# Operations Research II

# Second Semester

For the 3rd class students / Mathematics Department / College of Science for Women

**Prepared by:**
Dr.Najwa Raheem Mustafa
Mathematics Department
College of Science for Women
University of Baghdad

# Preface

These lecture notes are for the course “Operations Research II” for the 3rd grade- second semester in Mathematics Department / College of Science for Women /Baghdad University.

The author claims no originality. These lecture notes are collected from references listed in the “Bibliography”.

# Contents

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# Ch. 1: Assignment Problem

The ***assignment problem*** may be defined as follows: Given $n$ facilities and $n$ jobs and given the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job so as to optimize the given measures of effectiveness. The assignment problem is a special case of transportation problem.

Table(1.1) represents the assignment of $n$ facilities(machines) to $n$ jobs, $c_{ij}$ is the cost of assigning $ith$ facility to $jth$ job and $x_{ij}$ represents the assignment of $ith$ facility to $jth$ job. If $ith$ facility can be assigned to $jth$ job,$x_{ij} = 1$, otherwise zero. The matrix is called the ***cost matrix***.

| | | **Jobs** | | | | |
|---|---|---|---|---|---|---|
| | | **1** | **2** | **…** | **n** | $a_i$ **(Supply)** |
| **Facilities** | **1** | $c_{11}$ | $c_{12}$ | … | $c_{12}$ | 1 |
| | **2** | $c_{21}$ | $c_{22}$ | … | $c_{2n}$ | 1 |
| | **⋮** | ⋮ | ⋮ | … | | ⋮ |
| | **n** | $c_{n1}$ | $c_{n2}$ | … | $c_{nn}$ | 1 |
| | $b_j$**(Demand)** | 1 | 1 | … | 1 | |

**Table (1.1)**

## 1.1 Mathematical Representation of the Assignment Model

Mathematically, the assignment model can be expressed as follows:

Let

$$x_{ij} = \begin{cases} 0, if\ the\ ith\ facility\ is\ not\ assigned\ to\ jth\ job \\ 1, if\ the\ ith\ facility\ is\ assigned\ \ to\ jth\ job \end{cases}$$

Then, the model is given by:

$$min \quad Z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\ x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ij}\ x_{ij}$$

$$S.t. \quad \sum_{j=1}^{n} x_{ij} = 1 \qquad i = 1,2,\dots,n$$

$$\sum_{i=1}^{n} x_{ij} = 1 \qquad j = 1,2,\dots,n$$

$$x_{ij} = 0\ \ or\ 1 \qquad i = 1,2,\dots,n\,;\ \ j = 1,2,\dots,n \ \ (\text{or } x_{ij} = {x_{ij}}^2)$$

The technique used for solving assignment model makes use of two theorems:

**Theorem (1.1)**

In an assignment problem, if we add or subtract a constant to every element of a row (or column) in the cost matrix, then an assignment which minimizes the total cost on one matrix also minimizes the total cost on the other matrix.

**Theorem (1.2)**

If all $c_{ij} \geq 0$ and we can find a set $x_{ij} = x_{ij}^*$ such that $\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\ x_{ij}^* = 0$, then this solution is optimal.

The above two theorems indicates that if one can create a new $c_{ij}$ matrix with zero entries, and if these zero elements, or a subset thereof, contains feasible solution, then this feasible solution is the optimal solution.

## 1.2 The Hungarian Method

The ***Hungarian method*** (or ***reduced matrix method***) was developed by D. König, a Hungarian mathematician. The method consists of the following steps:

***Step 1: Prepare a square matrix***. Add dummy rows (columns) if needed (rows (columns) with zero cost).

***Step 2: Reduce the matrix***. Subtract the smallest element of each row from all the elements of the row. So there will be at least one zero in each row. Examine if there is at least one zero in each column. If not, subtract the smallest element of the column(s) not containing zero from all the elements of the column. This step reduces the elements of the matrix until zeros, ***called zero opportunity costs***, are obtained in each column.

***Step 3: Check whether an optimal assignment can be made in the reduced matrix or not***. For this:

a) Examine rows successively until a row with exactly one unmarked zero is obtained. Make an assignment to this single zero by marking square (□) around it. Cross (X) all other zeros in the same column as they will not be considered for making any more assignment in that column. Proceed in this way until all rows have been examined.

b) Now examine columns successively until a column with exactly one unmarked zero is found. Make an assignment there by marking square (□) around it and cross (X) any other zeros in the same row. Proceed in this way until all columns have been examined.

In case there is no row or column containing single unmarked zero (they contain more than one unmarked zero), mark square (□) around it arbitrarily and cross

(X) all other zeros in its row and column. Proceed in this manner till there is no unmarked zero left in the cost matrix.

Repeat sub-steps (a) and (b) till one of the following two cases occur:

i) There is one assignment in each row and in each column. In this case the optimal assignment can be made in the current solution. The minimum number of lines crossing all zeros is $n$, the order of the matrix.

ii) There is some row and/or column without assignment. In this case optimal assignment cannot be made in the current solution. The minimum number of lines crossing all zeros has to be obtained in this case by following step 4.

***Step 4: Find the minimum number of lines crossing all zeros***. This consists of the following sub-steps:

a) Mark (√ ) the rows that do not have assignments.

b) Mark (√ ) the columns (not already marked) that have zeros in marked rows.

c) Mark (√ ) the rows (not already marked) that have assignment in the marked columns.

d) Repeat sub-steps (b) and (c) till no more rows or columns can be marked.

e) Draw straight lines through all unmarked rows and marked columns. This gives the minimum number of lines crossing all zeros.

***Step 5: Iterate towards the optimal solution***. Examine the uncovered elements. Select the smallest element and subtract it from all the uncovered elements. Add this smallest element to every element that lies at the intersection of two lines. Leave the remaining elements of the matrix without change. This yields a new basic feasible solution.

***Step 6:*** Repeat steps 3 through 5 successively until the minimum number of lines crossing all zeros becomes equal to $n$, the order of the matrix. In such a case every row and column will have one assignment. This indicates that an optimal solution has been obtained. The total cost associated with this solution is obtained by adding the original costs of the assigned cells.

## Example (1.1):

A machine tool company decides to make four subassemblies through four contractors. Each contractor is to receive only one subassembly. The cost of each subassembly is determined by the bids submitted by each contractor and is shown in the following table in millions of Iraqi dinars.

1) Formulate the mathematical model for the problem.

2) Assign the different subassemblies to contractors to minimize the total cost.

| | | Contractors | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** |
| **Subass-emblies** | **1** | 15 | 13 | 14 | 17 |
| | **2** | 11 | 12 | 15 | 13 |
| | **3** | 13 | 12 | 10 | 11 |
| | **4** | 15 | 17 | 14 | 16 |

**Solution:**

1) Let $x_{ij} = \begin{cases} 0, if\ the\ ith\ subassembly\ \ is\ not\ assigned\ to\ jth\ contractor \\ 1, if\ the\ ith\ subassembly\ is\ assigned\ \ to\ jth\ contractor \end{cases}$

Then, the model is given by:

$$min \quad Z = \sum_{i=1}^{4}\sum_{j=1}^{4} \ c_{ij}\ x_{ij} = \sum_{j=1}^{4}\sum_{i=1}^{4} \ c_{ij}\ x_{ij}$$

S.t.

$$\left.\begin{array}{l} x_{11} + x_{12} + x_{13} + x_{14} = 1 \\ x_{21} + x_{22} + x_{23} + x_{24} = 1 \\ x_{31} + x_{32} + x_{33} + x_{34} = 1 \\ x_{41} + x_{42} + x_{43} + x_{44} = 1 \end{array}\right\} \text{Constraints on subassemblies}$$

$$\left.\begin{array}{l} x_{11} + x_{21} + x_{31} + x_{41} = 1 \\ x_{12} + x_{22} + x_{32} + x_{42} = 1 \\ x_{13} + x_{23} + x_{33} + x_{43} = 1 \\ x_{14} + x_{24} + x_{34} + x_{44} = 1 \end{array}\right\} \text{Constraints on contractors}$$

$$x_{ij} = 0 \ \ or\ 1 \qquad\qquad i = 1,2,3,4\ ; \ \ j = 1,2,3,4 \ \ (\text{or } x_{ij} = x_{ij}{}^{2})$$

2) We will reduce the matrix; the smallest element in the first row is 13, so we subtract 14 from all elements of the first row. Similarly for the remaining three rows. This gives the following matrix:

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 2 | 0 | 1 | 4 |
| **2** | 0 | 1 | 4 | 2 |
| **3** | 3 | 2 | 0 | 1 |
| **4** | 1 | 3 | 0 | 2 |

Each row contains at least one zero. The last column does not contain any zero , the we subtract the smallest element in that column (which is 1) from all the elements of the column. This gives the following matrix:

| | **1** | **2** | **3** | **4** |
|---|---|---|---|---|
| **1** | 2 | 0 | 1 | 3 |
| **2** | 0 | 1 | 4 | 1 |
| **3** | 3 | 2 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| **4** | 1 | 3 | 0 | 1 |

The assignment is given in the following matrix:

| | | **Contractors** | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** |
| **Subassemblies** | **1** | 2 | $\boxed{0}$ | 1 | 3 |
| | **2** | $\boxed{0}$ | 1 | 4 | 1 |
| | **3** | 3 | 2 | ~~0~~ | $\boxed{0}$ |
| | **4** | 1 | 3 | $\boxed{0}$ | 1 |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

Subassembly 1 is assigned to contractor 2

Subassembly 2 is assigned to contractor 1

Subassembly 3 is assigned to contractor 4

Subassembly 4 is assigned to contractor 3

And the minimum total cost is:

$$Z_{min} = (13 + 11 + 11 + 14) \times 10^6 = 49000000 \text{ ID}$$

## Example (1.2):

Four different jobs can be done on four different machines. The matrix below gives the cost in dolars of producing job $i$ on machine $j$.

| | | **Machines** | | | |
|---|---|---|---|---|---|
| | | **$M_1$** | **$M_2$** | **$M_3$** | **$M_4$** |
| **Jobs** | **$J_1$** | 5 | 7 | 11 | 6 |
| | **$J_2$** | 8 | 5 | 9 | 6 |
| | **$J_3$** | 4 | 7 | 10 | 7 |
| | **$J_4$** | 10 | 4 | 8 | 3 |

How should the jobs be assigned to the machines so that the total cost is minimized?

**Solution:**

Reducing the matrix involves the following steps:

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 5 | 7 | 11 | 6 |
| $J_2$ | 8 | 5 | 9 | 6 |
| $J_3$ | 4 | 7 | 10 | 7 |
| $J_4$ | 10 | 4 | 8 | 3 |

$\Rightarrow$

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 0 | 2 | 6 | 1 |
| $J_2$ | 3 | 0 | 4 | 1 |
| $J_3$ | 0 | 3 | 6 | 3 |
| $J_4$ | 7 | 1 | 5 | 0 |

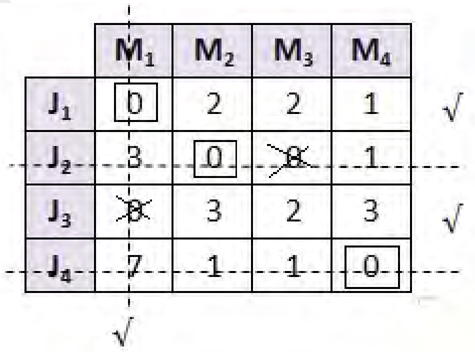
The third column does not contain a zero, then we subtract 4 (the smallest element of the third column) from all the elements of that column. This gives the following matrix:

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | 0 | 2 | 2 | 1 |
| $J_2$ | 3 | 0 | 0 | 1 |
| $J_3$ | 0 | 3 | 2 | 3 |
| $J_4$ | 7 | 1 | 1 | 0 |

The assignment is given in the following matrix:

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | $\boxed{0}$ | 2 | 2 | 1 |
| $J_2$ | 3 | $\boxed{0}$ | $\cancel{0}$ | 1 |
| $J_3$ | $\cancel{0}$ | 3 | 2 | 3 |
| $J_4$ | 7 | 1 | 1 | $\boxed{0}$ |

Row 3 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:



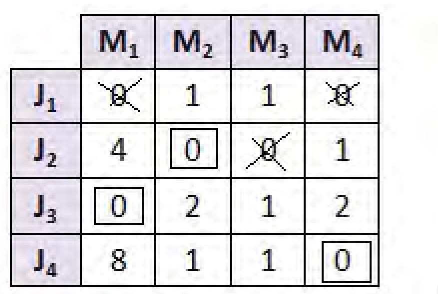
| | $M_1$ | $M_2$ | $M_3$ | $M_4$ | |
|---|---|---|---|---|---|
| $J_1$ | $\boxed{0}$ | 2 | 2 | 1 | √ |
| $J_2$ | 3 | $\boxed{0}$ | $\cancel{0}$ | 1 | |
| $J_3$ | $\cancel{0}$ | 3 | 2 | 3 | √ |
| $J_4$ | 7 | 1 | 1 | $\boxed{0}$ | |
| | √ | | | | |

The minimum number of lines crossing all zeros is $3 \neq n (n = 4 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5, the matrix will be:

| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|

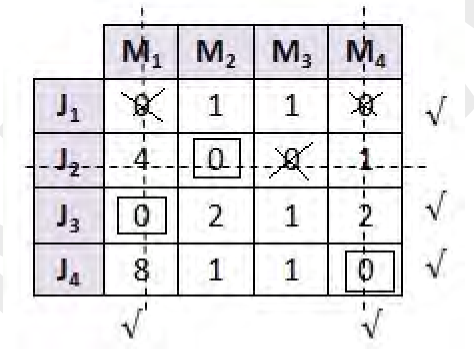
| | | | | |
|---|---|---|---|---|
| **J₁** | 0 | 1 | 1 | 0 |
| **J₂** | 4 | 0 | 0 | 1 |
| **J₃** | 0 | 2 | 1 | 2 |
| **J₄** | 8 | 1 | 1 | 0 |

The assignment is given in the following matrix:



| | **M₁** | **M₂** | **M₃** | **M₄** |
|---|---|---|---|---|
| **J₁** | $\cancel{0}$ | 1 | 1 | $\cancel{0}$ |
| **J₂** | 4 | $\boxed{0}$ | $\cancel{0}$ | 1 |
| **J₃** | $\boxed{0}$ | 2 | 1 | 2 |
| **J₄** | 8 | 1 | 1 | $\boxed{0}$ |

Row 1 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

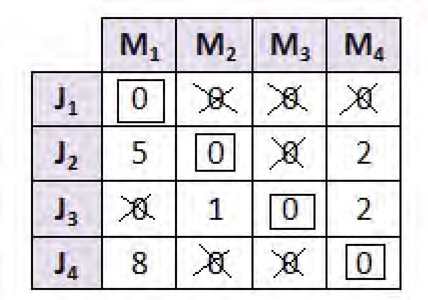


| | **M₁** | **M₂** | **M₃** | **M₄** | |
|---|---|---|---|---|---|
| **J₁** | $\cancel{0}$ | 1 | 1 | $\cancel{0}$ | √ |
| **J₂** | 4 | $\boxed{0}$ | $\cancel{0}$ | 1 | |
| **J₃** | $\boxed{0}$ | 2 | 1 | 2 | √ |
| **J₄** | 8 | 1 | 1 | $\boxed{0}$ | √ |
| | √ | | | √ | |

The minimum number of lines crossing all zeros is $3 \neq n(n = 4 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5, the matrix will be:

| | **M₁** | **M₂** | **M₃** | **M₄** |
|---|---|---|---|---|
| **J₁** | 0 | 0 | 0 | 0 |
| **J₂** | 5 | 0 | 0 | 2 |
| **J₃** | 0 | 1 | 0 | 2 |
| **J₄** | 8 | 0 | 0 | 0 |

The assignment is given in the following matrix:



|  | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ |
| $J_2$ | 5 | $\boxed{0}$ | $\cancel{0}$ | 2 |
| $J_3$ | $\cancel{0}$ | 1 | $\boxed{0}$ | 2 |
| $J_4$ | 8 | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

$J_1$ is assigned to $M_1$

$J_2$ is assigned to $M_2$

$J_3$ is assigned to $M_3$

$J_4$ is assigned to $M_4$

And the minimum total cost is: $Z_{min} = 5 + 5 + 10 + 3 = 23\ \$$

## 1.3 Variations of the Assignment Problem

### 1.3.1 Non-square Matrix (Unbalanced Assignment Problem)

Such a problem is found when the number of facilities is not equal to the number of jobs. Since the Hungarian method of solution requires a square matrix, dummy facilities or jobs may be added and zero costs is assigned to the corresponding cells of the matrix. These cells are then treated the same way as the real cost cells during the solution procedure.

### Example (1.3):

A company has one surplus truck in each of the cities A, B, C, D and E and one dificit truck in each of the cities 1, 2, 3, 4, 5 and 6. The distance between the cities in kilometres is shown in the matrix below. Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by vehicles is minimum.

|  | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 12 | 10 | 15 | 22 | 18 | 8 |
| B | 10 | 18 | 25 | 15 | 16 | 12 |
| C | 11 | 10 | 3 | 8 | 5 | 9 |
| D | 6 | 14 | 10 | 13 | 13 | 12 |
| E | 8 | 12 | 11 | 7 | 13 | 10 |

**Solution:**

The matrix is non-square, so we add a dummy city with surplus vehicle.Since there is no distance associated with it, the corresponding cell values are made all zeros.

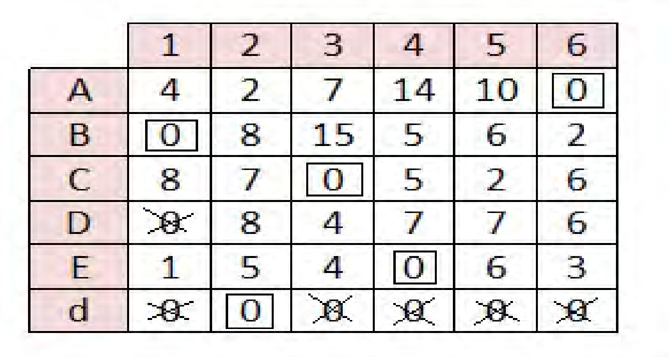
|  | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 12 | 10 | 15 | 22 | 18 | 8 |
| B | 10 | 18 | 25 | 15 | 16 | 12 |
| C | 11 | 10 | 3 | 8 | 5 | 9 |
| D | 6 | 14 | 10 | 13 | 13 | 12 |
| E | 8 | 12 | 11 | 7 | 13 | 10 |
| d | 0 | 0 | 0 | 0 | 0 | 0 |

|  | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 12 | 10 | 15 | 22 | 18 | 8 |
| B | 10 | 18 | 25 | 15 | 16 | 12 |
| C | 11 | 10 | 3 | 8 | 5 | 9 |
| D | 6 | 14 | 10 | 13 | 13 | 12 |
| E | 8 | 12 | 11 | 7 | 13 | 10 |
| d | 0 | 0 | 0 | 0 | 0 | 0 |

⇒

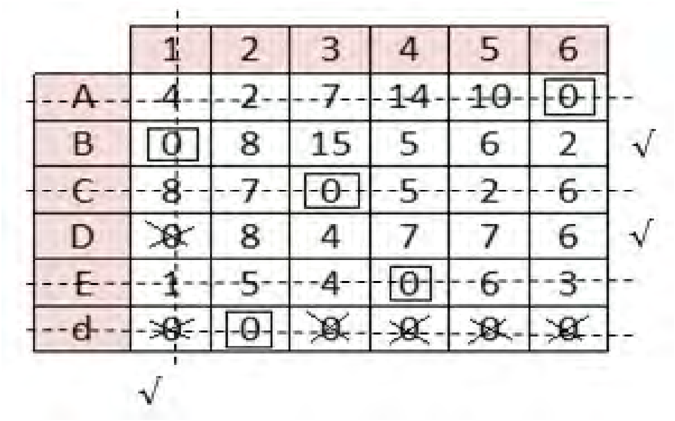
|  | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 4 | 2 | 7 | 14 | 10 | 0 |
| B | 0 | 8 | 15 | 5 | 6 | 2 |
| C | 8 | 7 | 0 | 5 | 2 | 6 |
| D | 0 | 8 | 4 | 7 | 7 | 6 |
| E | 1 | 5 | 4 | 0 | 6 | 3 |
| d | 0 | 0 | 0 | 0 | 0 | 0 |

The assignment is given in the following matrix:



|  | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 4 | 2 | 7 | 14 | 10 | $\boxed{0}$ |
| B | $\boxed{0}$ | 8 | 15 | 5 | 6 | 2 |
| C | 8 | 7 | $\boxed{0}$ | 5 | 2 | 6 |
| D | $\xcancel{0}$ | 8 | 4 | 7 | 7 | 6 |
| E | 1 | 5 | 4 | $\boxed{0}$ | 6 | 3 |
| d | $\xcancel{0}$ | $\boxed{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ |

Row 4 and column 5 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

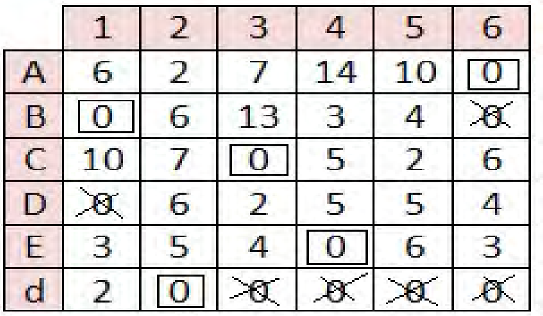


| | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| A | 4 | 2 | 7 | 14 | 10 | $\boxed{0}$ | |
| B | $\boxed{0}$ | 8 | 15 | 5 | 6 | 2 | √ |
| C | 8 | 7 | $\boxed{0}$ | 5 | 2 | 6 | |
| D | $\xcancel{0}$ | 8 | 4 | 7 | 7 | 6 | √ |
| E | 1 | 5 | 4 | $\boxed{0}$ | 6 | 3 | |
| d | $\xcancel{0}$ | $\boxed{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | |
| | √ | | | | | | |

The minimum number of lines crossing all zeros is $5 \neq n(n = 6 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 2. By applying step 5 the matrix will be:

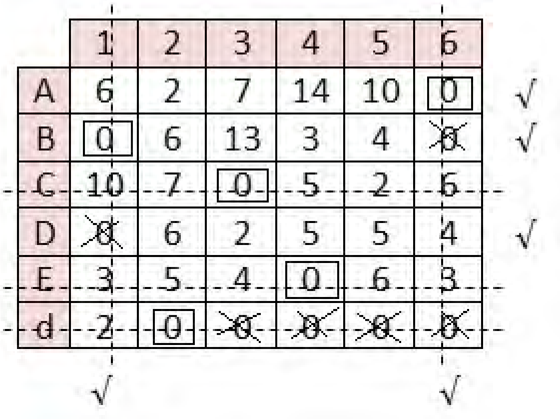
| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 6 | 2 | 7 | 14 | 10 | 0 |
| B | 0 | 6 | 13 | 3 | 4 | 0 |
| C | 10 | 7 | 0 | 5 | 2 | 6 |
| D | 0 | 6 | 2 | 5 | 5 | 4 |
| E | 3 | 5 | 4 | 0 | 6 | 3 |
| d | 2 | 0 | 0 | 0 | 0 | 0 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 6 | 2 | 7 | 14 | 10 | $\boxed{0}$ |
| B | $\boxed{0}$ | 6 | 13 | 3 | 4 | $\xcancel{0}$ |
| C | 10 | 7 | $\boxed{0}$ | 5 | 2 | 6 |
| D | $\xcancel{0}$ | 6 | 2 | 5 | 5 | 4 |
| E | 3 | 5 | 4 | $\boxed{0}$ | 6 | 3 |
| d | 2 | $\boxed{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ | $\xcancel{0}$ |

