Algebraic Properties of Differentiable functions on $\mathbb{R}^n$. ①

In this part we study the algebraic properties of differentiable functions defined on $\mathbb{R}^n$ or on a subset of $\mathbb{R}^n$; for example, the addition of two differentiable functions and multiplication of differentiable function on $A \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$ by the differentiable function defined on A into $\mathbb{R}$ are still differentiable functions. Fortuntely, the answer is yes and to do this we have the following remarks.

Remarks

(i) If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation then T is bounded, i.e. $\exists M > 0$ such that $\| Tx \| \leq M \| x \|$ for all $\underset{\wedge}{x} \in \mathbb{R}^n$, the proof of this remark can be found in linear algebra books.

(ii) If $T_1, T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are linear transformation then $T_1 \pm T_2 : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T_1 \pm T_2 \, (x) = T_1(x) \pm T_2(x)$ is a linear transformation, too. To do this

$$\begin{aligned} T_1 \pm T_2 \, (\alpha x + \beta y) &= T_1 (\alpha x + \beta y) \pm T_2 (\alpha x + \beta y) \\ &= \alpha T_1(x) + \beta T_1(y) \pm (\alpha T_2(x) + \beta T_2(y)) \\ &= \alpha \, (T_1 \pm T_2)_{(x)} + \beta \, (T_1 \pm T_2)_{(y)} \end{aligned}$$

i.e. $T_1 \pm T_2$ is a linear.

Theorem 1

Let $A \subseteq \mathbb{R}^n$ and $\underset{\sim}{c}$ be an interior point of A. If $f, g : A \to \mathbb{R}^m$ are differentiable at $\underset{\sim}{c}$ then $f \pm g$ are differentiable at $\underset{\sim}{c}$ and $D(f \pm g)|_{\underset{\sim}{c}} = Df|_{\underset{\sim}{c}} \pm Dg|_{\underset{\sim}{c}}$.

Proof

1. Since $Df|_{\underset{\sim}{c}}$ and $Dg|_{\underset{\sim}{c}}$, the derivative of f and g at $\underset{\sim}{c}$ respectivly, are linear then $Df|_{\underset{\sim}{c}} \pm Dg|_{\underset{\sim}{c}}$ are linear, too.

2. $\forall \epsilon > 0 \quad \exists \delta_f(\underset{\sim}{c}, \epsilon) > 0$ such that

$$\| f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \| \le \frac{\epsilon}{2} \| \underset{\sim}{x} - \underset{\sim}{c} \| \quad \ldots\ldots \quad (1)$$

whenever $\| \underset{\sim}{x} - \underset{\sim}{c} \| \le \delta_f(\underset{\sim}{c}, \epsilon)$.

$\exists \delta_g(\underset{\sim}{c}, \epsilon) > 0$ such that

$$\| g(\underset{\sim}{x}) - g(\underset{\sim}{c}) - Dg|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \| \le \frac{\epsilon}{2} \| \underset{\sim}{x} - \underset{\sim}{c} \| \quad \ldots\ldots \quad (2)$$

whenever $\| \underset{\sim}{x} - \underset{\sim}{c} \| \le \delta_g(\underset{\sim}{c}, \epsilon)$.

Define $\delta = \min\{ \delta_f(\underset{\sim}{c}, \epsilon), \delta_g(\underset{\sim}{c}, \epsilon) \}$, for $\| \underset{\sim}{x} - \underset{\sim}{c} \| < \delta$ we have $\| \underset{\sim}{x} - \underset{\sim}{c} \| \le \delta_f(\underset{\sim}{c}, \epsilon)$ and $\| \underset{\sim}{x} - \underset{\sim}{c} \| \le \delta_g(\underset{\sim}{c}, \epsilon)$.

