solution of the following ILPP

$$\begin{aligned} max \quad & Z = 5x_1 + 6x_2 \\ S.t \quad & 2x_1 + 3x_2 \leq 18 \\ & 2x_1 + x_2 \leq 12 \\ & x_1 + x_2 \leq 8 \\ & x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$

Solution:

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$$\begin{aligned} max \quad & Z - 5x_1 - 6x_2 = 0 \\ S.t \quad & 2x_1 + 3x_2 + s_1 = 18 \\ & 2x_1 + x_2 + s_2 = 12 \\ & x_1 + x_2 + s_3 = 8 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Let $x_1 = x_2 = 0,$ then $s_1 = 18, s_2 = 12, s_3 = 8,$ and $Z = 0$ and

B.V.	x_1	x_2	s_1	s_2	s_3	b	
s_1	2	3	1	0	0	18	
s_2	2	1	0	1	0	12	
s_3	1	1	0	0	1	8	
Z	−5	$\boxed{-6}$	0	0	0	0	
x_2	2/3	1	1/3	0	0	6	
s_2	4/3	0	−1/3	1	0	6	
s_3	1/3	0	−1/3	0	1	2	
Z	$\boxed{-1}$	0	2	0	0	36	
x_2	0	1	1/2	−1/2	0	3	3
x_1	1	0	−1/4	3/4	0	9/2	$4\frac{1}{2}$
s_3	0	0	−1/4	−1/4	1	1/2	$\frac{1}{2}$

Z	0	0	7/4	3/4	0	81/2

The non-integer optimal solution is $x_1 = \frac{9}{2}, x_2 = 3$, and $Z_{max} = 40\frac{1}{2}$. To construct Gomory's constraint, select x_1-row which has the greatest fractional part ½ (s_3-row also has the fractional part ½ , but a decision variable is preferred than slack variable).

$$1.x_1 + 0.x_2 - \frac{1}{4}.s_1 + \frac{3}{4}s_2 + 0.s_3 = \frac{9}{2}$$

Or $(1+0)x_1 + \left(-1+\frac{3}{4}\right)s_1 + \left(0+\frac{3}{4}\right)s_2 = 4 + \frac{1}{2}$

By using equation (2), the Gomory's constraint (cut) is:

$$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} \qquad \Rightarrow \quad -\frac{3}{4}s_1 - \frac{3}{4}s_2 + s_4 = -\frac{1}{2}$$

The modified table after inserting this equation becomes:

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	**b**	
x_2	0	1	1/2	$-1/2$	0	0	3	
x_1	1	0	$-1/4$	3/4	0	0	9/2	
s_3	0	0	$-1/4$	$-1/4$	1	0	1/2	
← s_4	0	0	$-3/4$	$\boxed{-3/4}$	0	1	$-1/2$ ←	
Z	0	0	7/4	3/4	0	0	81/2	
x_2	0	1	1	0	0	$-2/3$	10/3	$3\frac{1}{3}$
x_1	1	0	-1	0	0	1	4	4
s_3	0	0	0	0	1	$-1/3$	2/3	$\frac{2}{3}$
s_2	0	0	1	1	0	$-4/3$	2/3	$\frac{2}{3}$
Z	0	0	1	0	0	1	40	

($\left|\frac{7/2}{-3/4}\right| = 4.7, \left|\frac{3/4}{-3/4}\right| = 1$) . x_2 has a non-integer value (10/3). Since the fractional part of s_2 and s_3 are equal (=2/3), then from equation (1):

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } s_2 - \text{equation} = \frac{2/3}{2/3} = 1$$

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } s_3 - \text{equation} = \frac{2/3}{2/3} = 1$$

Since both ratios are equal, we choose s_2 arbitrarily to construct second Gomory's cut as follows:

$$1.s_1 + 1.s_2 - \frac{4}{3}s_4 = \frac{2}{3}$$

Or $(1+0)s_1 + (1+0)s_2 + \left(-2+\frac{2}{3}\right)s_4 = 0 + \frac{2}{3}$

Then from equation (2):

$s_5 = \frac{2}{3}s_4 - \frac{2}{3} \quad \Rightarrow \quad -\frac{2}{3}s_4 + s_5 = -\frac{2}{3}$

The modified table after inserting this equation becomes

B.V.	x_1	x_2	s_1	s_2	s_3	s_4	s_5	b
x_2	0	1	1	0	0	$-2/3$	0	10/3
x_1	1	0	-1	0	0	1	0	4
s_3	0	0	0	0	1	$-1/3$	0	2/3
s_2	0	0	1	1	0	$-4/3$	0	2/3
← s_5	0	0	0	0	0	$\boxed{-2/3}$	1	$-2/3$ ←
Z	0	0	1	0	0	1	0	40
x_2	0	1	1	0	0	0	-1	4
x_1	1	0	-1	0	0	0	1/2	3
s_3	0	0	0	0	1	0	$-1/2$	1
s_2	0	0	1	1	0	0	-2	2
s_4	0	0	0	0	0	1	$-3/2$	1
Z	0	0	1	0	0	0	3/2	39

The optimal solution is: $x_1 = 3, x_2 = 4,$ and $Z_{max} = 39.$

Example (3.10):

Discuss, graphically, the effect of the cuts in example (3.9) on the feasible solutions space.

Solution:

For $2x_1 + 3x_2 = 18 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (9, 0) is the intersection point with the x_1 – axis.

For $2x_1 + x_2 = 12 \;\; \Rightarrow$ if $x_1 = 0$ then (0, 12) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (6, 0) is the intersection point with the x_1 – axis.

For $x_1 + x_2 = 8 \;\; \Rightarrow$ if $x_1 = 0$ then (0, 8) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the x_1 – axis.

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$\cancel{2x_1} + 3x_2 = 18$

$\cancel{\mp} 2x_1 \mp x_2 = \mp 12$

$\Rightarrow 2x_2 = 6 \Rightarrow x_2 = 3 \xRightarrow{(constr.2)} x_1 = \frac{12-3}{2} = 4.5$

The feasible solution region space is the shaded area OABC in figure (3.4) whose corners are the points $O(0,0)$, $A(0,6)$, $B(4.5,3)$, and $C(6,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,6)$	$Z = 5 \times 0 + 6 \times 6 = 36$
$B(4.5,3)$	$Z = 5 \times 4.5 + 6 \times 3 = 40.5$ *
$C(6,0)$	$Z = 5 \times 6 + 6 \times 0 = 30$

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 4.5$, $x_2 = 3$, and $Z_{max} = 40.5$.

The first cut is: $0 \le s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2}$ ...(*), that is:

$-\frac{3}{4}s_1 - \frac{3}{4}s_2 \le -\frac{1}{2}$ .... (**)

From the first and second constraints in the standard form:

$s_1 = 18 - 2x_1 - 3x_2$

$s_2 = 12 - 2x_1 - x_2$

Substitute in (**), then:

$-\frac{3}{4}(18 - 2x_1 - 3x_2) - \frac{3}{4}(12 - 2x_1 - x_2) \le -\frac{1}{2}$

$\frac{-54-36}{4} + \frac{3}{2}x_1 + \frac{9}{4}x_2 + \frac{3}{2}x_1 + \frac{3}{4}x_2 \le -\frac{1}{2}$

$\Longrightarrow 3x_1 + 3x_2 \le 22$

$3x_1 + 3x_2 = 22 \Rightarrow$ if $x_1 = 0$ then (0, 7.3) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (7.3, 0) is the intersection point with the x_1 – axis.

The first cut intersects the first constraint in the point (4,10/3), this solution is not optimal. The second constraint is: $0 \le s_5 = \frac{2}{3}s_4 - \frac{2}{3}$, that is

$-\frac{2}{3}s_4 \le -\frac{2}{3}$ ...(***)

From equation (*):

$s_4 = \frac{3}{4}s_1 + \frac{3}{4}s_2 - \frac{1}{2} = \frac{3}{4}(18 - 2x_1 - 3x_2\,) + \frac{3}{4}(12 - 2x_1 - x_2\,) - \frac{1}{2}$

$s_4 = 22 - 3x_1 - 3x_2$

Substitute s_4 in (***) , the result is the second cut in terms of x_1 and x_2:

$2x_1 + 2x_2 \leq 14$

$2x_1 + 2x_2 = 14$ $\Rightarrow$ if $x_1 = 0$ then (0, 7) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (7, 0) is the intersection point with the x_1 – axis.

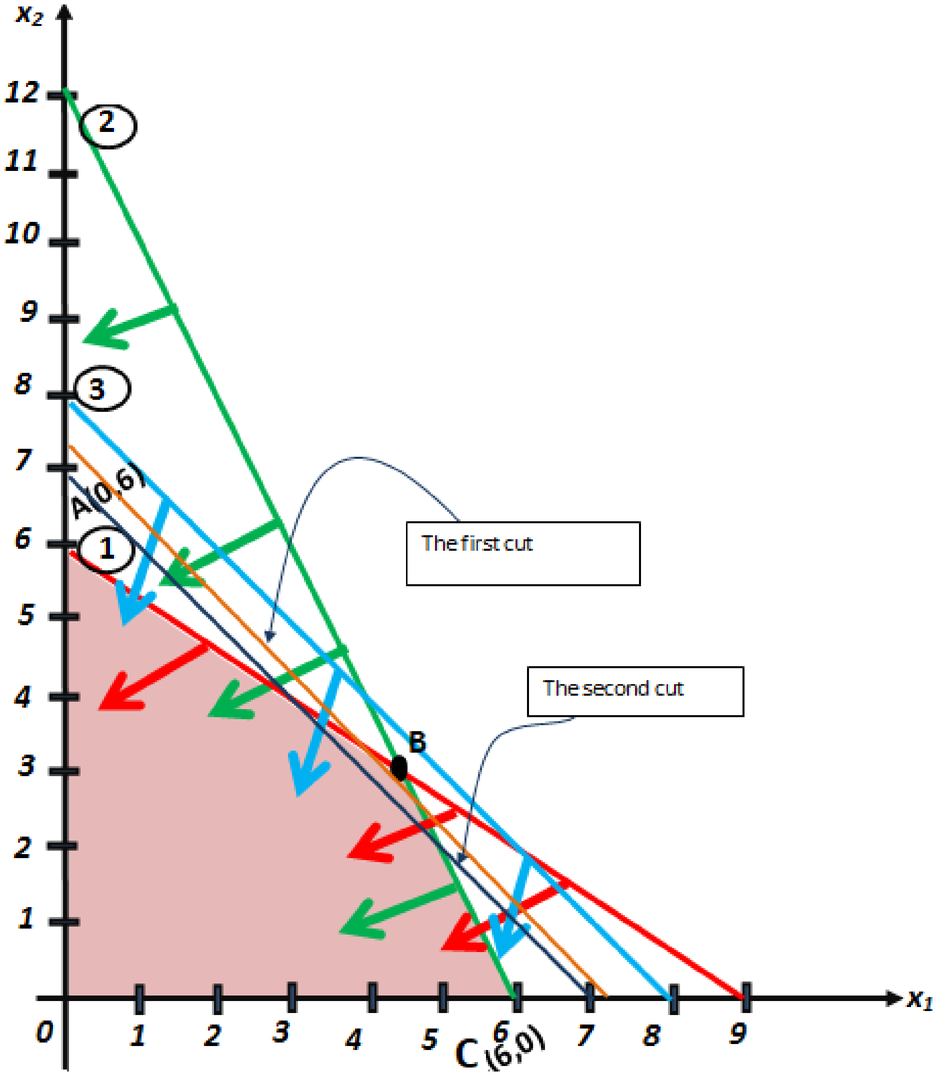


Figure (3.4)

The second cut passes through the point C(4,3) which is the optimal solution. Each cut neglecting a part of the feasible solutions set as we see in figures (3.4) and (3.5). Figure (3.5) shows the parts of the feasible solution set in which the extreme point exists.