Row 4 and column 5 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

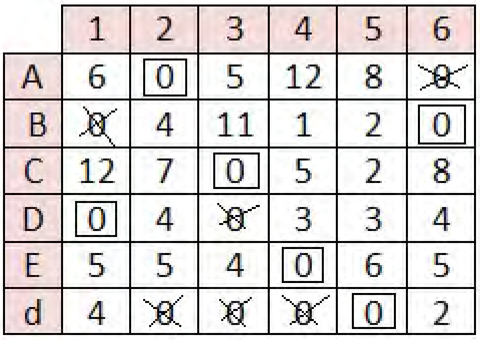


| | 1 | 2 | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|---|---|
| A | 6 | 2 | 7 | 14 | 10 | $\boxed{0}$ | √ |
| B | $\boxed{0}$ | 6 | 13 | 3 | 4 | $\cancel{0}$ | √ |
| C | 10 | 7 | $\boxed{0}$ | 5 | 2 | 6 | |
| D | $\cancel{0}$ | 6 | 2 | 5 | 5 | 4 | √ |
| E | 3 | 5 | 4 | $\boxed{0}$ | 6 | 3 | |
| d | 2 | $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | |
| | √ | | | | | √ | |

The minimum number of lines crossing all zeros is $5 \neq n(n = 6 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 2. By applying step 5 the matrix will be:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 6 | 0 | 5 | 12 | 8 | 0 |
| B | 0 | 4 | 11 | 1 | 2 | 0 |
| C | 12 | 7 | 0 | 5 | 2 | 8 |
| D | 0 | 4 | 0 | 3 | 3 | 4 |
| E | 5 | 5 | 4 | 0 | 6 | 5 |
| d | 4 | 0 | 0 | 0 | 0 | 2 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 6 | $\boxed{0}$ | 5 | 12 | 8 | $\cancel{0}$ |
| B | $\cancel{0}$ | 4 | 11 | 1 | 2 | $\boxed{0}$ |
| C | 12 | 7 | $\boxed{0}$ | 5 | 2 | 8 |
| D | $\boxed{0}$ | 4 | $\cancel{0}$ | 3 | 3 | 4 |
| E | 5 | 5 | 4 | $\boxed{0}$ | 6 | 5 |
| d | 4 | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ | 2 |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

City A should supply the vehicle to city 2
City B should supply the vehicle to city 6
City C should supply the vehicle to city 3
City D should supply the vehicle to city 1
City E should supply the vehicle to city 4
Minimum distance traveled $= 10 + 12 + 3 + 6 + 7 = 38\ km$

No truck supplid to city 5

## Example (1.4):

Solve the following assignment problem for minimal optimal cost:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| I | 9 | 14 | 19 | 15 |
| II | 7 | 17 | 20 | 19 |
| III | 9 | 18 | 21 | 18 |
| IV | 10 | 12 | 18 | 19 |
| V | 10 | 15 | 21 | 16 |

**Solution:**

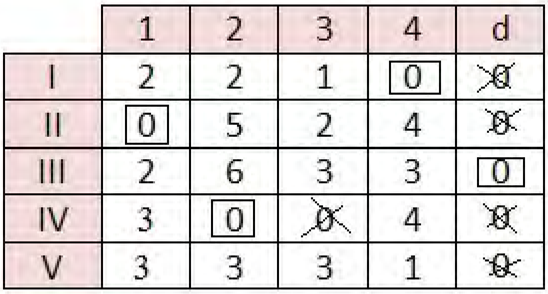
The matrix is non-square, so we add a dummy job with cell values are made all zeros.

| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 9 | 14 | 19 | 15 | 0 |
| II | 7 | 17 | 20 | 19 | 0 |
| III | 9 | 18 | 21 | 18 | 0 |
| IV | 10 | 12 | 18 | 19 | 0 |
| V | 10 | 15 | 21 | 16 | 0 |

⇒

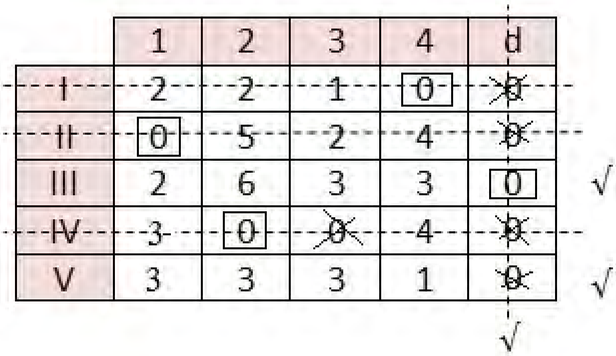
| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 2 | 2 | 1 | 0 | 0 |
| II | 0 | 5 | 2 | 4 | 0 |
| III | 2 | 6 | 3 | 3 | 0 |
| IV | 3 | 0 | 0 | 4 | 0 |
| V | 3 | 3 | 3 | 1 | 0 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 2 | 2 | 1 | $\boxed{0}$ | $\cancel{0}$ |
| II | $\boxed{0}$ | 5 | 2 | 4 | $\cancel{0}$ |
| III | 2 | 6 | 3 | 3 | $\boxed{0}$ |
| IV | 3 | $\boxed{0}$ | $\cancel{0}$ | 4 | $\cancel{0}$ |
| V | 3 | 3 | 3 | 1 | $\cancel{0}$ |

Row 5 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

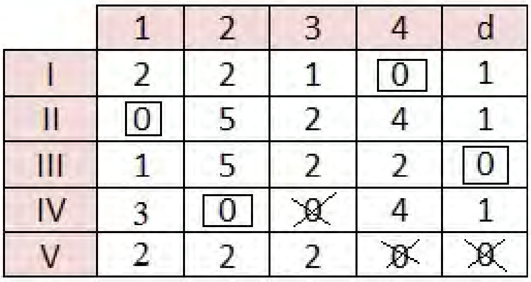


| | 1 | 2 | 3 | 4 | d | |
|---|---|---|---|---|---|---|
| I | 2 | 2 | 1 | $\boxed{0}$ | $\cancel{0}$ | |
| II | $\boxed{0}$ | 5 | 2 | 4 | $\cancel{0}$ | |
| III | 2 | 6 | 3 | 3 | $\boxed{0}$ | √ |
| IV | 3 | $\boxed{0}$ | $\cancel{0}$ | 4 | $\cancel{0}$ | |
| V | 3 | 3 | 3 | 1 | $\cancel{0}$ | √ |
| | | | | | √ | |

The minimum number of lines crossing all zeros is $4 \neq n(n = 5 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5 the matrix will be:

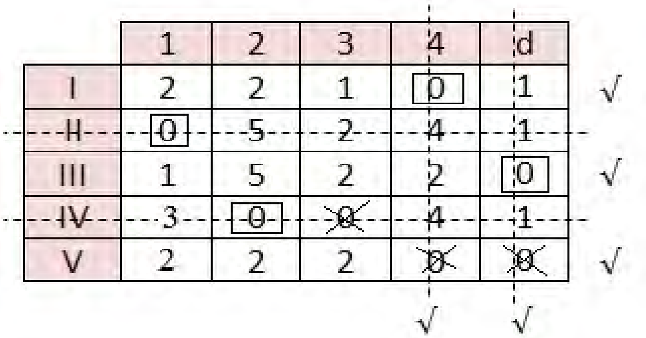
| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 2 | 2 | 1 | 0 | 1 |
| II | 0 | 5 | 2 | 4 | 1 |
| III | 1 | 5 | 2 | 2 | 0 |
| IV | 3 | 0 | 0 | 4 | 1 |
| V | 2 | 2 | 2 | 0 | 0 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 2 | 2 | 1 | $\boxed{0}$ | 1 |
| II | $\boxed{0}$ | 5 | 2 | 4 | 1 |
| III | 1 | 5 | 2 | 2 | $\boxed{0}$ |
| IV | 3 | $\boxed{0}$ | $\cancel{0}$ | 4 | 1 |
| V | 2 | 2 | 2 | $\cancel{0}$ | $\cancel{0}$ |

Row 5 and column 3 are without any assignment; hence we proceed as follows to find the minimum number of lines crossing all zeros:

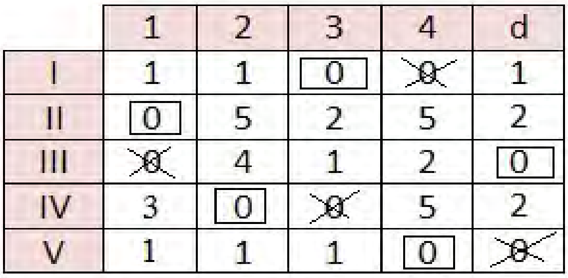


| | 1 | 2 | 3 | 4 | d | |
|---|---|---|---|---|---|---|
| I | 2 | 2 | 1 | $\boxed{0}$ | 1 | √ |
| II | $\boxed{0}$ | 5 | 2 | 4 | 1 | |
| III | 1 | 5 | 2 | 2 | $\boxed{0}$ | √ |
| IV | 3 | $\boxed{0}$ | $\cancel{0}$ | 4 | 1 | |
| V | 2 | 2 | 2 | $\cancel{0}$ | $\cancel{0}$ | √ |
| | | | | √ | √ | |

The minimum number of lines crossing all zeros is $4 \neq n(n = 5 \text{ here})$ . Hence the optimal assignment is not possible in the current solution. The smallest element in the cells that do not have a line through is 1. By applying step 5 the matrix will be:

| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 1 | 1 | 0 | 0 | 1 |
| II | 0 | 5 | 2 | 5 | 2 |
| III | 0 | 4 | 1 | 2 | 0 |
| IV | 3 | 0 | 0 | 5 | 2 |
| V | 1 | 1 | 1 | 0 | 0 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 | d |
|---|---|---|---|---|---|
| I | 1 | 1 | $\boxed{0}$ | $\cancel{0}$ | 1 |
| II | $\boxed{0}$ | 5 | 2 | 5 | 2 |
| III | $\cancel{0}$ | 4 | 1 | 2 | $\boxed{0}$ |
| IV | 3 | $\boxed{0}$ | $\cancel{0}$ | 5 | 2 |
| V | 1 | 1 | 1 | $\boxed{0}$ | $\cancel{0}$ |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

I is assigned to 3

II is assigned to 1

IV is assigned to 2

V is assigned to 4

The minimum cost $= 19 + 7 + 12 + 16 = 54$ units. III is not assigned.

### 1.3.2 Maximization Problem

Sometimes the assignment problem may deal with the maximization of the objective function. The maximization problem has to be changed to minimization before the Hungarian method may be applied. This transformation may be done in either of the following two ways:

a) By subtracting all the elements from the largest element of the matrix.

b) By multiplying the matrix elements by $-1$.

The hungarian method can then be applied to this equivalent minimization problem to obtain the optimal solution.

### Example (1.5):

A company has a team of four salesmen and there are four districts where the company wants to start business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in hundreds of thousends of dinars for each salesman in each district is as below:

| | | District | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Slesman | A | 16 | 10 | 14 | 11 |
| | B | 14 | 11 | 15 | 15 |
| | C | 15 | 15 | 13 | 12 |
| | D | 13 | 12 | 14 | 15 |

Find the assignment of salesmen to various districts which will yield maximum profit.

**Solution:**

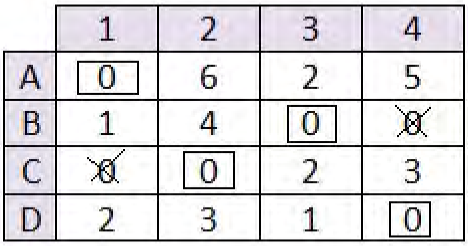
As the given problem is of a maximization type, it has to be changed to minimization type before solving it by the Hungarian method. This is achieved by subtracting all the elements of the matrix from the largest element (16), the equivalent matrix is:

| | | District | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 |
| Slesman | A | 0 | 6 | 2 | 5 |
| | B | 2 | 5 | 1 | 1 |
| | C | 1 | 1 | 3 | 4 |
| | D | 3 | 4 | 2 | 1 |

The Hungarian method can now be applied, the reduced matrix is:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | 0 | 6 | 2 | 5 |
| B | 1 | 4 | 0 | 0 |
| C | 0 | 0 | 2 | 3 |
| D | 2 | 3 | 1 | 0 |

The assignment is given in the following matrix:



| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| A | $\boxed{0}$ | 6 | 2 | 5 |
| B | 1 | 4 | $\boxed{0}$ | $\cancel{0}$ |
| C | $\cancel{0}$ | $\boxed{0}$ | 2 | 3 |
| D | 2 | 3 | 1 | $\boxed{0}$ |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

A is assigned to 1

B is assigned to 3

C is assigned to 2

D is assigned to 4

The maximum profit $= (16 + 15 + 15 + 15) \times 10^5 = 6100000 \text{ ID}.$

## Example (1.6):

Solve the following assignment problem for maximal optimal profit:

| | I | II | III | IV | V |
|---|---|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| 1 | 40 | 40 | 35 | 25 | 50 |
| 2 | 42 | 30 | 16 | 25 | 27 |
| 3 | 50 | 48 | 40 | 60 | 50 |
| 4 | 20 | 19 | 20 | 18 | 25 |
| 5 | 58 | 60 | 59 | 55 | 53 |
| 6 | 45 | 52 | 38 | 50 | 49 |

**Solution:**

Making the matrix a square matrix, then subtract all the elements of the matrix from the largest element ( 60). The following tables are obtained:

| | I | II | III | IV | V | d |
|---|---|---|---|---|---|---|
| 1 | 40 | 40 | 35 | 25 | 50 | 0 |
| 2 | 42 | 30 | 16 | 25 | 27 | 0 |
| 3 | 50 | 48 | 40 | 60 | 50 | 0 |
| 4 | 20 | 19 | 20 | 18 | 25 | 0 |
| 5 | 58 | 60 | 59 | 55 | 53 | 0 |
| 6 | 45 | 52 | 38 | 50 | 49 | 0 |

⇒

| | I | II | III | IV | V | d |
|---|---|---|---|---|---|---|
| 1 | 20 | 20 | 25 | 35 | 10 | 60 |
| 2 | 18 | 30 | 44 | 35 | 33 | 60 |
| 3 | 10 | 12 | 20 | 0 | 10 | 60 |
| 4 | 40 | 41 | 40 | 42 | 35 | 60 |
| 5 | 2 | 0 | 1 | 5 | 7 | 60 |
| 6 | 15 | 8 | 22 | 10 | 11 | 60 |

The reduced matrix is:

| | I | II | III | IV | V | d |
|---|---|---|---|---|---|---|
| 1 | 10 | 10 | 15 | 25 | 0 | 50 |
| 2 | 0 | 12 | 26 | 17 | 15 | 42 |
| 3 | 10 | 12 | 20 | 0 | 10 | 60 |
| 4 | 5 | 6 | 5 | 7 | 0 | 25 |
| 5 | 2 | 0 | 1 | 5 | 7 | 60 |
| 6 | 7 | 0 | 14 | 2 | 3 | 52 |

⇒

| | I | II | III | IV | V | d |
|---|---|---|---|---|---|---|
| 1 | 10 | 10 | 14 | 25 | 0 | 25 |
| 2 | 0 | 12 | 25 | 17 | 15 | 17 |
| 3 | 10 | 12 | 19 | 0 | 10 | 35 |
| 4 | 5 | 6 | 4 | 7 | 0 | 0 |
| 5 | 2 | 0 | 0 | 5 | 7 | 35 |
| 6 | 7 | 0 | 13 | 2 | 3 | 27 |

The assignment is given in the following matrix:

| | I | II | III | IV | V | d |
|---|---|---|---|---|---|---|
| 1 | 10 | 10 | 14 | 25 | $\boxed{0}$ | 25 |
| 2 | $\boxed{0}$ | 12 | 25 | 17 | 15 | 17 |
| 3 | 10 | 12 | 19 | $\boxed{0}$ | 10 | 35 |
| 4 | 5 | 6 | 4 | 7 | $\cancel{0}$ | $\boxed{0}$ |
| 5 | 2 | $\cancel{0}$ | $\boxed{0}$ | 5 | 7 | 35 |
| 6 | 7 | $\boxed{0}$ | 13 | 2 | 3 | 27 |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

1 is assigned to V
2 is assigned to I
3 is assigned to IV
5 is assigned to III
6 is assigned to II. 4 is not assigned.

The maximum profit $= 50 + 42 + 60 + 59 + 52 = 263$ units .

### 1.3.3 Restrictions on Assignment

Sometimes technical, space, legal or other restrictions do not permit the assignment of a particular facility to a particular job. Such problems can be solved by assigning a very heavy cost ( infinite cost) to the corresponding cell. Such a job will then be automatically excluded from further consideration (making assignment).

### Example (1.7):

Four new machines $M_1$, $M_2$, $M_3$ and $M_4$ are to be placed in a machine shop. There are five vacant places A, B, C, D and E available. Because of the limited space, machine $M_2$ cannot be placed at C and $M_3$ cannot be placed at A.The assignment cost of machine $i$ to place $j$ in thousands of dolars is shown below:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | 4 | 6 | 10 | 5 | 6 |
| $M_2$ | 7 | 4 | -- | 5 | 4 |
| $M_3$ | -- | 6 | 9 | 6 | 2 |
| $M_4$ | 9 | 3 | 7 | 2 | 3 |

Find the optimal assignment schedule.

**Solution:**

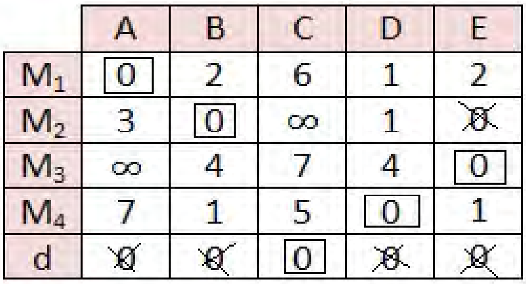
As the given matrix is non-square, we add a dummy machine and associate zero cost with the corresponding cells. As machine $M_2$ cannot be placed at C and $M_3$ cannot be placed at A, we assign infinite cost ($\infty$ ) in cells ($M_2$,C) and ($M_3$,A), resulting the following matrix:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | 4 | 6 | 10 | 5 | 6 |
| $M_2$ | 7 | 4 | $\infty$ | 5 | 4 |
| $M_3$ | $\infty$ | 6 | 9 | 6 | 2 |
| $M_4$ | 9 | 3 | 7 | 2 | 3 |
| d | 0 | 0 | 0 | 0 | 0 |

The reduced matrix is:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | 0 | 2 | 6 | 1 | 2 |
| $M_2$ | 3 | 0 | $\infty$ | 1 | 0 |
| $M_3$ | $\infty$ | 4 | 7 | 4 | 0 |
| $M_4$ | 7 | 1 | 5 | 0 | 1 |
| d | 0 | 0 | 0 | 0 | 0 |

The assignment is given in the following matrix:



| | A | B | C | D | E |
|---|---|---|---|---|---|
| $M_1$ | $\boxed{0}$ | 2 | 6 | 1 | 2 |
| $M_2$ | 3 | $\boxed{0}$ | $\infty$ | 1 | $\cancel{0}$ |
| $M_3$ | $\infty$ | 4 | 7 | 4 | $\boxed{0}$ |
| $M_4$ | 7 | 1 | 5 | $\boxed{0}$ | 1 |
| d | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ |

Since there is one assignment in each row and in each column, the optimal assignment can be made in the current solution. The optimal assignment is:

$M_1$ is assigned to place A
$M_2$ is assigned to place B
$M_3$ is assigned to place E
$M_4$ is assigned to place D

There is no machine assigned to place C.

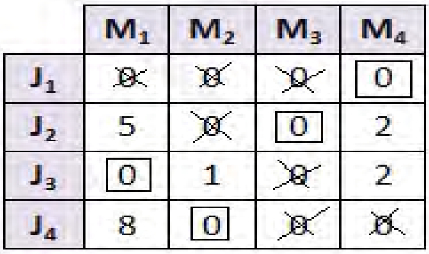
The assignment cost $= (4+4+2+2)\times 1000 = 12000$ \$ .

### 1.3.4 Alternate Optimal Solutions

Sometimes, it is possible to have two or more ways to strike off all zero elements in the reduced matrix for a given problem. In such cases, there will be alternate optimal solutions with the same cost.Alternate optimal solutions offer a great flexibility to the management since it can select the one which is more suitable to its requirement.

### Example (1.8):

Recall example (1.2), the optimal solution obtained is not unique. For example, we can make the following assignment:



| | $M_1$ | $M_2$ | $M_3$ | $M_4$ |
|---|---|---|---|---|
| $J_1$ | $\cancel{0}$ | $\cancel{0}$ | $\cancel{0}$ | $\boxed{0}$ |
| $J_2$ | 5 | $\cancel{0}$ | $\boxed{0}$ | 2 |
| $J_3$ | $\boxed{0}$ | 1 | $\cancel{0}$ | 2 |
| $J_4$ | 8 | $\boxed{0}$ | $\cancel{0}$ | $\cancel{0}$ |

Without change in the optimal cost (23$).

## Exercises 1 (In addition to the text book exercises)

Find the optimal assignment for the following:

| | I | II | III | IV | V |
|---|---|---|---|---|---|
| 1 | 11 | 17 | 8 | 16 | 20 |
| 2 | 9 | 7 | 12 | 6 | 15 |
| 3 | 13 | 16 | 15 | 12 | 16 |
| 4 | 21 | 24 | 17 | 28 | 26 |
| 5 | 14 | 10 | 12 | 11 | 15 |

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 6 | 5 | 1 | 6 |
| 2 | 2 | 5 | 3 | 7 |
| 3 | 3 | 7 | 2 | 8 |
| 4 | 7 | 7 | 5 | 9 |
| 5 | 12 | 8 | 8 | 6 |
| 6 | 6 | 9 | 5 | 10 |

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| A | 19 | 15 | -- | 16 | 13 | 22 |
| B | 13 | -- | 15 | -- | 21 | 14 |
| C | 15 | 17 | 19 | 20 | 12 | 18 |
| D | 20 | 22 | 16 | 18 | 17 | -- |
| E | -- | 16 | 14 | 19 | 18 | 15 |

Find the optimal assignment for the following assignment problem to maximize the profit.

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | 5 | 11 | 10 | 12 | 4 |
| B | 2 | 4 | 6 | 3 | 5 |
| C | 3 | 12 | 5 | 14 | 6 |
| D | 6 | 14 | 4 | 11 | 7 |
| E | 7 | 9 | 8 | 12 | 5 |

# Ch. 2: Game Theory

The theory of games (or game theory or competitive strategies) is a mathematical theory that deals with the general feature of competitive situations. This theory is helpful when two or more opponents (individuals, companies,… etc.) with conflicting objectives try to make decision. In such situations, a decision made by one decision-maker affects the decision made by one or more of the remaining decision-makers and the final outcome depend on the decision of all parties.

