Chapter 1

Functions

In this chapter we review the definition of functions and their graphs, how they are combined and transformed, and ways they can be classified. We review the trigonometric functions, and we discuss misrepresentations that can occur when using calculators and computers to obtain a function's graph.

1.1 Functions and Their graphs

Functions are the major tools for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description; we will use all four representations throughout this book. This section reviews these function ideas.

Definition. *(Function)*

A function from a set D to a set Y is a rule that assigns a single (unique) element of Y to each element of D.

Definition. *(Domain and range of a function)*

Let f be a function from D to Y.

1. *D (the set of all possible input values) is called the domain of the function and is denoted by $Dom(f)$.*
2. *Y is called the codomain of the function f.*

3. *The set of all values $f(x)$ as x varies throughout D is called the range of the function and is denoted by $Ran(f)$.*

Remark. *The range of a function from D to Y may not include every element in the set Y. In other words, $Ran(f) \subseteq Y$.*

Example 1.

Notations.

1. *We will denote functions by $f, g, \cdots$.*
2. *If f is a function from a set D to a set Y and $x \in D$, then the value of x under f is denoted by $f(x)$.*
3. *To say that y is a function of x, we write $y = f(x)$.*

   x is called the independent variable.

   y is called the dependent variable.

Graphs of functions

The graph of a function f is the graph of the equation $y = f(x)$; that is the set of all points (x, y) whose coordinates satisfy the equation $y = f(x)$.

In order to graph functions, we make a table of xy–pairs that satisfy the equation $y = f(x)$, then we plot the points (x, y) whose coordinates appear in the table, and draw a smooth curve through the plotted points.

Example

Find the domain and range then graph the following functions:

Increasing and decreasing functions

Definition. *(Increasing and decreasing functions)*
Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. *f is said to be increasing on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.*
2. *f is said to be decreasing on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.*

Remarks.

1. *It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points x_1 and x_2 in I with $x_1 < x_2$.*
2. *Since we use $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is strictly increasing or decreasing on I.*
3. *The graph of an increasing function rises or climbs as you move from left to right while the graph of a decreasing functions descends or falls as you move from left to right.*

Example 1.

Example

Sketch the graph of the following functions, and determine the intervals of increasing and decreasing:

Even and odd functions

Definition. *(Even and odd functions)*

1. *A function* $y = f(x)$ *is an even function if* $f(-x) = f(x)$ *for every* $x \in Dom(f)$.

2. *A function* $y = f(x)$ *is an odd function if* $f(-x) = -f(x)$ *for every* $x \in Dom(f)$.

Example 1.

Example

a) $f(x)=x^2$

b) $f(x)=x$

c) $f(x)=x^2+1$

d) $f(x)=x+1$

Common Functions

A variety of important types of functions are frequently encountered in calculus. We briefly describe them here.

Polynomials and Rational Functions

Definition. *(Polynomials)*
Let n be a nonnegative integer and let $a_n,\ a_{n-1}, \cdots,\ a_1,\ a_0$ be real numbers (constants). Then a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

is a polynomial. $a_n,\ a_{n-1}, \cdots,\ a_1,\ a_0$ are the coefficients of the polynomial

Definition. *If* $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *is a polynomial such that* $a_n \neq 0$, *then* n *is called the degree of the polynomial.*

Remarks.

1. *If* $p(x)$ *is a polynomial function, then* $Dom(p) = \mathbb{R}$.
2. *A polynomial of degree* 0, $p(x) = a_0$, *is also called a constant function.*
3. *A polynomial of degree* 1, $p(x) = a_1 x + a_0$, *is also called a linear function.*
4. *A polynomial of degree* 2, $p(x) = a_2 x^2 + a_1 x + a_0$, *is also called a quadratic function.*

Definition. *(Rational functions)*

A rational function is a function of the form $f(x) = \dfrac{p(x)}{q(x)}$*, where* $p(x)$ *and* $q(x)$ *are polynomials.*

Remark. *The domain of a rational function* $f(x) = \dfrac{p(x)}{q(x)}$ *is* $\mathbb{R} - \{x : q(x) = 0\}$.

Power functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function.

Remarks.

1. *If* $a = n$ *is a positive integer, then* $Dom(f) = \mathbb{R}$.

   If n *is even, then* $Ran(f) = [0, \infty)$ *and if* n *is odd, then* $Ran(f) = \mathbb{R}$.

2. *If* $a = n$ *a negative integer, then* $Dom(f) = \mathbb{R} - \{0\}$.

3. *If* $a = \frac{n}{m}$ *is a rational number, then we apply the domain convention to find* $Dom(f)$.

Algebraic functions

Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots) is an algebraic function.

Trigonometric functions

The six basic trigonometric functions are reviewed in section 1.3.

Exponential and logarithmic functions

We study exponential and logarithmic functions in Calculus B.