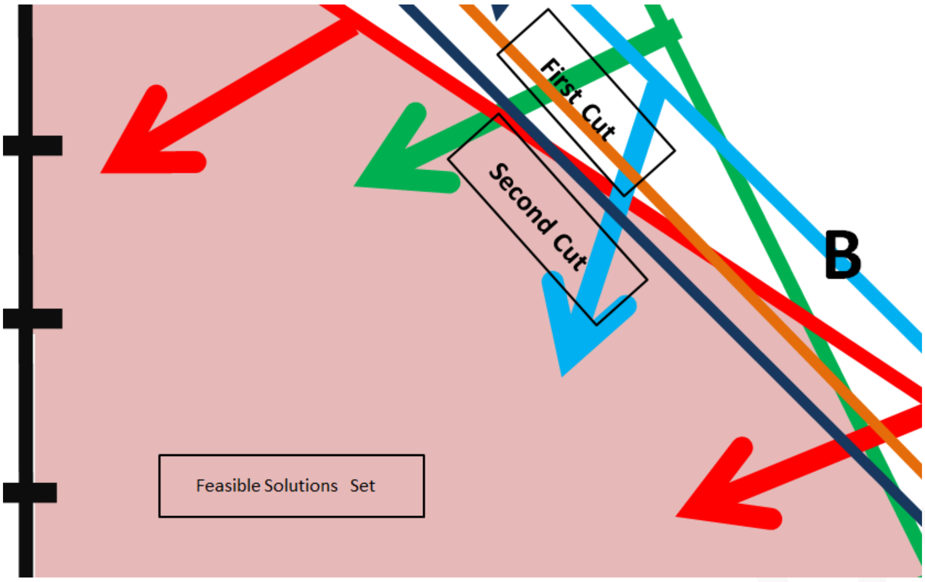


Figure (3.5)

Example (3.11):

Find the optimal solution of the following ILPP

$$max \quad Z = -4x_1 + 5x_2$$

$$S.t \quad -\frac{3}{5}x_1 + \frac{3}{5}x_2 \leq \frac{6}{5}$$

$$2x_1 + 4x_2 \leq 12$$

$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

Solution:

First of all, multiplying the first constraint by (5), so it will be:

$$-3x_1 + 3x_2 \leq 6$$

The standard form of the LPP (with modification in the objective function and ignoring integrality condition) is:

$$max \quad Z + 4x_1 - 5x_2 = 0$$

$$S.t \quad -3x_1 + 3x_2 + s_1 = 6$$

$$2x_1 + 4x_2 + s_2 = 12$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Let $x_1 = x_2 = 0$, then $s_1 = 6, s_2 = 12$, and $Z = 0$ and

B.V.	x_1	x_2	s_1	s_2	b
s_1	-3	3	1	0	6
s_2	2	4	0	1	12
Z	4	-5	0	0	0
x_2	-1	1	1/3	0	2

s_2	6	0	−4/3	1	4	
Z	$\boxed{-1}$	0	5/3	0	10	
x_2	0	1	1/9	1/6	8/3	$2\frac{2}{3}$
x_1	1	0	−2/9	1/6	2/3	$\frac{2}{3}$
Z	0	0	13/9	1/6	32/3	

The non-integer optimal solution is $x_1 = \frac{2}{3}, x_2 = \frac{8}{3}$, and $Z_{max} = \frac{32}{3}$. Since the fractional part of x_1 and x_2 are equal (=2/3), then from equation (1):

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } x_1 - \text{equation} = \frac{2/3}{(7/9)+(1/6)} = 12/17$$

$$\frac{f_i}{\sum_{j=1}^{n} f_{ij}} \text{ for } x_2 - \text{equation} = \frac{2/3}{(1/9)+(1/6)} = 12/5$$

x_2 –equation is selected as the source row and Gomory's cut is:

$$1.x_2 + \frac{1}{9}s_1 + \frac{1}{6}s_2 = \frac{8}{3}$$

Or $(1+0)x_2 + (0+\frac{1}{9})s_1 + (0+\frac{1}{6})s_2 = 2 + \frac{2}{3}$

Then from equation (2):

$$s_3 = \frac{1}{9}s_1 + \frac{1}{6}s_2 - \frac{2}{3} \quad \Longrightarrow \quad -\frac{1}{9}s_1 - \frac{1}{6}s_2 + s_3 = -\frac{2}{3}$$

The modified table after inserting this equation becomes

B.V.	x_1	x_2	s_1	s_2	s_3	b
x_2	0	1	1/9	1/6	0	8/3
x_1	1	0	−2/9	1/6	0	2/3
s_3	0	0	−1/9	$\boxed{-1/6}$	1	−2/3
Z	0	0	13/9	1/6	0	32/3
x_2	0	1	0	0	1	2
x_1	1	0	−3/9	0	1	0
s_2	0	0	6/9	1	−6	4
Z	0	0	4/3	0	1	10

The optimal solution is $x_1 = 0, x_2 = 2,$ and $Z_{max} = 10$.

Exercises 3.3 (In addition to the text book exercises)

Find the optimal solution of the following ILPP:

1. $max \quad Z = 14x_1 + 20x_2$

$S.t \quad -2x_1 + 6x_2 \leq 12$

$14x_1 + 2x_2 \leq 70$

$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

2. $max \quad Z = 2x_1 + 10x_2 + x_3$

$S.t \quad 5x_1 + 2x_2 + x_3 \leq 15$

$2x_1 + x_2 + 7x_3 \leq 20$

$x_1 + 3x_2 + 2x_3 \leq 25$

$x_1, x_2, x_3 \geq 0 \text{ and } \text{ are integers}$

3.3.2 Branch-and-Bound (B&B) Method

The first B&B algorithm was developed in 1960 by A.H.Land and A.G.Doig for the general mixed and pure ILPP. In this method also, the problem is first solved as a continuous LPP ignoring the integrality condition. Assume a maximization (minimization) problem, set an initial lower (upper) bound $Z = -\infty \ (\infty)$ on the optimum objective value of ILPP. Set $i = 0$.

***Step 1:* (Fathoming / bounding)**. Select Z_i , the next subproblem to be examined. Solve Z_i , and attempt to fathom it using one of three conditions:

a) The optimal Z-value of Z_i cannot yield a better objective value than the current lower bound.

b) Z_i yields a better feasible integer solution than the current lower bound.

c) Z_i has no feasible solution.

Two cases will arise:

a) If Z_i is fathomed and a better solution is found, update the lower bound. If all subproblems have been fathomed, stop; the optimum ILPP is associated with the current finite lower bound. If no finite lower bound exists, the problem has no feasible solution. Else, set $i = i + 1$, and repeat step 1.

b) If Z_i is not fathomed, go to step 2 for branching.

***Step 2:* (branching)**. Select one of the integer variables x_j, whose optimum value x_j^* is not integer. Eliminate the region:

$$[x_j^*] < x_j < [x_j^*] + 1$$

By creating two LP subproblems that correspond to:

$$x_j \leq [x_j^*] \text{ and } x_j \geq [x_j^*] + 1$$

x_j is called the ***branching variable***. These two conditions are mutually execlusive and when applied separately to the continuous LPP, form two different subproblems.Thus the original problem is branched (or partitioned) into two subproblems (also called ***nodes***). Geometrically, it means that the branching process eliminates that portion of the feasible region that contains no feasible integer solution. Set $i = i + 1$, and go to step 1.

Example (3.12):

Find the optimal solution of the following ILPP:

$$max \quad Z = x_1 + x_2$$
$$S.t \quad 2x_1 + 5x_2 \leq 16$$
$$6x_1 + 5x_2 \leq 30$$
$$x_1, x_2 \geq 0, x_1 \text{ and } x_2 \text{ are integers}$$

Solution:

For $2x_1 + 5x_2 = 16 \Rightarrow$ if $x_1 = 0$ then (0, 3.2) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (8, 0) is the intersection point with the x_1 – axis.

For $6x_1 + 5x_2 = 30 \Rightarrow$ if $x_1 = 0$ then (0, 6) is the intersection point with the x_2 – axis.

And if $x_2 = 0$ then (5, 0) is the intersection point with the x_1 – axis.

The graphical representation is:

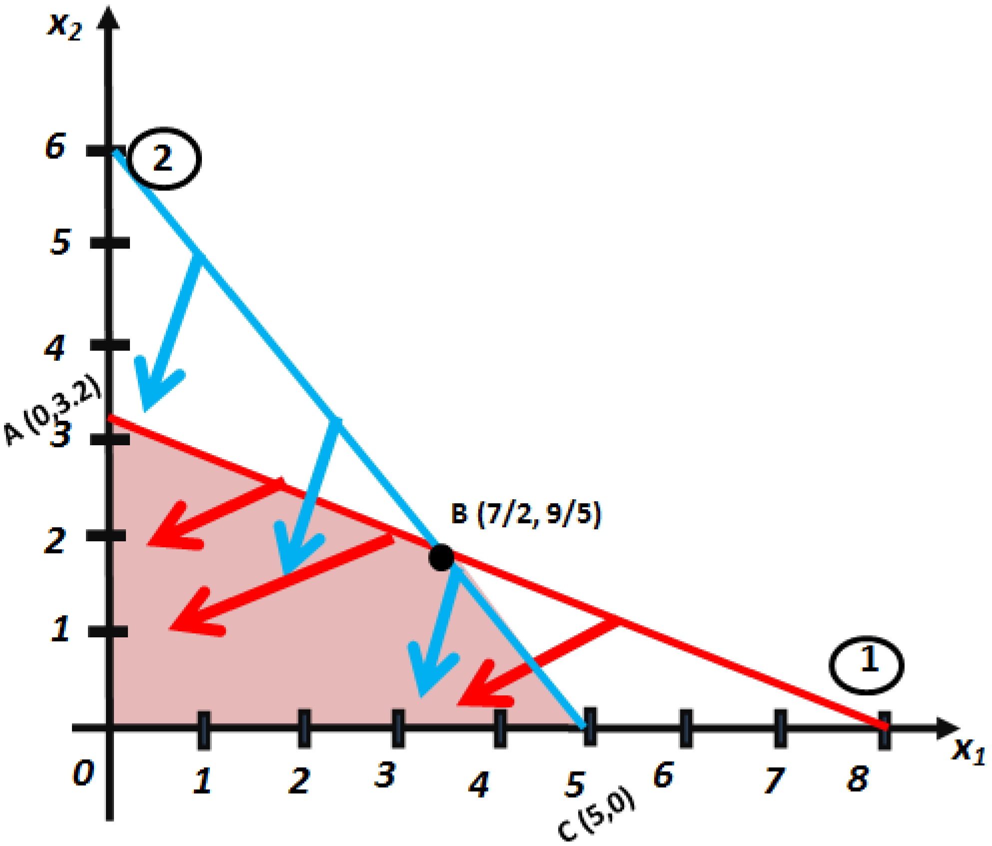


Figure (3.6)

The point B is resulting from the intersection of the lines representing the first and the second constraints, so we use these constraints to find the coordinates of B.

$$\begin{array}{l} 2x_1 + \cancel{5x_2} = 16 \\ \mp 6x_1 \mp \cancel{5x_2} = \mp 30 \\ \hline \end{array}$$

$$\Rightarrow -4x_1 = -14 \Rightarrow x_1 = 7/2 \xrightarrow{(constr.2)} x_2 = \frac{16-7}{5} = 9/5$$

The feasible solution region space is the shaded area OABC in figure (3.6) whose corners are the points $O(0,0)$, $A(0,3.2)$, $B(7/2,9/5)$, and $C(5,0)$.