The game theory is based on the ***minimax principle*** put forward by J. von Neuman (1903-1957) which implies that each competitor will act so as to minimize his maximum loss (or maximize his minimum gain) or achieve the best of the worst. The theory does not describe how a game should be played; it describes only the procedure and principles by which plays should be selected.

## 2.1 Characteristics of the Game

A competitive game has the following characteristics:

a) There is finite number of participants or competitors. If the number of participants is 2, the game is called ***two- person game***; for number greater than two, it is called ***n-person game***.

b) Each participant has a list of finite number of possible courses of actions available to him. The list may not be the same for each participant.

c) Each participant knows all the possible choices available to others but does not know which of them is going to be chosen by them.

d) A ***play*** is said to occur when each of the participants chooses one of the courses of actions available to him. The choices are assumed to be made simultaneously so that no participant knows the choices made by others until he has decided his own.

e) Every combination of courses of actions determines an outcome which results in gains of the participants. The gain (***payoff***) may be positive, negative or zero. Negative gain is called ***loss***.

f) The gain of a participant depends not only on his own actions but also on those of others.

g) The gains of each and every play are fixed and specified in advance and are known to each player.

h) The players make individual decisions without direct communication.

## 2.2 Definitions

**Definition (2.1):**

A ***game*** is an activity between two or more persons, involving action by each one of them according to a set of rules which results in some gain ( +ve, −ve or zero) for each.

**Definition (2.2):**

Each participant or competitor playing a game is called a ***player***.

**Definition (2.3):**

A ***strategy*** is a predetermined rule by which a player decides his course of action from his list of courses of actions during the game. To decide a particular strategy the player needs to know the other's strategy.

**Definition (2.4):**

A ***pure strategy*** is the decision rule to always select a particular course of action.

**Definition (2.5):**

***Mixed strategy*** is the decision, in advance of all plays, to choose a course of action for each play in accordance with some probability distribution. Thus, a mixed strategy is a selection among pure strategies with some fixed probabilities.

**Definition (2.6):**

The strategy that puts the player in the most preferred position irrespective of the strategy of his opponents is called an ***optimal strategy***. Any deviation from this strategy would reduce his payoff.

**Definition (2.7):**

***Zero-sum game*** is a game in which the sum of payments to all the players, after the play of the game, is zero. In such a game, the gain of players that win is exactly equal the loss of players that lose.

**Definition (2.8):**

***Two-person zero-sum game*** is a game involving only two players in which the gain of one player equals the loss of the other. It is also called a ***rectangular game*** or ***matrix game*** because the payoff matrix is rectangular in form.

**Definition (2.9):**

A ***nonzero- game*** is a game in which a third party receives or makes some payment.

### Definition (2.10):

***Payoff*** (***gain*** or ***game***) ***matrix*** is the table showing the amounts received by the player named at the left-hand-side after all possible plays of the game. The payment is made by the player named at the top of the table.

In a two-person zero-sum game, the cell entries in B's payoff matrix will be the negative of the corresponding cell entries in A's payoff matrix. A is called ***maximizing player*** as he would try to maximize his gains, while B is called ***minimizing player*** as he would try to minimize his losses.

**Player B**

| **Player A** | **1** | **2** | **…** | ***j*** | **…** | ***n*** |
|---|---|---|---|---|---|---|
| **1** | $a_{11}$ | $a_{12}$ | … | $a_{1j}$ | … | $a_{1n}$ |
| **2** | $a_{21}$ | $a_{22}$ | … | $a_{2j}$ | … | $a_{2n}$ |
| **⋮** | ⋮ | ⋮ | | ⋮ | | ⋮ |
| ***i*** | $a_{i1}$ | $a_{i2}$ | … | $a_{ij}$ | … | $a_{in}$ |
| **⋮** | ⋮ | ⋮ | | ⋮ | | ⋮ |
| ***m*** | $a_{m1}$ | $a_{m2}$ | … | $a_{mj}$ | … | $a_{mn}$ |

**A's payoff matrix**

**Player B**

| **Player A** | 1 | 2 | … | $j$ | … | $n$ |
|---|---|---|---|---|---|---|
| 1 | $-a_{11}$ | $-a_{12}$ | … | $-a_{1j}$ | … | $-a_{1n}$ |
| 2 | $-a_{21}$ | $-a_{22}$ | … | $-a_{2j}$ | … | $-a_{2n}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ | | ⋮ |
| $i$ | $-a_{i1}$ | $-a_{i2}$ | … | $-a_{ij}$ | … | $-a_{in}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ | | ⋮ |
| $m$ | $-a_{m1}$ | $-a_{m2}$ | … | $-a_{mj}$ | … | $-a_{mn}$ |

**B's payoff matrix**

Thus the sum of payoff matrices for A and B is a null matrix. Then, we shall usually omit B's payoff matrix; keeping in mind that it is just the negative of A's payoff matrix. That is if $a_{ij} > 0$, it is a gain for player A, $a_{ij} < 0$, it is a gain for player B, $a_{ij} = 0$, players gain nothing.

## 2.3 Rule 1: Look for a Pure Strategy (Saddle Point)

The steps required to detect a saddle point:

1) At the right of each row, write the row minimum and ring the largest of them (***maximin***).
2) At the bottom of each column, write the column maximum and ring the smallest of them (***minimax***).
3) If minimax = maximin, the cell where the corresponding row and column meet is a ***saddle point*** (***equilibrium point***) and the element in that cell is the value of the game, the game is called ***stable game***.
4) If minimax ≠ maximin , there is no saddle point and the value of the game lies between these two values.
5) If there are more than one saddle points then there will be more than one solution, each solution corresponding to each saddle point.

### Example (2.1):

In a game of matching coins, the payoff matrix is given in the following table. Determine the best strategies for each player and the value of the game>

| | | | |
|---|---|---|---|
| | | B | |
| | | H | T |
| A | H | 0 | 5 |
| | T | −2 | 0 |

**Solution:**

First, we search for a saddle point:

| | | | | |
|---|---|---|---|---|
| | | B | | |
| | | H | T | min |
| A | H | 0 | 5 | 0 |
| | T | −2 | 0 | -2 |
| | max | 0 | 5 | |

Minimax=0, maximin=0. Since minimax=maximin, then there is a saddle point (1,1)[means first strategy of A and first strategy of B].

Optimal strategy for player A :( 1, 0)

Optimal strategy for player B :( 1, 0)

The value of the game V=0

### Example (2.2):

Does the following game have a saddle point?

| | | | | |
|---|---|---|---|---|
| | | | B | |
| | | $B_1$ | $B_2$ | $B_3$ |
| A | $A_1$ | 0 | 7 | 6 |
| | $A_2$ | 3 | 12 | 1 |

**Solution:**

| | | | | | |
|---|---|---|---|---|---|
| | | | B | | |
| | | $B_1$ | $B_2$ | $B_3$ | min |
| A | $A_1$ | 0 | 7 | 6 | 0 |
| | $A_2$ | 3 | 12 | 1 | 1 |
| | max | 3 | 12 | 6 | |

Minimax=3, maximin=1. Since minimax≠maximin, then there is no saddle point.

## 2.4 Rule 2: Reduce the Game

If no pure strategy exists, the next step is to eliminate certain strategies (rows and/or columns) by dominance. The resulting game can be solved for some mixed strategy. The ***dominance rules*** are:

***For rows:*** The row $i$ dominating row $k$ if : $a_{ij} \geq a_{kj}\, , j = 1, \dots, n.$

***For columns:*** The column $j$ dominating column $k$ if : $a_{ij} \leq a_{ik}\, , i = 1, \dots, m.$

## Example (2.3):

Two players P and Q play a game. Each of them has to choose one of the three colors, white (W), black (B), and red (R) independently of the other. Thereafter the colors are compared. If both P and Q have chosen white (W,W), neither wins anything. The payoff matrix is shown below. Does the game have a saddle point? If not reduce the game.

$$\begin{array}{cc|c|c|c|}
 & & & P & \\
 & & W & B & R \\
\hline
 & W & 0 & -2 & 7 \\
\hline
Q & B & 2 & 5 & 6 \\
\hline
 & R & 3 & -3 & 8 \\
\hline
\end{array}$$

## Solution:

$$\begin{array}{cc|c|c|c|c}
 & & & P & & \\
 & & W & B & R & \text{min} \\
\cline{3-5}
 & W & 0 & -2 & 7 & -2 \\
\cline{3-5}
Q & B & 2 & 5 & 6 & \textcircled{2} \\
\cline{3-5}
 & R & 3 & -3 & 8 & -3 \\
\cline{3-5}
 & \text{max} & \textcircled{3} & 5 & 8 &
\end{array}$$

Minimax=3, maximin=2. Since minimax≠maximin, then there is no saddle point. $2 \le V \le 3$. To reduce the matrix: the first column dominating the third column ($0 < 7, 2 < 6, 3 < 8$ ). The resulting matrix is:

$$\begin{array}{cc|c|c|}
 & & P & \\
 & & W & B \\
\hline
 & W & 0 & -2 \\
\hline
Q & B & 2 & 5 \\
\hline
 & R & 3 & -3 \\
\hline
\end{array}$$

The second row dominating the first row ($2 > 0, 5 > -2$ ). The resulting matrix is:

$$\begin{array}{cc|c|c|}
 & & P & \\
 & & W & B \\
\hline
Q & B & 2 & 5 \\
\hline
 & R & 3 & -3 \\
\hline
\end{array}$$

## Remark (2.1)

Sometimes the previous dominance rules are not useful; in this case we can use the ***average rule***:

***For rows:*** The rows $i$ and $k$ dominating row $h$ if every element in the average of rows $i$ and $k$ is greater than or equal the corresponding element in row $h$.

***For columns:*** The columns $j$ and $k$ dominating column $h$ if every element in the average of columns $j$ and $k$ is smaller than or equal the corresponding element in column $h$.

### Example (2.4):

Consider the following game:

| | | B | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| | **1** | 6 | 1 | 3 |
| **A** | **2** | 0 | 9 | 7 |
| | **3** | 2 | 3 | 4 |

This game has no saddle point, since:

| | | B | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | min |
| | **1** | 6 | 1 | 3 | 1 |
| **A** | **2** | 0 | 9 | 7 | 0 |
| | **3** | 2 | 3 | 4 | ② |
| | max | ⑥ | 9 | 7 | |

Minimax=6, maximin=2, minimax≠maximin. $2 \le V \le 6$. The game cannot be reduced by dominance rules. The average of A's first and second strategy is: $\left(\frac{6+0}{2},\frac{1+9}{2},\frac{3+7}{2}\right) = (3{,}5{,}5)$ . By comparing each element in the average with the corresponding element in the third row: $3 > 2,\ 5 > 3, 5 > 4$. The resulting matrix will be:

| | | B | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| **A** | **1** | 6 | 1 | 3 |
| | **2** | 0 | 9 | 7 |

## 2.5 Rule 3: Solve for a Mixed Strategy

In case where there is no saddle point and dominance has been used to reduce the game matrix, players will use mixed strategies. Such games are called ***unstable games***.

## 2.6 Mixed Strategies for 2 x 2 Games

### 2.6.1 Arithmetic method ( Odds Method)

It provides an easy method for finding the optimum strategies for each player in a 2 x 2 game without a saddle point. It consists of the following steps:

i) Subtract the two digits in column 1 and write the difference under column 2, ignoring sign.
ii) Subtract the two digits in column 2 and write the difference under column 1, ignoring sign.
iii) Similarly proceed for the two rows, write the results to the right of each row.

These values are called oddments.

iv) To find the frequency (probability) in which the players must use their courses of action in their optimum strategy, divide the oddment of each player on the sum of his oddments.
v) The value of the game result by multiplying the elements of a row or column by the probabilities corresponding to these elements.

## Example (2.5):

Consider the game in example (2.3), find the optimal strategy for each player and the value of the game.

**Solution:**

The game is reduced to a 2 x 2 game which we must check the existence of a saddle point:

| | | | | |
|---|---|---|---|---|
| | | **P** | | |
| | | **W** | **B** | min |
| **Q** | **B** | 2 | 5 | ② |
| | **R** | 3 | −3 | −3 |
| | max | ③ | 5 | |

Minimax=3, maximin=2. Since minimax≠maximin, then there is no saddle point and $2 \le V \le 3$. Then:

| | | | | | |
|---|---|---|---|---|---|
| | | **P** | | | |
| | | **W** | **B** | | |
| **Q** | **B** | 2 | 5 | 6 | 6/9 |
| | **R** | 3 | −3 | 3 | 3/9 |
| | | 8 | 1 | | |
| | | 8/9 | 1/9 | | |

Optimal strategy for player P is : (8/9, 1/9, 0)

Optimal strategy for player Q is : (0, 6/9, 3/9)

To obtain the value of the game:

By using Q's oddments:

V= $\frac{2\times6+3\times3}{9} = 21/9$ when Q plays B

V= $\frac{5\times6-3\times3}{9}=21/9$ when Q plays R

By using P’s oddments:

V= $\frac{2\times8+5\times1}{9}=21/9$ when P plays W

V= $\frac{3\times8-3\times1}{9}=21/9$ when P plays B

## Remark (2.2)

The above values of V are equal only if sum of the oddments vertically and horizontally are equal.

## Example (2.6):

In a game of matching coins, the payoff matrix is given in the following table. Determine the best strategies for each player and the value of the game>

| | | B | |
|---|---|---|---|
| | | **H** | **T** |
| **A** | **H** | 2 | −1 |
| | **T** | −1 | 0 |

### Solution:

First, we search for a saddle point:

| | | B | | |
|---|---|---|---|---|
| | | **H** | **T** | min |
| **A** | **H** | 2 | −1 | −1 |
| | **T** | −1 | 0 | −1 |
| | max | 2 | 0 | |

Minimax=0, maximin= −1. Since minimax≠ maximin, then there is no saddle point and $-1\le V\le 0$.

| | | B | | | |
|---|---|---|---|---|---|
| | | **H** | **T** | | |
| **A** | **H** | 2 | −1 | 1 | 1/4 |
| | **T** | −1 | 0 | 3 | 3/4 |
| | | 1 | 3 | | |
| | | 1/4 | 3/4 | | |

Optimal strategy for player A is : (1/4, 3/4)

Optimal strategy for player B is : (1/4, 3/4)

V=$\frac{2\times1-1\times3}{4}=-1/4$, that is B is the winner.

## Example (2.7):

Find the optimal strategy of each player and the value of the following game:

|  |  | B |  |  |  |
|---|---|---|---|---|---|
|  |  | I | II | III | IV |
| A | 1 | 3 | 2 | 4 | 0 |
|  | 2 | 3 | 4 | 2 | 4 |
|  | 3 | 4 | 2 | 4 | 0 |
|  | 4 | 0 | 4 | 0 | 8 |

**Solution:**

|  |  | B |  |  |  |  |
|---|---|---|---|---|---|---|
|  |  | I | II | III | IV | min |
| A | 1 | 3 | 2 | 4 | 0 | 0 |
|  | 2 | 3 | 4 | 2 | 4 | 2 |
|  | 3 | 4 | 2 | 4 | 0 | 0 |
|  | 4 | 0 | 4 | 0 | 8 | 0 |
|  | max | 4 | 4 | 4 | 8 |  |

Minimax=4, maximin=2. Since minimax$\neq$ maximin, then there is no saddle point and $2 \leq V \leq 4$. Then we try to reduce the matrix:

|  |  | B |  |  |  |
|---|---|---|---|---|---|
|  |  | I | II | III | IV |
| A | 1 | 3 | 2 | 4 | 0 |
|  | 2 | 3 | 4 | 2 | 4 |
|  | 3 | 4 | 2 | 4 | 0 |
|  | 4 | 0 | 4 | 0 | 8 |

$\xRightarrow{R1\ vs\ R3}$

|  |  | B |  |  |  |
|---|---|---|---|---|---|
|  |  | I | II | III | IV |
| A | 2 | 3 | 4 | 2 | 4 |
|  | 3 | 4 | 2 | 4 | 0 |
|  | 4 | 0 | 4 | 0 | 8 |

No saddle point

$\xRightarrow{CI\ vs\ CIII}$

|  |  | B |  |  |
|---|---|---|---|---|
|  |  | II | III | IV |
| A | 2 | 4 | 2 | 4 |
|  | 3 | 2 | 4 | 0 |
|  | 4 | 4 | 0 | 8 |

No saddle point

$\xRightarrow{CII\ vs\ (CIII+CIV)/2}$

|  |  | B |  |
|---|---|---|---|
|  |  | III | IV |
| A | 2 | 2 | 4 |
|  | 3 | 4 | 0 |
|  | 4 | 0 | 8 |

No saddle point

$\xRightarrow{R2\ vs\ (R3+R4)/2}$

|  |  | B |  |
|---|---|---|---|
|  |  | III | IV |
| A | 3 | 4 | 0 |
|  | 4 | 0 | 8 |

The last matrix has no saddle point ( maximin=0, minimax=4), then:

|  |  | **B** |  |  |  |
|---|---|---|---|---|---|
|  |  | **III** | **IV** |  |  |
| **A** | **3** | 4 | 0 | 8 | 2/3 |
|  | **4** | 0 | 8 | 4 | 1/3 |
|  |  | 8 | 4 |  |  |
|  |  | 2/3 | 1/3 |  |  |

Optimal strategy for player A is: ( 0, 0, 2/3,1/3)

Optimal strategy for player B is: ( 0, 0, 2/3,1/3)

V= $\frac{4\times2+0\times1}{3}=8/3$

## Example (2.8):

Reduce the following game and find the optimal strategy of each player and the value of the following game:

|  |  | **B** |  |  |  |  |
|---|---|---|---|---|---|---|
|  |  | **1** | **2** | **3** | **4** | **5** |
| **A** | **I** | 1 | 3 | 2 | 7 | 4 |
|  | **II** | 3 | 4 | 1 | 5 | 6 |
|  | **III** | 6 | 5 | 7 | 6 | 5 |
|  | **IV** | 2 | 0 | 6 | 3 | 1 |

**Solution:**

$\xRightarrow{RIV\ vs\ RIII}$

|  |  | **B** |  |  |  |  |
|---|---|---|---|---|---|---|
|  |  | **1** | **2** | **3** | **4** | **5** |
| **A** | **I** | 1 | 3 | 2 | 7 | 4 |
|  | **II** | 3 | 4 | 1 | 5 | 6 |
|  | **III** | 6 | 5 | 7 | 6 | 5 |

$\xRightarrow{\substack{C4\ vs\ C2\\ C5\ vs\ C2}}$

|  |  | **B** |  |  |
|---|---|---|---|---|
|  |  | **1** | **2** | **3** |
| **A** | **I** | 1 | 3 | 2 |
|  | **II** | 3 | 4 | 1 |
|  | **III** | 6 | 5 | 7 |

$\xRightarrow{\substack{RI\ vs\ RIII\\ RII\ vs\ RIII}}$

|  |  | B |  |  |
|---|---|---|---|---|
|  |  | 1 | 2 | 3 |
| **A** | **III** | 6 | 5 | 7 |

Optimal strategy for player A is: ( 0, 0, 1,0)

Optimal strategy for player B is: ( 0, 1, 0,0,0) [ B must play strategy 2 in order to minimize his losses]

V=5

## Example (2.9):

A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in the following table represent wages increase while negative sign represents wage reduction. What are the optimal strategies for the company as well as the union and what is the value of the game?

| | | Union strategies | | | |
|---|---|---|---|---|---|
| | | $U_1$ | $U_2$ | $U_3$ | $U_4$ |
| Company strategies | $C_1$ | +0.25 | +0.27 | +0.35 | −0.02 |
| | $C_2$ | +0.20 | +0.16 | +0.08 | +0.08 |
| | $C_3$ | +0.14 | +0.12 | +0.15 | +0.13 |
| | $C_4$ | +0.30 | +0.14 | +0.19 | +0.00 |

## Solution:

Since in a game matrix, player to its left is a maximizing player and the one at the top is a minimizing player, the above table is transposed and rewritten as the following table since company's interest is to minimize the wage increase while union's interest is to get the maximum wage increase.

| | | Company strategies | | | |
|---|---|---|---|---|---|
| | | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
| Union strategies | $U_1$ | 0.25 | 0.2 | 0.14 | 0.3 |
| | $U_2$ | 0.27 | 0.16 | 0.12 | 0.14 |
| | $U_3$ | 0.35 | 0.08 | 0.15 | 0.19 |
| | $U_4$ | −0.02 | 0.08 | 0.13 | 0.00 |

First, we must look for a saddle point:

| | | Company strategies | | | | |
|---|---|---|---|---|---|---|
| | | $C_1$ | $C_2$ | $C_3$ | $C_4$ | min |
| Union strategies | $U_1$ | 0.25 | 0.2 | 0.14 | 0.3 | 0.14 |
| | $U_2$ | 0.27 | 0.16 | 0.12 | 0.14 | 0.12 |
| | $U_3$ | 0.35 | 0.08 | 0.15 | 0.19 | 0.08 |
| | $U_4$ | −0.02 | 0.08 | 0.13 | 0.00 | −0.02 |
| | max | 0.35 | 0.2 | 0.15 | 0.3 | |

Maximin=0.14, minimax=0.15, since maximin $\neq$ minimax, then there is no saddle point and $0.14 \leq V \leq 0.15$

$\overset{U_3 vs. U_4}{\Longrightarrow}$

| | | Company strategies | | | |
|---|---|---|---|---|---|
| | | $C_1$ | $C_2$ | $C_3$ | $C_4$ |
| Union strategies | $U_1$ | 0.25 | 0.2 | 0.14 | 0.3 |
| | $U_2$ | 0.27 | 0.16 | 0.12 | 0.14 |
| | $U_3$ | 0.35 | 0.08 | 0.15 | 0.19 |

There is no saddle point

$\overset{\substack{C_1 vs. C_2 \\ C_4 vs. C_3}}{\Longrightarrow}$

| | | Company strategies | |
|---|---|---|---|
| | | $C_2$ | $C_3$ |
| Union strategies | $U_1$ | 0.2 | 0.14 |
| | $U_2$ | 0.16 | 0.12 |
| | $U_3$ | 0.08 | 0.15 |

There is no saddle point

$\xRightarrow{U_2 vs. U_1}$

| | | Company strategies | | |
|---|---|---|---|---|
| | | **C₂** | **C₃** | min |
| **Union strategies** | **U₁** | 0.2 | 0.14 | 0.14 |
| | **U₃** | 0.08 | 0.15 | 0.08 |
| | max | 0.2 | 0.15 | |

There is no saddle point.

| | | Company strategies | | | |
|---|---|---|---|---|---|
| | | **C₂** | **C₃** | | |
| **Union strategies** | **U₁** | 0.2 | 0.14 | 0.07 | 7/13 |
| | **U₃** | 0.08 | 0.15 | 0.06 | 6/13 |
| | | 0.01 | 0.12 | | |
| | | 1/13 | 12/13 | | |

Optimal strategy for the company: (0, 1/13, 12/13,0)

Optimal strategy for the union: (7/13, 0, 6/13, 0)

The value of the game is V= $\frac{0.2*7+0.08*6}{13}=\frac{1.88}{13}=0.145$

## 2.6.2 Algebraic Method for Finding Optimum Strategies and Game Value

Consider the following 2 x 2 game:

| | | B | | |
|---|---|---|---|---|
| | | **B₁** | **B₂** | |
| **A** | **A₁** | a | b | $x$ |
| | **A₂** | c | d | $1-x$ |
| | | $y$ | $1-y$ | |

While applying this method it is assumed that $x$ represents the fraction of time (frequency) for which player A uses strategy 1 and $(1-x)$ represents the fraction of time (frequency) for which player A uses strategy 2. Then the value of the game:

$$V = a*x+c*(1-x)=b*x+d*(1-x)$$

Solve these equations to find the value of $x$. Similarly $y$ and $(1-y)$ represents the fraction of time (frequency) for which player B uses strategies 1 and 2 respectively. Then the value of the game:

$$V = a*y+b*(1-y)=c*y+d*(1-y)$$

Solve these equations to find the value of $y$.