3. $\| (f \pm g)(\underset{\sim}{x}) - (f \pm g)(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} \pm Dg|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \|$

$= \| (f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} (\underset{\sim}{x} - c)) \pm (g(\underset{\sim}{x}) - g(\underset{\sim}{c}) - Dg|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \|$

$\le \| f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \| \pm \| g(\underset{\sim}{x}) - g(\underset{\sim}{c}) - Dg|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \|$

$\le \epsilon \| \underset{\sim}{x} - \underset{\sim}{c} \|$ by using (1) and (2)

$\therefore$ $f \pm g$ are differentiable at $\underset{\sim}{c}$ and $D f \pm g|_{\underset{\sim}{c}} = Df|_{\underset{\sim}{c}} \pm Dg|_{\underset{\sim}{c}}$. (3)

Remark

If $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation, $S : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a linear, $\gamma \in \mathbb{R}$ and $\underset{\sim}{z}_0 \in \mathbb{R}^m$ then

$$S(\,)\underset{\sim}{z}_0 + \gamma T(\,) : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad S(\,)\underset{\sim}{z}_0 + \gamma T(\,)_{(\underset{\sim}{x})} = S(\underset{\sim}{x})\underset{\sim}{z}_0 + \gamma T(\underset{\sim}{x})$$

is linear transformation. To do this

$$S(\,)\underset{\sim}{z}_0 + \gamma T(\,)_{(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})} = S(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})\underset{\sim}{z}_0 + \gamma T(\alpha \underset{\sim}{x} + \beta \underset{\sim}{y})$$

$$= (\alpha S(\underset{\sim}{x}) + \beta S(\underset{\sim}{y}))\underset{\sim}{z}_0 + \gamma(\alpha T(\underset{\sim}{x}) + \beta T(\underset{\sim}{y}))$$

$$= \alpha(S(\underset{\sim}{x})\underset{\sim}{z}_0 + \gamma T(\underset{\sim}{x})) + \beta(S(\underset{\sim}{y})\underset{\sim}{z}_0 + \gamma T(\underset{\sim}{y}))$$

$$= \alpha(S(\,)\underset{\sim}{z}_0 + \gamma T(\,))_{(\underset{\sim}{x})} + \beta(S(\,)\underset{\sim}{z}_0 + \gamma T(\,))_{(\underset{\sim}{y})}$$

i.e. $S(\,)\underset{\sim}{z}_0 + \gamma T(\,)$ is linear transformation.

Theorem 2

If $\varphi : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}^m$ are differentiable at $\underset{\sim}{c} \in A^\circ$ (interior of A) then $\varphi f : A \longrightarrow \mathbb{R}^m$, $\varphi f(\underset{\sim}{x}) = \varphi(\underset{\sim}{x}) f(\underset{\sim}{x})$ is differentiable at $\underset{\sim}{c}$ and $D\varphi f|_{\underset{\sim}{c}} = D\varphi|_{\underset{\sim}{c}}(\,) f(\underset{\sim}{c}) + \varphi(c)\, Df|_{c}(\,)$

Proof :

1. $\forall \epsilon > 0$ $\exists \delta_f(\underset{\sim}{c}, \epsilon) > 0$ such that

$$\| f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}}(\underset{\sim}{x} - \underset{\sim}{c}) \| \le \epsilon \| \underset{\sim}{x} - \underset{\sim}{c} \|$$

whenever $\| \underset{\sim}{x} - \underset{\sim}{c} \| \le \delta_f(\underset{\sim}{c}, \epsilon)$. .... (1)

f is continuous at $\underset{\sim}{c}$ i.e. $\exists \delta_f^c(\underset{\sim}{c}, \epsilon) > 0$

$\| f(\underset{\sim}{x}) - f(\underset{\sim}{c}) \| \leq \epsilon$ whenever $\| \underset{\sim}{x} - \underset{\sim}{c} \| \leq \delta_f^{*}(\underset{\sim}{c}, \epsilon)$ ..... (2)

and $\exists M > 0$ s.t $\| f(\underset{\sim}{x}) \| < M$ for $\| \underset{\sim}{x} - \underset{\sim}{c} \| \leq \delta_f^{*}(\underset{\sim}{c}, \epsilon)$.