Corner	Value of Z
O(0,0)	Z=0
$A(0,3.2)$	$Z = 0 + 3.2 = 3.2$
$B(7/2,9/5)$	$Z = \frac{7}{2} + \frac{9}{5} = \frac{53}{10} = 5.3$ *
$C(5,0)$	$Z = 5 + 0 = 5$

From the table, we see that the greatest value of Z occurs at corner B, then $x_1 = 7/2$, $x_2 = 9/5$, and $Z_{max} = 5.3$. The solution is not optimal, then choose $x_1 = 3.5$ as a branching variable, $3 \leq x_1 \leq 4$. Construct two new problems by adding the constructs: $x_1 \leq 3, x_1 \geq 4$.

Subproblem Z_1:

$$\begin{array}{ll} max & Z = x_1 + x_2 \\ S.t & 2x_1 + 5x_2 \leq 16 \\ & 6x_1 + 5x_2 \leq 30 \\ & x_1 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

x_1 and x_2 are integers

The solution is: $x_1 = 3$, $x_2 = 2, Z_{max} = 5$

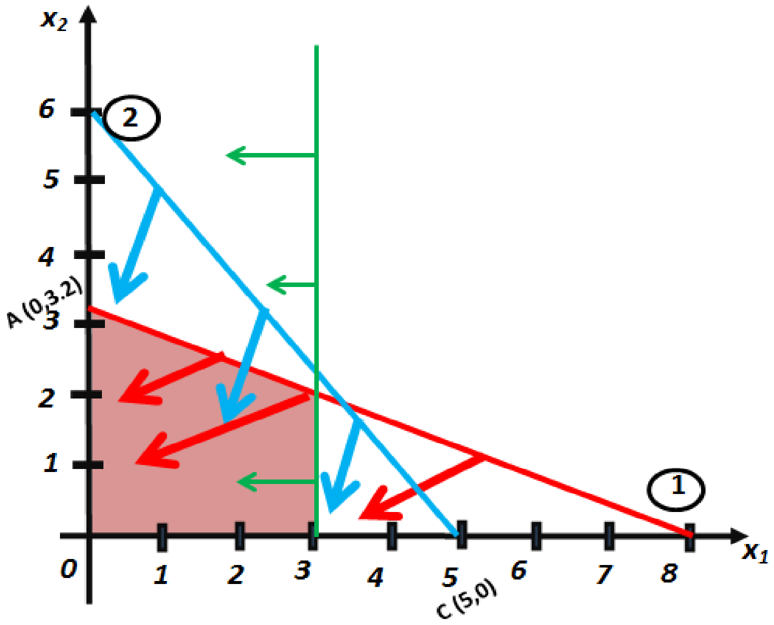


Figure (3.7)

Subproblem Z_2:

$$\begin{aligned} max\quad & Z = x_1 + x_2 \\ S.t\quad & 2x_1 + 5x_2 \le 16 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4 \\ & x_1, x_2 \ge 0\text{ , } x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$

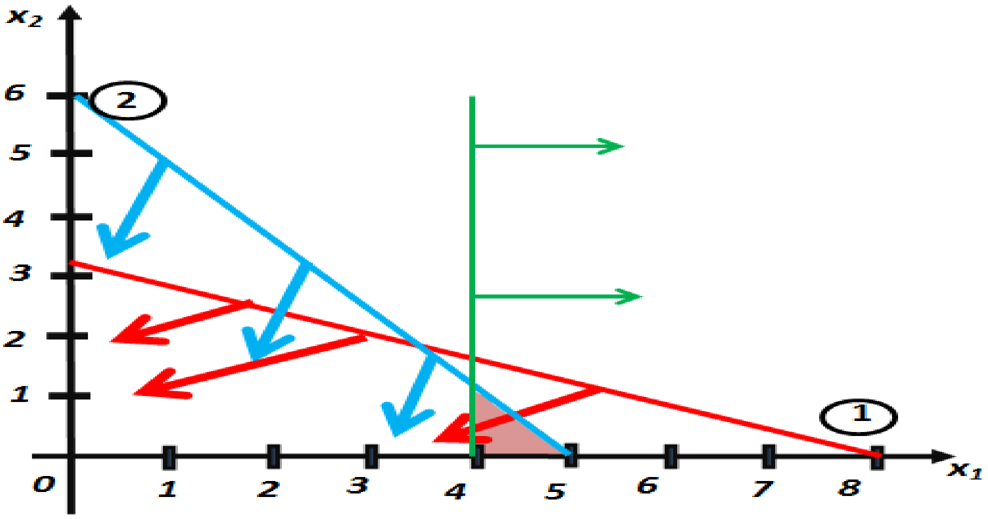


Figure (3.8)

The solution is $x_1 = 4, x_2 = \frac{6}{5} = 1.2$, and $Z_{max} = \frac{26}{5} = 5.2$.

Since the solution of subproblem Z_1 are integers, there is no need to branch subproblem Z_1(subproblem Z_1 is fathomed). The lower bound is now $Z_{max} = 5$. We branch from subproblem Z_2. Since $1 \le x_2 \le 2$, then choose x_2 as a branching variable. Construct two new problems by adding the constructs: $x_2 \le 1, x_2 \ge 2$.

Subproblem Z_3 :

$$\begin{aligned} max\quad & Z = x_1 + x_2 \\ S.t\quad & 2x_1 + 5x_2 \le 16 \\ & 6x_1 + 5x_2 \le 30 \\ & x_1 \ge 4 \\ & x_2 \le 1 \\ & x_1, x_2 \ge 0\text{ , } x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$

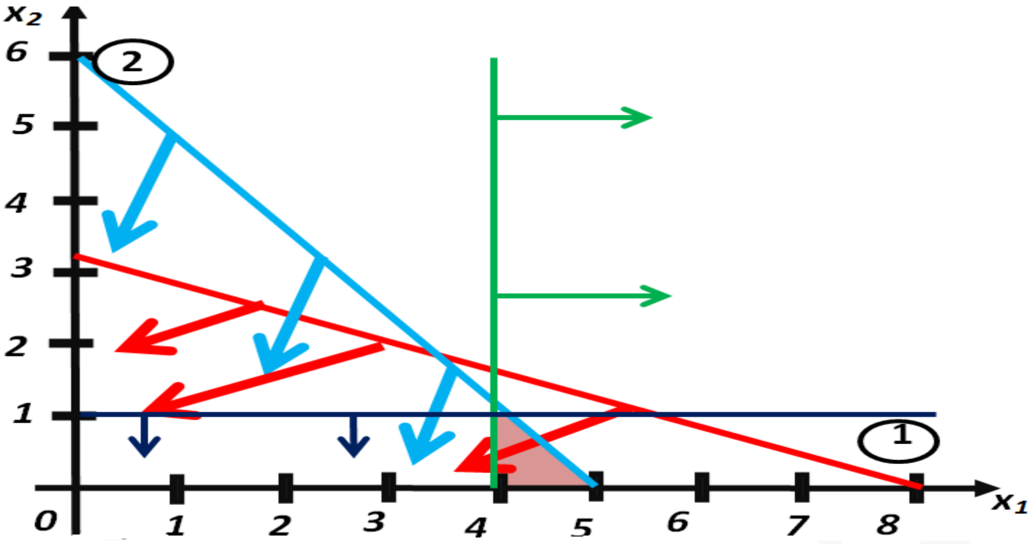


Figure (3.9)

The solution is: $x_1 = 25/6, x_2 = 1,$ and $Z_{max} = \frac{31}{6} = 5.17.$

Subproblem Z_4:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 4$

$x_2 \geq 2$

$x_1, x_2 \geq 0$, x_1 and x_2 are integers

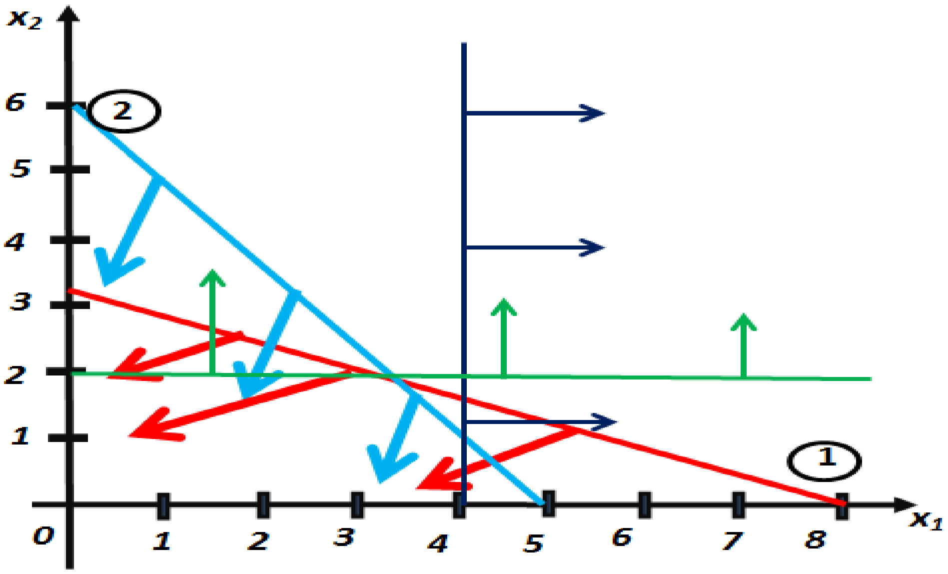


Figure (3.10)

There is no feasible solution.

Subproblem Z_4 is fathomed. Since the solution of subproblem Z_3 is $Z_{max} = \frac{31}{6} = 5.17$, which is not inferior to the lower bound. Therefore it can be branched from subproblem Z_3 into further subproblems. Since x_1 is the only fractional valued variable. Since $4 \leq x_1 \leq 5$ construct two new problems by adding the constructs: $x_1 \leq 4, x_1 \geq 5$.

Subproblem Z_5:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 = 4$

$x_2 \leq 1$

$x_1, x_2 \geq 0$

x_1 and x_2 are integers

The solution is: $x_1 = 4$ $, x_2 = 1$, and $Z_{max} = 5$.

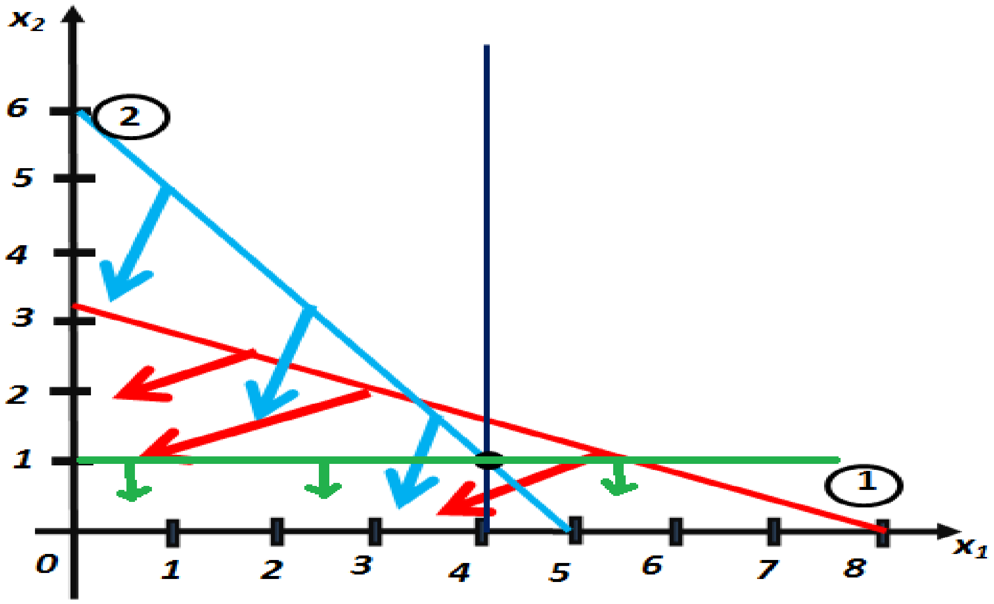


Figure (3.11)

Subproblem Z_6:

$max \quad Z = x_1 + x_2$

$S.t \quad 2x_1 + 5x_2 \leq 16$

$6x_1 + 5x_2 \leq 30$

$x_1 \geq 5$

$$x_2 \leq 1$$
$$x_1, x_2 \geq 0$$

x_1 and x_2 are integers

The solution is: $x_1 = 5$, $x_2 = 0$, and $Z_{max} = 5$.