### Example (2.10):

Two armies are at war. Army A has two airbases, one of which is thrice as valuable as the other. Army B can destroy an undefended airbase, but it can destroy only one of them. Army A can also defend only one of them. Find the best strategy for A to minimize his losses and find the optimal strategy for B.

**Solution:**

Since both armies have only two possible courses of action, the gain matrix for the game is:

| | | | Army A | |
|---|---|---|---|---|
| | | | **1 Defend the smaller airbase** | **2 Defend the larger airbase** |
| **Army B** | **1** | **Attack the smaller airbase** | 0 | 1 |
| | **2** | **Attack the larger airbase** | 3 | 0 |

First, we check for the existence of a saddle point:

| | | Army A | | |
|---|---|---|---|---|
| | | **1** | **2** | **min** |
| **Army B** | **1** | 0 | 1 | 0 |
| | **2** | 3 | 0 | 0 |
| **max** | | 3 | 1 | |

Maximin=0, minimax=1. Since minimax $\neq$ maximin, then there is no saddle point and $0 \leq V \leq 1$.

Let $x$ and $(1-x)$ represents the fraction of time (frequency) for which player B uses strategies 1 and 2 respectively. Then the value of the game:

$$V = 0 \times x + 3 \times (1-x) = 1 \times x + 0 \times (1-x)$$

$$\Rightarrow 3 - 3x = x \Rightarrow 3 = 4x \ \Rightarrow x = \frac{3}{4} \Rightarrow 1 - x = \frac{1}{4}$$

Similarly let $y$ and $(1-y)$ represents the fraction of time (frequency) for which player A uses strategies 1 and 2 respectively. Then the value of the game:

$$V = 0 \times y + 1 \times (1-y) = 3 \times y + 0 \times (1-y)$$

$$\Rightarrow 1 - y = 3y \Rightarrow 1 = 4y \ \Rightarrow y = \frac{1}{4} \Rightarrow 1 - y = \frac{3}{4}$$

The optimal strategy for player A : (1/4, 3/4)

The optimal strategy for player B : (3/4, 1/4)

The value of the game $V = 0 \times \frac{1}{4} + 1 \times \frac{3}{4} = \frac{3}{4}$

## Exercise 2.1 (in addition to text book exercises)

Find the optimum strategies for each player and the value of the games:

1-

| | | B | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| **A** | **1** | −1 | −2 | 8 |
| | **2** | 7 | 5 | −1 |
| | **3** | 6 | 0 | 12 |

2- Two breakfast food manufacturers, A and B are competing for an increased market share. The payoff matrix, represented in the following table, shows the increase in market share for A and decrease in in market share for B:

| | | B | | | |
|---|---|---|---|---|---|
| | | **Give gifts** | **Decrease price** | **Maintain present strategy** | **Increase advertising** |
| **A** | **Give gifts** | 2 | −2 | 4 | 1 |
| | **Decrease price** | 6 | 1 | 12 | 3 |
| | **Maintain present strategy** | −3 | 2 | 0 | 6 |
| | **Increase advertising** | 2 | −3 | 7 | 1 |

Find the optimal strategies for both manufacturers and the value of the game.

## 2.7 Mixed Strategies for 2 x $n$ or $m$ x 2 Games

These are games in which one player has only two courses of action open to him while his opponent may have any number. If the game has no saddle point and cannot be reduced to a 2 x 2 game, it can be still solved by method of subgames or graphical method.

### 2.7.1 Method of Subgames for 2 x $n$ or $m$ x 2 Games

This method subdivides the given 2 x $n$ or $m$ x 2 game into a number of 2 x 2 games, each of which is then solved and then the optimal strategies are determined. If $k = n$ (for 2 x $n$ games) or $k = m$ (for $m$ x 2 games), then the number of subgames is: $\frac{k!}{2!(k-2)!}$ .

### Example (2.11):

Find the optimal strategy for each player and the value of the following game:

| | | B | | |
|---|---|---|---|---|
| | | **1** | **2** | **3** |
| **A** | **1** | 275 | −50 | −75 |
| | **2** | 125 | 130 | 150 |

**Solution:**

First we search for a saddle point:

| | | B | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **min** |
| **A** | **1** | 275 | −50 | −75 | −75 |
| | **2** | 125 | 130 | 150 | 125 |
| | **max** | 275 | 130 | 150 | |

There is no saddle point and $125 \leq V \leq 130$. The game cannot be reduced. This game can be thought as three 2 x 2 games.

**Subgame 1:**

| | | | | |
|---|---|---|---|---|
| | | **B** | | |
| | | **1** | **2** | **min** |
| **A** | **1** | 275 | −50 | −50 |
| | **2** | 125 | 130 | 125 |
| | **max** | 275 | 130 | |

There is no saddle point, then:

| | | | | | |
|---|---|---|---|---|---|
| | | **B** | | | |
| | | **1** | **2** | | |
| **A** | **1** | 275 | −50 | 5 | 1/66 |
| | **2** | 125 | 130 | 325 | 65/66 |
| | | 180 | 150 | | |
| | | 36/66 | 30/66 | | |

The strategy for A: (1/66, 65/66)

The strategy for B: (36/66, 30/66, 0)

The value of the game: $V = \frac{275\times1+125\times65}{66} = 127.3$

**Subgame 2:**

| | | | | |
|---|---|---|---|---|
| | | **B** | | |
| | | **1** | **3** | **min** |
| **A** | **1** | 275 | −75 | −75 |
| | **2** | 125 | 150 | 125 |
| | **max** | 275 | 150 | |

There is no saddle point, then:

| | | | | | |
|---|---|---|---|---|---|
| | | **B** | | | |
| | | **1** | **3** | | |
| **A** | **1** | 275 | −75 | 25 | 1/15 |
| | **2** | 125 | 150 | 350 | 14/15 |
| | | 225 | 150 | | |
| | | 9/15 | 6/15 | | |

The strategy for A: (1/15, 14/15)

The strategy for B: (9/15, 0, 6/15)

The value of the game: $V = \frac{275\times1+125\times14}{15} = 135$

## Subgame 3:

| | | B | | |
|---|---|---|---|---|
| | | 2 | 3 | min |
| A | 1 | −50 | −75 | −75 |
| | 2 | 130 | 150 | 130 |
| | max | 130 | 150 | |

There is a saddle point (2, 2), thus:

The strategy for A: (0, 1)

The strategy for B: (0, 1, 0)

The value of the game: $V = 130$

Since player B has the flexibility to play any two of the courses of action available to him, he will play those strategies for which his loss is minimum. AS the value of all subgames are positive, player A is the winner. Hence Player B will play subgame 1 for which the loss is minimum, i.e. 127.3. The complete solution of the problem is:

The optimal strategy for A: (1/66, 65/66)

The optimal strategy for B: (36/66, 30/66, 0)

The value of the game: $V = 127.3$

## 2.7.2 Graphical Method for 2 x $n$ or $m$ x 2 Games

Graphical method is applicable to only those games in which one of the players has two strategies only. The advantage of this method is that it is relatively fast. It reduces the 2 x $n$ or $m$ x 2 game to 2 x 2 game and the game can then be solved by the methods discussed earlier. The resulting solution is also the solution of the original problem.

## Example (2.12):

Solve the game given in the following table:

| | | B | | | |
|---|---|---|---|---|---|
| | | B₁ | B₂ | B₃ | B₄ |
| A | A₁ | 19 | 6 | 7 | 5 |
| | A₂ | 7 | 3 | 14 | 6 |
| | A₃ | 12 | 8 | 18 | 4 |
| | A₄ | 8 | 7 | 13 | −1 |

### Solution:

First, we must search for a saddle point:

$$\begin{array}{cc|c|c|c|c|c}
 & & & & \mathbf{B} & & \\
 & & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \mathbf{B_4} & \mathbf{min} \\
\hline
 & \mathbf{A_1} & 19 & 6 & 7 & 5 & \boxed{5} \\
\hline
\mathbf{A} & \mathbf{A_2} & 7 & 3 & 14 & 6 & 3 \\
\hline
 & \mathbf{A_3} & 12 & 8 & 18 & 4 & 4 \\
\hline
 & \mathbf{A_4} & 8 & 7 & 13 & -1 & -1 \\
\hline
 & \text{max} & 19 & 8 & 18 & \boxed{6} &
\end{array}$$

There is no saddle point and $5 \leq V \leq 6$. Columns B$_1$ and B$_3$ are dominated by column B$_2$, then the reduced matrix will be:

$$\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{B_2} & \mathbf{B_4} \\
\hline
 & \mathbf{A_1} & 6 & 5 \\
\hline
\mathbf{A} & \mathbf{A_2} & 3 & 6 \\
\hline
 & \mathbf{A_3} & 8 & 4 \\
\hline
 & \mathbf{A_4} & 7 & -1 \\
\hline
\end{array}$$

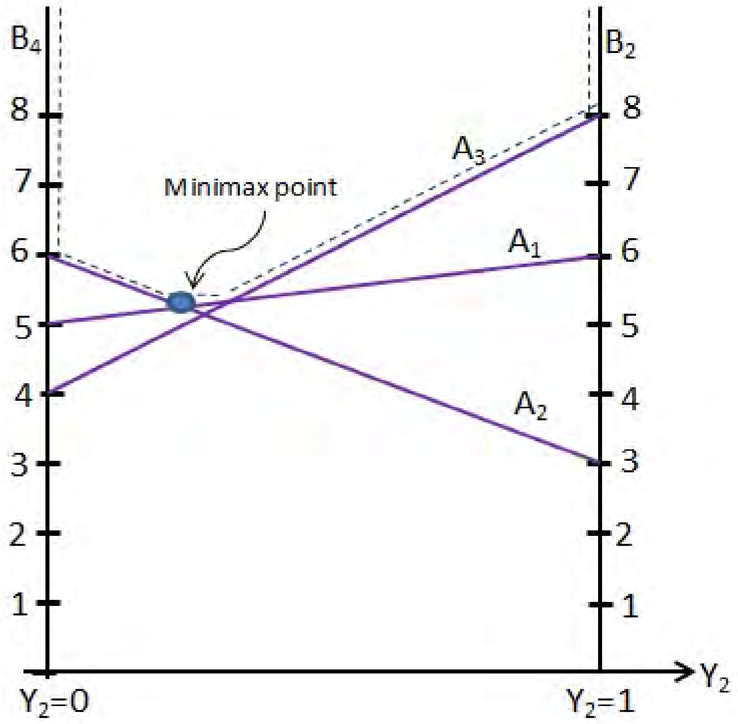
Row A$_3$ dominates row A$_4$ and the reduced matrix will be:

$$\begin{array}{cc|c|c|}
 & & \mathbf{B} & \\
 & & \mathbf{B_2} & \mathbf{B_4} \\
\hline
 & \mathbf{A_1} & 6 & 5 \\
\hline
\mathbf{A} & \mathbf{A_2} & 3 & 6 \\
\hline
 & \mathbf{A_3} & 8 & 4 \\
\hline
 & & y_2 & y_4 = 1 - y_2
\end{array}$$

Let A$_1$, A$_2$, and A$_3$ be the strategies which A mixes with probabilities $x_1, x_2$, and $x_3$ respectively and B$_2$, B$_4$ be the strategies which B mixes with probabilities $y_2$ and $y_4 = 1 - y_2$.When B adopts strategy B$_2$, $y_2 = 1$ and the probability with which he will adopt strategy B$_4$, i.e. $y_4 = 0$. B's expected Payoffs corresponding to A's pure strategies are given below:

| **A's pure strategies** | **B's expected Payoffs** |
|---|---|
| **A$_1$** | $6y_2 + 5y_4 = 6y_2 + 5(1 - y_2) = y_2 + 5$ |
| **A$_2$** | $3y_2 + 6y_4 = 3y_2 + 6(1 - y_2) = -3y_2 + 6$ |
| **A$_3$** | $8y_2 + 4y_4 = 8y_2 + 4(1 - y_2) = 4y_2 + 4$ |

These three lines can be plotted as functions of $y_2$ as follows: draw two lines B$_2$ and B$_4$ parallel to each other one unit apart and mark a scale on each of them. To represent A's first strategy, A$_1$, join mark 5 on B$_4$ (when $y_2 = 0$ ) to 6 on B$_2$ (when $y_2 = 1$ ). Similarly for other A's strategies, A$_2$ and A$_3$, and bound the figure from above as shown since B is a minimization player.



Since player B wishes to minimize his maximum expected losses, the two lines which intersect at the lowest point of the upper bound show the two courses of action A should choose in his best strategy, i.e. $A_1$ and $A_2$. Thus, we can reduce the 3 x 2 game to the following 2 x 2 game which has no saddle point:

| | | B | | | |
|---|---|---|---|---|---|
| | | **$B_2$** | **$B_4$** | | |
| **A** | **$A_1$** | 6 | 5 | 3 | 3/4 |
| | **$A_2$** | 3 | 6 | 1 | 1/4 |
| | | 1 | 3 | | |
| | | 1/4 | 3/4 | | |

The optimal strategies are: A (3/4, 1/4, 0, 0), B (0, 1/4, 0, 3/4)

The value of the game is: $V = \frac{6\times1+5\times3}{4} = \frac{21}{4}$

## Example (2.13):

Solve the following 2 x 5 game:

| | | | | B | | |
|---|---|---|---|---|---|---|
| | | **$B_1$** | **$B_2$** | **$B_3$** | **$B_4$** | **$B_5$** |
| **A** | **$A_1$** | −5 | 5 | 0 | −1 | 8 |
| | **$A_2$** | 8 | −4 | −1 | 6 | −5 |

**Solution:**

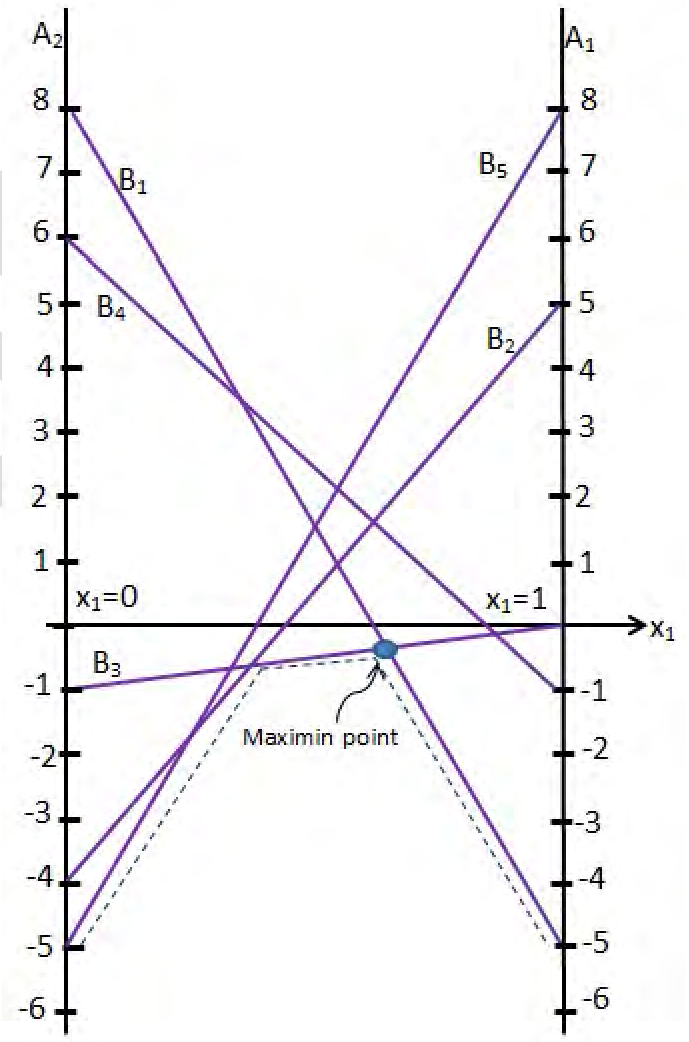
First, we must look for a saddle point; it does not exist in this problem.

$$\begin{array}{cc|c|c|c|c|c|ccc}
 & & & & \mathbf{B} & & & & & \\
 & & \mathbf{B_1} & \mathbf{B_2} & \mathbf{B_3} & \mathbf{B_4} & \mathbf{B_5} & & \text{min} & \\
\hline
\mathbf{A} & \mathbf{A_1} & -5 & 5 & 0 & -1 & 8 & & -5 & x_1 \\
\hline
 & \mathbf{A_2} & 8 & -4 & -1 & 6 & -5 & & -5 & x_2 = 1 - x_1 \\
\hline
 & \mathbf{max} & 8 & 5 & 0 & 6 & 8 & & & 
\end{array}$$

In this problem, the matrix cannot be reduced to a smaller matrix. The A's expected payoffs corresponding to B's pure strategies are:

| B's pure strategies | A's expected payoffs |
|---|---|
| 1 | $-5x_1 + 8x_2 = -5x_1 + 8(1 - x_1) = -13x_1 + 8$ |
| 2 | $5x_1 - 4x_2 = 5x_1 - 4(1 - x_1) = 9x_1 - 4$ |
| 3 | $0x_1 - 1x_2 = -(1 - x_1) = x_1 - 1$ |
| 4 | $-1x_1 + 6x_2 = -1x_1 + 6(1 - x_1) = -7x_1 + 6$ |
| 5 | $8x_1 - 5x_2 = 8x_1 - 5(1 - x_1) = 13x_1 - 5$ |

The five lines can be plotted as a function of $x_1$ as follows: draw two lines A₁ and A₂ parallel to each other one unit apart and mark a scale on each of them. To represent B's first strategy, B₁, join mark 8 on A₂ (when $x_1 = 0$ ) to −5 on A₁ (when $x_1 = 1$ ) and so on. Bound the figure from below as shown since A is a maximization player.



Since player A wishes to maximize his minimum expected payoff, the two lines which intersect at the highest point of the lower bound show the two courses of action B should choose in his best strategy, i.e. $B_1$ and $B_3$. Thus, we can reduce the 2 x 5 game to the following 2 x 2 game which has no saddle point:

| | | **B** | | | |
|---|---|---|---|---|---|
| | | **$B_1$** | **$B_3$** | | |
| **A** | **$A_1$** | −5 | 0 | 9 | 9/14 |
| | **$A_2$** | 8 | −1 | 5 | 5/14 |
| | | 1 | 13 | | |
| | | 1/14 | 13/14 | | |

The optimal strategies are: A (9/14, 5/14), B (1/14, 0, 13/14, 0, 0)

The value of the game is: $V = \frac{-5\times1+0\times13}{14} = \frac{-5}{14}$

## Exercise 2.2 (in addition to text book exercises)

Solve the following game in two ways:

| | | B | |
|---|---|---|---|
| | | 1 | 2 |
| | 1 | 3 | −1 |
| | 2 | 0 | 5 |
| A | 3 | 7 | −2 |
| | 4 | −3 | 4 |
| | 5 | 6 | 2 |

## 2.8 Mixed strategies for 3 x 3 Game – Method of Matrices

If the game has no saddle point and it reduced to a 3 x 3 matrix, the game can be solved by the method of matrices. The steps of this method are as follows:

***Step 1:*** subtract 2nd row from the 1st and 3rd row from the 2nd and write down the values below the matrix.

***Step 2:*** similarly, subtract each column from the column to its left (i.e. subtract 2nd column from the 1st and 3rd column from the 2nd ) and write down the values to the right of the matrix.

***Step 3:*** Calculate the oddments for $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, and $B_3$. The oddment of each strategy is the determinant of the numbers calculated in steps 1 and 2 , after neglecting the strategy numbers. Write down these elements to the right and down the table, neglecting their signs.

***Step 4:*** If the sum of the oddments of the players are equal, then there is a solution to the game; if not, this method fails.

***Step 5:*** For each player calculate the probability in which he uses his strategies by dividing his oddments on the sum of oddments.