2. $\exists \delta_\varphi(\underset{\sim}{c}, \epsilon) > 0$ such that $\| \varphi(\underset{\sim}{x}) - \varphi(\underset{\sim}{c}) - D\varphi|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \| \leq \epsilon \| \underset{\sim}{x} - \underset{\sim}{c} \|$

whenever $\| \underset{\sim}{x} - \underset{\sim}{c} \| \leq \delta_\varphi(\underset{\sim}{c}, \epsilon)$. ..... (4)

3. From remark we have $\varphi(\underset{\sim}{c}) Df|_{\underset{\sim}{c}} (\ ) + D\varphi|_{\underset{\sim}{c}} (\ ) f(\underset{\sim}{c})$

is linear transformation on $\mathbb{R}^n$ into $\mathbb{R}^m$.

4. $\delta = \min \{ \delta_f(\underset{\sim}{c}, \epsilon), \delta_f^{*}(c, \epsilon), \delta_\varphi(\underset{\sim}{c}, \epsilon) \} > 0$, $\| \underset{\sim}{x} - \underset{\sim}{c} \| \leq \delta$

$\| \varphi f(\underset{\sim}{x}) - \varphi f(\underset{\sim}{c}) - (\varphi(\underset{\sim}{c}) Df|_c (\ ) + D\varphi|_c (\ ) f(\underset{\sim}{c}))(\underset{\sim}{x} - \underset{\sim}{c}) \|$

$= \| \varphi(\underset{\sim}{x}) f(\underset{\sim}{x}) - \varphi(\underset{\sim}{c}) f(\underset{\sim}{c}) - (\varphi(c) Df|_c (\underset{\sim}{x} - \underset{\sim}{c}) - D\varphi|_c (\underset{\sim}{x} - \underset{\sim}{c}) f(\underset{\sim}{c})$

$= \| (\varphi(\underset{\sim}{x}) - \varphi(\underset{\sim}{c}) - D\varphi|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c})) f(\underset{\sim}{c}) + D\varphi|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \{ f(\underset{\sim}{x}) - f(\underset{\sim}{c}) \}$

$+ \varphi(x) \{ f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \} \|$

$\leq \| \varphi(\underset{\sim}{x}) - \varphi(\underset{\sim}{c}) - D\varphi|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \| \| f(\underset{\sim}{c}) \| + \| D\varphi|_c (\underset{\sim}{x} - \underset{\sim}{c}) \| \| f(\underset{\sim}{x}) -$

$+ \| \varphi(x) \| \| f(\underset{\sim}{x}) - f(\underset{\sim}{c}) - Df|_{\underset{\sim}{c}} (\underset{\sim}{x} - \underset{\sim}{c}) \|$

By using (1) ~~and~~, (2), (3) and (4) we get

$\| \varphi f(x) - \varphi f(\underset{\sim}{c}) - (\varphi(\underset{\sim}{c}) Df|_{\underset{\sim}{c}} (\ ) + D\varphi|_{\underset{\sim}{c}} (\ ) f(\underset{\sim}{c})) \| \leq k \epsilon \| \underset{\sim}{x} - \underset{\sim}{c} \|$

$\leq M_\varphi$

where $k = \| f(\underset{\sim}{c}) \| + M_1 + \| \varphi(\underset{\sim}{x}) \|$, where $M_1 > 0$ s.t

$\| D\varphi(\underset{\sim}{x} - \underset{\sim}{c}) \| \leq M_1 \| \underset{\sim}{x} - \underset{\sim}{c} \|$.

$\therefore \varphi f$ is differentiable at $\underset{\sim}{c}$ and $D\varphi f|_{\underset{\sim}{c}} (\ ) = \varphi(c) Df|_c (\ ) + D\varphi$