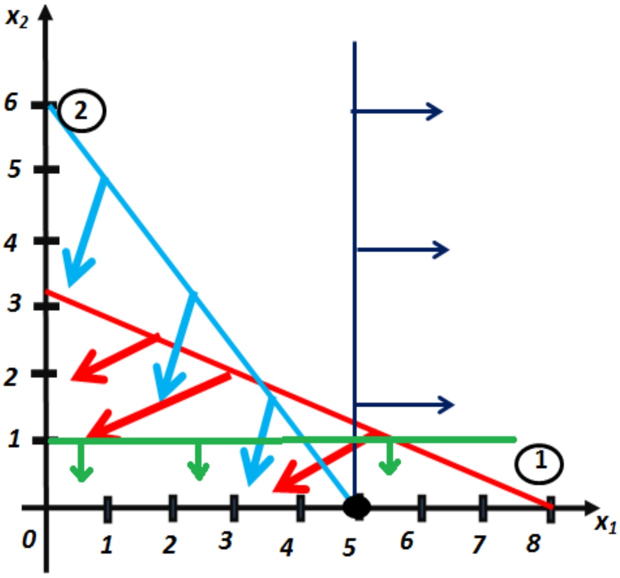


Figure (3.12)

There is more than one solution to this problem, they are:

$x_1 = 3, x_2 = 2,$ and $Z_{max} = 5$

$x_1 = 4, x_2 = 1,$ and $Z_{max} = 5$

$x_1 = 5, x_2 = 0,$ and $Z_{max} = 5$

Figure (3.13) summarize the generated subproblems in the form of a tree.

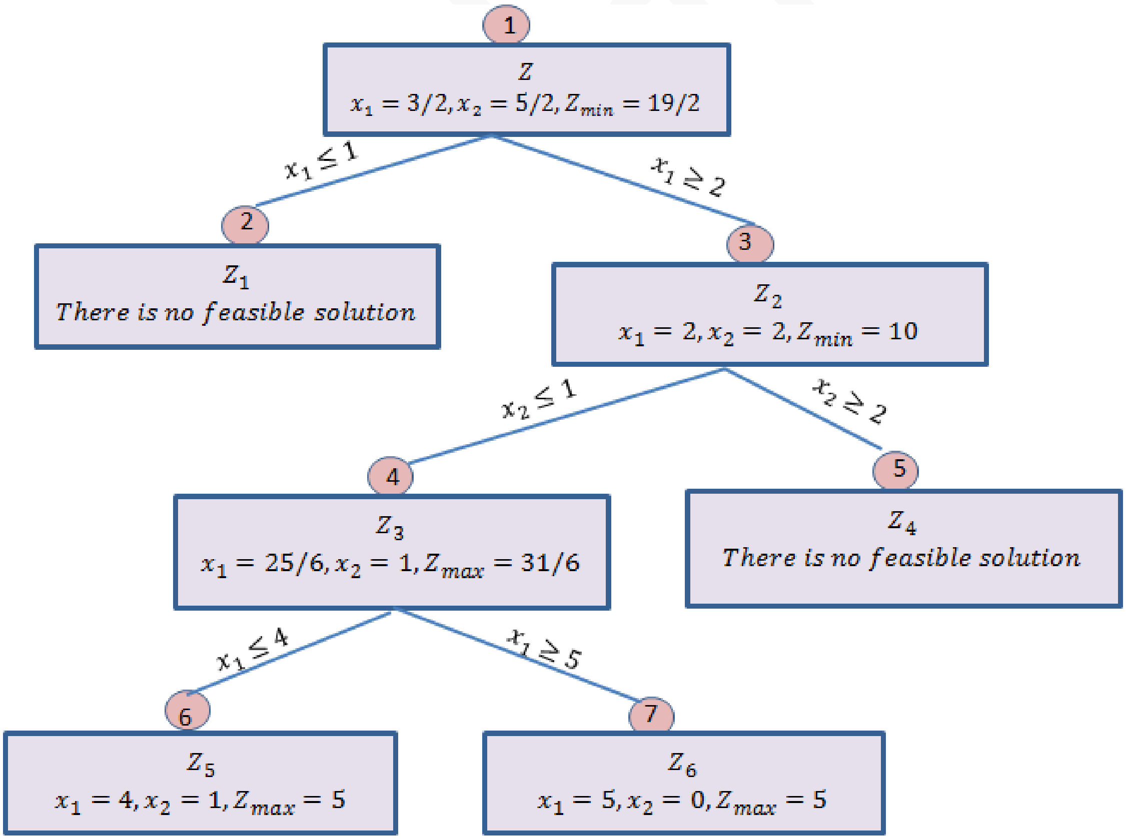


Figure (3.13)

Example (3.13):

Find the optimal solution of the following ILPP:

$min \quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Solution:

The standard form of the ILPP is:

$min \quad Z = 3x_1 + 2x_2 + MR_1 + MR_2 + MR_3$

$S.t \quad x_1 + x_2 + R_1 = 4$

$x_1 + 3x_2 - s_1 + R_2 = 6$

$5x_1 + 3x_2 - s_2 + R_3 = 15$

$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1$ and x_2 are integers

From the constraints:

$R_1 = 4 - x_1 - x_2$

$R_2 = 6 - x_1 - 3x_2 + s_1$

$R_3 = 15 - 5x_1 - 3x_2 + s_2$

Substitute $R_1, R_2,$ and R_3 in the Z=equation and rearrange Z-equation, the standard form will be:

$min \quad Z + (-3+7M)x_1 + (-2 + 7M)x_2 - Ms_1 + Ms_2 = 25M$

$S.t \quad x_1 + x_2 + R_1 = 4$

$x_1 + 3x_2 - s_1 + R_2 = 6$

$5x_1 + 3x_2 - s_2 + R_3 = 15$

$x_1, x_2, s_1, s_2, R_1, R_2, R_3 \geq 0, x_1$ and x_2 are integers

Let $x_1 = x_2 = s_1 = s_2 = 0, then\ R_1 = 4, R_3 = 6, R_3 = 15, Z = 25M.$

B.V.	x_1	x_2	s_1	s_2	R_1	R_2	R_3	b
R_1	1	1	0	0	1	0	0	4
R_2	1	3	−1	0	0	1	0	6
R_3	$\boxed{5}$	3	0	−1	0	0	1	15
Z	−3+7M	−2+7M	−M	−M	0	0	0	25M

B.V.	x_1	x_2	s_1	s_2	R_1	R_2	b
R_1	0	2/5	0	1/5	1	0	1
R_2	0	$\boxed{12/5}$	−1	1/5	0	1	3

x_1	1	3/5	0	−1/5	0	0	3
Z	0	−1/5+14/5M	−M	$-\frac{3}{5}+2/5M$	0	0	4M+9

B.V.	x_1	x_2	s_1	s_2	R_1	**b**
R_1	0	0	$\boxed{1/6}$	1/6	1	1/2
x_2	0	1	−5/12	1/12	0	5/4
x_1	1	0	1/4	−1/4	0	9/4
Z	0	0	$-\frac{1}{12}+1/6M$	$-\frac{7}{12}+1/6M$	0	1/2M+37/4

B.V.	x_1	x_2	s_1	s_2	**b**
s_1	0	0	1	1	3
x_2	0	1	0	1/2	5/2
x_1	1	0	0	−1/2	3/2
Z	0	0	0	−1/2	19/2

$\therefore\ x_1 = 3/2, x_2 = 5/2, and\ Z_{min} = 19/2$. The solution is not optimal, we need two constraints $x_1 \leq 1\ and\ x_1 \geq 2$.

Subproblem Z_1:

$min\quad Z = 3x_1 + 2x_2$

$S.t\qquad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1 \leq 1$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $x_1 + s_3 = 1$

x_1 is a basic variable, so we substitute x_1 from the table:

$x_1 + 0.x_2 + 0.s_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Rightarrow\ \frac{3}{2} + \frac{1}{2}s_2 + s_3 = 1\quad \Rightarrow\ \frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

B.V.	x_1	x_2	s_1	s_2	s_3	**b**
s_1	0	0	1	1	0	3
x_2	0	1	0	1/2	0	5/2
x_1	1	0	0	−1/2	0	3/2

s_3	0	0	0	$\boxed{1/2}$	1	−1/2
Z	0	0	0	−1/2	0	19/2

There is no feasible solution

Subproblem Z_2:

$min \quad Z = 3x_1 + 2x_2$

$S.t \quad x_1 + x_2 = 4$

$x_1 + 3x_2 \geq 6$

$5x_1 + 3x_2 \geq 15$

$x_1 \geq 2$

$x_1, x_2 \geq 0, x_1$ and x_2 are integers

Either we solve the above problem in the usual way from the beginning or by using the last table. The additional constraint is written as: $-x_1 \leq -2$ $\Rightarrow -x_1 + s_3 = -2$

x_1 is a basic variable, so we substitute x_1 from the table:

$x_1 - \frac{1}{2}s_2 = \frac{3}{2}$, then: $x_1 = \frac{3}{2} + \frac{1}{2}s_2$

$\Longrightarrow \quad -\frac{3}{2} - \frac{1}{2}s_2 + s_3 = -2 \quad \Longrightarrow -\frac{1}{2}s_2 + s_3 = \frac{-1}{2}$

B.V.	x_1	x_2	s_1	s_2	s_3	**b**
s_1	0	0	1	1	0	3
x_2	0	1	0	1/2	0	5/2
x_1	1	0	0	−1/2	0	3/2
s_3	0	0	0	$\boxed{-1/2}$	1	−1/2
Z	0	0	0	−1/2	0	19/2
s_1	0	0	1	0	2	3
x_2	0	1	0	0	1	2
x_1	1	0	0	0	−1	2
s_2	0	0	0	1	−2	1
Z	0	0	0	0	−1	10

$\therefore$ the optimal solution is $x_1 = 2, x_2 = 2, and \; Z_{min} = 10.$

Figure (3.14) summarize the generated subproblems in the form of a tree.