## Example (2.14):

Solve the following game:

$$\begin{array}{cc|c|c|c|} & & & \mathbf{B} & \\ & & \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \hline & \mathbf{1} & 7 & 1 & 7 \\ \hline \mathbf{A} & \mathbf{2} & 9 & -1 & 1 \\ \hline & \mathbf{3} & 5 & 7 & 6 \\ \hline \end{array}$$

**Solution:**

$$\begin{array}{cc|c|c|c|c} & & & \mathbf{B} & & \\ & & \mathbf{1} & \mathbf{2} & \mathbf{3} & \text{min} \\ \hline & \mathbf{1} & 7 & 1 & 7 & 1 \\ \hline \mathbf{A} & \mathbf{2} & 9 & -1 & 1 & -1 \\ \hline & \mathbf{3} & 5 & 7 & 6 & \textcircled{5} \\ \hline & \text{max} & 9 & \textcircled{7} & 7 & \end{array}$$

There is no saddle point and $5 \leq V \leq 7$. The matrix cannot be reduced, then:

$$\begin{array}{cc|c|c|c|cc} & & & \mathbf{B} & & & \\ & & \mathbf{1} & \mathbf{2} & \mathbf{3} & & \\ \hline & \mathbf{1} & 7 & 1 & 7 & 6 & -6 \\ \hline \mathbf{A} & \mathbf{2} & 9 & -1 & 1 & 10 & -2 \\ \hline & \mathbf{3} & 5 & 7 & 6 & -2 & 1 \\ \hline & & -2 & 2 & 6 & & \\ & & 4 & -8 & -5 & & \end{array}$$

The oddments are:

$$\text{Oddment for } A_1 = \left|\begin{vmatrix} 10 & -2 \\ -2 & 1 \end{vmatrix}\right| = 6$$

$$\text{Oddment for } A_2 = \left|\begin{vmatrix} 6 & -6 \\ -2 & 1 \end{vmatrix}\right| = 6$$

$$\text{Oddment for } A_3 = \left|\begin{vmatrix} 6 & -6 \\ 10 & -2 \end{vmatrix}\right| = 48$$

$$\text{Oddment for } B_1 = \left|\begin{vmatrix} 2 & 6 \\ -8 & -5 \end{vmatrix}\right| = 38$$

$$\text{Oddment for } B_2 = \left|\begin{vmatrix} -2 & 6 \\ 4 & -5 \end{vmatrix}\right| = 14$$

$$\text{Oddment for } B_3 = \left|\begin{vmatrix} -2 & 2 \\ 4 & -8 \end{vmatrix}\right| = 8$$

sum of oddments for $A = 6 + 6 + 48 = 60$, sum of oddments for $B = 38 + 14 + 8 = 60$ . Then:

| | | | B | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | | |
| | **1** | 7 | 1 | 7 | 6 | 3/30 |
| **A** | **2** | 9 | $-1$ | 1 | 6 | 3/30 |
| | **3** | 5 | 7 | 6 | 48 | 24/30 |
| | | 38 | 14 | 8 | | |
| | | 19/30 | 7/30 | 4/30 | | |

The optimal strategies are:

A (3/30, 3/30, 24/30), B (19/30, 7/30, 4/30)

The value of the game: $V = \frac{7\times3+9\times3+5\times24}{30} = \frac{168}{30} = \frac{28}{5}$

## Exercise 2.3 (in addition to text book exercises)

Solve the following game:

| | | | B | |
|---|---|---|---|---|
| | | 1 | 2 | 3 |
| | **1** | 1 | $-1$ | $-1$ |
| **A** | **2** | $-1$ | $-1$ | 3 |
| | **3** | $-1$ | 2 | $-1$ |

## 2.9 Method of Linear Programming

Game theory bears a strong relationship to linear programming, since every finite two-person zero-sum game can be expressed as a linear program and vice versa. Linear programming is the most general method of solving any two-person zero-sum game. Consider the following game:

| | | | | Player B | | | |
|---|---|---|---|---|---|---|---|
| | | **1** | **2** | **…** | $\boldsymbol{j}$ | **…** | $\boldsymbol{n}$ |
| **Player A** | **1** | $a_{11}$ | $a_{12}$ | … | $a_{1j}$ | … | $a_{1n}$ |
| | **2** | $a_{21}$ | $a_{22}$ | … | $a_{2j}$ | … | $a_{2n}$ |
| | **⋮** | ⋮ | ⋮ | | ⋮ | | ⋮ |
| | $\boldsymbol{i}$ | $a_{i1}$ | $a_{i2}$ | … | $a_{ij}$ | … | $a_{in}$ |
| | **⋮** | ⋮ | ⋮ | | ⋮ | | ⋮ |
| | $\boldsymbol{m}$ | $a_{m1}$ | $a_{m2}$ | … | $a_{mj}$ | … | $a_{mn}$ |

Let $p_1, p_2, \dots, p_m$ and $q_1, q_2, \dots, q_n$ be the probabilities by which A and B respectively select their strategies and let V be the value of the game. Consider the game from A’s point of view, A is trying to maximize V, that is:

$$a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \geq V$$
$$a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \geq V$$
$$\vdots$$

$a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \geq V$
$p_1 + p_2 + \cdots + p_m = 1$
$p_i \geq 0 \qquad i = 1,2, \dots, m$

Since $V > 0$, then divide by V, the above system will be:

$$a_{11}\frac{p_1}{V} + a_{21}\frac{p_2}{V} + \cdots + a_{m1}\frac{p_m}{V} \geq 1$$
$$a_{12}\frac{p_1}{V} + a_{22}\frac{p_2}{V} + \cdots + a_{m2}\frac{p_m}{V} \geq 1$$
$$\vdots$$
$$a_{1n}\frac{p_1}{V} + a_{2n}\frac{p_2}{V} + \cdots + a_{mn}\frac{p_m}{V} \geq 1$$
$$\frac{p_1}{V} + \frac{p_2}{V} + \cdots + \frac{p_m}{V} = \frac{1}{V}$$
$$\frac{p_i}{V} \geq 0 \qquad i = 1,2, \dots, m$$

Let $x_i = \frac{p_i}{V}, i = 1,2, \dots, m$. Since A is trying to maximize *V*, i.e. minimize *1/V*, then let $Z = \frac{1}{V} = x_1 + x_2 + \cdots + x_m$ , the LPP will be:

$$\begin{aligned}
&min \qquad Z = x_1 + x_2 + \cdots + x_m \\
&S.t. \qquad a_{11}x_1 + a_{21}x_2 + \cdots + a_{m1}x_m \geq 1 \\
&\qquad\qquad a_{12}x_1 + a_{22}x_2 + \cdots + a_{m2}x_m \geq 1 \\
&\qquad\qquad \vdots \\
&\qquad\qquad a_{1n}x_1 + a_{2n}x_2 + \cdots + a_{mn}x_m \geq 1 \\
&\qquad\qquad x_i \geq 0 \qquad i = 1,2, \dots, m
\end{aligned}$$

In a similar way, we can write the LP model for the player B, which is, in fact, the dual of the LP model for player A. That is:

$$\begin{aligned}
&max \qquad W = y_1 + y_2 + \cdots + y_n \\
&S.t. \qquad a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n \leq 1 \\
&\qquad\qquad a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n \leq 1 \\
&\qquad\qquad \vdots \\
&\qquad\qquad a_{m1}y_1 + a_{m2}y_2 + \cdots + a_{mn}y_n \leq 1 \\
&\qquad\qquad y_j \geq 0 \qquad j = 1,2, \dots, n \qquad (\text{ where } y_j = \frac{q_j}{V}, j = 1,2, \dots, n)
\end{aligned}$$

By the duality principal, the optimal solution of one problem automatically yields the optimal solution of the other.

### Example (2.15):

Use linear programming to solve the following game:

$$\begin{array}{cc|c|c|c|}
 & & & \mathbf{B} & \\
 & & \mathbf{1} & \mathbf{2} & \mathbf{3} \\
\hline
 & \mathbf{1} & -1 & 1 & 1 \\
\hline
\mathbf{A} & \mathbf{2} & 2 & -2 & 2 \\
\hline
 & \mathbf{3} & 3 & 3 & -3 \\
\hline
\end{array}$$

Solution:

$$\begin{array}{cc|c|c|c|c}
 & & & \mathbf{B} & & \\
 & & \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{min} \\
\hline
 & \mathbf{1} & -1 & 1 & 1 & \bigcirc\!\!\!\!\!\!-1 \\
\hline
\mathbf{A} & \mathbf{2} & 2 & -2 & 2 & -2 \\
\hline
 & \mathbf{3} & 3 & 3 & -3 & -3 \\
\hline
 & \mathbf{max} & 3 & 3 & \bigcirc\!\!\!\!\!\!2 & \\
\end{array}$$

There is no saddle point , $-1 \le V \le 2$, and the game cannot be reduced to a smaller game. Player A's linear program:

$$\begin{aligned}
&min && Z = x_1 + x_2 + x_3 \\
&S.t. && -x_1 + 2x_2 + 3x_3 \ge 1 \\
& && x_1 - 2x_2 + 3x_3 \ge 1 \\
& && x_1 + 2x_2 - 3x_3 \ge 1 \\
& && x_1, x_2, x_3 \ge 0
\end{aligned}$$

The dual of A's LP; which is B's linear program is:

$$\begin{aligned}
&max && W = y_1 + y_2 + y_3 \\
&S.t. && -y_1 + y_2 + y_3 \le 1 \\
& && 2y_1 - 2y_2 + 2y_3 \le 1 \\
& && 3y_1 + 3y_2 - 3y_3 \le 1 \\
& && y_1, y_2, y_3 \ge 0
\end{aligned}$$

The standard form of the last LPP (with modification in the objective function) is:

$$\begin{aligned}
&max && W - y_1 - y_2 - y_3 = 0 \\
&S.t. && -y_1 + y_2 + y_3 + s_1 = 1 \\
& && 2y_1 - 2y_2 + 2y_3 + s_2 = 1 \\
& && 3y_1 + 3y_2 - 3y_3 + s_3 = 1 \\
& && y_1, y_2, y_3, s_1, s_2, s_3 \ge 0
\end{aligned}$$

Let $y_1 = y_2 = y_3 = 0$, then $s_1 = s_2 = s_3 = 1$

| **Basic Var.'s** | $y_1$ ↓ | $y_2$ | $y_3$ | $s_1$ | $s_2$ | $s_3$ | **Solution** | |
|---|---|---|---|---|---|---|---|---|
| $s_1$ | −1 | 1 | 1 | 1 | 0 | 0 | 1 | |
| $s_2$ | 2 | −2 | 2 | 0 | 1 | 0 | 1 | 1/2 |
| ← $s_3$ | $\boxed{3}$ | 3 | −3 | 0 | 0 | 1 | 1 | 1/3 |
| **W** | −1 | −1 | −1 | 0 | 0 | 0 | 0 | |

| Basic Var.’s | $y_1$ | $y_2$ | $y_3$ ↓ | $s_1$ | $s_2$ | $s_3$ | Solution |
|---|---|---|---|---|---|---|---|
| $s_1$ | 0 | 2 | 0 | 1 | 0 | 1/3 | 4/3 |
| ← $s_2$ | 0 | −4 | $\boxed{4}$ | 0 | 1 | −2/3 | 1/3 |
| $y_1$ | 1 | 1 | −1 | 0 | 0 | 1/3 | 1/3 |
| **W** | 0 | 0 ↓ | −2 | 0 | 0 | 1/3 | 1/3 |
| ← $s_1$ | 0 | $\boxed{2}$ | 0 | 1 | 0 | 1/3 | 4/3 |
| $y_3$ | 0 | −1 | 1 | 0 | 1/4 | −1/6 | 1/12 |
| $y_1$ | 1 | 0 | 0 | 0 | 1/4 | 1/6 | 5/12 |
| **W** | 0 | −2 | 0 | 0 | 1/2 | 0 | 1/2 |
| $y_2$ | 0 | 1 | 0 | 1/2 | 0 | 1/6 | 2/3 |
| $y_3$ | 0 | 0 | 1 | 1/2 | 1/4 | 0 | 3/4 |
| $y_1$ | 1 | 0 | 0 | 0 | 1/4 | 1/6 | 5/12 |
| **W** | 0 | 0 | 0 | 1 | 1/2 | 1/3 | 11/6 |

$\Rightarrow W_{max} = Z_{min} = \frac{11}{6} \Rightarrow V = \frac{6}{11}. x_1 = 1, x_2 = \frac{1}{2}, x_3 = \frac{1}{3}$. Since $p_i = x_i V, i = 1,2,3, then: p_1 = x_1 V = 1 * \frac{6}{11} = \frac{6}{11}, p_2 = x_2 V = \frac{1}{2} * \frac{6}{11} = \frac{3}{11}, p_3 = x_3 V = \frac{1}{3} * \frac{6}{11} = \frac{2}{11}$.

$y_1 = \frac{5}{12}, y_2 = \frac{2}{3}, y_3 = \frac{3}{4}$. Since $q_j = y_j V, j = 1,2,3, then: q_1 = y_1 V = \frac{5}{12} * \frac{6}{11} = \frac{5}{22}, q_2 = y_2 V = \frac{2}{3} * \frac{6}{11} = \frac{8}{22}, q_3 = y_3 V = \frac{3}{4} * \frac{6}{11} = \frac{9}{22}$.

∴ The optimal strategy for player A: ( 6/11, 3/11, 2/11)

The optimal strategy for player B: ( 5/22, 8/22, 9/22)

The value of the game: V=6/11

## Exercise 2.4 (in addition to text book exercises)

Solve the following games by linear programming:

|  |  | **B** |  |  |
|---|---|---|---|---|
|  |  | **1** | **2** | **3** |
|  | **1** | 0 | 2 | 2 |
| **A** | **2** | 3 | −1 | 3 |
|  | **3** | 4 | 4 | −2 |

|  |  | **B** |  |  |  |
|---|---|---|---|---|---|
|  |  | **1** | **2** | **3** | **4** |
|  | **1** | 3 | −2 | 1 | 4 |
|  | **2** | 2 | 3 | −5 | 0 |
| **A** | **3** | −1 | 2 | −2 | 2 |
|  | **4** | −3 | −5 | 4 | 1 |

# Ch.3: Network Models

Many operations research situations can be modeled and solved as networks such as determination of the shortest route between two cities, design of an offshore natural-gas pipeline.

## 3.1 Network Logic

Some of the terms commonly used in networks are defined below.

### Definition (3.1):

An ***activity*** is physically identifiable part of a project which requires time and resources for its execution. An activity is represented by an ***arc*** or ***arrow***, the tail of which represents the start and the head, the finish of the activity.

### Definition (3.2):

The beginning and end points of an activity are called ***events*** or ***nodes***. Event is a point in time and does not consume any resources. It is represented by a circle.

### Definition (3.3):

An unbroken chain of activity arrows connecting the initial event to some other event is called a ***path***.

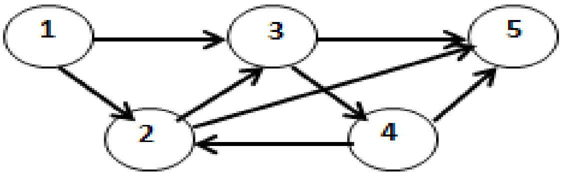
### Definition (3.4):

A ***network*** is the graphical representation of logically and sequentially connected nodes and arcs (arrows) representing activities and events of a project. The notion for describing a network is (N, A), where N is the set of nodes, and A is the set of arcs.

Associated with each network is a ***flow***, e.g. oil products flow in pipeline and automobile flow in highway. The maximum flow in a network can be finite or infinite, depending on the capacity of its arcs.

### Example (3.1):

The network in the following figure is described as:



N= {1, 2, 3, 4, 5}

A= { (1,2), (1,3), (2,3), (2,5), (3,4), (3,5), (4,2), (4,5)}

Each of: 1-2-5, 1-2-3-4-5 are paths between nodes 1 and 5.

## Definition (3.5):

An arc is said to be ***directed*** or ***oriented*** if it allows positive flow in one direction only. A directed network has all directed arcs.

## Definition (3.6):

An activity which only determines the dependency of one activity on the other, but does not consume any time is called ***dummy activity***. Dummies are usually represented by dotted line arrows.

## Definition (3.7):

A path forms a ***cycle*** or a ***loop*** if it connects a node back to itself through other nodes.

## Example (3.2):

In example (3.1): 2-3-4-2 is a cycle

## 3.2 Remarks

1- The length, shape and direction of the arrow have no relation to the size of the activity.

2- An arrow (activity) directed from node 1 to node 2 can be denoted either by (1, 2) or by 12 or by 1-2 or simply by a letter, e.g. A.

3- For each activity $(i, j), i < j$ .

4- Each activity is represented by one and only one arc.

5- Each activity must have a tail and head event.

6- No two or more activities may have the same tail and head events. In this case dummy activities must be used.

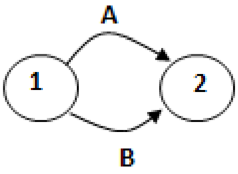
7- In a network diagram there should be only one initial event and one end event.

8- An activity must end before its successor begins.

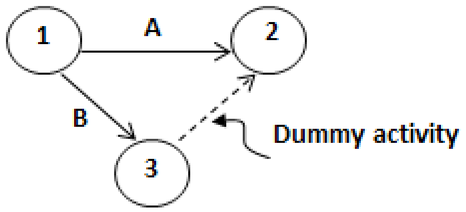
9- An activity occurs only once, that is loops are not allowed.

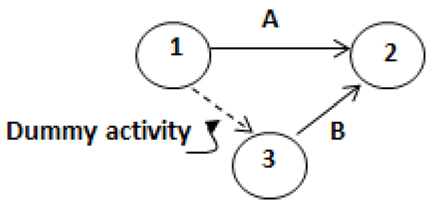
## Example (3.3):

Consider the following:



Both activities A and B are joining nodes 1 and 2, this is not allowed, thus we insert a dummy activity as follows:

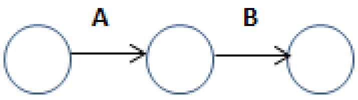




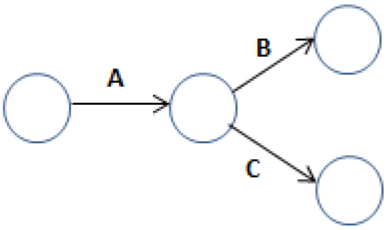
## Example (3.4):

Consider the following:

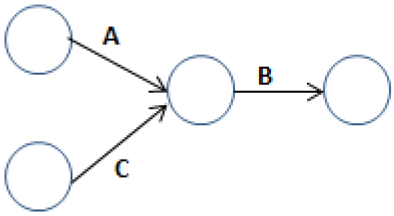
1- Activity A must end before activity B begins.



2- Activity A must end before activities B and C begins.



3- Activities A and C must ends before activity B begins.



## 3.3 The Critical Path Method (CPM)

The end result in CPM is a time schedule for the project. To achieve this goal, special computations are carried out to produce the following information:

1- Total duration needed to complete the project.
2- Classification of the activities of the project as critical and noncritical.

### Definition (3.8):

An activity is ***critical*** if its start and finish times are predetermined (fixed).An activity is ***noncritical*** if it can be scheduled in a time span greater than its duration, permitting flexible start and finish times (within limits).

A delay in the start time of a critical activity definitely causes a delay in the completion of the entire project.

To carry out the necessary computations let:

$ES_j =$ Earliest start time (Earliest occurrence time) of node (event) $j$ (it will be denoted by □ in network)

$LS_j =$ Latest finish time (Latest occurrence time) of node (event) $j$ (it will be denoted by Δ in network)

$D_{ij} =$ Duration of activity $(i, j)$

The critical path calculations involve two passes: the ***forward pass*** determines the earliest start time of events, and the ***backward pass*** determines the Latest finish time of events.

### Forward pass (Earliest start times)

The computations start at node 1 and advance recursively to node $n$.

***Initial step:*** Set $ES_1 = 0$.

***General step j:*** Given that nodes $p_1, p_2, \dots,$ and $p_m$ are linked directly to node $j$ by incoming activities $(p_1, j), (p_2, j), \dots,$ and $(p_m, j)$ and that the earliest occurrence times of events (nodes) $p_1, p_2, \dots,$ and $p_m$ have already been computed, then the earliest occurrence time of event $j$ is computed as:

$$ES_j = max \left\{ES_{p_1} + D_{p_1 j}, ES_{p_2} + D_{p_2 j}, \dots, ES_{p_m} + D_{p_m j}\right\}$$

The forward pass is complete when $ES_n$ at node $n$ has been computed. By definition, $ES_j$ is the longest path (duration) to node $j$.

### Backward pass (Latest start times)

The computations start at node $n$ and ends at node 1.

***Initial step:*** Set $LS_n = ES_n$ to indicate that latest occurrence of the last node equals the duration of the project.

***General step j:*** Given that nodes $p_1, p_2, \dots,$ and $p_m$ are linked directly to node $j$ by outgoing activities $(j, p_1), (j, p_2), \dots,$ and $(j, p_m)$ and that the latest occurrence times of events (nodes) $p_1, p_2, \dots,$ and $p_m$ have already been computed, then the latest occurrence time of event $j$ is computed as:

$$LS_j = min \left\{LS_{p_1} - D_{j p_1}, LS_{p_2} - D_{j p_2}, \dots, LS_{p_m} - D_{j p_m}\right\}$$

The backward pass is complete with $LS_1 = 0$ at node 1.

Based on the preceding computations, the activity $(i, j)$ will be ***critical*** if it satisfies three conditions:

1- $LS_i = ES_i$
2- $LS_j = ES_j$
3- $LS_j - ES_i = D_{ij}$ (or equivalently: $LS_j - LS_i = ES_j - ES_i = D_{ij}$)

## Example (3.5):

Determine the finishing time and the critical path for the following project network. All the durations are in days.

### Solution:

### Forward pass

**Node 1:** Let $ES_1 = 0$

**Node 2:** $ES_2 = ES_1 + D_{12} = 0 + 5 = 5$

**Node 3:** $ES_3 = max\ \{ES_1 + D_{13},\ ES_2 + D_{23}\} = max\{0 + 6, 5 + 3\} = 8$

**Node 4:** $ES_4 = ES_2 + D_{24} = 5 + 8 = 13$

**Node 5:** $ES_5 = max\ \{ES_3 + D_{35},\ ES_4 + D_{45}\} = max\{8 + 2, 13 + 0\} = 13$

**Node 6:** $ES_6 = max\ \{ES_3 + D_{36},\ ES_4 + D_{46}, ES_5 + D_{56}\} = max\{8 + 11, 13 + 1, 13 + 12\} = 25$

The finishing time of the project is 25 days.

### Backward pass

**Node 6:** Let $LS_6 = ES_6 = 25$

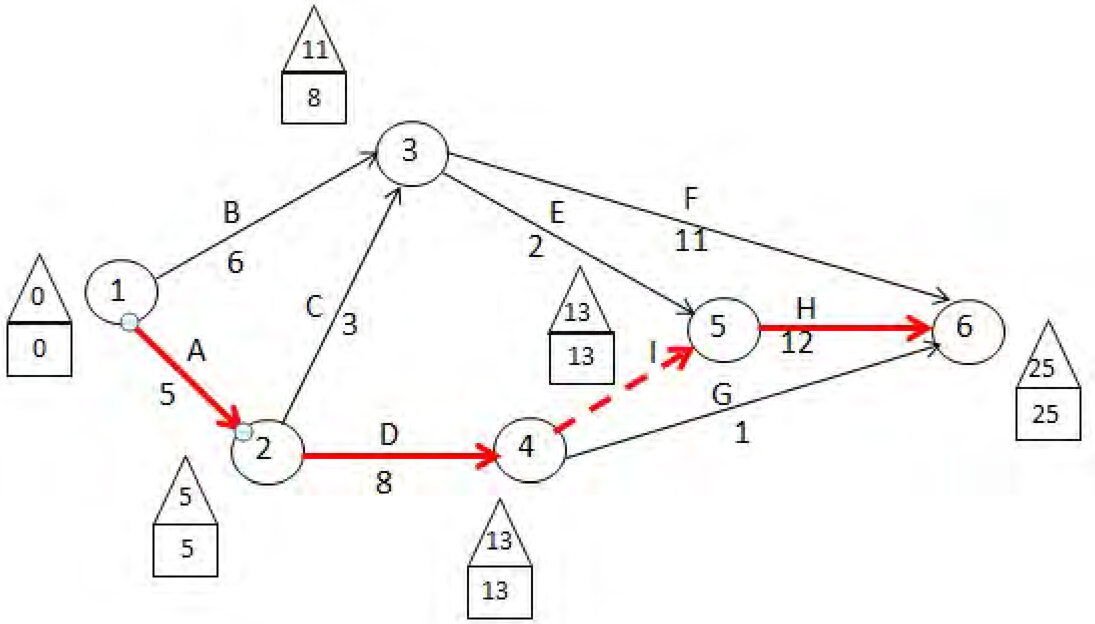
**Node 5:** $LS_5 = LS_6 - D_{56} = 25 - 12 = 13$

**Node 4:** $LS_4 = min\ \{LS_5 - D_{45},\ LS_6 - D_{46}\} = min\{13 - 0, 25 - 1\} = 13$

**Node 3:** $LS_3 = min\ \{LS_5 - D_{35},\ LS_6 - D_{36}\} = min\{13 - 2, 25 - 11\} = 11$

**Node 2:** $LS_2 = min\ \{LS_3 - D_{23},\ LS_4 - D_{24}\} = min\{11 - 3, 13 - 8\} = 5$

**Node 1:** $LS_1 = min\ \{LS_2 - D_{12},\ LS_3 - D_{13}\} = min\{5 - 5, 11 - 6\} = 0$



Then the critical activities are A, D, I, and H (or equivalently: (1,2), (2,4), (4,5), and (5,6)) and the critical path is: 1-2-4-5-6.