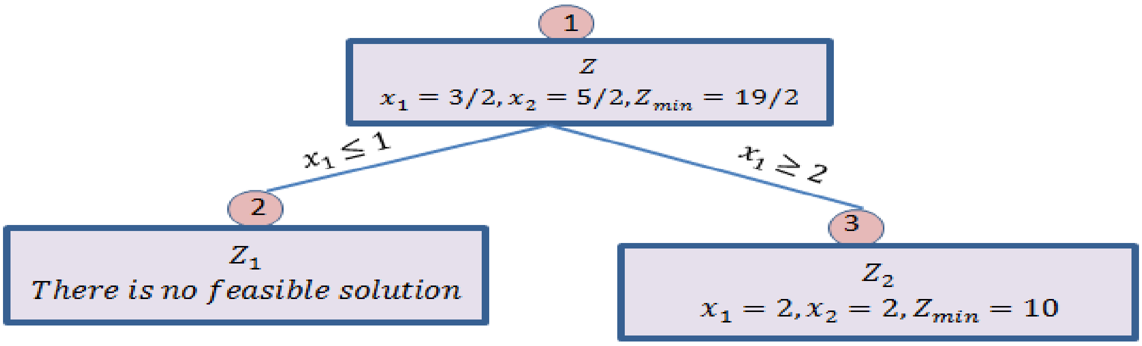


Figure (3.14)

Exercises 3.4 (In addition to the text book exercises)

Use B & B method to solve the following ILPP:

1. $$\begin{aligned} max \quad & Z = 9x_1 + 3x_2 + 9x_3 \\ S.t \quad & -\ 3x_1 + 6x_2 + 3x_3 \le 12 \\ & 12\ x_1 - 9x_3 \le 18 \\ & 3\ x_1 - 9x_2 + 6x_3 \le 9 \\ & x_1, x_2, x_3 \ge 0, x_1 \text{ and } x_3 \text{ are integers} \end{aligned}$$

2. $$\begin{aligned} min \quad & Z = 5x_1 + 4x_2 \\ S.t \quad & x_1 + x_2 \le 5 \\ & 10x_1 + 6x_2 \le 45 \\ & x_1, x_2 \ge 0, x_1 \text{ and } x_2 \text{ are integers} \end{aligned}$$

Ch. 4: Transportation Problem

4.1 Definition of the Transportation Problem

The ***transportation problem*** is a special case of linear programming in which the objective is to transport a homogeneous commodity from various origins or factories to different destinations or markets at a total minimum cost.

Suppose that there are m sources and n destinations. Let a_i be the number supply units available at source i $(i = 1, 2, \dots, m)$ and b_j be the number demand units required at destination j $(j = 1, 2, \dots, n)$. Let c_{ij} represent the unit transportation cost for transporting the units from source i to destination j. The objective is to determine the number of units to be transported from source i to destination j so that the total transportation cost is minimum. In addition, the supply limits at the source and the demand requirements at the destination must be satisfied exactly.

If x_{ij} $(x_{ij} \geq 0)$ is the number of units shipped from source I to destination j, the equivalent LP model will be:

Find $x_{ij}(i = 1,2, \dots, m\, ; j = 1,2, \dots, n)$ in order to

$$min \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} \; c_{ij} \; x_{ij}$$

$$S.t. \quad \sum_{j=1}^{n} x_{ij} = a_i \qquad i = 1,2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \qquad j = 1,2, \dots, n$$

$$x_{ij} \geq 0 \qquad i = 1,2, \dots, n\, ; \; j = 1,2, \dots, m$$

The two sets of constraints will be ***consistent***, i.e., the system will be in ***balance*** if:

$$\sum_{i=1}^{m} a_{i=} \sum_{j=1}^{n} b_j$$

The consistency condition is necessary and sufficient condition for a transportation problem to have a feasible solution. The above information can be put in the form of a general matrix shown below. This table is called the ***transportation matrix***. In the table (4.1), $c_{ij}, i = 1,2, \dots, m; j = 1,2, \dots, n$, is the unit shipping cost from the ith origin (source) to the jth destination, x_{ij} is the quantity shipped from the ith origin to the jth destination, a_i is the supply available at origin i and b_j is the demand at destination j.

		Destinations						
		1	**2**	**…**	**j**	**…**	**n**	**Supply**
Sources(or Origins)	**1**	c_{11} x_{11}	c_{12} x_{12}	…	c_{1j} x_{1j}	…	c_{1n} x_{1n}	a_1
	2	c_{21} x_{21}	c_{22} x_{22}	…	c_{2j} x_{2j}	…	c_{2n} x_{2n}	a_2
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	i	c_{i1} x_{i1}	c_{ij} x_{i2}	…	c_{ij} x_{ij}	…	c_{in} x_{in}	a_i
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	m	c_{m1} x_{m1}	c_{m2} x_{m2}	…	c_{mj} x_{mj}	…	c_{mn} x_{mn}	a_m
	Demand	b_1	b_2	…	b_j	…	b_n	

Table (4.1)

Definition (4.1):

An allocation is said to satisfy the ***rim requirements***, *i.e.*, it must satisfy availability constraints and requirement constraints.

4.2 Solution of the Transportation Model

The steps to solve transportation problem are:

Step 1: Make a transportation model.

Step 2: Find a basic feasible solution.

Step 3: Perform optimality test.

Step 4: Iterate toward an optimal solution.

Step 5: Repeat steps 3-4 until optimal solution is reached.

Step 1: Make a Transportation Model

This consists in expressing supply from origins, requirements at destinations and cost of shipping from origins to destinations in the form of a cost matrix.

A check is made to find if the problem is balanced, if not add a dummy origin or destination to balance the supply and demand.

Example (4.1): A dairy firm has three plants located throughout a state. Daily milk production at each plant is as follows:

Plant 1: 6 million liters

Plant 2: 1 million liters, and

Plant 3: 10 million liters

Each day the firm must fulfill the needs of its four distribution centers. Milk requirements at each center are as follows:

Distribution center 1: 7 million liters

Distribution center 2: 5 million liters

Distribution center 3: 3 million liters, and

Distribution center 4: 2 million liters

Cost of shipping of one million liter of milk from each plant to each distribution center is given in the following table in hundreds of Iraqi dinars:

		Distribution centers			
		1	**2**	**3**	**4**
Plants	**1**	2	3	11	7
	2	1	0	6	1
	3	5	8	15	9

a) Construct the cost table.

b) Formulate the mathematical model of the problem.

Solution:

a) The cost table is:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3	11	7	6
	2	1	0	6	1	1
	3	5	8	15	9	10
Requirement		7	5	3	2	17

$\sum_{i=1}^{3} a_{i=}\ \ 6+1+10=17$,$\sum_{j=1}^{4} b_j = 7+5+3+2=17$,i.e. the constraints are consistent.

b) Let $x_{ij}, i = 1,2,3; j = 1,2,3,4$ denotes the quantity of units to be transported from each origin to each destination (i.e., x_{ij} are decision variables), then the objective function is to minimize the cost of transportation.

i.e. $\min \quad Z = 2x_{11} + 3x_{12} + 11x_{13} + 7x_{14} + x_{22} + 6x_{23} + x_{24} + 5x_{31} + 8x_{32} + 15x_{33} + 9x_{34}$

In general, if c_{ij} is the unit cost of shipping from the ith source to the jth destination, the mathematical LP model is:

$$\begin{aligned}
min \quad & Z = \textstyle\sum_{i=1}^{3}\sum_{j=1}^{4} \; c_{ij}\; x_{ij} \\
S.t. \quad & x_{11} + x_{12} + x_{13} + x_{14} = 6 \\
& x_{21} + x_{22} + x_{23} + x_{24} = 1 \\
& x_{31} + x_{32} + x_{33} + x_{34} = 10 \\
& x_{11} + x_{21} + x_{31} = 7 \\
& x_{12} + x_{22} + x_{32} = 5 \\
& x_{13} + x_{23} + x_{33} = 3 \\
& x_{14} + x_{24} + x_{34} = 2 \\
& x_{ij} \geq 0 \qquad\qquad i = 1,2,3\,; \;\; j = 1,2,3,4
\end{aligned}$$

Step 2: Find a Basic Feasible Solution

There are many methods for finding the basic feasible solution; three of them are described below:

4.2.1 North-West Corner Method (NWCM)

This rule may be stated as follows:

a) Start in the north-west corner of the transportation matrix framed in step 1, i.e. cell (1,1). Compare a_1 and b_1:
  i) If $a_1 < b_1$, set $x_{11} = a_1$, compute the balance supply and demand and proceed to cell (2,1) (i.e. proceed vertically).
  ii) If $b_1 < a_1$, set $x_{11} = b_1$, compute the balance supply and demand and proceed to cell (1,2) (i.e. proceed horizontally).
  iii) If $a_1 = b_1$, set $x_{11} = a_1 = b_1$, compute the balance supply and demand and proceed to cell (2,2) (i.e. proceed diagonally). Also make a zero allocation to the least cost cell in a_1/b_1.

b) Continue in the same manner, step by step, away from the north-west corner until, finally, a value is reached in the south-east corner.

Example (4.2):

For the transportation problem in example (4.1):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 6	3	11	7	~~6~~
	2	1 1	0 0	6	1	~~1~~
	3	5	8 5	15 3	9 2	~~10~~ ~~5~~ ~~2~~
Requirement		~~7~~ ~~1~~	~~5~~	~~3~~	~~2~~	17

$$Z = (6 \times 2 + 1 \times 1 + 5 \times 8 + 3 \times 15 + 2 \times 9) \times 100 = 11600\ ID$$

4.2.2 Least-Cost Method

This method consists in allocating as much as possible in the lowest cost cell/cells and then further allocation is done in the cell/cells with second lowest cost and so on. In case of tie among the cost, select the cells where allocation of more number of units can be made.

Example (4.3):

For the transportation problem in example (4.1):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 6	3	11	7	~~6~~
	2	1	0 1	6	1	~~1~~
	3	5 1	8 4	15 3	9 2	~~10~~ ~~9~~ ~~5~~ ~~3~~
Requirement		~~7~~ ~~1~~	~~5~~ ~~4~~	~~3~~	~~2~~	17

$$Z = (6 \times 2 + 1 \times 5 + 4 \times 8 + 3 \times 15 + 2 \times 9) \times 100 = 11200\ ID$$

4.2.3 Vogel's Approximation Method (VAM)

Vogel's approximation method (or penalty method) makes effective use of the cost information and yields a better initial solution than obtained by other methods. This method consists of the following sub-steps:

i) Write down the cost matrix. Enter the difference between the smallest and the second smallest element in each column below the corresponding column and the difference between the smallest and the second smallest element in each row to the right of the row.

ii) Select the row or column with the greatest difference and allocate as much as possible within the restriction of the rim condition to the lowest cost cell in the row or column selected.

In case of a tie among the highest penalties, select the row or column having minimum cost. In case of tie in the minimum cost also, select the cell which can have maximum allocation. If there is a tie among maximum allocation cells also, select the cell arbitrarily for allocation. Following these rules yields the best possible initial basic feasible solution and reduces the number of iterations required to reach the optimal solution.

iii) Cross out the row or column completely satisfied by the allocation just made.

iv) Repeat steps i to iii until all assignments have been made.

Example (4.4):

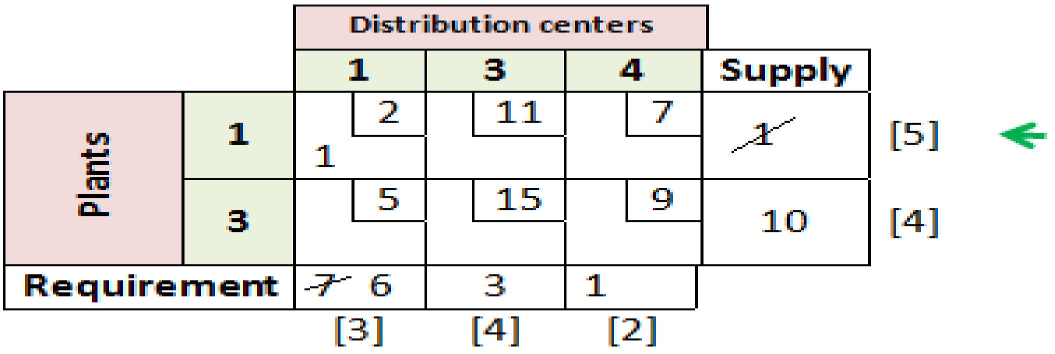
For the transportation problem in example (4.1), the cost matrix with penalties is shown below:

		Distribution centers					
		1	2	3	4	Supply	
Plants	1	2	3	11	7	6	[1]
	2	1	0	6	1 1	~~1~~	[1]
	3	5	8	15	9	10	[3]
Requirement		7	5	3	~~2~~ 1	17	
		[1]	[3]	[5]	[6] ▲		

The greatest penalty is [6], so we choose the 4th column and allocate as much as possible (i.e. 1) to cell (2,4)(the cell with smallest cost in the 4th column). Supply of plant 2 is completely satisfied, so row 2 is crossed out and the shrunken matrix with penalties and allocation is as below:

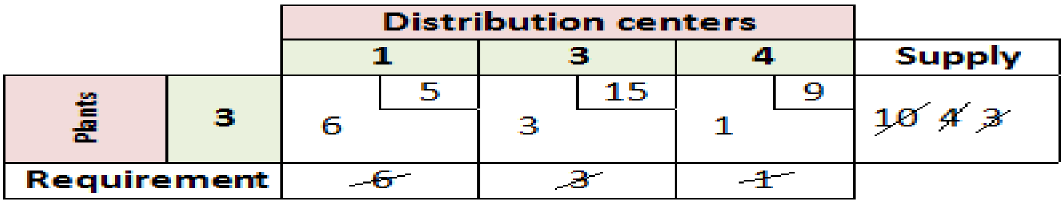
		Distribution centers					
		1	2	3	4	Supply	
Plants	1	2	3 5	11	7	~~6~~ 1	[1]
	3	5	8	15	9	10	[3]
Requirement		7	~~5~~	3	1		
		[3]	[5] ▲	[4]	[2]		

The greatest penalty is [5], so we choose the 2nd column and allocate as much as possible (i.e. 5) to cell (1,4)(the cell with smallest cost in the 2nd column). Requirement of distribution center 2 is completely satisfied, so column 2 is crossed out and the shrunken matrix with penalties and allocation is as below:



		Distribution centers					
		1	3	4	Supply		
Plants	1	2 1	11	7	~~1~~	[5]	←
	3	5	15	9	10	[4]	
Requirement		~~7~~ 6	3	1			
		[3]	[4]	[2]			

The greatest penalty is [5], so we choose the 1st row and allocate as much as possible (i.e. 1) to cell (1,1)(the cell with smallest cost in the 1st row). Supply of plant 1 is completely satisfied, so row 1 is crossed out and the shrunken matrix is as below:



		Distribution centers			
		1	3	4	Supply
Plants	3	5 6	15 3	9 1	~~10~~ ~~4~~ ~~3~~
Requirement		~~6~~	~~3~~	~~1~~	

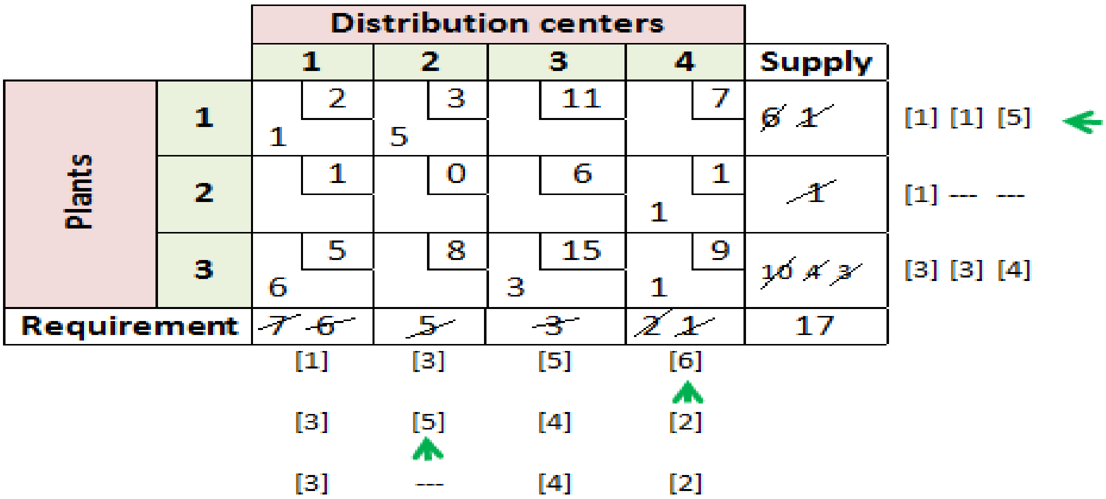
It is possible to find row difference, but it is not possible to find column difference. Therefore, the remaining allocations are made by following the least-cost method. The transportation matrix will be:

		Distribution centers				
		1	2	3	4	Supply
Plants	1	2 1	3 5	11	7	6
	2	1	0	6	1 1	1
	3	5 6	8	15 3	9 1	10
Requirement		7	5	3	2	17

$$Z = (1 \times 2 + 5 \times 3 + 1 \times 1 + 3 \times 15 + 6 * 5 + 1 \times 9) \times 100 = 10200\ ID$$

Remark (4.1):

It is possible to all the previous steps in one table as follows:



		Distribution centers							
		1	2	3	4	Supply			
Plants	1	2 1	3 5	11	7	~~6~~ ~~1~~	[1]	[1]	[5]
	2	1	0	6	1 1	~~1~~	[1]	---	---
	3	5 6	8	15 3	9 1	~~10~~ ~~4~~ ~~3~~	[3]	[3]	[4]
Requirement		~~7~~ ~~6~~	~~5~~	~~3~~	~~2~~ ~~1~~	17			
		[1]	[3]	[5]	[6]				
		[3]	[5]	[4]	[2]				
		[3]	---	[4]	[2]				

Remark (4.2):

Vogel method yields the best initial solution, and the north-west corner method yields the worst.

Step 3: Perform Optimality Test

Make an optimality test to find whether the obtained feasible solution is optimal or not. An optimality test can be performed only on that feasible solution in which:

1) Number of allocations is $m + n - 1$, where m is the number of rows and n is the number of columns.
2) These $m + n - 1$ allocations should be in independent positions, i.e. it is impossible to increase or decrease any allocation without either changing the position of allocation or violating the row and column restrictions. A simple rule for allocations to be in independent position is that it is impossible to travel from any allocation, back to itself by a series of horizontal and vertical jumps from one occupied cell to another, without a direct reversal of route, or simply do not form a closed loop.

To check optimality we must find empty cells evaluation, if there is at least one cell with negative evaluation, and then the current solution is not optimal.

4.2.4 The Stepping-Stone Method

Starting from the chosen empty cell, trace a path in the matrix consisting of a series of alternate horizontal and vertical lines. The path begins and terminates in the chosen cell. All other corners of the path lie in the cells for which allocations have been made. The path may skip over any number of occupied or vacant cells. Mark the corner of the path in the chosen vacant cell as positive and other corners of the path alternately $-ve, +ve, -ve$ and so on.

Allocate 1 unit to the chosen cell; subtract and add 1 unit from the cells at the corners of the path, maintaining the row and column requirements. The net change in the total cost resulting from this adjustment is called the ***evaluation*** of the chosen empty cell. In a transportation problem involving m rows and n columns, the total number of empty cells will be $m.n-(m+n-1)=(m-1)(n-1)$. Therefore, there are $(m-1)(n-1)$ evaluations which must be calculated.

Example (4.5):

We will check optimality of basic feasible solution obtained by Vogel's method in example (4.4):

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2 1	3 5	11	7	6
	2	1	0	+1 6	−1 1 1	1
	3	5 6	8	−1 15 3	9 1 +1	10
Requirement		7	5	3	2	17

1) Number of allocations=6, $m=3, n=4, m+n-1=3+4-1=6$ =number of allocations.
2) These allocations are independent in positions.

To find the evaluation of the empty cell (2,3) for example, the closed path the begins and end in cell (2,3) is explained in the table above. To allocate 1 unit in cell (2,3), we must subtract, add, subtract 1 unit from cells (2,4), (3,4), and (3,3) respectively. For each empty cell, the closed path that start and end with the empty cell and whose other corners are allocated cells and the evaluation of the empty cell(in hundreds of dinars) are as follows:

Cell (1, 3): $(1,3) \rightarrow (3,3) \rightarrow (3,1) \rightarrow (1,1)$

Evaluation of cell (1, 3) $= c_{13} - c_{33} + c_{31} - c_{11} = 11 - 15 + 5 - 2 = -1$

Cell (1, 4): $(1,4) \rightarrow (3,4) \rightarrow (3,1) \rightarrow (1,1)$

Evaluation of cell (1, 4) $= c_{14} - c_{34} + c_{31} - c_{11} = 7 - 9 + 5 - 2 = +1$

Cell (2, 1): $(2,3) \rightarrow (2,4) \rightarrow (3,4) \rightarrow (3,1)$

Evaluation of cell (2, 1) $= c_{23} - c_{24} + c_{34} - c_{31} = 1 - 1 + 9 - 5 = +4$

Cell (2, 2): $(2,2) \rightarrow (2,4) \rightarrow (3,4) \rightarrow (3,1) \rightarrow (1,1) \rightarrow (1,2)$

Evaluation of cell (2, 2) $= c_{22} - c_{24} + c_{34} - c_{31} + c_{11} - c_{12} = 0 - 1 + 9 - 5 + 2 - 3 = +2$

Cell (2, 3): $(2,3) \rightarrow (2,4) \rightarrow (3,4) \rightarrow (3,3)$

Evaluation of cell (2, 3) $= c_{23} - c_{24} + c_{34} - c_{33} = 6 - 1 + 9 - 15 = -1$

Cell (3, 2): $(3,2) \rightarrow (3,1) \rightarrow (1,1) \rightarrow (1,2)$

Evaluation of cell (3, 3) $= c_{32} - c_{31} + c_{11} - c_{12} = 8 - 5 + 2 - 3 = +2$

Since the evaluation of cells (1,3) and (2,3) are negative, then the current solution is not optimal.

4.2.5 The Modified Distribution (MODI) Method

It is also called the ***$u - v$ method***. This method calculates cell evaluation of all unoccupied cells simultaneously. Thus it provides considerable time saving over the stepping-stone method. It consists of the following sub-steps:

Sub-step 1: Set-up a cost matrix containing the unit costs associated with the cells for which allocations have been made.

Sub-step 2: Introduce dual variables corresponding to the supply and demand constraints. If there are m origins and n distinations, there will be $m + n$ dual variables. Let $u_i(i = 1,2, \dots, m)$ and $v_j(j = 1,2, \dots, n)$ be the dual variables corresponding to supply and demand constraints. Variables $u_i$and v_j are such that $u_i + v_j = c_{ij}$. Therefore, enter a set of numbers $u_i(i = 1,2, \dots, m)$ along the left of the matrix and $v_j(j = 1,2, \dots, n)$ across the top of the matrix so that their sums equal the costs entered in sub-step 1. Assume one of them equal to zero and find their values.

Sub-step 3: Fill the vacant cells with the sum of u_i and v_j.

Sub-step 4: Subtract the cell values of the matrix of sub-step 3 from the original cost matrix. The resulting matrix is called the ***cell evaluation matrix*** (***CEM***).

Sub-step 5: Signs of the values in the cell evaluation matrix indicates wether optimal solution has been obtained or not. The sign have the following significance:

a) A negative value in an unoccupied cell indicates that a better solution can be obtained by allocating units to this cell.

b) A positive value in an unoccupied cell indicates that a poorer solution will result by allocating units to this cell.

c) A zero value in an unoccupied cell indicates that another solution of the same value can be obtained by allocating units to this cell.