## Example (3.6):

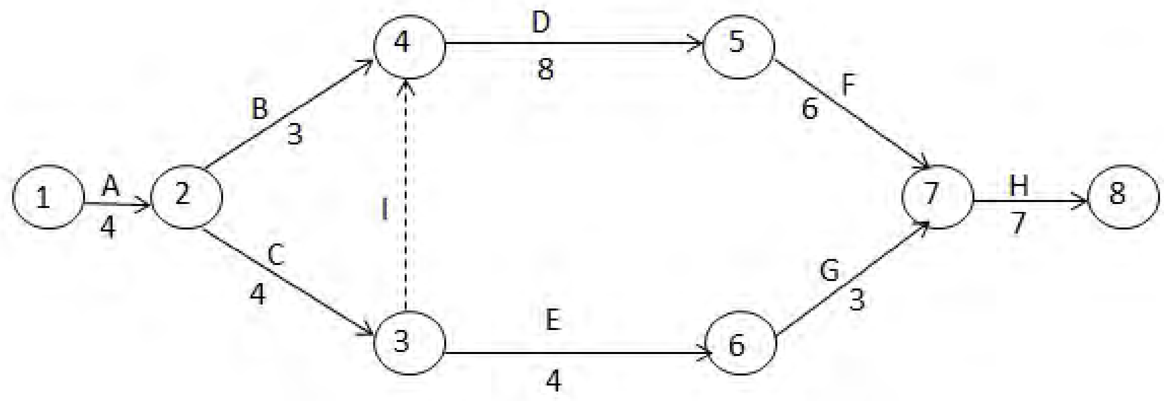
A project to produce radios requires the following activities according to times marked by each of them.

| Activity | Description | Preceding activity | Duration/Days |
|---|---|---|---|
| A | Study the desired marketing specifications | -- | 4 |
| B | Develop designs and geometric shapes | A | 3 |
| C | Provide the machinery and basic supplies for the production | A | 4 |
| D | Provide the manpower needed for the production | B, C | 8 |
| E | The organization of production lines within the plant | C | 4 |
| F | Training of workers on manufacturing processes | D | 6 |
| G | Provide the secondary supplies for the production | E | 3 |
| H | Production | G, F | 7 |

Determine the finishing time and the critical path for the following project network. All the durations are in days.

**Solution:**

The project network is:



**Forward pass**

**Node 1:** Let $ES_1 = 0$

**Node 2:** $ES_2 = ES_1 + D_{12} = 0 + 4 = 4$

**Node 3:** $ES_3 = ES_2 + D_{23} = 4 + 4 = 8$

**Node 4:** $ES_4 = max\ \{ES_2 + D_{24},\ ES_3 + D_{34}\} = max\{4 + 3, 8 + 0\} = 8$

**Node 5:** $ES_5 = ES_4 + D_{45} = 8 + 8 = 16$

**Node 6:** $ES_6 = ES_3 + D_{36} = 8 + 4 = 12$

**Node 7:** $ES_7 = max\ \{ES_5 + D_{57},\ ES_6 + D_{67}\} = max\{16 + 6, 12 + 3\} = 22$

**Node 8:** $ES_8 = ES_7 + D_{78} = 22 + 7 = 29$

The finishing time of the project is 29 days.

### Backward pass

**Node 8:** Let $LS_8 = ES_8 = 29$

**Node 7:** $LS_7 = LS_8 - D_{78} = 29 - 7 = 22$

**Node 6:** $LS_6 = LS_7 - D_{67} = 22 - 3 = 19$

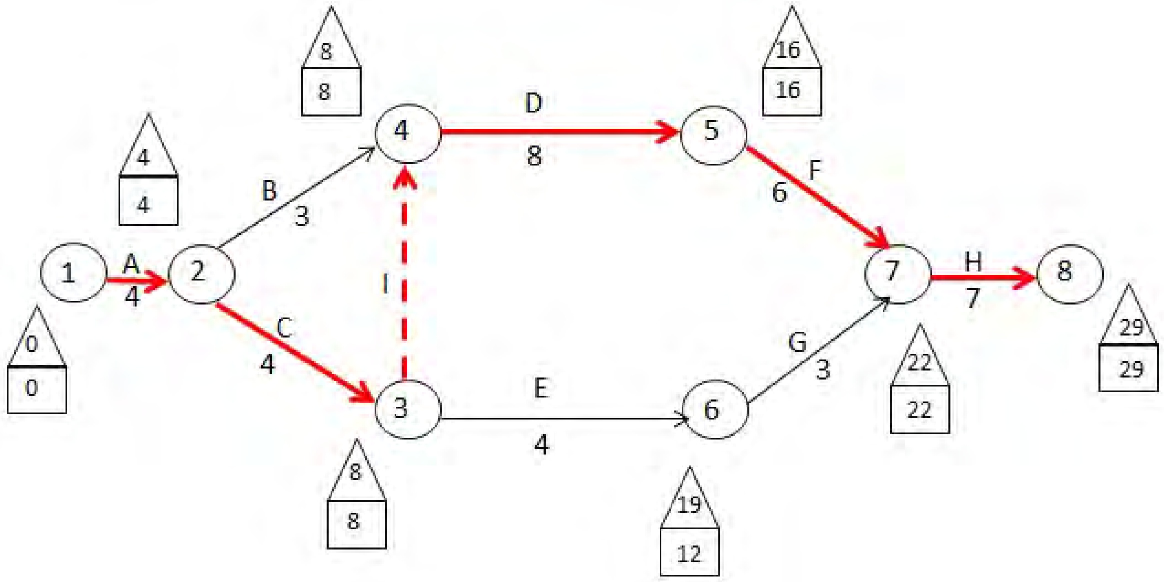
**Node 5:** $LS_5 = LS_7 - D_{57} = 22 - 6 = 16$

**Node 4:** $LS_4 = LS_5 - D_{45} = 16 - 8 = 8$

**Node 3:** $LS_3 = min\ \{LS_4 - D_{34},\ LS_6 - D_{36}\} = min\{8 - 0, 19 - 4\} = 8$

**Node 2:** $LS_2 = min\ \{LS_3 - D_{23},\ LS_4 - D_{24}\} = min\{8 - 3, 8 - 4\} = 4$

**Node 1:** $LS_1 = LS_2 - D_{12} = 4 - 4 = 0$

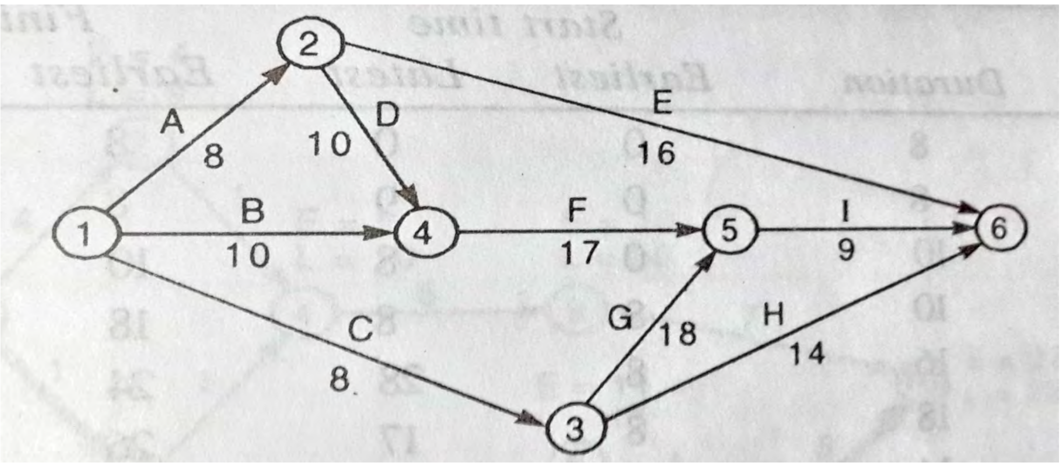


Then the critical activities are A, C, I, D, F and H (or equivalently: (1,2), (2,3), (3,4), (4,5),(5,7) and (7,8)) and the critical path is: 1-2-3-4-5-7-8.

## Exercise 3.1 (in addition to text book exercises)

Determine the finishing time and the critical path for each of the following project networks.

1- Duration in days.



2- The R and D department is planning to bid on a large project for the development of a new communication system for commercial planes. The accompanying table shows the activities, times and sequence required.

| Activity | Immediate predecessor | Time / weeks |
|---|---|---|
| A | --- | 3 |
| B | A | 2 |
| C | A | 4 |
| D | A | 4 |
| E | B | 6 |
| F | C , D | 6 |
| G | D, F | 2 |
| H | D | 3 |
| I | E, G, H | 3 |

Draw the network diagram. Determine the finishing time and the critical path for the network.

## 3.4 Program Evaluation and Review Technique (PERT)

PERT differs from CPM in that it assumes probabilistic duration times based on three estimates:

***The optimistic time***, ***a***, which occurs when execution goes extremely well.

***The most likely time***, ***m***, which occurs when execution is done under normal conditions.

***The pessimistic time***, ***b***, which occurs when execution goes extremely poorly.

The most likely time, $m$, falls in the range $(a, b)$. Based on the estimates, the average duration time, $\bar{D}$, and variance, $v = \sigma^2$, are approximated as:

$$\bar{D} = \frac{a + 4m + b}{6}$$

$$\sigma^2 = v = \left(\frac{b-a}{6}\right)^2$$

CPM calculations can be applied directly, with $\bar{D}$, replacing the single estimate *D*. To find the probability of completing the project in time *S*, we calculate:

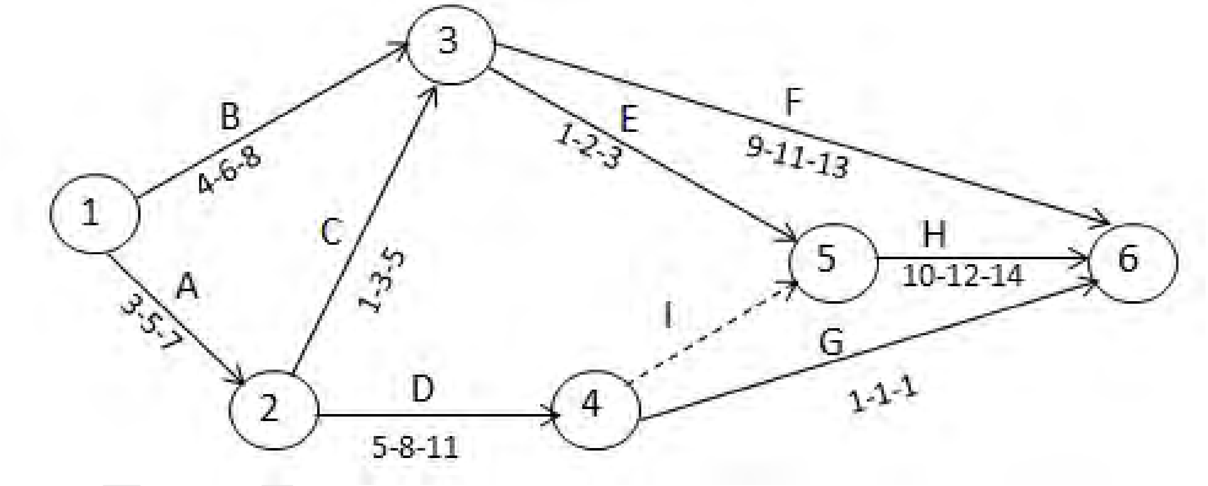
$$z = \frac{S - FT}{\sigma}$$

Where FT is the finishing time, $\sigma = \sqrt{\sum_{critical\ path} \sigma^2}$ . The probability is then read from the standard normal probability distribution table for the value of z calculated above.

## Example (3.7):

Determine for the following network:

1- The finishing time.
2- The critical path.
3- The probability that the project will be completed in a) $S_1 = 30, S_2 = 21$, and $S_3 = 29$ days .



## Solution:

We must calculate expected times as follows:

| **Activity** | $\boldsymbol{\bar{D}_{ij}}$ |
|---|---|
| A or (1,2) | $\bar{D}_{12} = \frac{3+20+7}{6} = 5$ |
| B or (1,3) | $\bar{D}_{13} = \frac{4+24+8}{6} = 6$ |
| C or (2,3) | $\bar{D}_{23} = \frac{1+12+5}{6} = 3$ |
| D or (2,4) | $\bar{D}_{24} = \frac{5+32+11}{6} = 8$ |

| | |
|---|---|
| E or (3,5) | $\overline{D}_{35} = \frac{1+8+3}{6} = 2$ |
| F or (3,6) | $\overline{D}_{36} = \frac{9+44+13}{6} = 11$ |
| I or (4,5) | $\overline{D}_{45} = \frac{0+0+0}{6} = 0$ |
| G or (4,6) | $\overline{D}_{46} = \frac{1+4+1}{6} = 1$ |
| H or (5,6) | $\overline{D}_{56} = \frac{10+48+14}{6} = 12$ |

**Forward pass**

**Node 1:** Let $ES_1 = 0$

**Node 2:** $ES_2 = ES_1 + \overline{D}_{12} = 0 + 5 = 5$

**Node 3:** $ES_3 = max\ \{ES_1 + \overline{D}_{13},\ ES_2 + \overline{D}_{23}\} = max\{0 + 6, 5 + 3\} = 8$

**Node 4:** $ES_4 = ES_2 + \overline{D}_{24} = 5 + 8 = 13$

**Node 5:** $ES_5 = max\ \{ES_3 + \overline{D}_{35},\ ES_4 + \overline{D}_{45}\} = max\{8 + 2, 13 + 0\} = 13$

**Node 6:** $ES_6 = max\ \{ES_3 + \overline{D}_{36},\ ES_4 + \overline{D}_{46}, ES_5 + \overline{D}_{56}\} = max\{8 + 11, 13 + 1, 13 + 12\} = 25$

1- The finishing time of the project is 25 days.

**Backward pass**

**Node 6:** Let $LS_6 = ES_6 = 25$

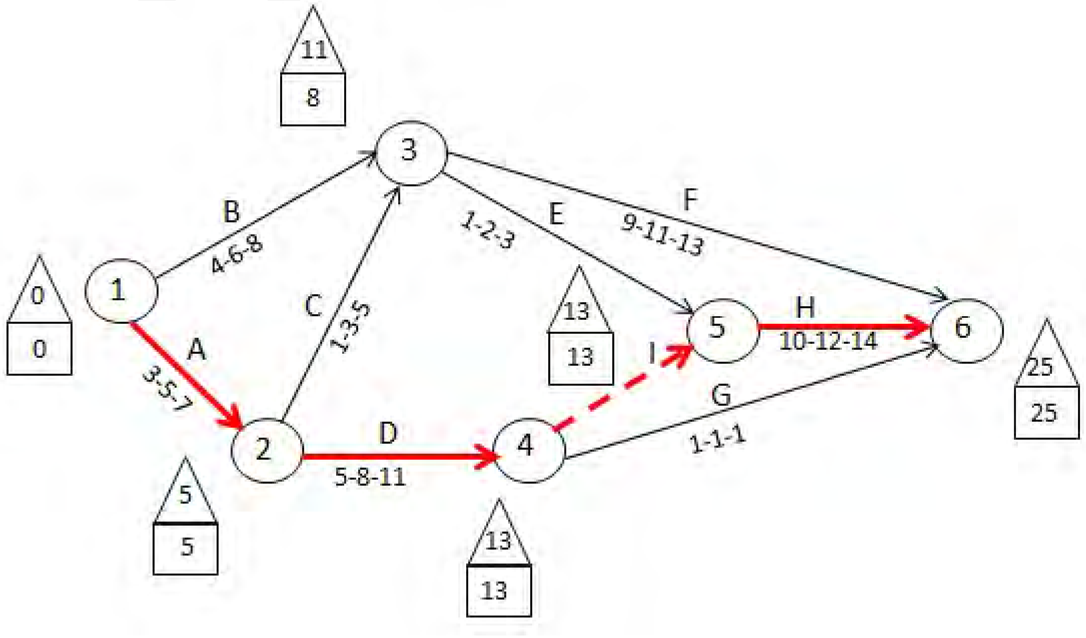
**Node 5:** $LS_5 = LS_6 - \overline{D}_{56} = 25 - 12 = 13$

**Node 4:** $LS_4 = min\ \{LS_5 - \overline{D}_{45},\ LS_6 - \overline{D}_{46}\} = min\{13 - 0, 25 - 1\} = 13$

**Node 3:** $LS_3 = min\ \{LS_5 - \overline{D}_{35},\ LS_6 - \overline{D}_{36}\} = min\{13 - 2, 25 - 11\} = 11$

**Node 2:** $LS_2 = min\ \{LS_3 - \overline{D}_{23},\ LS_4 - \overline{D}_{24}\} = min\{11 - 3, 13 - 8\} = 5$

**Node 1:** $LS_1 = min\ \{LS_2 - \overline{D}_{12},\ LS_3 - \overline{D}_{13}\} = min\{5 - 5, 11 - 6\} = 0$



2- Then the critical activities are A, D, I, and H (or equivalently: (1,2), (2,4), (4,5), and (5,6)) and the critical path is: 1-2-4-5-6.

3- To calculate the variance for critical activities:

| **Activity** | $\boldsymbol{\sigma_{ij}{}^2 = v_{ij}}$ |
|---|---|
| A or (1,2) | $\sigma_{12}{}^2 = \left(\frac{7-3}{6}\right)^2 = 0.444$ |
| D or (2,4) | $\sigma_{24}{}^2 = \left(\frac{11-5}{6}\right)^2 = 1$ |
| I or (4,5) | $\sigma_{46}{}^2 = \left(\frac{0-0}{6}\right)^2 = 0$ |
| H or (5,6) | $\sigma_{56}{}^2 = \left(\frac{14-10}{6}\right)^2 = 0.444$ |

$$\sigma = \sqrt{\sigma_{12}{}^2 + \sigma_{24}{}^2 + \sigma_{45}{}^2 + \sigma_{56}{}^2} = \sqrt{0.444 + 1 + 0 + 0.444} = \sqrt{1.888} = 1.37$$

$$z_1 = \frac{S_1 - FT}{\sigma} = \frac{30-25}{1.37} = 3.65 \implies p(z_1 \leq 30) = 0.9999 = 99.99\%$$

$$z_2 = \frac{S_2 - FT}{\sigma} = \frac{21-25}{1.37} = -2.92 \implies p(z_2 \leq 21) = 1 - 0.9983 = 0.0017 = 0.17\%$$

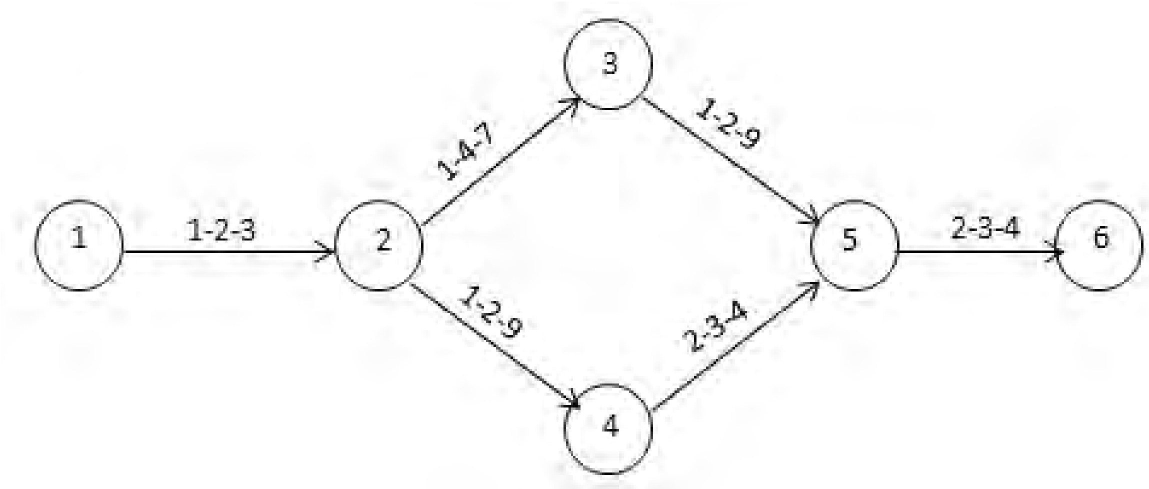
$$z_3 = \frac{S_3 - FT}{\sigma} = \frac{29-25}{1.37} = 2.92 \implies p(z_3 \leq 29) = 0.9983 = 99.83\%$$

$$[p(z \leq -g) = p(z \geq g) = 1 - p(z \leq g)]$$

## Example (3.8):

Determine for the following network:

1- The finishing time.
2- The critical path.
3- The probability that the project will be completed in a) $S_1 = 12, S_2 = 14$, and $S_3 = 10$ days .



**Solution:**

We must calculate expected times as follows:

| **Activity** | $\bar{D}_{ij}$ |
|---|---|
| (1,2) | $\bar{D}_{12} = \frac{1+8+3}{6} = 2$ |
| (2,3) | $\bar{D}_{23} = \frac{1+16+7}{6} = 4$ |
| (2,4) | $\bar{D}_{24} = \frac{1+8+9}{6} = 3$ |
| (3,5) | $\bar{D}_{35} = \frac{1+8+9}{6} = 3$ |
| (4,5) | $\bar{D}_{45} = \frac{2+12+4}{6} = 3$ |
| (5,6) | $\bar{D}_{56} = \frac{2+12+4}{6} = 3$ |

**Forward pass**

**Node 1:** Let $ES_1 = 0$

**Node 2:** $ES_2 = ES_1 + \bar{D}_{12} = 0 + 2 = 2$

**Node 3:** $ES_3 = ES_2 + \bar{D}_{23} = 2 + 4 = 6$

**Node 4:** $ES_4 = ES_2 + \bar{D}_{24} = 2 + 3 = 5$

**Node 5:** $ES_5 = max\ \{ES_3 + \bar{D}_{35},\ ES_4 + \bar{D}_{45}\} = max\{6 + 3, 5 + 3\} = 9$

**Node 6:** $ES_6 = ES_5 + \bar{D}_{56} = 9 + 3 = 12$

1- The finishing time of the project is 12 days.