Example (4.6):

We will check optimality of basic feasible solution obtained by Vogel's method in example (4.4). Since number of allocations= $6 = m + n - 1$ and they are in independent positions, then we can check optimality. The cost matrix of allocated cells is:

2	3		
			1
5		15	9

$\Longrightarrow$ Entering $u_i (i = 1,2,3)$ and $v_j (j = 1,2,3,4)$ such that:

$u_1 + v_1 = 2$, $u_1 + v_2 = 3$, $u_2 + v_4 = 1$, $u_3 + v_1 = 5$, $u_3 + v_3 = 15$, $u_3 + v_4 = 5$. Let $v_1 = 0$, then: $u_1 = 2$, $v_2 = 1$, $u_3 = 5$, $v_3 = 10$, $v_4 = 4$, $u_2 = -3$.

	v_j	v_1	v_2	v_3	v_4
	u_i	0	1	10	4
u_1	2	2	3		
u_2	-3				1
u_3	5	5		15	9

$\Longrightarrow$

v_j / u_i	0	1	10	4
2	☒	☒	12	6
-3	-3	-2	7	☒
5	☒	6	☒	☒

$u_i + v_j$

$\Rightarrow$

☒	☒	11-12	7-6
1+3	0+2	6-7	☒
☒	8-6	☒	☒

$c_{ij} - (u_i + v_j)$

$\Longrightarrow$

☒	☒	-1	1
4	2	-1	☒
☒	2	☒	☒

C.E.M.

Since the evaluation of cells (1, 3) and (2, 3) is –ve , then the current solution is not optimal.

Step 4: Iterate Toward an Optimal Solution

This involves the following sub-steps:

Sub-step 1: From the cell evaluation matrix, identify the cell with the most negative evaluation. This is the rate by which total transportation cost can be reduced if one unit is allocated to this cell. This cell is called the ***identified cell***. In case of tie in the cell evaluation, the cell which maximum allocation can be made is selected.

Sub-step 2: Write down again the initial basic feasible solution, check mark (√) the identified cell.

Sub-step 3: Trace a closed path in the matrix. This closed path consists of vertical and horizontal lines (not diagonal) begin and terminate in the identified cell and all other corners of the path lie in the allocated cell only.

Sub-step 4: Mark the identified cell as positive and each occupied cell at the corners of the path alternately −ve, +ve, −ve and so on.

Sub-step 5: Make a new allocation in the identified cell by entering the smallest allocation on the path that has been assigned a −ve sign. Add and subtract this new allocation from the cells at the corners of the path, maintaining the row and column requirements.

Step 5: Repeat Steps 3-4 Until Optimal Solution is Reached

Repeat steps 3 and 4 until an optimal solution is reached.

Example (4.7):

After we check optimality of basic feasible solution obtained by Vogel's method in example (4.4) and find that this solution is not optimal. Then we proceed to find the optimal solution. Choose cell (1, 3) as the identified cell, then:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	– 1	5	+ √		6
	2				1	1
	3	+ 6		3 –	1	10
Requirement		7	5	3	2	17

The smallest element in the corners with negative sign is 1, so add and subtract 1 from the cells at the corners of the path. The matrix will be:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3 5	11 1	7	6
	2	1	0	6	1 1	1
	3	5 7	8	15 2	9 1	10
Requirement		7	5	3	2	17

Since number of allocations$= 6 = m + n - 1$ and they are in independent positions, then we can check optimality.

v_j / u_i	1	3	11	5
0		3	11	
-4				1
4	5		15	9

v_j / u_i	1	3	11	5
0	1	☒	☒	5
-4	-3	-1	7	☒
4	☒	7	☒	☒

$u_i + v_j$ ⇒

$2-1$	☒	☒	$7-5$
$1+3$	$0+1$	$6-7$	☒
☒	$8-7$	☒	☒

$c_{ij} - (u_i + v_j)$ ⇒

1	☒	☒	2
4	1	-1	☒
☒	1	☒	☒

C.E.M.

The new solution is not optimal, Choose cell (2, 3) as the identified cell, then:

		Distribution centers				
		1	2	3	4	Supply
Plants	1		5	1		6
	2			+ √	1 −	1
	3	7		− 2	1 +	10
Requirement		7	5	3	2	17

The smallest element in the corners with negative sign is 1, so add and subtract 1 from the cells at the corners of the path. The matrix will be:

		Distribution centers				
		1	2	3	4	Supply
Plants	1	$\boxed{2}$	$\boxed{3}$ 5	$\boxed{11}$ 1	$\boxed{7}$	6
	2	$\boxed{1}$	$\boxed{0}$	$\boxed{6}$ 1	$\boxed{1}$	1
	3	$\boxed{5}$ 7	$\boxed{8}$	$\boxed{15}$ 1	$\boxed{9}$ 2	10
Requirement		7	5	3	2	17

Since number of allocations$= 6 = m + n - 1$ and they are in independent positions, then we can check optimality.

$$\begin{array}{c|c|c|c|c|}
{}^{v_j}_{u_i} & 0 & 2 & 10 & 4 \\ \hline
1 & & 3 & 11 & \\ \hline
-4 & & & 6 & \\ \hline
5 & 5 & & 15 & 9 \\ \hline
\end{array}$$

$$\begin{array}{c|c|c|c|c|}
{}^{v_j}_{u_i} & 0 & 2 & 10 & 4 \\ \hline
1 & 1 & \times & \times & 5 \\ \hline
-4 & -4 & -2 & \times & 0 \\ \hline
5 & \times & 7 & \times & \times \\ \hline
\end{array}
\Rightarrow
\begin{array}{|c|c|c|c|}
\hline
2-1 & \times & \times & 7-5 \\ \hline
1+4 & 0+2 & \times & 1+0 \\ \hline
\times & 8-7 & \times & \times \\ \hline
\end{array}
\Rightarrow
\begin{array}{|c|c|c|c|}
\hline
1 & \times & \times & 2 \\ \hline
5 & 2 & \times & 1 \\ \hline
\times & 1 & \times & \times \\ \hline
\end{array}$$

$u_i + v_j$ $\qquad$ $c_{ij} - (u_i + v_j)$ $\qquad$ C.E.M.

Since all the elements of the cell evaluation matrix are positive, then the optimal solution is:

		Distribution centers				
		1	**2**	**3**	**4**	**Supply**
Plants	**1**	2	3 5	11 1	7	6
	2	1	0	6 1	1	1
	3	5 7	8	15 1	9 2	10
Requirement		7	5	3	2	17

And the transportation cost is:

$$Z = (5 * 3 + 1 * 11 + 1 * 6 + 7 * 5 + 1 * 15 + 2 * 9) * 100 = 10000 \quad \text{ID}$$

4.3 The Unbalanced Transportation Problem

In many real life situations, the total availability may not be equal to the total demand, i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, such problems are called ***Unbalanced Transportation Problem***. In these problems either some resources will remain unused or some requirements will remain unfilled. Since a feasible solution exists only for a balanced problem, it is necessary that the total availability be made equal to the total demand.

1) If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$: we add a dummy resource, the costs of this resource are set equal to zero.
2) If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$: we add a dummy destination, the costs of this destination are set equal to zero.

The supply (demand) of the dummy resource (destination) is:$\left|\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j\right|$.

Example (4.8):

Find the optimal solution of the following transportation problem:

		Stores				
		1	**2**	**3**	**4**	**Supply**
Factories	**1**	4	6	8	13	50
	2	13	11	10	8	70
	3	14	4	10	13	30
	4	9	11	13	8	50
Requirement		25	35	105	20	

Solution:

$\sum_{i=1}^{4} a_i = 50 + 70 + 30 + 50 = 200$, $\sum_{j=1}^{4} b_j = 25 + 35 + 105 + 20 = 185$, then the problem is unbalanced. Therefore, we create a dummy destination. The associated cost coefficients are taken as zero. The cost matrix becomes

		Stores					
		1	**2**	**3**	**4**	**d**	**Supply**
Factories	**1**	4	6	8	13	0	50
	2	13	11	10	8	0	70
	3	14	4	10	13	0	30
	4	9	11	13	8	0	50
Requirement		25	35	105	20	15	200

$\sum_{i=1}^{4} a_i = 200$, $\sum_{j=1}^{5} b_j = 200$, then the problem is balanced. We shall use Vogel's approximation method to find the initial feasible solution. The cost matrix with penalties is shown below:

		Stores						
		1	**2**	**3**	**4**	**d**	**Supply**	
Factories	**1**	4	6	8	13	0	50	[4]
	2	13	11	10	8	0	70	[8]
	3	14	4	10	13	0	30	[4]
	4	9	11	13	8	0 15	~~50~~ 35	[8] ←
Requirement		25	35	105	20	~~15~~ 0	200	
		[5]	[2]	[2]	[0]	[0]		

The greatest penalty is [8], so we choose the 4th row and allocate as much as possible (i.e. 15) to cell (4, 5)(the cell with smallest cost in the 4th row). Requirement of store d is completely satisfied, so column d is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores					
		1	**2**	**3**	**4**	**Supply**	
Factories	**1**	4	6	8	13	50	[2]
	2	13	11	10	8	70	[2]
	3	14	4 30	10	13	~~30~~ 0	[6] ←
	4	9	11	13	8	35	[1]
Requirement		25	~~35~~ 5	105	20	200	
		[5]	[2]	[2]	[0]		

The greatest penalty is [6], so we choose the 3rd row and allocate as much as possible (i.e. 30) to cell (3, 2)(the cell with smallest cost in the 3rd row). Supply of factory 3 is completely satisfied, so row 3 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores					
		1	**2**	**3**	**4**	**Supply**	
Factories	**1**	4 25	6	8	13	~~50~~ 25	[2]
	2	13	11	10	8	70	[2]
	4	9	11	13	8	35	[1]
Requirement		~~25~~ 0	5	105	20	200	
		[5]	[5]	[2]	[0]		
		↑					

The greatest penalty is [5], so we choose the 1st column and allocate as much as possible (i.e. 25) to cell (1, 1)(the cell with smallest cost in the 1st column). Requirement of store 1 is completely satisfied, so column 1 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores				
		2	3	4	Supply	
Factories	1	6 5	8	13	~~25~~ 20	[2]
	2	11	10	8	70	[2]
	4	11	13	8	35	[3]
Requirement		~~5~~ 0	105	20	200	
		[5] ↑	[2]	[0]		

The greatest penalty is [5], so we choose the 2nd column and allocate as much as possible (i.e. 5) to cell (1, 2)(the cell with smallest cost in the 2nd column). Requirement of store 2 is completely satisfied, so column 2 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores			
		3	4	Supply	
Factories	1	8 20	13	~~20~~ 0	[5]←
	2	10	8	70	[2]
	4	13	8	35	[5]
Requirement		~~105~~ 85	20	200	
		[2]	[0]		

The greatest penalty is [5], so we choose the 1st row and allocate as much as possible (i.e. 20) to cell (1, 3)(the cell with smallest cost in the 1st row). Supply of factory 1 is completely satisfied, so row 1 is crossed out and the shrunken matrix with penalties and allocation is as below:

		Stores			
		3	4	Supply	
Factories	2	10	8	70	[2]
	4	13	8 20	~~35~~ 15	[5] ←
Requirement		85	~~20~~ 0	200	
		[3]	[0]		

The greatest penalty is [5], so we choose the 4th row and allocate as much as possible (i.e. 20) to cell (4, 4)(the cell with smallest cost in the 4th row). Requirement of store 4 is completely satisfied, so column 4 is crossed out and the shrunken matrix with allocation is as below(according to least cost):