**Backward pass**

**Node 6:** Let $LS_6 = ES_6 = 12$

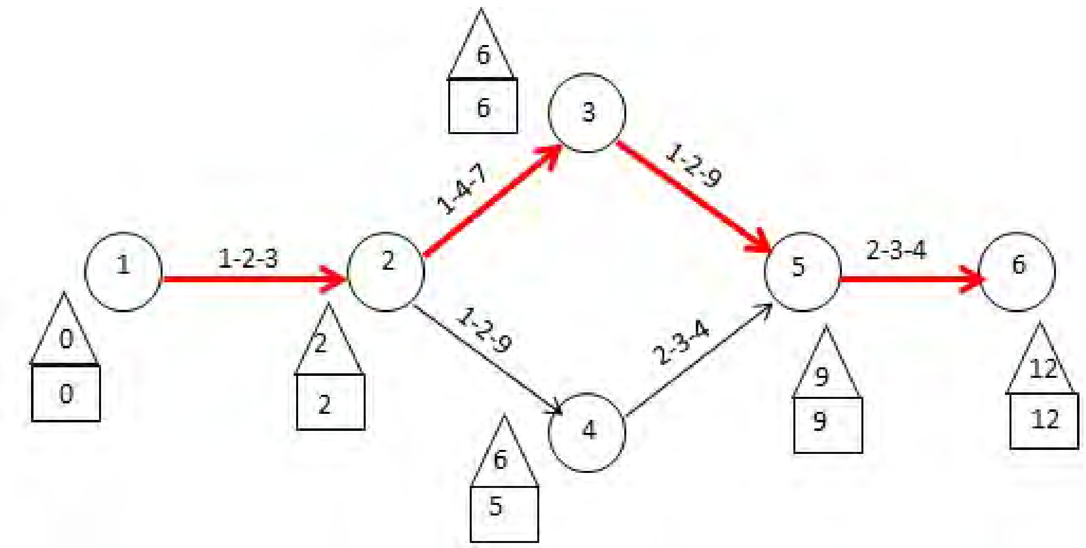
**Node 5:** $LS_5 = LS_6 - \bar{D}_{56} = 12 - 3 = 9$

**Node 4:** $LS_4 = LS_5 - \bar{D}_{45} = 9 - 3 = 6$

**Node 3:** $LS_3 = LS_5 - \bar{D}_{35} = 9 - 3 = 6$

**Node 2:** $LS_2 = min\ \{LS_3 - \bar{D}_{23},\ LS_4 - \bar{D}_{24}\} = min\{6 - 4, 6 - 3\} = 2$

**Node 1:** $LS_1 = LS_2 - \overline{D}_{12} = 2 - 2 = 0$



2- Then the critical activities are (1,2), (2,3), (3,5), and (5,6) and the critical path is: 1-2-3-5-6.

3- To calculate the variance for critical activities:

| **Activity** | $\boldsymbol{\sigma_{ij}{}^2 = v_{ij}}$ |
|---|---|
| (1,2) | $\sigma_{12}{}^2 = \left(\frac{3-1}{6}\right)^2 = 1/9$ |
| (2,3) | $\sigma_{23}{}^2 = \left(\frac{7-1}{6}\right)^2 = 1$ |
| (3,5) | $\sigma_{35}{}^2 = \left(\frac{9-1}{6}\right)^2 = 16/9$ |
| (5,6) | $\sigma_{56}{}^2 = \left(\frac{4-2}{6}\right)^2 = 1/9$ |

$$\sigma = \sqrt{\sigma_{12}{}^2 + \sigma_{23}{}^2 + \sigma_{35}{}^2 + \sigma_{56}{}^2} = \sqrt{\frac{1}{9} + 1 + \frac{16}{9} + \frac{1}{9}} = 1.73$$

$$z_1 = \frac{S_1 - FT}{\sigma} = \frac{12-12}{1.73} = 0 \implies p(z_1 \leq 12) = 0.5000 = 50\%$$

$$z_2 = \frac{S_2 - FT}{\sigma} = \frac{14-12}{1.73} = 1.16 \implies p(z_2 \leq 14) = 0.877 = 87.7\%$$

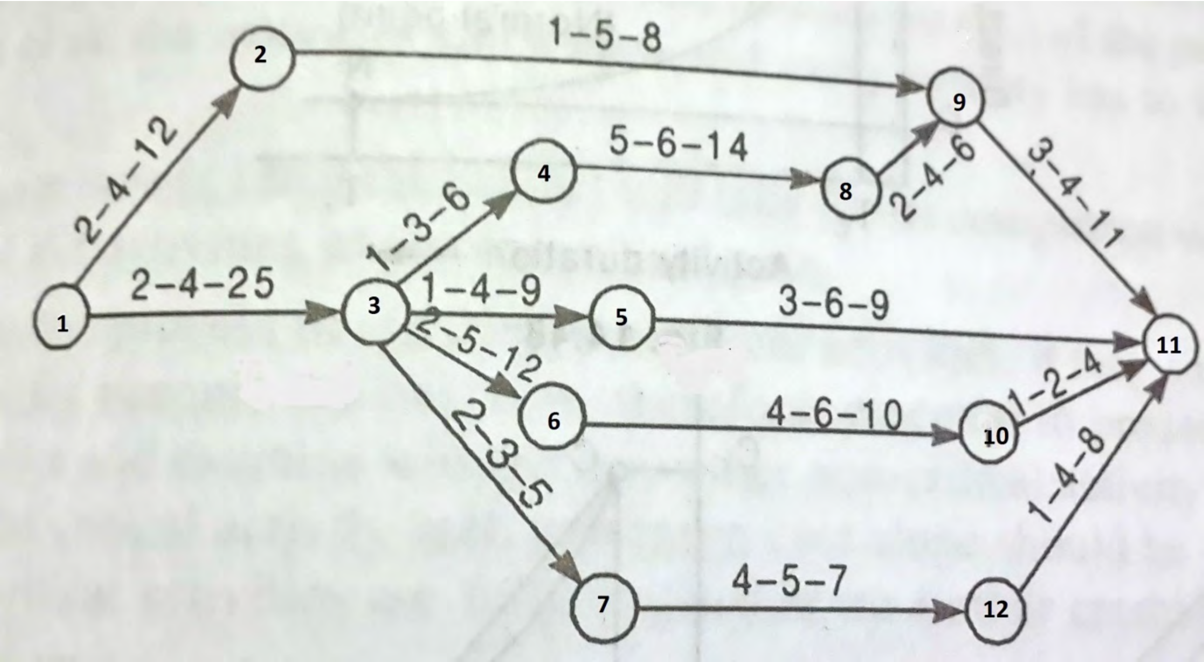
$$z_3 = \frac{S_3 - FT}{\sigma} = \frac{10-12}{1.73} = -1.16 \implies p(z_3 \leq 10) = 1 - 0.877 = 0.123 = 12.3\%$$

## Exercise 3.2 (in addition to text book exercises)

Determine for the following network:

1- The finishing time.
2- The critical path.

3- The probability that the project will be completed in a) $S_1 = 32, S_2 = 27, \text{and } S_3 = 20 \text{ days}$ .



### STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| **0.1** | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56749 | .57142 | .57535 |
| **0.2** | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| **0.3** | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| **0.4** | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| **0.5** | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| **0.6** | .72575 | .72907 | .73237 | .73565 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| **0.7** | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| **0.8** | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| **0.9** | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| **1.0** | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| **1.1** | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| **1.2** | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| **1.3** | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| **1.4** | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| **1.5** | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| **1.6** | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| **1.7** | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| **1.8** | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| **1.9** | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| **2.0** | .97725 | .97778 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| **2.1** | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| **2.2** | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| **2.3** | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| **2.4** | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| **2.5** | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| **2.6** | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| **2.7** | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| **2.8** | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| **2.9** | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| **3.0** | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| **3.1** | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| **3.2** | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| **3.3** | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| **3.4** | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| **3.5** | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| **3.6** | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| **3.7** | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| **3.8** | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| **3.9** | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |

# Ch.4: Machine Scheduling Problem

Suppose that $m$ machines $M_i(i = 1, \dots, m)$ have to process $n$ jobs $j(j = 1, \dots, n)$. A ***schedule*** is for each job an allocation of one or more time intervals to one or more machines. A schedule is ***feasible*** if at any time, there is at most one job on each machine; each job is run on at most one machine. A schedule is optimal if it minimizes (or maximizes) a given optimality criterion. A scheduling problem type can be specified using three- field classification $\alpha / \beta / \gamma$ composed of machine environment, the job characteristics, and the optimality criterion.

## 4.1 Job Data

Let *n* denote the number of jobs. The following data is specified for each job *j* ( *j=1, 2,…, n* ):

| | |
|---|---|
| $p_{ij}$ | A processing time of its *i*th operation, *i=1, 2, …,* $m_j$ , where $m_j$ is the number of operations on job *j* .If $m_j$*=1*, we shall write $p_j$ instead of $p_{ij}$ . |
| $r_j$ | A release date on which job *j* become available for processing. |
| $d_j$ | A due date, the time by which job *j* ideally be completed. |
| $\tilde{d}_j$ | A deadline, the time by which *j* must be completed. |
| $w_j$ | The weight of job *j* representing the importance of job *j* relative to another job. |
| $f_j$ | A non-decreasing real cost function measuring the cost $f_j(t)$ incurred if job *j* completed at time *t*. |

In general $p_{ij}, d_j, r_j, \tilde{d}_j$ and $w_j$ are given positive integer constants.

## 4.2 Machine environment

The first field $\alpha = \alpha_1 \alpha_2$ represents the machine environment. If $\alpha_1 \in \{\phi, P, Q, R\}$, each job *j* consists of a single operation which can be processed on any machine $M_i$. Let $p_{ij}$ denote the time to process job *j* on $M_i$.

$\alpha_1 = \phi$ : ***Single machine***, there is only one machine, $p_{ij} = p_j$ for all *j*.

$\alpha_1 = P$ : ***Identical parallel machines***; there are multiple machines operate at the same speed, $p_{ij} = p_j$ (*i=1, 2, …, m*).

$\alpha_1 = Q$: ***Uniform parallel machines***; there are multiple machines, each machine $M_i$ has its own speed $v_i$, $p_{ij} = p_j / v_i$ for all $M_i$ and jobs *j*.

$\alpha_1 = R$: ***Unrelated parallel machines***; there are multiple machines with different job-related speeds, that is the processing times are unrelated. If machine $M_i$ runs job *j* with a job-dependent speed $v_{ij}$, $p_{ij} = p_j / v_{ij}$ for all $M_i$ and jobs *j*.

In parallel machine environment, a job can be processed in any of the m machines.

If $\alpha_1 \in \{J, F, O\}$, each job *j* consists of a set of operations $\{O_{1j}, O_{2j}, \ldots, O_{m_j j}\}$.

$\alpha_1 = J$: ***Job-shop***; each job *j* consists of a chain of operations $\{O_{1j}, O_{2j}, \ldots, O_{m_j j}\}$, which must be processed in that order. Each operation $O_{ij}$ must be processed on a designated machine for $p_{ij}$ units of time. The order in which operations are processed is fixed by the ordering of the chain, but the order may be different for different jobs.

$\alpha_1 = F$ : ***Flow-shop***; is a special case of job-shop, each job *j* consists of a chain of operations $\{O_{1j}, O_{2j}, \ldots, O_{mj}\}$, where $O_{ij}$ is to be processed on machine $M_i$ for $p_{ij}$ units of time. The order of the operations is the same for every job.

$\alpha_1 = O$: ***Open-shop***; each job *j* composed of a chain of operations $\{O_{1j}, O_{2j}, \ldots, O_{mj}\}$, where $O_{ij}$ is to be processed on $M_i$ for $p_{ij}$ units of time. The order in which operations are executed is arbitrary.

$\alpha_2 \in \{\phi\} \cup \aleph$, where $\aleph$ is the set of natural numbers.

$\alpha_2 \in \aleph$ : *m*, the number of machines, is constant and equal to $\alpha_2$.

$\alpha_2 = \phi$: *m* is variable.

## 4.3 Job Characteristics

The second field $\beta \in \{\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6\}$ indicates certain job characteristics which are defined as follows:

$\beta_1 \in \{pmtn, \phi\}$

$\beta_1 = pmtn$: Preemptions are allowed, the processing of any job can be interrupted at no cost and resumed at a later time on any machine or at the same time on a different machine.

$\beta_1 = \phi$: Preemptions are not allowed, once a job is started on a machine, the job occupies that machine until it is finished.

$\beta_2 \in \{ prec, tree, chain, \phi \}$

$\beta_2 = prec$: A general precedence relation $\prec$ exists between the jobs, that is, if $i \prec j$ (job $i$ precedes job $j$ ), then job $i$ must be completed before job $j$ can be started.

$\beta_2 = tree$: A precedence tree describes the precedence relation between jobs , that is each vertex in the associated graph has outdegree or indegree of at most one.

$\beta_2 = chain$: Precedence constraints between jobs are of chain-type where each vertex in the associated graph has outdegree and indegree of at most one.

$\beta_2 = \phi$ : There is no precedence relation for the jobs; jobs are independent.

$\beta_3 \in \{ r_j, \phi \}$

$\beta_3 = r_j$: Jobs have release dates.

$\beta_3 = \phi$: $r_j=0$, ( $j=1,2,\ldots,n$ ); all jobs are released at the same time.

$\beta_4 \in \{ \tilde{d}_j, \phi \}$

$\beta_4 = \tilde{d}_j$: Jobs have deadlines.

$\beta_4 = \phi$: No deadlines are specified.

$\beta_5 \in \{ p_{ij} = 1, pl \le p_{ij} \le pu, \phi \}$

$\beta_5 = p_{ij} = 1$: Each operation has a unit processing time.

$\beta_5 = pl \le p_{ij} \le pu$: Processing times are bounded below by $pl$ and above by $pu$ .

$\beta_5 = \phi$: No bounds on processing times.

$\beta_6 \in \{ s_f, \phi \}$

$\beta_6 = s_f$ : There are sequence independent family set-up times, jobs are subdivided into families and a set-up time is incurred whenever there is a switch from processing a job in a family to a job in another family

$\beta_6 = \phi$ : There are no set-up times.

## 4.4 Optimality Criteria

The third field $\gamma$ defines the optimality criterion or the objective, the value which is to be optimized (minimized). Given a schedule, the following can be computed for each job $j$:

$C_j$ The completion time, the time at which the processing of job $j$ is completed.

$F_j$ The flow time, the time job $j$ spends in the system, $F_j = C_j - r_j$.

$L_j$ The lateness, $L_j = C_j - d_j$, the amount of time by which the completion time of job j exceed its due date. Lateness can be negative if job j finishes earlier than its due date.

$T_j$ The tardiness, $T_j = max\ \{L_j, 0\}$.

$E_j$ The earliness, $E_j = max\ \{-L_j, 0\}$.

$U_j$ The unit penalty, a unit penalty of job $j$ if it fails to meet its deadline.

$U_j = 0$ if $C_j \leq d_j$ , $U_j = 1$ otherwise.

The cost $f_j$ for each job j usually takes one of the variables described above or the product of the weight $w_j$ with one of the variables. The optimality criterion can be any function of the costs $f_j, j = 1, \ldots, n$. Common optimality criteria are usually in the form:

1. $f = f_{\max} = \max\{f_j | j = 1,\ldots,n\}$.
2. $f = \sum f_j$ .

The following objective functions have frequently been chosen to be minimized.

$f = \Sigma(w_j)C_j$ : The total (weighted) completion time.

Introducing due dates $d_j$ ( *j=1,…,n* ) we have the following objective functions:

$f = C_{max}$ : The maximum completion time (makespan)

$f = L_{max} = \underset{j}{max}\{L_j\}$: The maximum lateness.

$f = T_{max} = \underset{j}{max}\{T_j\}$: The maximum tardiness.

$f = \Sigma T_j$ : The total tardiness.

$f = \Sigma U_j$ : The total number of late jobs.

We may also choose to minimize:

$f = \sum w_j T_j$ : The total weighted tardiness.

$f = \sum w_j U_j$ : The total weighted number of late jobs.

$f = \sum w_j E_j$ : The total weighted earliness.

**Example (4.1):**

$1/r_j / \sum w_j C_j$ is the problem of minimizing the total weighted completion time on single machine subject to non-trivial release date.

$P3/pmtn, prec / L_{max}$ is the problem of minimizing maximum lateness on three identical parallel machines subject to general precedence constraint, allowing preemption.

**Example (4.2):**

Consider the following schedule:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 7 | 3 | 2 | 9 | 5 | 1 | 2 | 6 |
| $d_j$ | 5 | 13 | 20 | 5 | 30 | 21 | 29 | 25 |

Then to calculate the total completion time, maximum lateness, total earliness, total tardiness, and the total number of late jobs:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 7 | 3 | 2 | 9 | 5 | 1 | 2 | 6 |
| $d_j$ | 5 | 13 | 20 | 5 | 30 | 21 | 29 | 25 |
| $C_j$ | 7 | 10 | 12 | 21 | 26 | 27 | 29 | 35 |
| $L_j$ | 2 | $-3$ | $-8$ | 16 | $-4$ | 6 | 0 | 10 |
| $E_j$ | 0 | 3 | 8 | 0 | 4 | 0 | 0 | 0 |
| $T_j$ | 2 | 0 | 0 | 16 | 0 | 6 | 0 | 10 |

Then: $\sum C_j = 7 + 10 + 12 + 21 + 26 + 27 + 29 + 35 = 167, L_{max} = 16, \sum E_j = 15, \sum T_j = 34, \sum U_j = 4.$

## 4.5 Single Machine Scheduling Problems

### 4.5.1 $1 / / \sum C_j$ Problem

This is the problem of sequencing $n$ jobs on a single machine to minimize the total completion time. This problem is solved by the SPT (shortest processing time) rule. The jobs are sequenced in non-decreasing order of processing times $P_j$.

### Example (4.3):

Solve the following $1//\sum C_j$ problem:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 7 | 3 | 2 | 9 | 5 | 1 | 2 | 6 |

To minimize $\sum C_j$ , we use the SPT rule as follows:

| $j$ | 6 | 3 | 7 | 2 | 5 | 8 | 1 | 4 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 1 | 2 | 2 | 3 | 5 | 6 | 7 | 9 |
| $C_j$ | 1 | 3 | 5 | 8 | 13 | 19 | 26 | 35 |

Then by SPT rule: $\sum C_j = 1 + 3 + 5 + 8 + 13 + 19 + 26 + 35 = 110$. That is the optimal schedule is s= (6,3,7,2,5,8,1,4) with $\sum C_j = 110$.

## 4.5.2 $1\,/\,/\sum w_jC_j$ Problem

This is the problem of sequencing $n$ jobs on a single machine to minimize the weighted total completion time. This problem is solved by the SWPT (shortest weighted processing time) rule. The jobs are sequenced in non-decreasing order of processing times $P_j/w_j$.

### Example (4.4):

Consider the following schedule:

| $j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P_j$ | 6 | 10 | 12 | 18 | 4 |
| $w_j$ | 2 | 4 | 3 | 3 | 4 |

To minimize $\sum w_jC_j$ , we must first find $P_j/w_j$ for each job $j$:

| $j$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $P_j$ | 6 | 10 | 12 | 18 | 4 |
| $w_j$ | 2 | 4 | 3 | 3 | 4 |
| $P_j/w_j$ | 3 | 2.5 | 4 | 6 | 1 |

Then, use the SWPT rule as follows:

| $j$ | 5 | 2 | 1 | 3 | 4 |
|---|---|---|---|---|---|
| $P_j/w_j$ | 1 | 2.5 | 3 | 4 | 6 |
| $P_j$ | 4 | 10 | 6 | 12 | 18 |
| $w_j$ | 4 | 4 | 2 | 3 | 3 |
| $C_j$ | 4 | 14 | 20 | 32 | 50 |
| $w_jC_j$ | 16 | 56 | 40 | 96 | 150 |

Then by SWPT: $\sum w_jC_j = 358$. That is the optimal schedule is s= (5,2,1,3,4) with $\sum w_jC_j = 358$ ( $\sum w_jC_j = 498$ for the original sequence).

### 4.5.3 1 / / $L_{max}$ Problem

This is the problem of sequencing $n$ jobs on a single machine to minimize the maximum lateness. This problem is solved by the EDD (earliest due date) rule. The jobs are sequenced in non-decreasing order of due dates $d_j$.

**Example (4.5):**

Consider the following schedule:

| $j$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $P_j$ | 4 | 5 | 3 | 2 |
| $d_j$ | 7 | 8 | 5 | 4 |

To minimize $L_{max}$ we use the EDD rule:

| $j$ | 4 | 3 | 1 | 2 |
|---|---|---|---|---|
| $P_j$ | 2 | 3 | 4 | 5 |
| $d_j$ | 4 | 5 | 7 | 8 |
| $C_j$ | 2 | 5 | 9 | 14 |
| $L_j$ | $-2$ | 0 | 2 | 6 |

$\therefore L_{max} = 6$(for the original schedule $L_{max} = 10$). The optimal schedule is s = (4,3,1,2) with $L_{max} = 6$.

### 4.5.4 1 / / $\sum U_j$ Problem

This is the problem of sequencing $n$ jobs on a single machine to minimize the number of late jobs (minimize the total unit penalties). This problem is solved by Moore algorithm. Let $E$ denote the set of early jobs and $L$ denote the set of late jobs. The jobs of $E$ are sequenced in EDD rule followed by the jobs of $L$.

## Moore (and Hodgson) Algorithm

***Step 1:*** Number the jobs in EDD order. Set $E = \phi\ , L = \phi\ ,\ k = 0, t = 0$.

***Step 2:*** Let $k = k + 1$. If $k > n$ go to step 4.

***Step 3:*** Let $t = t + P_k$ and $E = E \cup \{k\}$. If $t \leq d_k$ go to step 2. If $t > d_k$ , find $j \in E$ with $P_j$ as large as possible and let $t = t - P_j\ , E = E - \{j\}, L = L \cup \{j\}$. Go to step 2.

***Step 4:*** $E$ is the set of early jobs and $L$ is the set of late jobs.

## Example (4.6):

Minimize $\sum U_j$ for the following schedule:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 5 | 3 | 1 | 8 | 4 | 7 | 5 | 3 |
| $d_j$ | 12 | 32 | 10 | 18 | 23 | 27 | 15 | 24 |

To minimize $\sum U_j$ we use Moore algorithm:

| $j$ | 3 | 1 | 7 | 4 | 5 | 8 | 6 | 2 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 1 | 5 | 5 | 8 | 4 | 3 | 7 | 3 |
| $d_j$ | 10 | 12 | 15 | 18 | 23 | 24 | 27 | 32 |
| $C_j$ | 1 | 6 | 11 | 19 | | | | |
| $C_j$ | 1 | 6 | 11 | * | 15 | 18 | 25 | 28 |

$\therefore \sum U_j = 1, E = \{3,1,7,5,8,6,2\}, L = \{4\}$. The optimal schedule is: s= (3,1,7 ,5,8,6,2,4) (in the original schedule $\sum U_j = 3$) .

## Example (4.7):

Minimize $\sum U_j$ for the following schedule:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 4 | 2 | 7 | 6 | 4 | 7 | 5 | 5 |
| $d_j$ | 12 | 27 | 10 | 15 | 30 | 22 | 8 | 28 |

**Solution:**

To minimize $\sum U_j$ we use Moore's algorithm:

| $j$ | 7 | 3 | 1 | 4 | 6 | 2 | 8 | 5 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 5 | 7 | 4 | 6 | 7 | 2 | 5 | 4 |
| $d_j$ | 8 | 10 | 12 | 15 | 22 | 27 | 28 | 30 |
| $C_j$ | 5 | 12 | | | | | | |
| $C_j$ | 5 | * | 9 | 15 | 22 | 24 | 29 | |
| $C_j$ | 5 | * | 9 | 15 | * | 17 | 22 | 26 |

**Remark:** 5th job (Job 6) is selected although it is early since it has the greatest $P_j$ among all jobs in $E$.

$\therefore \sum U_j = 2, E = \{7,1,4,2,8,5\}, L = \{3,6\}$.The optimal schedule is: s=(7,1,4,2,8,5,3,6). Also, s=(7,1,4,2,8,5,6,3) is an optimal schedule.