		Stores	
		3	**Supply**
Factories	**2**	10 70	~~70~~ 0
	4	13 15	~~15~~ 0
Requirement		~~85~~ ~~15~~ 0	200

That is the initial feasible solution is:

		Stores					
		1	**2**	**3**	**4**	**d**	**Supply**
Factories	**1**	4 25	6 5	8 20	13	0	50
	2	13	11	10 70	8	0	70
	3	14	4 30	10	13	0	30
	4	9	11	13 15	8 20	0 15	50
Requirement		25	35	105	20	15	200

Since number of allocations$= 8 = m + n - 1$ and they are in independent positions, then we can check optimality. The sub-steps are:

$$\begin{array}{c|c|c|c|c|c|}
v_j \backslash u_i & 0 & 2 & 4 & -1 & -9 \\ \hline
4 & 4 & 6 & 8 & & \\ \hline
6 & & & 10 & & \\ \hline
2 & & 4 & & & \\ \hline
9 & & & 13 & 8 & 0 \\ \hline
\end{array}
\Rightarrow
\begin{array}{c|c|c|c|c|c|}
v_j \backslash u_i & 0 & 2 & 4 & -1 & -9 \\ \hline
4 & \boxtimes & \boxtimes & \boxtimes & 3 & -5 \\ \hline
6 & 6 & 8 & \boxtimes & 5 & -3 \\ \hline
2 & 2 & \boxtimes & 6 & 1 & -7 \\ \hline
9 & 9 & 11 & \boxtimes & \boxtimes & \boxtimes \\ \hline
\end{array}
\Rightarrow
\begin{array}{|c|c|c|c|c|}
\hline
\boxtimes & \boxtimes & \boxtimes & 10 & 5 \\ \hline
7 & 3 & \boxtimes & 3 & 3 \\ \hline
12 & \boxtimes & 4 & 12 & 7 \\ \hline
0 & 0 & \boxtimes & \boxtimes & \boxtimes \\ \hline
\end{array}$$

$u_i + v_j$ C.E.M $(c_{ij} - (u_i + v_j))$

Since cell values are positive, then the first feasible solution is optimal and

$$Z = 25 * 4 + 5 * 6 + 20 * 8 + 70 * 10 + 30 * 4 + 15 * 13 + 20 * 8 + 15 * 0$$
$$= 1465$$

4.4 Degeneracy in Transportation Problem

In transportation problem with m origins and n destinations if a basic feasible solution has less than $m + n - 1$ allocations (occupied cells), the problem is said to be a degenerate transportation problem. Degeneracy can occur in the initial solution or during some subsequent iteration.

In this case we allocate an infinitesimally but positive value ϵ to vacant cell (cells) with least cost so that there are exactly $m+n-1$ allocated cells in independent positions and the procedure can then be continued in the usual manner. Subscripts are used when more than one such letter is required (f.e. $\epsilon_1, \epsilon_2,$ etc.). These ϵ's are treated like any other positive basic variable and are kept in the transportation matrix until temporary degeneracy is removed or until the optimal solution is reached, whichever occurs first. At this point we set each $\epsilon = 0$. Notice that ϵ is infinitesimally small and hence its effect can be neglected when it is added to or subtracted from a positive value (f.e. $10 + \epsilon = 10, 5 - \epsilon = 5, \epsilon + \epsilon = 2\epsilon, \epsilon - \epsilon = 0$). Consequently, they do not alter the physical nature of the original set of allocations but do help in carrying out further computations such as optimality test.

Example (4.9):

Find the optimal solution of the following transportation problem.

		Destinations						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
Origins	**1**	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	

Solution:

$\sum_{i=1}^{4} a_i = 5 + 6 + 2 + 9 = 22, \sum_{j=1}^{6} b_j = 4 + 4 + 6 + 2 + 4 + 2 = 22$, then the system is balanced. The initial basic feasible solution by using Vogel's approximation method is:

		Destinations							
		1	2	3	4	5	6	Supply	
Origins	1	9	12	9 5	6	9	10	~~5~~ 0	[3] [3] [0] [0] [0] [0]
	2	7	3 4	7	7	5	5 2	~~6~~ ~~4~~ 0	[2] [2] [2] [4]← -- --
	3	6 1	5	9 1	11	3	11	~~2~~ ~~1~~ 0	[2] [2] [2] [1] [3] [3]←
	4	6 3	8	11	2 2	2 4	10	~~9~~ ~~7~~ ~~3~~ 0	[0] [0] [4]← [2] [5]← --
Requirement		~~4~~ ~~1~~ 0	~~4~~ 0	~~6~~ ~~1~~ 0	~~2~~ 0	~~4~~ 0	~~2~~ 0		
		[0]	[2]	[2]	[4]	[1]	[5]		
		[0]	[2]	[2]	[4]	[1]	↑		
		[0]	[2]	[2]	↑	[1]			
		[0]	[2]	[2]	--	--	--		
		[0]	--	[0]	--	--	--		
		[3]	--	[0]	--	--	--		

Since number of allocations $= 8 \neq 9 = m + n - 1 (m = 4, n = 6)$, then we must select unoccupied cell with least cost and set an infinitesimal allocation to it. The unoccupied cell (3, 5) has the least cost, but this cell form a closed loop with cells (3, 1), (4, 1), and (4, 5). There are two next higher cost cell ((2, 5) and (3, 2)), allocation in either of these cells does not result a closed loop. Let us choose cell (2, 5) and allocate ϵ to it. The matrix will be:

		Destinations						
		1	2	3	4	5	6	Supply
Origins	1	9	12	9 5	6	9	10	5
	2	7	3 4	7	7	5 ϵ	5 2	6
	3	6 1	5	9 1	11	3	11	2
	4	6 3	8	11	2 2	2 4	10	9
Requirement		4	4	6	2	4	2	

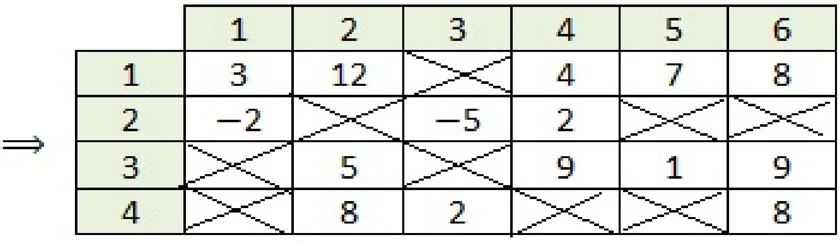
Since number of allocations$= 9 = m + n - 1$ and they are in independent positions, then we can check optimality. The sub-steps are:

v_j / u_i	0	−6	3	−4	−4	−4
6			9			
9		3			5	5
6	6		9			
6	6			2	2	

$\Rightarrow$

v_j / u_i	0	−6	3	−4	−4	−4
6	6	0	☒	2	2	2
9	9	☒	12	5	☒	☒
6	☒	0	☒	2	2	2
6	☒	0	9	☒	☒	2

$u_i + v_j$



⟹

	1	2	3	4	5	6
1	3	12	✕	4	7	8
2	−2	✕	−5	2	✕	✕
3	✕	5	✕	9	1	9
4	✕	8	2	✕	✕	8

C.E.M($c_{ij}-(u_i+v_j)$)

Since cells (2, 1) and (2, 3) have negative values, then the current feasible solution is not optimal. Cell (2, 3) has the most negative value, then:

	1	2	3	4	5	6
1			5			
2		4	+ √		ϵ −	2
3	+ 1		− 1			
4	− 3			2	4 +	

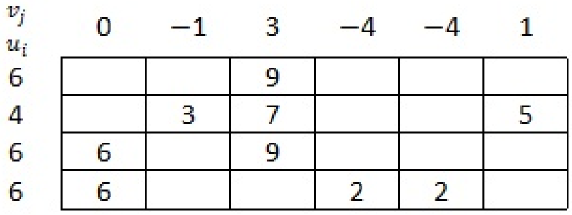
⟹

	1	2	3	4	5	6
1			5			
2		4	ϵ		$\epsilon-\epsilon$	2
3	$1+\epsilon$		$1-\epsilon$			
4	$3-\epsilon$			2	$4+\epsilon$	

⟹

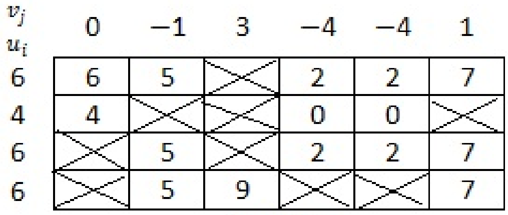
	1	2	3	4	5	6
1			5			
2		4	ϵ			2
3	$1+\epsilon$		$1-\epsilon$			
4	$3-\epsilon$			2	$4+\epsilon$	

Since number of allocations$=9=m+n-1$ and they are in independent positions, then we can check optimality. The sub-steps are:



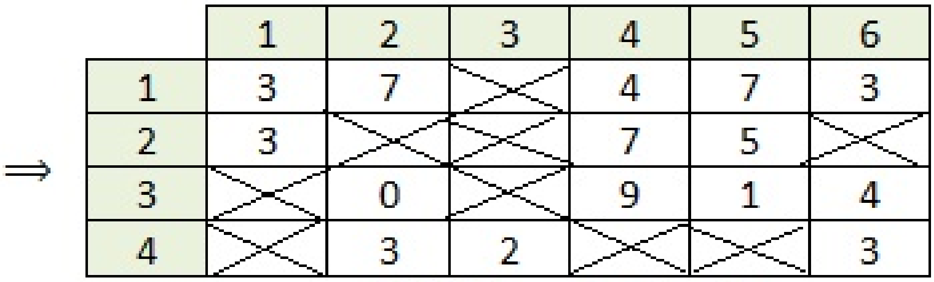
v_j u_i	0	−1	3	−4	−4	1
6			9			
4		3	7			5
6	6		9			
6	6			2	2	

⟹



v_j u_i	0	−1	3	−4	−4	1
6	6	5	✕	2	2	7
4	4	✕	✕	0	0	✕
6	✕	5	✕	2	2	7
6	✕	5	9	✕	✕	7

u_i+v_j



⟹

	1	2	3	4	5	6
1	3	7	✕	4	7	3
2	3	✕	✕	7	5	✕
3	✕	0	✕	9	1	4
4	✕	3	2	✕	✕	3

C.E.M($c_{ij}-(u_i+v_j)$)

Since all the elements of the cell evaluation matrix are positive, then the optimal solution is (considering $\epsilon = 0$):

		Destinations						
		1	**2**	**3**	**4**	**5**	**6**	**Supply**
Origins	**1**	9	12	9 5	6	9	10	5
	2	7	3 4	7	7	5	5 2	6
	3	6 1	5	9 1	11	3	11	2
	4	6 3	8	11	2 2	2 4	10	9
Requirement		4	4	6	2	4	2	

The cost is:

$$Z = 5*9+4*3+2*5+1*6+1*9+3*6+2*2+4*2 = 112 \text{ units}$$

Exercises 4.1 (In addition to the text book exercises)

Find the optimal solution of the following transportation problems:

1:

		Destinations				
		1	**2**	**3**	**4**	**Supply**
Origins	**1**	90	90	100	110	200
	2	50	70	130	85	100
Requirement		75	100	100	30	

2:

	1	**2**	**3**	**4**	**5**	**Supply**
1	8	6	2	4	12	80
2	10	4	6	8	10	60
3	6	10	12	6	4	40
4	4	8	8	10	6	20
Requirement	60	60	30	40	10	