## Example (4.8):

Minimize $\sum U_j$ for the following schedule:

| $j$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 4 | 3 | 1 | 5 | 2 | 3 | 1 | 3 |
| $d_j$ | 7 | 6 | 4 | 7 | 9 | 6 | 4 | 5 |

**Solution:**

To minimize $\sum U_j$ we use Moore's algorithm:

| $j$ | 3 | 7 | 8 | 2 | 6 | 1 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|
| $P_j$ | 1 | 1 | 3 | 3 | 3 | 4 | 5 | 2 |
| $d_j$ | 4 | 4 | 5 | 6 | 6 | 7 | 7 | 9 |
| $C_j$ | 1 | 2 | 5 | 8 | | | | |
| $C_j$ | 1 | 2 | * | 5 | 8 | | | |
| $C_j$ | 1 | 2 | * | * | 5 | 9 | | |
| $C_j$ | 1 | 2 | * | * | 5 | * | 10 | |
| $C_j$ | 1 | 2 | * | * | 5 | * | * | 7 |

$\therefore \sum U_j = 4, E = \{3,7,6,5\}, L = \{8,2,1,4\}$.The optimal schedule is: s=(3,7,6,5,8,2,1,4).

# Ch.5 Inventory Models

The inventory deals with stocking an item to meet fluctuations in demand. The inventory problem involves placing and receiving orders of given sizes periodically. The basis for the decision is a model that balances the cost of capital resulting from holding too much inventory against the penalty cost resulting from inventory shortage. The problem reduces to controlling the inventory level by devising an ***inventory policy*** that answers two questions:

1. How much to order?
2. When to order?

The basis for answering these questions is the minimization of the following inventory cost function:

$$\begin{pmatrix} Total \\ inventory \\ cost \end{pmatrix} = \begin{pmatrix} Purchasing \\ cost \end{pmatrix} + \begin{pmatrix} Setup \\ cost \end{pmatrix} + \begin{pmatrix} Holding \\ cost \end{pmatrix} + \begin{pmatrix} Shortage \\ cost \end{pmatrix}$$

*1.* ***Purchasing cost*** is the price per unit of an inventory item. At times the item is offered at a discount if the order size exceeds a certain amount, which is a factor in deciding how much to order.

*2.* ***Setup cost*** represents the fixed charge incurred when an order is placed regardless of its size. This includes salaries, transportation cost, insurance, etc.

*3.* ***Holding cost*** represents the cost of maintaining inventory in stock. It includes the interest on capital, the cost of storage, maintenance, and handling.

*4.* ***Shortage cost*** is the penalty incurred when we run out of stock. It includes potential loss of income, disruption in production, and the more subjective cost of loss in customer's goodwill.

An inventory system may be based on ***periodic review*** (e.g., ordering every week or every month). Alternatively, the system may be based on ***continuous review***, where a new order is placed when the inventory level drops to a certain level, called the ***reorder point***.

## 5.1 Role of Demand in the Development of Inventory Models

In general, the analytic complexity of inventory models depends on whether the demand for an item is deterministic or probabilistic. Within either category, the demand may or may not vary with time. For example, the consumption of natural gas used in heating homes is seasonal. Though this seasonal pattern repeats itself annually, the same-month consumption may vary from year to year, depending, for example, on the severity of weather.

In practical situations the demand pattern in an inventory model may assume one of four types:

1. Deterministic and constant (static) with time.
2. Deterministic and variable (dynamic) with time.
3. Probabilistic and stationary over time.
4. Probabilistic and non-stationary over time.

This categorization assumes the availability of data that are representative of future demand. Demand is usually probabilistic, but in some cases the simpler deterministic approximation may be acceptable. The complexity of the inventory problem does not allow the development of a general model that covers all possible situations.

## 5.2 Static Economic-Order-Quantity (EOQ) Models

### 5.2.1 Classic EOQ Model (Constant-Rate Demand, no Shortage)

The simplest of the inventory models involves constant-rate demand with instantaneous order replenishment and no shortage. Define:

$y$ = Order quantity (number of units)

$D$ = Demand rate (units per unit time)

$t_0$ = Ordering cycle length (time units)

The inventory level follows the pattern explained if Figure (5.1). When the inventory reaches zero level, an order of size $y$ units is received instantaneously. The stock is then depleted uniformly at the constant demand rate $D$.

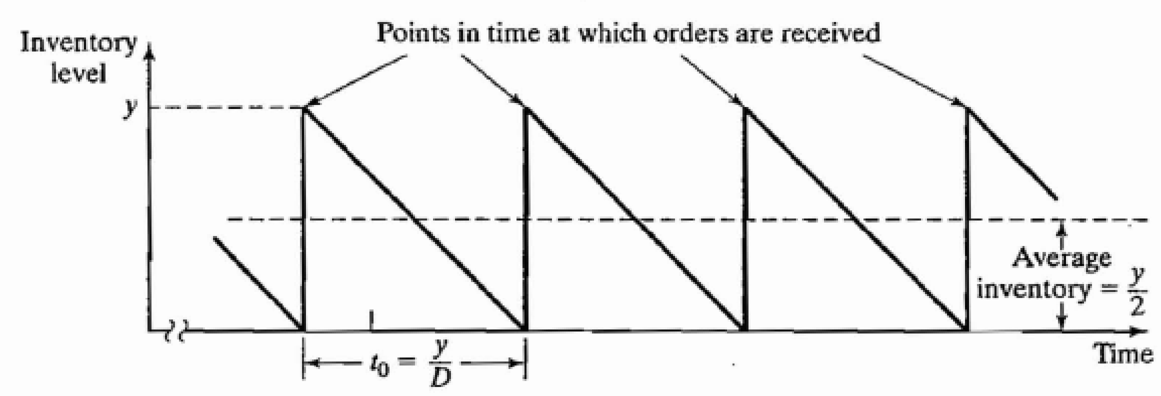


**Figure (5.1)**

The ordering cycle for this pattern is:

$$t_0 = \frac{y}{D}\ time\ units$$

The cost model requires two cost parameters:

$K$ = Setup cost associated with the placement of an order (monetary units per order)

$h$ = Holding cost (monetary units per inventory unit per unit time)

Given that the average inventory level is $\frac{y}{2}$, the total ***cost per unit time*** (***TCU***) is thus computed as

$TCU(y)$ = Setup cost per unit time + Holding cost per unit time

$$= \frac{Setup\ cost + Holding\ cost\ per\ cycle\ t_0}{t_0}$$

$$= \frac{K+h(\frac{y}{2})t_0}{t_0} = \frac{K}{(\frac{y}{D})} + h\left(\frac{y}{2}\right) = \frac{KD}{y} + \frac{hy}{2}$$

The optimum value of the order quantity $y$ is determined by minimizing $(y)$ . Assuming $y$ is continuous, a necessary condition for optimality is*:*

$$\frac{dTCU(y)}{dy} = -\frac{KD}{y^2} + \frac{h}{2} = 0$$

The condition is also sufficient because $TCU(y)$ is convex.

The solution of the equation yields the $EOQ\ \ y^*$ as

$$y^* = \sqrt{\frac{2KD}{h}}$$

Thus, the optimum inventory policy for the proposed model is

$$Order\ y^* = \sqrt{\frac{2KD}{h}}\ units\ every\ t_o^* = \frac{y^*}{D}\ time\ units$$

Actually, a new order need not be received at the instant it is ordered. Instead, a positive ***lead time***, $L$, may occur between the placement and the receipt of an order. In this case, the ***reorder point*** occurs when the inventory level drops to $LD$ units. Sometimes, it is assumed that the lead time $L$ is less than the cycle length $t_0^*$ ,which may not be the case in general. To account for this situation, we define the ***effective lead time*** as

$$L_e = L - nt_0^*$$

where $n$ is the largest integer not exceeding $\frac{L}{t_0^*}$ . The reorder point occurs at $L_eD$ units, and the inventory policy can be restated as:

Order the quantity $y^*$ whenever the inventory level drops to $L_eD$ units

## Example (5.1):

Neon lights on the U of A campus are replaced at the rate of 100 units per day. The physical plant orders the neon lights periodically. It costs 100 $ to initiate a purchase order. A neon light kept in storage is estimated to cost about 0.02 $ per day. The lead time between placing and receiving an order is 12 days. Determine the optimal inventory policy for ordering the neon lights.

### Solution:

From the data of the problem, we have:

$D$ = 100 units per day

$K$ = 100 $ per order

$h$ = 0.02 $ per unit per day

$L$ = 12 days

Thus,

$$y^* = \sqrt{\frac{2KD}{h}} = \sqrt{\frac{2\times100\times100}{0.02}} = 1000 \text{ neon light}$$

The associated cycle length is:

$$t_0^* = \frac{y^*}{D} = \frac{1000}{100} = 10 \; days$$

Because the lead time $L = 12$ days exceeds the cycle length $t_0^*$ (= 10 days) , we must compute $L_e$. The number of integer cycles included in $L$ is

$$n = \left(Largest\ integer\ \leq \frac{L}{t_0^*}\right) = \left(Largest\ integer\ \leq \frac{12}{10}\right) = 1$$

Thus,

$$L_e = L - nt_0^* = 12 - 1\times10 = 2 \text{ days}$$

The reorder point thus occurs when the inventory level drops to

$$L_eD = 2\ X\ 100 = 200 \text{ neon lights}$$

The inventory policy for ordering the neon lights is:

*Order 1000 units whenever the inventory level drops to 200 units.*

The daily inventory cost associated with the proposed inventory policy is:

$$TCU(y) = \frac{KD}{y} + \frac{hy}{2} = \frac{100\times100}{1000} + 0.02\left(\frac{1000}{2}\right) = 20\ \$/\ \text{day}$$

## Exercise 5.1 (in addition to text book exercises)

A carpenter orders 48000 unit of an item yearly. The order costs 800$ and the holding cost is 10 cents per item monthly. The lead time between placing and receiving an order is 4 month. Determine the optimal inventory policy.

### 5.2.2 Manufacturing Model, no Shortage

In previous discussed models we have assumed that the replenishment time is zero and the items are procured in one lot. But in real practice, particularly in manufacturing model, items are produced on a machine at a finite rate per unit of time; hence we cannot say the replenishment time as zero. Here we assume that the replenishment rate is finite say at the rate of $\alpha$ units per unit of time. Let:

$y =$ Order quantity (number of units)

$D =$ Demand quantity (units per unit time)

$P =$ Production quantity (units per unit time) $(P > D)$

$t_0 =$ Ordering cycle length (time units)

$K =$ Setup cost associated with the placement of an order (monetary units per order)

$h =$ Holding cost (monetary units per inventory unit per unit time)

Figure (5.2) shows variation of inventory with time

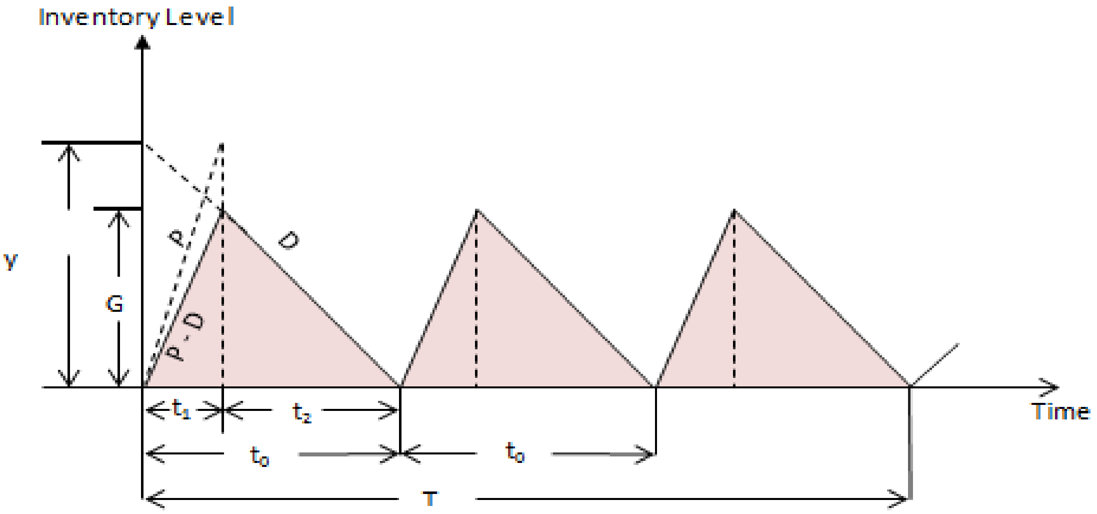


**Figure (5.2)**

Here each production run of length $t$ consists of two parts $t_1$ and $t_2$ , where:

i) $t_1$ is the time during which the stock is building up at a rate $P - D$ units per unit time.

ii) $t_2$ is the time during which there is no production (for supply or replenishment) and inventory is decreasing at a constant demand rate $D$ per unit time.

Let $G =$ the maximum inventory available at the end of time $t_1$ which is expected to be consumed during the remaining period $t_2$ at the demand rate D.

$TCU(y)/circle \; =$ Setup cost per unit time + Holding cost per unit time

$$= K + h\frac{G}{2}t_1 + h\frac{G}{2}t_2 = K + h\frac{G}{2}(t_1 + t_2)$$

Since $t_0 = t_1 + t_2$ , then:

$$TCU(y)/circle \; = K + h\frac{G}{2}t_0$$

From right-angled triangle: $t_1 = \frac{G}{P-D} \Longrightarrow G = t_1(P - D)$

$\Longrightarrow G = \frac{y}{P}(P - D) = y(1 - \frac{D}{P})$

Let: $b = 1 - \frac{D}{P}$, then $G = yb$

$\Longrightarrow TCU(y)/circle = K + h\frac{yb}{2}t_0$

$\Longrightarrow TCU(y) = \frac{K}{t_0} + h\frac{yb}{2} = \frac{KD}{y} + h\frac{yb}{2}$

The optimum value of the order quantity $y$ is determined by minimizing $(y)$ . Assuming $y$ is continuous, a necessary condition for optimality is*:*

$$\frac{dTCU(y)}{dy} = -\frac{KD}{y^2} + \frac{hb}{2} = 0$$

*Then:* $y^* = \sqrt{\frac{2KD}{hb}}$

Thus, the optimum inventory policy for the proposed model is

$Order\ y^* = \sqrt{\frac{2KD}{hb}}\ units\ whenever\ the\ inventory\ level\ drops\ to\ G^* = y^*b$

## Example (5.2):

A manufacturer must supply 10000 units of an item to a car factory daily. He can produce 25000 units daily; the holding cost of each unit is 2 cents per year and the fixed cost of production is 18 $. Determine the optimal number of produced items (no shortage) then find the total inventory cost for a year and the optimum inventory policy.

**Solution:**

From the data of the problem, we have:

D=10000 units per day

P=25000 units per day

h=0.02/360 $ per day

K=18 $ per cycle

$$b = 1 - \frac{D}{P} = 1 - \frac{10000}{25000} = \frac{3}{5}$$

$$y^* = \sqrt{\frac{2KD}{hb}} = \sqrt{\frac{2 \times 18 \times 10000}{\frac{0.02}{360} \times \frac{3}{5}}} = 10400\, unit$$

$$cost = \frac{KD}{y^*} + h\frac{y^*b}{2} = \frac{18 \times 10000}{10400} + \frac{0.02}{360} \times \frac{10400 \times \frac{3}{5}}{2} = 1224\ \$/\text{day}$$

$Then\ cost\ per\ year = 1224 \times 360 = 440640\ \$$

$G = y^*b = 10400 \times \frac{3}{5} = 6240\ unit$ . Then the optimal inventory policy is: Produce 10400 units when the inventory level drops to 6240 unit.

## Exercise 5.2 (in addition to text book exercises)

A company has a demand of 12000 units / year for an item and it can produce 2000 such items per month. The cost of one setup is 400 $ and the holding cost / unit / month is 0.15 $. Find the optimum lot size and the total cost per year.

## 5.3 Probabilistic inventory models

The models previously discussed are only artificial since in practical situations demand is hardly known precisely. In most situations demand is probabilistic since only probability distribution of future demand, rather than the exact value of demand itself, is known. The probability distribution of future demand is usually determined from the data collected from past experience. In such situations we choose policies that minimize the expected costs rather than the actual costs.

### 5.3.1 Instantaneous Demand, Setup Cost Zero, Stock Levels Discrete and Lead Time Zero

This model deals with the inventory situation of items that require one time purchase only. Perishable items such that cut flowers, cosmetics, spare parts, seasonal items such as calendars and diaries, etc. fall under this category.

In this model the item is ordered at the beginning of the period to meet the demand during that period, the demand being instantaneous as well as discrete in nature. At the end of the period, there are two types of cost involved: over-stocking cost and under-stocking cost. They represent opportunity losses incurred when the number of units stocked is not exactly equal to the number of units actually demanded. Let:

$D$ = Discrete demand rate with probability $P_D$

$y_m$ = Discrete stock level for time interval $t_0$

$t_0$ = Ordering cycle length

$C_1$ = Over-stocking cost (over-ordering cost). This is opportunity loss associated with each unit left unsold.

$= C + C_h - V$

$C_2$ = Under-stocking cost (under-ordering cost). This is opportunity loss due to not meeting the demand.

$= S - C - C_h/2 + C_s$

Where $C$ is the unit cost price, $C_h$ the unit carrying (holding) cost, $C_s$ the unit shortage cost, $S$ the unit selling price and $V$ is the salvage value. If value of any parameter is not given, it is taken as zero.

Production is assumed to be instantaneous and lead time is negligibly small. The problem is to determine the optimal inventory level $y_m$, where $D \leq y_m$(there is no shortage) or $D > y_m$ ( shortage occur).

Then the optimal order quantity $y_m^*$ is determined when value of cumulative probability distribution exceeds the ratio $\frac{C_2}{C_1+C_2}$ by computing:

$$P_{D \leq y_m - 1} \leq \frac{C_2}{C_1 + C_2} \leq P_{D \leq y_m}$$

## Example (5.3):

A trader stocks a particular seasonal product at the beginning of the season and cannot reorder: the item costs him 25 $ and he sells it at 50 $ each. For any item that cannot be met on demand, the trader has estimated a goodwill cost of 15 $. Any item unsold will have a salvage value of 10 $. Holding cost during the period is estimated to be 10 % of the price. The probability of demand is as follows:

| Units stocked | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Probability of demand | 0.35 | 0.25 | 0.20 | 0.15 | 0.05 |

Determine the optimal number of items to be stocked.

### Solution:

Here: $= 25\$, S = 50\$, C_h = 0.10 \times 25 = 2.5\$, C_s = 15\$, V = 10\$$ .

$\therefore\ C_1 = C + C_h - V = 25 + 2.5 - 10 = 17.5\ \$$

$C_2 = S - C - \frac{C_h}{2} + C_s = 50 - 25 - \frac{2.5}{2} + 15 = 38.75\ \$$

Cumulative probability of demand is now calculated:

| Units stocked | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Probability of demand | 0.35 | 0.25 | 0.20 | 0.15 | 0.05 |
| Cumulative probability of demand $\sum_{D=0}^{y_m} P_D$ | 0.35 | 0.60 | 0.80 | 0.95 | 1.00 |

Now: $\frac{C_2}{C_1+C_2} = \frac{38.75}{17.5+38.75} = 0.69.$

Since $0.60 < 0.69 < 0.80,$ then $3 < y_m < 4$. Then $y_m^* = 4$ units.

## Example (5.4):

A newspaper boy buys papers for 5 ¢ each and sells them for 6 ¢ each. He cannot return unsold newspapers. Daily demand D for newspapers follows the distribution:

| $D$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|
| $P_D$ | 0.05 | 0.15 | 0.40 | 0.20 | 0.10 | 0.05 | 0.05 |

If each day's demand is independent of the previous day's, how many papers should be ordered each day?

**Solution:**

Here: $= 0.05\$, S = 0.06\$, C_h = 0\ \$, C_s = 0\ \$, V = 0\ \$$.

$\therefore\ C_1 = C + C_h - V = 0.05\ \$$

$C_2 = S - C - \frac{C_h}{2} + C_s = 0.06 - 0.05 = 0.01\ \$$

Cumulative probability of demand is now calculated:

| $D$ | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|
| $P_D$ | 0.05 | 0.15 | 0.40 | 0.20 | 0.10 | 0.05 | 0.05 |
| $\sum_{D=0}^{y_m} P_D$ | 0.05 | 0.20 | 0.60 | 0.80 | 0.90 | 0.95 | 1.00 |

Now: $\frac{C_2}{C_1+C_2} = \frac{0.01}{0.01+0.05} = \frac{1}{6} = 0.167.$

Since $0.05 < 0.167 < 0.20,$ then $10 < y_m < 11$. Then $y_m^* = 11$ newspapers.

### 5.3.2 Instantaneous Demand, Setup Cost Zero, Stock Levels Continuous and Lead Time Zero

In this model, all conditions are the same as model in 5.3.1 except that the stock levels are continuous. Therefore, probability $f(D)dD$ will be used instead of $P_D$, where f$(D)$ is the probability density function of the demand rate $D$.

Then the optimal order quantity $y_m^*$ is determined when value of cumulative probability distribution exceeds the ratio $\frac{C_2}{C_1+C_2}$ by computing:

$$\int_{D=0}^{y_m} f(D)dD = \frac{C_2}{C_1 + C_2}$$

## Example (5.5):

A baking company sells one of its types of cakes by weight. It makes profit of 95 ¢ a pound on every pound of cake sold on the day it is baked. It disposes all cakes not sold on the day they are baked at loss of 15 ¢ a pound. If demand is known to have a probability density function:

$$f(D) = 0.03 - 0.0003\,D$$

Find the optimum amount of cake the company should bake daily.

**Solution:**

Penalty cost / unit of oversupply, $C_1 = 0.15$ \$

Penalty cost / unit of undersupply, $C_2 = 0.95$ \$

Using the relation: $\int_{D=0}^{y_m} f(D)dD = \frac{C_2}{C_1+C_2}$, we get:

$$\int_0^{y_m}(0.03 - 0.0003D)dD = \frac{0.95}{0.15+0.95} = \frac{0.95}{1.1} = 0.8636$$

$$0.03y_m - 0.00015y_m^2 = 0.8636 \qquad (\times 10^5)$$

$$3000y_m - 15y_m^2 = 86360 \qquad (\div 15)$$

$$200y_m - y_m^2 = 5757$$

$$y_m^2 - 200y_m + 5757 = 0,$$

$$y_m = \frac{200 \pm \sqrt{(200)^2 - 4\times 5757}}{2} = 165.15 \text{ or } 34.84 \text{ pounds}$$

$y_m = 165.15$ pounds is not feasible since the given probability distribution of D is not applicable above 100 pounds.

$\therefore\ y_m^* = 34.85$ pounds

## Exercise 5.3 (in addition to text book exercises)

1: The probability distribution of monthly sales of certain item is as follows:

| Monthly sales | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| Probability | 0.01 | 0.06 | 0.25 | 0.35 | 0.20 | 0.03 | 0.10 |

The cost of carrying inventory is 30 \$ per unit per month and the cost of unit shortage is 70 \$ per month. Determine the optimum stock level which minimizes the total expected cost.

2: A baking company sells one of its types of cakes by weight. It makes profit of 50 ¢ a pound on every pound of cake sold on the day it is baked. It disposes all cakes not sold on the day they are baked at loss of 12 ¢ a pound. If the demand is known to be rectangular between 2000 and 3000 pounds, determine the optimum daily amount baked.( In a rectangular(or uniform) distribution all values within a range between a and b are equally likely. The probability density is: 1 / (b - a))